Original Article

Valuation of reverse convertibles in the variance gamma economy

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ABSTRACT **Prior research on structured products has demonstrated that equity-linked notes (ELNs) sold to retail investors in initial public offerings are typically issued at above their fair market value. A particular type of ELN – reverse convertibles – embed down-and-in put options and offer investors relatively high coupon payments in exchange for bearing some of the downside risk of the equity underlying the note. We analytically study the magnitude of the overpricing of reverse convertibles – one of the most popular structured products on the market today – within a stochastic volatility model.We extend the current literature to include analytical valuation formulas within a model of stochastic volatility – the variance gamma (VG) model. We show that these complex notes are even more overpriced than previously estimated when stochastic volatility is taken into account. As a result of their complex payoffs and the lack of a secondary market to correct the mispricing,**

reverse convertible notes continue to be sold at prices substantially in excess of their fair market value.

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INTRODUCTION

In this article we present new valuation formulas for structured products with path-dependent payoffs incorporating stochastic volatility. The methodology used in this article could be applied to any structured product with an embedded barrier option to determine the impact of volatility skew on product valuation. We study reverse convertibles mainly because they are among the most popular structured products and because they provide insight into the pricing and sales practices of other equity-linked structured products.

Reverse convertibles are short-term notes whose principal repayment is linked to a stock, an index or a basket of stocks. If the reference security's price or level falls below a pre-specified level – called the 'trigger price' or 'protection level' – during the term of the note, investors may receive substantially less than the face-value of the notes. Reverse convertibles are fundamentally notes composed of a coupon paying bond and an embedded short put option.

Thus, while buy and hold investors in traditional short-term notes are only exposed to the issuer's credit risk, investors in reverse convertibles are also exposed to the risk of a decline in the price of the reference security. Investors in reverse convertibles are partially compensated for the risk of the embedded short put options with higher periodic coupons. The risk of these embedded options was realized by many investors in late 2008 and early 2009 as some notes matured after substantial stock market declines.

By valuing over 2000 reverse convertibles issued between 2001 and 2010, Deng *et al* (2010) report that the average fair value of the products was just 93 per cent of the offering price. Szymanowska *et al* (2009) and Hernández *et al* (2007) find a similar level of mispricing for this type of structured product. Henderson and Pearson (2010) estimate that investors who purchased an aggregate of $2 billion of short-term Stock Participation Accreting Redemption Quarterly Securities (SPARQS) reverse convertibles from Morgan Stanley in 69 offerings from 2001 to 2005 paid on average 8 per cent more than the securities' true value.

Regulators have been paying attention to these products as well. The Financial Industry Regulatory Authority (FINRA) Chairman Richard Ketchum commented, 'Reverse convertibles are complex investments which, like many structured products, often entail significant risk of loss. For the typical retail investor, for instance, it would be unwise to put a significant portion of life savings into riskier structured products such as reverse convertibles'.[1] FINRA followed by issuing an 'Investor Alert'[2] and a 'Regulatory Notice'[3] to highlight these concerns.

Reverse convertibles accounted for approximately 6.0 per cent ($2.34 billion) of SEC-registered structured notes in 2012, 12.0 per cent ($5.46 billion) in 2011 and 13.7 per cent ($6.76 billion) in 2010 according to Bloomberg Briefs. Reverse convertibles stand as third most popular type of SEC-registered notes, behind equity-linked notes (ELNs) and securities tied to rates.[4] Much like reverse convertibles, ELNs are debt instruments that are tied to the the underlying equity and expose investors to the issuer's default risk. Typically ELNs are principal-protected and the payoff at maturity depends upon the return of the underlying equity and a participation rate.[5] Rate-linked notes have a payoff that depend upon interest rate(s). These notes could be tied to the value of a particular interest rate, the difference of two interest rates or on the steepness of a given yield curve.

Asset return volatilities are stochastic (Engel, 2004). Analysis of asset return distributions suggests that infinite-activity jump specifications are well suited to capture the daily market value fluctuations of many financial assets (Wu, 2007). The VG model that we consider in this article is an infinite-activity pure jump process.[6] Though there are many models of stochastic volatility, we chose to work with the VG model because of its analytic tractability and the parsimony of its underlying model. The VG process can be thought of as a geometric Brownian motion with a gamma time-change and therefore could be seen as a generalization of the Black–Scholes (BS) model. The gamma time-change is also the most simple continuous state-space analog of the Poisson process (Jaimungal, 2004). The VG model is minimal as it leads to a characterization of the two most important inherent biases in the BS formula – lack of skewness and excess kurtosis – using only one extra parameter (see Black and Scholes, 1973 and Merton, 1976).

The VG model was first introducted in Madan and Seneta (1990) and was later generalized to the Carr, Geman, Madan, Yor (CGMY) model in Carr *et al* (2002) to include a set of models within a convenient parameterization. Recently, the VG model has been extended and applied to study corporate defaults (Fiorani *et al,* 2010). Closed-form solutions for European options in other stochastic volatility models have been presented in the literature. See, for example, a Heston stochastic volatility model (Heston, 1993), which assumes dynamic variance follows a Cox, Ingersoll, Ross (CIR) process. Madan and Milne (1991) price European call options in the symmetric VG economy for individuals with varying degrees of risk aversion. Fourier-based valuation methods on European call options have been applied to the VG model by Carr and Madan (1999).

Hernández *et al* (2007) present valuation formulas for several types of reverse convertibles with embedded barrier options within the BS model and find significant underpricing of these instruments. Baule and Tallau (2011) studied the valuation of a related type of structured product – bonus certificates – in the context of the Heston model. Bonus certificates are similar in structure to reverse convertibles. The main difference is that bonus certificates embeds a down-and-out put option rather than the down-and-in put option embedded in reverse convertibles. Baule and Tallau (2011) used simulations to price the bonus certificates under the Heston model. Lipton (2001) presents a closed-form valuation of barrier options using a Green's function approach within a special case of the Heston model where the asset price and volatility are assumed to be uncorrelated. There is no closed-form solution for barrier options in the generic Heston model. Several approaches to the valuation of barrier options can be found in the literature, including Wiener–Hopf factorization theory Kudryavtsev and Levendorskiĭ (2009), finite-difference methods Itkin and Carr (2010) and Fourier-based approaches (Fang and Oosterlee, 2011). For an overview of the valuation of exotic options within Lévy models in general, see Schoutens (2006).

The main contributions of our article are as follows. We derive the first closed-form valuation of reverse convertibles within the VG model. More precisely, we provide a valuation formula for the down-and-in (barrier) put option embedded within the reverse convertible. We first derive a reflection lemma specific to the stochastic volatility model being considered. The reflection lemma maps the path-dependent payoff for a barrier option to a path-independent payoff, conditional on a fixed gamma time. Using this reflection lemma, we derive a solution for the barrier option within VG stochastic volatility model. We isolate the pricing effect of the stochastic volatility assumption by comparing the results of the VG model and the BS model. We priced approximately 1800 reverse convertibles issued by several large investment banks in 2010 and 2011. We calibrated each model using options data collected for the underlying asset and analyzed the impact of volatility skew on the value of reverse convertibles. We find that estimates of the issue date overpricing in reverse convertibles increases when we incorporate stochastic volatility.

This article is organized as follows. We begin with a section introducing the VG model in detail, including the valuation of European call options. We then present our analytic valuations, based upon a new lemma we present, of path-dependent options within the VG model. We calibrate the model with options data and then price reverse convertibles using the calibrated model. The final section is reserved for our conclusions.

VG MODEL

Model specifications

The VG model is a pure-jump process that can also be written as a Brownian motion with a gamma time-change. The first author to consider the effect of discontinuous evolution of stock prices was Merton (1976). Monroe (1978) showed that any semimartingale has a representation as a time-changed Brownian motion. Monroe's result implies that the log-returns of the underlying asset are normally distributed with respect to financial time (which is positively correlated to business-activity time). The financial clock ticks more quickly than the observable clock in times of high business activity and ticks more slowly than the observable clock in times of low business activity (Jaimungal, 2004).

Let $b(t; \theta, \sigma)$ be a Brownian motion with drift θ and volatility σ and let $\gamma_t(1, \nu)$ be a gamma process with mean rate 1 and variance rate ν. The VG process can then be written as

$$X_t(\sigma, \nu, \theta) = b(\gamma_t(1, \nu); \theta, \sigma).$$

The spot price S_t at time t is given by the risk neutral process,

$$\begin{aligned} S_t = S_0 \exp(&X_t(\sigma, \nu, \theta) \\ &+ (r-q)t + (t/\nu)\log\Omega) \end{aligned}$$

where $\Omega = (1-\theta\nu-\sigma^2\nu)/2$. The probability density function for the stock price is given by,

$$\begin{aligned} f_{X_t}(X) = \int_0^\infty &\frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(X-\theta g)^2}{2\sigma^2 g}\right) \\ &\times \left(\frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}\right) dg \end{aligned} \tag{1}$$

where Γ is the usual gamma function. Conditioning on g, the distribution becomes a normal distribution with mean θg and variance $\sigma^2 g$.

The Fourier approach to option valuation within Lévy models has been advocated by Carr and Madan (1999). We present the valuation of European put options in the Appendix B and explain why this approach falls short in the valuation of path-dependent options considered in this article. The characteristic function, defined as the Fourier transform of the probability distribution function $\phi_{X_t}(u) = E[\exp(iuX_t)]$, is given by

$$\phi_{X_t}(u) = \left(1 - i\theta\nu u + \frac{\sigma^2\nu}{2}u^2\right)^{-t/\nu}. \qquad (2)$$

In the limit ν approaches zero, one recovers the characteristic function corresponding to the BS probability distribution.

The VG process can also be written as a difference of two independent increasing gamma processes,

$$X_t(\sigma,\nu,\theta) = \gamma_t^+(\mu_+,\nu_+) - \gamma_t^-(\mu_-,\nu_-),$$

where

$$\mu_\pm = 12\sqrt{\theta^2 + 2\frac{\sigma^2}{\nu}} \pm \theta/2,$$
$$\nu_\pm = (\mu_\pm)^2\nu.$$

The proof of these equations is trivial once one derives the characteristic function for the VG process. For an explicit proof, see Madan *et al* (1998). These formulae imply positive and negative jumps in the price of an underlying asset have separate distributions (arrive at different rates).

Explicit expressions for the first four central moments of the return distribution are given by,

$$E[X_t] = \theta t$$
$$E\left[(X_t - E[X_t])^2\right] = \left(\theta^2\nu + \sigma^2\right)t$$
$$E\left[(X_t - E[X_t])^3\right] = \left(2\theta^3\nu^2 + 3\sigma^2\theta\nu\right)t$$
$$E\left[(X_t - E[X_t])^4\right] = \left(3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3\right)t + \left(3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2\right)t^2.$$

Notice that the sign of the skewness of the returns distribution is determined by θ. Furthermore, the skewness is only non-zero for $\nu > 0$.[7] Kurtosis beyond that which is present in the BS model can also be accounted for through this model as shown above.

European call option valuation

We present the European call option valuation here as there is some disagreement in the literature about the correct formula.[8] To fix ideas, we are considering a European call option expiring T years from valuation with strike price K and continuously compounded (constant) risk-free rate r. The underlying asset has initial price S_0 and (constant) continuously compounded dividend yield q. Conditional on the gamma time g, the call option value can be computed by evaluating the integral

$$c(g) = \int_{-\infty}^{\infty} \left(e^{-rT}\max(S_T - K, 0)\right) \frac{\exp\left(-\frac{(X-\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}}\,dX = S_0 e^{-rT}\int_k^{\infty}\left(e^z - e^k\right)\frac{\exp\left(-\frac{(z-(r-q)T-(T/\nu)\log\Omega-\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}}\,dz$$

Where $z = \log(S_T/S_0)$ and $k = \log(K/S_0)$. The conditional call option value is given by

$$c(g) = S_0 e^{-qT}\Omega^{T/\nu}e^{(\theta+\sigma^2/2)g}\Phi\left(\frac{d}{\sqrt{g}} + \frac{\theta+\sigma^2}{\sigma}\sqrt{g}\right) - Ke^{-rT}\Phi\left(\frac{d}{\sqrt{g}} + \frac{\theta}{\sigma}\sqrt{g}\right), \qquad (3)$$

where

$$d = 1/\sigma\left[\log\left(S_0/K\right) + (r-q)T + \frac{T}{\nu}\log\,\Omega\right]$$

and Φ is the standard normal cumulative distribution function.

The unconditional call option value is derived by integrating the conditional call option value as follows,

$$C(S_0; K, T) = \int_0^\infty c(g)\left(\frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}\right)dg.$$

In terms of the function Ψ defined in Madan *et al* (1998)

$$\Psi(a, b, c) = \int_0^\infty \Phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right)\left(\frac{u^{c-1}e^{-u}}{\Gamma(c)}\right)du, \tag{4}$$

the unconditional call option value can be written as

$$C(S_0; K, T) = S_0 e^{-qT}\Psi\left(d\sqrt{\frac{\Omega}{\nu}}, \frac{\theta+\sigma^2}{\sigma}\sqrt{\frac{\nu}{\Omega}}, \frac{T}{\nu}\right) - Ke^{-rT}\Psi\left(\frac{d}{\sqrt{\nu}}, \frac{\theta}{\sigma}\sqrt{\nu}, \frac{T}{\nu}\right), \tag{5}$$

where, again, $\Omega = (1-\theta\nu-\sigma^2\nu)/2$. Madan *et al* (1998) show that the function Ψ can be written in terms of degenerate hypergeometric functions of two variables and modified Bessel functions of the second kind. We refer the reader to their paper for explicit formulae for Ψ in terms of these special functions of mathematics.

REVERSE CONVERTIBLES IN THE VG ECONOMY

A reverse convertible pays the face-value of the note unless on some observation date the price of the asset linked to the note falls below a certain trigger price. To fix ideas, consider a \$1000 face-value reverse convertible with T years to expiration, trigger price H and initial underlying asset price S_0. Let τ be the first time the asset price falls below the trigger price. At maturity, the payoff of the reverse convertible is given by,

$$f(S_T) = \begin{cases} \$\,1000 & : \tau > T \\ \frac{\$\,1000}{S_0} \times \min\left(S_T, S_0\right) & : \tau \leqslant T \end{cases}$$

Figure 1 plots the payoff at maturity of the reverse convertible.

Reverse convertibles can be decomposed into three investments: a long position in a zero coupon bond, a long position in a series of coupon payments and a short position in a down-and-in (barrier) put option. The formula for a reverse exchangeable decomposed into these three components is presented in Hernández *et al* (2007).[9]

Assume the reverse convertible matures at time T, pays coupons C_i at time t_i for $i = 1, 2, \ldots, n$.

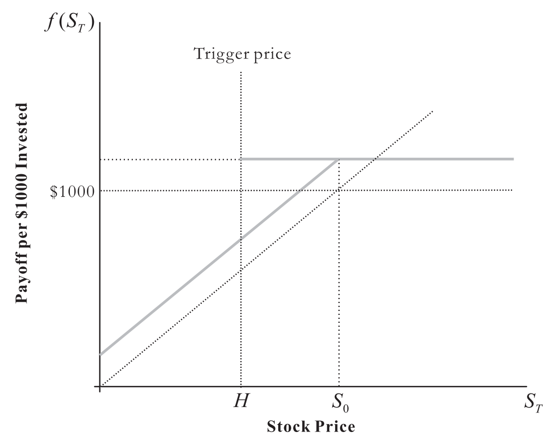

Figure 1: Payoff at maturity of a reverse convertible note with face-value \$1000, T years to expiration, trigger price H and valuation date asset price of S_0. The value of the payment at maturity for option prices between the initial asset price and the trigger price is dependent upon the price of the underlying asset during the observation period.

The value of a \$1000 face-value reverse exchangeable with trigger price H and valuation date asset value S_0 is given by

$$V(S_0, H, T) = \$\,1000 e^{-(r+\overline{D})T} + \sum_{i=1}^{n} C_i e^{-(r+\overline{D})t_i} - \frac{\$\,1000}{S_0} e^{-\overline{D}T} P_{di}(S_0; H, S_0, T),$$

where $\overline{D}$ is the CDS rate of the issuer and we have made the assumption that the strike of the down-and-in put option is S_0. Following Baule and Tallau (2011), we include the issuer's CDS rate in the pricing formula to reflect credit risk of the issuer.

We write the path-dependent payoff function above in terms of an equivalent static payoff because there is evidence that the static hedge of a portfolio with path-dependent options is preferred over a dynamic hedging because of lower transaction costs (Tompkins, 1997). We focus on the VG model for this article because we derive a reflection lemma that allows for the conversion of path-dependent payoffs to equivalent static payoffs. The lemma is similar to that found in Carr and Chou (1997).[10]

Lemma 1: *In a VG economy, suppose that X is a portfolio of European options expiring at time T, conditional on the gamma time-change, with payoff:*

$$X(S_T|g) = \begin{cases} f(S_T) & : S_T \in (A, B) \\ 0 & : S_T \notin (A, B) \end{cases} \tag{6}$$

For $H > 0$, let Y be a portfolio of European options with maturity T and payoff:

$$Y(S_T|g) = \begin{cases} (S_T H)^{p(g)} f(H^2/S_T) & : S_T \in (H^2/B, H^2/A) \\ 0 & : S_T \notin (H^2/B, H^2/A) \end{cases} \tag{7}$$

where the power p(g) is given by

$$p(g) = -2\left(\frac{\theta g + (r-q)T + \dfrac{T}{\nu}\log(1-\theta\nu-\sigma^2\nu/2)}{\sigma^2 g}\right) \tag{8}$$

and the parameters $\{\theta, \sigma, \nu\}$ characterize the VG economy, r is the constant continuously compounded risk-free rate and q is the constant continuously compounded dividend yield. Then X and Y have the same payoff whenever the spot equals H.

For a proof of the Lemma 1, see the Appendix A. This lemma allows us to write a dynamic payoff in terms of an equivalent static payoff. This new result facilitates the efficient valuation of barrier options within a model of stochastic volatility.

As reverse convertibles are structured products with an embedded down-and-in put option, we present the valuation formula for this type of barrier option. The value of any other type of barrier option, including double barrier options, can in principle be calculated based upon the methodology we develop below.

We consider a down-and-in put option on an asset with initial price S_0 characterized by a barrier H, strike $K \geqslant H$ and maturity T.[11] We are working in the VG economy with parameters θ, σ and ν, (constant) continuously compounded risk-free rate r and dividend yield q. Using the Lemma 1, the equivalent (static) payoff for the path-dependent down-and-in put option is

$$\hat{f}(S_T|g) = \begin{cases} 0 & : S_T \in (H, \infty) \\ f(S_T) + \left(\frac{S_T}{H}\right)^{p(g)} f(H^2/S_T) & : S_T \in (0, H) \end{cases} \tag{9}$$

where $f(S_T) = (K - S_T)^+$ is the usual European put option payoff. The down-and-in put option value, conditional on the gamma time g, is found by integrating the equivalent static payoff as follows:

$$P_{di}(S_0; K, H, T|g) = P_{di}(g) = e^{-rT} \int_{-\infty}^{\infty} \hat{f}(S_T|g) \frac{\exp\left(-\frac{(X-\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}} dX. \tag{10}$$

Equation (11) results from evaluating the integral in equation (10),

$$\begin{aligned} P_{di}(g) &= Ke^{-rT}\Phi\left(\frac{h}{\sqrt{g}} - \frac{\theta}{\sigma}\sqrt{g}\right) \\ &\quad - S_0 e^{-qT}\Omega^{T/\nu} e^{(1-\Omega)g/\nu}\Phi\left(\frac{h}{\sqrt{g}} - \frac{\theta+\sigma^2}{\sigma}\sqrt{g}\right) \\ &\quad + \left(\frac{S_0 e^{-qT}}{He^{-rT}}\right)^p Ke^{-rT}\Omega^{Tp/\nu} e^{(\theta p + p^2\sigma^2/2)g} \\ &\quad \times \left[\Phi\left(\frac{h}{\sqrt{g}} - \frac{\theta+\sigma^2 p}{\sigma}\sqrt{g}\right) - \Phi\left(\frac{h_1}{\sqrt{g}} - \frac{\theta+\sigma^2 p}{\sigma}\sqrt{g}\right)\right] \\ &\quad - \left(\frac{S_0 e^{-qT}}{He^{-rT}}\right)^{(p-1)} He^{-rT}\Omega^{T(p-1)/\nu} e^{(\theta(p-1)+(p-1)^2\sigma^2/2)g} \\ &\quad \times \left[\Phi\left(\frac{h}{\sqrt{g}} - \frac{\theta+\sigma^2(p-1)}{\sigma}\sqrt{g}\right) - \Phi\left(\frac{h_1}{\sqrt{g}} - \frac{\theta+\sigma^2(p-1)}{\sigma}\sqrt{g}\right)\right], \end{aligned} \tag{11}$$

where

$$h = \frac{\log\left(\frac{H}{S_0}\right) - (r-q)T - \frac{T}{\nu}\log\Omega}{\sigma}$$

$$h_1 = \frac{\log\left(\frac{H^2}{KS_0}\right) - (r-q)T - \frac{T}{\nu}\log\Omega}{\sigma} = h - \frac{\log(K/H)}{\sigma}$$

and $\Omega = (1-\theta\nu-\sigma^2\nu)/2$. We begin by simplifying the above equation using the definition of p wherever possible,

$$\begin{aligned} P_{di}(g) &= Ke^{-rT}\phi\left(\frac{h}{\sqrt{g}} - \frac{\theta}{\sigma}\sqrt{g}\right) \\ &\quad - S_0 e^{-qT}\Omega^{T/\nu} e^{(1-\Omega)g/\nu}\phi\left(\frac{h}{\sqrt{g}} - \frac{\theta+\sigma^2}{\sigma}\sqrt{g}\right) \\ &\quad + Ke^{-rT}\left(\frac{S_0}{H}\right)^p \left[\Phi\left(\frac{\bar{h}}{\sqrt{g}} + \frac{\theta}{\sigma}\sqrt{g}\right) - \Phi\left(\frac{\bar{h}_1}{\sqrt{g}} + \frac{\theta}{\sigma}\sqrt{g}\right)\right] \\ &\quad - S_0 e^{-qT}\Omega^{T/\nu} e^{(1-\Omega)g/\nu}\left(\frac{S_0}{H}\right)^{p-2} \\ &\quad \left[\Phi\left(\frac{\bar{h}}{\sqrt{g}} + \frac{\theta+\sigma^2}{\sigma}\sqrt{g}\right) - \Phi\left(\frac{\bar{h}_1}{\sqrt{g}} + \frac{\theta+\sigma^2}{\sigma}\sqrt{g}\right)\right], \end{aligned} \tag{12}$$

where

$$\bar{h} = \frac{\log\left(\frac{H}{S_0}\right) + (r-q)T + \frac{T}{\nu}\log\Omega}{\sigma}$$

$$\bar{h}_1 = \frac{\log\left(\frac{H^2}{KS_0}\right) + (r-q)T + \frac{T}{\nu}\log\Omega}{\sigma} = \bar{h} - \frac{\log(K/H)}{\sigma}.$$

The unconditional option value follows from evaluating an integral of the form

$$\overline{\Psi}(a,b,c,d)=\int_0^\infty \Phi\left(\frac{a}{\sqrt{g}}+b\sqrt{g}\right)e^{c/g}\left(\frac{g^{d-1}e^{-g}}{\Gamma(d)}\right)dg. \tag{13}$$

Here we use the series expansion of the exponential function to write the expression in terms of the function defined in equation (4),

$$\overline{\Psi}(a,b,c,d)=\sum_{n=0}^{\infty}\frac{c^n}{n!}\left(\frac{\Gamma(d-n)}{\Gamma(d)}\right)\Psi(a,b,d-n). \tag{14}$$

Each term in the series is a linear combination of hypergeometric functions and modified Bessel functions as given by Madan *et al* (1998).

We write the unconditional down-and-in put option value in terms of this function $\overline{\Psi}$ as

$$\begin{aligned}
P_{di}&(S_0;H,K,T)\\
&=Ke^{-rT}\Psi\left(\frac{h}{\sqrt{\nu}},-\frac{\theta}{\sigma}\sqrt{\nu},\frac{T}{\nu}\right)\\
&\quad-S_0e^{-qT}\Psi\left(h\sqrt{\frac{\Omega}{\nu}},-\frac{\theta+\sigma^2}{\sigma}\sqrt{\frac{\nu}{\Omega}},\frac{T}{\nu}\right)\\
&\quad+Ke^{-rT}\left(\frac{S_0}{H}\right)^{-2\theta/\sigma^2}\\
&\quad\left[\overline{\Psi}\left(\frac{\bar{h}}{\sqrt{\nu}},\frac{\theta}{\sigma}\sqrt{\nu},C_1,\frac{T}{\nu}\right)\right.\\
&\quad\left.-\overline{\Psi}\left(\frac{\bar{h}_1}{\sqrt{\nu}},\frac{\theta}{\sigma}\sqrt{\nu},C_1,\frac{T}{\nu}\right)\right]\\
&\quad-S_0e^{-qT}\left(\frac{S_0}{H}\right)^{-2\frac{\theta+\sigma^2}{\sigma^2}}\\
&\quad\left[\Psi\left(\bar{h}\sqrt{\frac{\Omega}{\nu}},\frac{\theta+\sigma^2}{\sigma}\sqrt{\frac{\nu}{\Omega}},C_2,\frac{T}{\nu}\right)\right.\\
&\quad\left.-\Psi\left(\bar{h}_1\sqrt{\frac{\Omega}{\nu}},\frac{\theta+\sigma^2}{\sigma}\sqrt{\frac{\nu}{\Omega}},C_2,\frac{T}{\nu}\right)\right]
\end{aligned} \tag{15}$$

where

$$C_1=\log\left(\frac{H^2}{S_0^2}\right)\left\{\frac{(r-q)T+T\nu\log\Omega}{\nu}\right\}$$
$$\text{and}\quad C_2=\Omega C_1.$$

This is an analytic solution for the value of a path-dependent option within the VG model.

VALUATION OF REVERSE CONVERTIBLES

Model calibration

Our calibration of the VG model follows closely that of Madan *et al* (1998). For the issue date of each reverse convertible, we calibrated the parameter σ in the BS model and the parameters $\{\sigma,\theta,\nu\}$ of the VG model based upon contemporaneously traded at-the-money put options of the underlying asset. To be precise, we used the traded European put options that were close to at-the-money. For example, if the underlying asset had a price of S_0 at closing the day before the valuation date, then we used options with strike prices in the range of $0.75S_0$ and $1.25S_0$. We considered options of all available maturities.

For the BS model, we determined the optimal implied volatility given the options data based upon the squared error between market prices and BS model prices. The implied volatilities we computed are very close to the at-the-money implied volatilities provided by other sources such as Bloomberg. For the VG model, we followed Madan *et al* (1998) and minimized the quantity,

$$k=\sqrt{\frac{1}{M}\sum_{i=1}^{M}(\log(w_i)-\log(\hat{w}_i))^2}$$

Table 1: Fit results for parameter values within the BS and VG models. We present the mean value for each parameter, along with the standard deviation

2*Parameter	Mean value	SD
Black–Scholes model		
σ	0.453	0.096
Variance gamma model		
σ	0.425	0.125
θ	−0.705	0.653
ν	0.108	0.077

where w_i is the observed market price of the ith option and $\hat{w}_i$ is the model price with error model $w_i = \hat{w}_i \exp(\eta\varepsilon_i - \eta^2/2)$ where $\{\varepsilon_i\}$ are normally distributed with unit variance. For put option prices in the VG model, we used either the integration approach as in the section 'European call option valuation' or a more efficient Fourier closed-form equation (19). To exclude non-positive parameters σ, ν in the VG model, we have used exponential transformations suggested in Madan *et al* (1998). For the results of the calibration, see Table 1.

VG and BS valuations

We collected 1817 reverse convertibles issued in 2010 and 2011 by four large investment banks. Barclays issued 1324, JP Morgan issued 404, UBS issued 74 and Morgan Stanley issued 15 of the notes. The average issue size for those notes is \$1 185 163.

After that we calibrated the parameters of the models and we obtained each issuer's CDS rates. As a proxy for risk-free rates, we used treasury yields of approximately the same maturity as the reverse convertible. We also obtained the dividend yield for each of the underlying assets as of the offering dates from Bloomberg.

As an example, consider the reverse convertible (linked to the stock price of Patriot Coal Corporation (PCX)) issued by Barclays on 21 January 2010 with total principal amount \$1 million to mature on 21 April 2010.[12] On 15 January 2010 pricing date, Barclays' CDS rate was 42 basis points, the risk-free rate was 0.25 per cent, the dividend yield of the underlying asset was 0 per cent and the asset price was \$20.20. The note paid a monthly coupon with annualized rate of 19 per cent. The trigger price during the 3-month period was \$14.14 – 70 per cent of the initial asset price. The calibrated implied volatility for the BS model is 64.18 per cent and the calibrated parameters for the VG model are ($\sigma = 66.01$ per cent, $\theta = -0.7799$, $\nu = 0.0500$). Given the calibrated parameters, we plot in Figure 2 the probability density function for the distribution of log returns ($R_T = \log(S_T/S_0)$).

In the BS model, the return distribution is normal. In the VG model, the returns distribution exhibits higher skewness and kurtosis (fatter tails).

The transformed equivalent payoff function $\hat{f}(x|g)$ for the VG model, $\hat{f}(x)$ for the BS model and $f(x)$ for a standard put option is illustrated in Figure 3. Both equivalent payoff functions are similar and truncated at the trigger price \$14.14. The equivalent payoff for the BS model is larger than the VG model between \$10 and \$14.

The value of the reverse convertible is determined by integrating the payoff function with the appropriate probability distribution function. For the VG model, there is one

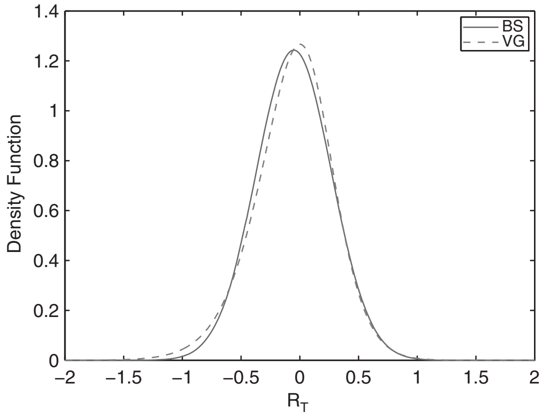

Figure 2: The probability density function for the Black–Scholes model and variance gamma model as a function of log returns ($R_T = \log(S_T/S_0)$) for the 3-month period.

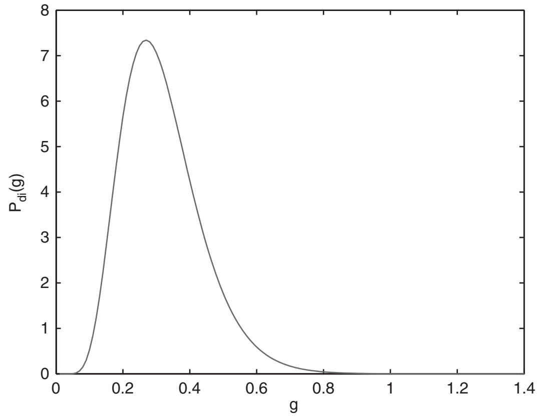

Figure 4: Conditional down-and-in put option value as a function of the g, $P_{di}(g)$. The ratio of the trigger price to the valuation date asset price is 70 per cent.

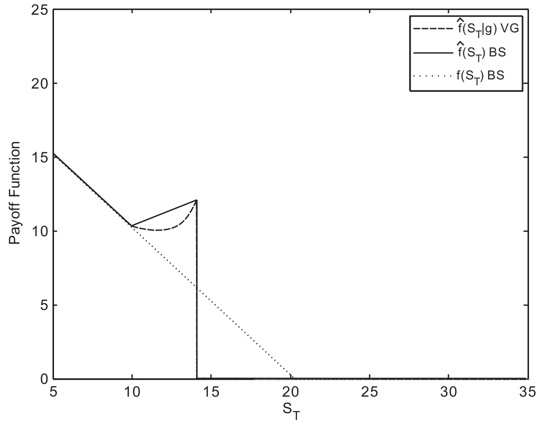

Figure 3: Static payoff function for $\hat{f}(x|g = 0.5)$ in the VG model along with $\hat{f}(x)$ in the Black–Scholes model.

Table 2: Results of reverse convertible evaluation by issuer for the Black–Scholes model and variance gamma model.The average BS and VG values are normalized to notes in units of $1000.

Issuer	*Number issued*	*Average Black–Scholes value ($)*	*Average variance gamma value ($)*
Barclays	1324	935.06	934.44
JP Morgan	404	960.56	957.65
UBS	74	959.63	958.46
Morgan Stanley	15	960.07	956.80

more layer of integration to remove the gamma time g. Conditional on g the price function $P_{di}(g)$ as in equation (11) is plotted in Figure 4. The price of the down-and-in put option is $1.94 in the BS model and $2.01 in the VG model. For each $1000 face-value reverse convertible note purchased, the investor received a note worth $949.86 as calculated in the BS model and $947.40 as calculated in the VG model.

Table 2 summarizes the main results of our valuation of the 1817 reverse convertibles in our sample. On average, an investor recieved \$941.94 within the BS model and \$940.74 within the VG model per \$1000 dollar invested. The net effect of stochastic volatility is to increase the value of the embedded barrier option and therefore to decrease the issue date fair market value of the structured product.

CONCLUDING REMARKS

We have directly integrated the probability distribution function with an equivalent payoff function to value the down-and-in put options embedded within reverse convertibles. An alternative method is a Fourier-based approach using readily available Lévy measures.[13] The benefit of using the Fourier-based approach is the applicability to many different stochastic volatility models. We have included a detailed derivation of the valuation of a European put option in the Appendix following closely the approach of Carr and Madan (1999). The Fourier-based approach has two limitations in this context. The first limitation is that the valuation of barrier options requires the derivation of 'incomplete' characteristic functions owing to the truncation of the payoff function at the trigger price. The second limitation is that we have only presented the reflection lemma for the VG economy, and would therefore need to derive a similar result for other stochastic volatility models. We see this as a direction for future research.

In this article, we studied the valuation of one of the most popular varieties of equity-linked structured products – reverse convertibles. We studied these products in the context of a particular model of stochastic volatility. Within the VG model, we derived a reflection lemma similar to that found in the BS model. We used this reflection lemma to convert the path-dependent payoff of the reverse convertible to a static payoff. We outlined a procedure for the valuation of barrier options commonly embedded in structured products and presented a closed-form valuation formula for down-and-in put options.

Using our new formula, we compared the value of reverse convertible notes issued between 2010 and 2011 predicted by the BS model and the VG model. The overall effect of stochastic volatility is to increase the value of the embedded put option and therefore decrease the value of the note. The average value of a \$1000 face-value reverse convertible note issued during this time period was \$941.90 in the BS model and \$940.70 in the VG model.

NOTES

1 www.finra.org/Newsroom/NewsReleases/2010/P120914.

2 www.finra.org/Investors/ProtectYourself/InvestorAlerts/Bonds/P120883.

3 FINRA Regulatory Notice 10–09, 'Reverse Convertibles', February 2010.

4 Bloomberg Brief: Structured Notes, 5 January 2012, 3 January 2013 and 10 January 2013.

5 'Principal-protected' in this context means, at a minimum, the security returns the amount invested (unless the issuer defaults).

6 These processes generate an infinite number of jumps within any finite interval.

7 The parameter ν is positive semi-definite. As ν tends to zero, the return distribution should match the normal distribution. In agreement with expectations, in this limit the distribution has vanishing skewness.

8 The formula for the European call option valuation takes the same form in most of the

literature; however, the dependence upon the model parameters varies widely. For three explicit examples of the discrepancies, see Madan *et al* (1998), Ballotta (2010)and Jaimungal (2004).

9 For a discussion of barrier options in the Black–Scholes model, see Hull (2011).

10 For a review of the use of static portfolios for the replication of portfolios with path-dependent payoffs, see Carr and Chou (1997).

11 Usually a reverse convertible has an embedded down-and-in put option with strike $K = S_0$. To keep our results general, we do not impose such a restriction.

12 See SEC filing http://www.sec.gov/Archives/edgar/data/312070/000119312510007443/d424b2.htm

13 For a veritable catalog of Lévy measures, see Wu (2007).

References

Ballotta, L. (2010) Efficient pricing of ratchet equity-index annuities in a variance-gamma economy. *North American Actuarial Journal* 14(3): 355–368.

Baule, R. and Tallau, C. (2011) The pricing of path-dependent structured financial retail products: The case of bonus certificates. *Journal of Derivatives* 18(4): 54–71.

Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy* 81(3): 637–654.

Carr, P. and Chou, A. (1997) Hedging complex barrier options *Journal of Business* 75(2): 305–332.

Carr, P. and Madan, D. (1999) Option valuation using the fast Fourier transform. *Journal of Computational Finance* 2(4): 61–73.

Carr, P., Geman, H., Madan, D. and Yor, M. (2002) The fine structure of asset returns: An empirical investigation. *Journal of Business* 75(2): 305–332.

Deng, G., McCann, C. and O'Neal, E. (2010) What TiVo and JPMorgan Teach Us about Reverse Convertibles. Working Paper.

Engel, R. (2004) Risk and volatility: Econometric models and financial practice. *American Economic Review* 94(3): 405–420.

Fang, F. and Oosterlee, C. (2011) A Fourier-based valuation method for Bermudan and barrier options under Heston's model. *SIAM Journal of Financial Mathematics* 2(1): 439–463.

Fiorani, F., Luciano, E. and Semeraro, P. (2010) Single and joint default in a structural model with purely discontinuous assets. *Quantitative Finance* 10(3): 249–263.

Henderson, B. and Pearson, N. (2010) The dark side of financial innovation: A case study of the pricing of a retail financial product. *Journal of Financial Economics* 100(2): 227–247.

Hernández, R., Lee, W. and Liu, P. (2007) An Economic Analysis of Reverse Exchangeable Securities: An Option-pricing Approach. Working Paper.

Heston, S. (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6(2): 327–343.

Hull, J. (2011) *Options, Futures and Other Derivatives*. New Jersey: Prentice Hall.

Itkin, A. and Carr, P. (2010) A finite-difference approach to the pricing of barrier options in stochastic skew models? Working paper.

Jaimungal, S. (2004) Pricing and Hedging Equity Indexed Annuities with Variance-gamma Deviates. Working Paper.

Kudryavtsev, O. and Levendorskiĭ , S. (2009) Fast and accurate pricing of barrier options under Lévy proccesses. *Finance and Stochastics* 13(1): 531–562.

Lipton, A. (2001) *Mathematical Methods for Foreign Exchange: A Financial Engineer's Approach*. World Scientific.

Madan, D. and Milne, F. (1991) Aoption pricing with VG martingale components. *Mathematical Finance* 1(4): 39–55.

Madan, D. and Seneta, E. (1990) The variance gamma (V.G.) model for share market returns. *Journal of Business* 63(4): 511–524.

Madan, D., Carr, P. and Chang, E. (1998) The variance gamma process and option pricing. *European Finance Review* 2: 79–105.

Merton, R. (1976) Option pricing when underlying stock distributions are discontinuous. *Journal of Financial Economics* 3: 125–144.

Monroe, I. (1978) Processes that can be embedded in Brownian motion. *Annals of Probability* 6(1): 42–56.

Schoutens, W. (2006) Exotic options under Lévy models: An overview. *Journal of Computational and Applied Mathematics* 189(1–2): 526–538.

Szymanowska, M., Ter Horst, J. and Veld, C. (2009) Reverse convertible bonds analyzed. *Journal of Futures Markets* 29(10): 895–919.

Tompkins, R. (1997) Static versus dynamic hedgings of exotic options: An evaluation of hedge performance via simulation. *Journal of Risk Finance* 3(4): 6–34.

Wu, L. (2007) Modeling financial security returns using Lévy processes. *Handbooks in Operations Research and Management Science* 15: 117–162.

APPENDIX A

Proof of lemma

Proof: Assuming the spot equals H, we can write the asset price as

$$S_T = H\exp(X+mT),$$

where $m=(r-q)+(1/\nu)\log(1-\theta\nu-\sigma^2\nu/2)$. The value of portfolio X is given by

$$V(X) = \int_A^B \int_0^\infty f(S_T)pdf(S_T|g)dg\frac{dS_T}{S_T}$$

$$= \int_{\log(A/H)-mT}^{\log(B/H)-mT} \int_0^\infty f(H\exp(X+mT)) \frac{\exp\left(\dfrac{-(X-\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}}\mu(g)dgdX \tag{A.1}$$

where $\mu(g)=(g/\nu)^{T/\nu-1}e^{-g/\nu}/\Gamma(T/\nu)$. Making the change of variable

$$H\exp(X+mT) = \frac{H^2}{H\exp(\hat{X}+mT)}$$

results in the following

$$\int_{\log(H/B)-mT}^{\log(H/A)-mT} \int_0^\infty f\left(\frac{H}{\exp(\hat{X}+mT)}\right) \frac{\exp\left(\dfrac{-(\hat{X}+2mT+\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}}\mu(g)dgd\hat{X}.$$

Completing the square in the exponent and simplifying the result brings us to

$$\Rightarrow \int_{\log(H/B)-mT}^{\log(H/A)-mT} \int_0^\infty f\left(\frac{H}{\exp(\hat{X}+mT)}\right) \left(\frac{\exp(\hat{X}+mT)}{H}\right)^{p(g)} \frac{\exp\left(\dfrac{-(\hat{X}-\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}}\mu(g)dgd\hat{X}$$

$$= \int_{H^2/B}^{H^2/A} \int_0^\infty f\left(\frac{H^2}{\hat{S}_T}\right)\left(\frac{\hat{S}_T}{H}\right)^{p(g)} pdf(\hat{S}_T|g)dg\frac{d\hat{S}_T}{\hat{S}_T}.$$

where

$$p(g) = -2\left(\frac{\theta g+(r-q)T+\frac{T}{\nu}\log(1-\theta\nu-\sigma^2\nu/2)}{\sigma^2 g}\right). \tag{A.2}$$

This result shows that portfolio Y and portfolio X have the same value whenever the spot equals H. □

APPENDIX B

Option valuation using Fourier transform

Here we present the Fourier transform method of the valuation of a plain-vanilla European put option in the VG model. In this section, we follow closely the approach of Carr and Madan (1999). Using the definition of the characteristic function,

$$\phi_T(u) = \int_{-\infty}^{\infty} e^{iuz}q_T(z)dz$$

$$= \left(1-i\theta\nu u+\frac{\sigma^2\nu u^2}{2}\right)^{-T/\nu},$$

one can write the value of the plain-vanilla European put option as

$$P_T(k) = \int_{-\infty}^{k} S_0 e^{-rT}(e^k - e^z) q_T(z) dz,$$

where $k = \log(K/S_0)$.

In the limit $K \gg S_0$, the put option approaches Ke^{-rT}. The put option pricing function is therefore not square-integrable and we need to regulate the divergent integral. Define the following,

$$p_T(k) \equiv e^{-\alpha k} P_T(k)$$

for $\alpha > 0$. The Fourier transform of $p_T(k)$ is then

$$\begin{aligned}\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} p_T(k) dk \to P_T(k)\\ &= \frac{e^{\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) dv.\end{aligned}$$

Upon explicit computation of the Fourier transform

$$\begin{aligned}\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} \int_{-\infty}^{k} S_0 e^{-rT}\left(e^{(1-\alpha)k} - e^{z-\alpha k}\right) q_T(z) dz dk\\ &= S_0 e^{-rT} \int_{-\infty}^{\infty} \int_{z}^{\infty} \left(e^{(1-\alpha+iv)k} - e^{z-(\alpha-iv)k}\right) q_T(z) dk dz\\ &= S_0 e^{-rT} \int_{-\infty}^{\infty} e^{(1-\alpha+iv)z} \left(\frac{1}{iv-\alpha} - \frac{1}{iv+1-\alpha}\right) q_T(z) dz\\ &= S_0 e^{-rT} \left(\frac{\phi_T(v-(1-a)i)}{(iv-a)(iv+1-a)}\right),\end{aligned} \tag{B.1}$$

one arrives at the following price for a plain-vanilla European put option

$$P_T(k) = \frac{e^{\alpha k}}{2\pi} \int_{-\infty}^{\infty} S_0 e^{-rT} \left(\frac{\phi_T(v-(1-\alpha)i) e^{-ivk}}{(iv-\alpha)(iv+1-\alpha)}\right) dv. \tag{B.2}$$

Taking a similar approach to determine a closed-form valuation for the price of a barrier option runs into trouble in a number of ways. For example, the domain of integration requires the derivation of an 'incomplete' characteristic function,

$$\begin{aligned}\phi^{(\gamma)}(u) &= E^{(\gamma)}[e^{iuX}]\\ &\equiv \int_0^{\infty} \int_{-\infty}^{\gamma} \frac{\exp\left(-\frac{(X-\theta g)^2 - 2i\sigma^2 guX}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}} dX\\ &\quad \left(\frac{g^{-T/\nu-1} e^{-g/\nu}}{\nu^{T/\nu}\Gamma(T/\nu)}\right) dg\\ &= \left(1 - i\theta\nu u + \frac{\sigma^2 \nu u^2}{2}\right)^{-T/\nu}\\ &\quad \times \int_0^{\infty} \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\gamma - \theta g - iu\sigma^2 g}{\sigma\sqrt{2g}}\right)\right)\\ &\quad \left(\frac{\bar{g}^{T/\nu-1} e^{-\bar{g}/\nu}}{\nu^{T/\nu}\Gamma(T/\nu)}\right) d\bar{g}\end{aligned}$$

where $\operatorname{erf}(x)$ is the error function and $\bar{g} = (1 - i\theta\nu u + u^2\sigma^2\nu/2)g$. This result, along with the fact that the equivalent static payoff function depends on the gamma time-change, makes finding a closed-form solution for the barrier option within the VG model unlikely using the Fourier transform approach.