Erratum

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Due to technical problems arising during the production stage of the above-cited article, several serious errors appeared in the printed version. To put matters right, we are now reprinting the corrected article on the following pages.

We sincerely apologise to both authors and to all readers for any inconvenience caused by the mistakes that appeared in the first printed version.

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Immersions of Surfaces in Spin^c-Manifolds with a Generic Positive Spinor

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Abstract. We define and discuss totally real and pseudoholomorphic immersions of real surfaces in a 4-manifold which, instead of an almost complex structure, carries only a "framed spin^c-structure," that is, a spin^c-structure with a fixed generic section of its positive half-spinor bundle. In particular, we describe all pseudoholomorphic immersions of closed surfaces in the 4-sphere with a standard framed spin structure.

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1. Introduction

Almost complex structures on real manifolds of dimension $2n$ are well-known to be, essentially, a special case of spin^c-structures. This amounts to a specific Lie-group embedding $\mathrm{U}(n) \to \mathrm{Spin}^c(2n)$ (see [6, p. 392] and Remark 7.1 below). The present paper deals with the case $n = 2$. The relation just mentioned then can also be couched in the homotopy theorists' language: *for a compact four-manifold M with a fixed* CW-*decomposition, a spin^c-structure over M is nothing else than an almost complex structure on the 2-skeleton of M, admitting an extension to its 3-skeleton; the extension itself is not a part of the data.* (Kirby [5] attributes the italicized comment to Brown.)

Our approach is explicitly geometric and proceeds as follows. Given an almost complex structure J on a 4-manifold M, one can always choose a Riemannian metric g compatible with J, thus replacing J by an almost Hermitian structure (J, g) on M. The latter may in turn be treated as a spin^c-structure on M with a fixed unit C^∞ section ψ of its positive half-spinor bundle σ^+ (cf. Section 7). Complex points of an immersion in M of any oriented real surface Σ then can be directly described in terms of ψ rather than J. Such a description is presented in the initial part of the paper, culminating in Theorem 12.1. Moreover, when Σ is compact, the net total number of complex points for a generic immersion $\Sigma \to M$ depends just on the underlying spin^c-structure (and not even on ψ); see Corollary 12.2.

Imitating the above description, we may define complex points of a real-surface immersion $\Sigma \to M$, where M is a 4-manifold with a spin^c-structure and ψ is *any* section of σ^+. This raises the question of finding a suitable additional condition on ψ, for which the corresponding notion of a complex point and the resulting classes of totally real and pseudoholomorphic immersions would be less restrictive than for almost complex structures, yet still interesting and reasonable. (As an "unreasonable" example, choose ψ to be the zero section; then *every* immersion is pseudoholomorphic.)

The condition we propose, in Section 13, consists in requiring ψ to be a C^∞ section of σ^+ transverse to the zero section, and defined only up to a positive functional factor; when such ψ is chosen, we speak of a *framed spin^c-structure*.

The notions of *totally real* and *pseudoholomorphic* immersions of oriented real surfaces Σ have straightforward extensions to the case where the receiving 4-manifold M is endowed with a framed spin^c-structure. We observe, in Section 14, that the former immersions have properties analogous to those in the almost-complex case, and classify the latter ones when M is S^4 with a "standard" framed spin structure and Σ is a closed surface (Section 20).

Unlike almost complex structures, framed spin^c-structures exist on every orientable 4-manifold. On the other hand, pseudoholomorphic immersions of surfaces have been used, with great success, to probe the topology of symplectic manifolds, especially in dimension 4. One may wonder if they could similarly be employed as a tool for studying more general 4-manifolds with framed spin^c-structures.

2. Preliminaries

"Planes" and "lines" always mean *vector* spaces. A real/complex vector space V is called *Euclidean*/*Hermitian* if dim $V < \infty$ and V carries a fixed positive-definite inner product (always written as $\langle,\rangle$, with $||$ denoting its associated norm). Such $\langle,\rangle$ is uniquely determined by Re $\langle,\rangle$ or $||$.

Remark 2.1. We identify oriented (real) Euclidean planes with (complex) Hermitian lines, so that both inner products have the same real part and norm, and multiplication by i is the positive rotation by the angle $\pi/2$.

All manifolds are assumed to be connected except when stated otherwise, while most real manifolds we deal with are oriented and of class C^∞. The orientations for orthogonal complements of real vector subspaces, Cartesian products, total spaces of locally trivial bundles, preimages of regular values of mappings, and (cf. (a) in Section 11) zero sets of transverse sections in vector bundles, are all obtained using the direct-sum convention; because the real dimensions involved are all even, no ambiguity arises even if the order of the summands is not specified. This is consistent with the convention which treates (almost) complex manifolds as *oriented* real manifolds, declaring that

$$(e_1, ie_1, \ldots, e_n, ie_n) \text{ is a positive-oriented real basis of a complex vector space with a complex basis } (e_1, \ldots, e_n). \tag{1}$$

We also deal with *inner-product spaces*, including *normed lines*, over the field **H** of quaternions. The **H**-sesquilinearity requirement imposed on an **H**-valued inner product $\langle , \rangle$ in a quaternion vector space W with $\dim W < \infty$ includes the condition $\langle px, qy \rangle = p\langle x, y\rangle\bar{q}$ for $x, y \in W$ and $p, q \in \mathbf{H}$. Using the inclusion $\mathbf{C} \subset \mathbf{H}$ for $\mathbf{C} = \mathrm{Span}_{\mathbf{R}}(1, \mathbf{i})$, we treat any such W as a complex Hermitian space, with the **C**-valued inner product having the same real part (or, equivalently, the same associated norm) as the original $\langle , \rangle$.

Denoting by **K** any of the scalar fields **R**, **C**, **H**, we let $G_m(W)$ stand for the Grassmannian manifold of all subspaces L with $\dim_{\mathbf{K}} L = m$ in a given vector space W over **K**, where $1 \leq m \leq \dim W < \infty$. Each tangent space $T_L[G_m(W)]$ of $G_m(W)$ then has a canonical real-isomorphic identification $T_L[G_m(W)] = \mathrm{Hom}_{\mathbf{K}}(L, W/L)$. Namely, $A \in \mathrm{Hom}_{\mathbf{K}}(L, W/L)$ corresponds to $d\pi_{\mathbf{w}}[RA\mathbf{w}] \in T_L[G_m(W)]$ for any basis $\mathbf{w} = (w_1, \ldots, w_m)$ of L and any right inverse R of the projection $W \to W/L$. Here π is the standard projection onto $G_m(W)$ of the *Stiefel manifold* $\mathrm{St}_m(W)$, that is, the open set in the mth Cartesian power W^m formed by all linearly independent systems $\mathbf{w} = (w_1, \ldots, w_m)$, and $RA\mathbf{w} = (RAw_1, \ldots, RAw_m)$, so that we have $RA\mathbf{w} \in W^m = T_{\mathbf{w}}[W^m] = T_{\mathbf{w}}[\mathrm{St}_m(W)]$.

Remark 2.2. Thus, writing any given vector in $T_L[G_m(W)]$ as $d\pi_{\mathbf{w}}\mathbf{v}$ with a fixed $\mathbf{w} \in \pi^{-1}(L) \subset \mathrm{St}_m(W)$ and a suitable $\mathbf{v} = (v_1, \ldots, v_m) \in W^m = T_{\mathbf{w}}[\mathrm{St}_m(W)]$, we can define $A \in \mathrm{Hom}_{\mathbf{K}}(L, W/L)$ associated as above with $d\pi_{\mathbf{w}}\mathbf{v} \in T_L[G_m(W)]$ to be the operator sending the basis $(w_1, \ldots, w_m)$ of L onto the ordered m-tuple $(v_1 + L, \ldots, v_m + L)$ in W/L.

Next, let $P(W) \approx \mathbf{K}\mathrm{P}^{n-1}$ be the projective space of all lines in a vector space W over **K** with $1 \leq \dim W < \infty$. As $P(W) = G_1(W)$, the above identification $T_L[G_m(W)] = \mathrm{Hom}_{\mathbf{K}}(L, W/L)$ now becomes

$$T_L[P(W)] = \mathrm{Hom}_{\mathbf{K}}(L, W/L) \qquad \text{for every } L \in P(W). \tag{2}$$

Remark 2.3. Given W, **K** as above, a co-dimension-one subspace V of W, and a vector $u \in W \backslash V$, let M' be the complement of $P(V)$ in $P(W)$. The mapping $\Theta : V \to M'$ defined by $\Theta(y) = \mathbf{K}w$ with $w = y + u$ then clearly is a C^∞ diffeomorphism, while Θ^{-1} represents a standard *projective coordinate system* in $P(W)$. Also, let $A = d\Theta_y v$ for any $y \in V$ and $v \in V = T_y V$. Using (2) with $L = \Theta(y)$ to treat A as an operator $L \to W/L$, we then have $Aw = v + L$, where $w = y + u$.

This is clear from Remark 2.2 for $m = 1$, as Θ is the restriction to V of the translation by u followed by the projection $\pi : W \setminus \{0\} \to P(W)$, and so $d\Theta_y v = d\pi_w v$ by the chain rule.

3. The Grassmannian of Oriented Planes

Let $G_2^+(W)$ be the Grassmannian of real oriented planes in a real vector space W with $\dim W < \infty$, so that $G_2^+(W)$ is a two-fold covering manifold of $G_2(W)$ as defined in Section 2 for $\mathbf{K} = \mathbf{R}$. Any inner product in W makes $G_2^+(W)$ an almost complex manifold, and so, by (1), $G_2^+(W)$ is *naturally oriented*. In fact, an oriented Euclidean plane $\mathcal{T} \in G_2^+(W)$ is a complex line (Remark 2.1), which turns the tangent space of $G_2^+(W)$ at $\mathcal{T}$, that is, $\mathrm{Hom}_{\mathbf{R}}(\mathcal{T}, W/\mathcal{T})$ (see Section 2), into a complex vector space.

Remark 3.1. If, in addition, W itself is the underlying real space of a complex vector space, $G_2^+(W)$ contains two disjoint, embedded complex manifolds $P(W)$, $\bar{P}(W)$, which are copies of the complex projective space of W consisting of all complex lines in W treated as real planes oriented, in the case of $P(W)$, by the complex-line orientation with (1), or, for $\bar{P}(W)$, by its opposite. The almost complex structure in $G_2^+(W)$, obtained as above from a Euclidean inner product in W which is the real part of a Hermitian inner product, then makes the differential of the inclusion $P(W) \to G_2^+(W)$ (or, $\bar{P}(W) \to G_2^+(W)$) complex-linear (or, respectively, antilinear) at each point. This is clear from (2) and its analogue for $T_L[G_m(W)]$ in Section 2.

For a 4-manifold M, the eight-dimensional Grassmannian manifold G_2^+M is the total space of the bundle over M with the fibres $G_2^+(T_yM)$, $y \in M$, so that, as a set, $G_2^+M = \{(y, \mathcal{T}) : y \in M \text{ and } \mathcal{T} \in G_2^+(T_yM)\}$. The *Gauss mapping* $F : \Sigma \to G_2^+M$ of any immersion $f : \Sigma \to M$ of an oriented real surface Σ then is given by $F(x) = (f(x), df_x(T_x\Sigma))$. If M is oriented, so is G_2^+M, due to the natural orientations of its fibres (see above).

If M is the underlying oriented 4-manifold of an almost complex surface, cf. (1), the oriented eight-dimensional manifold G_2^+M has two distinguished six-dimensional submanifolds Q and $\bar{Q}$, which form total spaces of $\mathbf{C}\mathrm{P}^1$-bundles over M. Their fibres over any $y \in M$ are $P(W)$ and $\bar{P}(W)$, defined as in Remark 3.1 for $W = T_yM$. Both Q and $\bar{Q}$ are orientable. We will always treat them as oriented manifold, choosing, however, the *opposites* of their natural orientations. Specifically, the orientations we use are the direct sums of the orientation with (1) on the base M and the orientations of the fibres $P(W)$ and $\bar{P}(W)$, with $W = T_yM$, which are opposite to their natural orientations of complex projective lines.

4. Complex Points of Immersed Real Surfaces

Any real plane $\mathcal{T}$ in a complex plane V (see Section 2) either is a complex line, or is *totally real* in the sense that $\mathrm{Span}_{\mathbf{C}}\mathcal{T} = V$.

Let M be an almost complex surface ($\dim_{\mathbf{R}} M = 4$). An immersion $f : \Sigma \to M$ of a real surface Σ is said to have a *complex point* at $x \in \Sigma$ if $\tau_x = df_x(T_x\Sigma)$ is a complex line in $T_{f(x)}M$. One calls *f totally real* or *pseudoholomorphic* if it has no complex points or, respectively, only complex points. Thus, f is totally real if and only if τ_x is totally real in $T_{f(x)}M$ for every $x \in \Sigma$. On the other hand, if f is pseudoholomorphic, Σ acquires a natural orientation, pulled back to $T_x\Sigma$ from τ_x by df_x for every $x \in \Sigma$. We say that f is a pseudoholomorphic immersion in M of an *oriented* real surface Σ if this orientation coincides with the one prescribed in Σ or, equivalently, if $F(\Sigma) \subset Q$ (with F, Q as at the end of Section 3). See [1, 4].

The *determinant bundle* $\det_{\mathbf{R}}\eta$ (or, $\det_{\mathbf{C}}\eta$) of a rank k real/complex vector bundle η is its highest exterior power $\eta^{\wedge k}$. For an immersion f of a real surface Σ in an almost complex surface M,

$$f^*c_1(\kappa) = 0 \text{ in } H^2(\Sigma, \mathbf{Z}) \text{ if } f \text{ is totally real and } \kappa = \det_{\mathbf{C}}TM \tag{3}$$

Then one also has a natural orientation-reversing vector-bundle isomorphism between the tangent bundle $T\Sigma$ and the normal bundle ν_f, that is,

$$\nu_f = \overline{T\Sigma} \text{ whenever } f \text{ is totally real and } \Sigma \text{ is oriented.} \tag{4}$$

These well-known facts follow since $f^*TM = \mathrm{Span}_{\mathbf{C}}\tau = \tau \oplus i\tau$, where τ stands for the subbundle $df(T\Sigma)$ of f^*TM. Thus, $\nu_f = i\tau$, which gives (4) as the multiplication by i is orientation-reversing (by (1)), while f^*TM coincides with the complexification $\tau^{\mathbf{C}}$ of $\tau \approx T\Sigma$, and so $f^*[\det_{\mathbf{C}}TM] = [\det_{\mathbf{R}}T\Sigma]^{\mathbf{C}}$. Now (3) is obvious: namely, $\det_{\mathbf{R}}T\Sigma$ is trivial if Σ is closed and orientable, and $H^2(\Sigma, \mathbf{Z}) = \{0\}$ otherwise.

If M is an almost complex surface, $f : \Sigma \to M$ is an *arbitrary* immersion of a *closed, oriented* real surface, and $\cdot$ denotes the intersection form in $H_2\big(G_2^+M, \mathbf{Z}\big)$, we may replace (3) by the following formula, proved later in Corollary 12.2: with Q,$\bar{Q}$ defined at the end of Section 3,

$$\int_\Sigma f^*c_1(\kappa) = ([Q] + [\bar{Q}]) \cdot F_*[\Sigma] \text{ for } \kappa = \det_{\mathbf{C}}TM, \tag{5}$$

where $F : \Sigma \to G_2^+M$ is the Gauss mapping of f (see Section 3).

When f is totally real, $Q \cap F(\Sigma) = \bar{Q} \cap F(\Sigma) = \emptyset$, and (5) becomes (3).

Expression $([Q]+[\bar{Q}])\cdot F_*[\Sigma]$ in (5) represents the *net total number of complex points* of *f*. In fact, given an immersion f of an oriented real surface Σ (closed or not) in an almost complex surface M, every complex point x of f is a *Q-complex point* or a *$\bar{Q}$-complex point*, in the sense that $F(x) \in Q$ or, respectively, $F(x) \in \bar{Q}$. The *index* of such an immersion *f*, at any isolated complex point $x \in \Sigma$, is defined to be the intersection index that the immersion $F : \Sigma \to G_2^+M$ has, at the isolated intersection point x, with the disconnected oriented 6-manifold $Q \cup \bar{Q} \subset G_2^+M$.

Thus, if Σ is closed and the immersion f is *generic* (that is, has only finitely many complex points), $\big([Q] + [\bar{Q}]\big) \cdot F_*[\Sigma]$ is the sum of indices of all complex points of f, while the separate contributions corresponding to Q-complex and $\bar{Q}$-complex points are $[Q] \cdot F_*[\Sigma]$ and $[\bar{Q}] \cdot F_*[\Sigma]$.

5. Spin^c(4)-Geometries

The terms “Hermitian plane/line” refer, as in Section 2, to complex vector spaces.

We treat the quaternion algebra **H** as a Hermitian plane, with the multiplication by complex scalars declared to be the *right* quaternion multiplication by elements of $\mathbf{C} = \mathrm{Span}_{\mathbf{R}}(1, i) \subset \mathbf{H}$, and with the inner product $\langle,\rangle$ making the **C**-basis 1, **j** orthonormal, so that it corresponds to the norm $| \ |$ with $|p|^2 = p\bar{p}$. Somewhat surprisingly,

$$1,\ \mathbf{i},\ \mathbf{j},\ \mathbf{k} \text{ form a } \textit{negative-oriented} \text{ real basis of } \mathbf{H}, \tag{6}$$

as the **C**-basis 1, **j** leads to the positive-oriented **R**-basis 1, **i**, **j**, **ji** (see (1)) and $\mathbf{ji} = -\mathbf{k}$. For $a, b, x, y, u, v \in \mathbf{C}$ we have $a\mathbf{j} = \mathbf{j}\bar{a}$, and so, in **H**,

$$(a + \mathbf{j}b)(x + \mathbf{j}y) = u + \mathbf{j}v \quad \text{if and only if} \quad \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{7}$$

DEFINITION 5.1. By a spin^c(4)-*geometry* we mean a triple $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ consisting of two Hermitian planes $\mathcal{S}^\pm$ and a Hermitian line $\mathcal{K}$, endowed with two fixed norm-preserving isomorphic identifications $[\mathcal{S}^\pm]^{\wedge 2} = \mathcal{K}$.

The inner products in $[\mathcal{S}^\pm]^{\wedge 2}$, used here, are induced by those of $\mathcal{S}^\pm$, via the formula $|\phi \wedge \chi|^2 = |\phi|^2 |\chi|^2 - |\langle \phi, \chi \rangle|^2$. Thus, instead of assuming that such identifications are given, we could require that there be skew-symmetric bilinear multiplications $\mathcal{S}^\pm \times \mathcal{S}^\pm \to \mathcal{K}$, which, written as $(\phi, \chi) \mapsto \phi \wedge \chi$, satisfy the last formula, or, equivalently, the condition $|\phi \wedge \chi| = |\phi|\,|\chi|$ whenever $\langle \phi, \chi \rangle = 0$.

An example of a spin^c(4)-geometry is $\mathcal{S}^\pm = \mathbf{C}^2$, $\mathcal{K} = \mathbf{C}$ with both skew-symmetric multiplications given by $\phi\chi = ad - bc$ for $\phi = (a, b)$, $\chi = (c, d)$. Any spin^c(4)-geometry $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ is equivalent to this one under an isomorphism obtained by choosing bases $(\phi^\pm, \chi^\pm)$ in $\mathcal{S}^\pm$ with

$$\phi^\pm, \chi^\pm \in \mathcal{S}^\pm, \quad |\phi^\pm| = |\chi^\pm| = 1, \quad \langle \phi^\pm, \chi^\pm \rangle = 0, \quad \phi^+ \wedge \chi^+ = \phi^- \wedge \chi^-. \tag{8}$$

Every spin^c(4)-geometry $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ gives rise to its associated *determinant mapping* det: $\mathrm{Hom}(\mathcal{S}^+, \mathcal{S}^-) \to \mathbf{C}$ such that, for $A : \mathcal{S}^+ \to \mathcal{S}^-$, the operator $A^{\wedge 2}$: $[\mathcal{S}^+]^{\wedge 2} \to [\mathcal{S}^-]^{\wedge 2}$ is the multiplication by det A in the line $[\mathcal{S}^+]^{\wedge 2} = [\mathcal{S}^-]^{\wedge 2} = \mathcal{K}$. Let us call $A \in \mathrm{Hom}(\mathcal{S}^+, \mathcal{S}^-)$ a *homothety* if $|A\phi| = |A||\phi|$ for some $|A| \geq 0$ and all $\phi \in \mathcal{S}^+$, and set

$$\mathcal{V} = \{\mathrm{A} \in \mathrm{Hom}(\mathcal{S}^+, \mathcal{S}^-) \colon A \text{ is a homothety and } \det A \in [0, \infty)\}. \tag{9}$$

Remark 5.2. Any fixed $\phi^\pm$, $\chi^\pm$ with (8) obviously make $\mathcal{V}$ correspond to the set of all 2×2 matrices appearing in (7), with $a, b, \in \mathbf{C}$. Thus, $\mathcal{V}$ is a real vector space, $\dim_{\mathbf{R}} \mathcal{V} = 4$, and $A \mapsto |A|$ is a Euclidean norm in $\mathcal{V}$, with $|A|^2 = \det A = |a|^2 + |b|^2$ if $a, b, \in \mathbf{C}$ represent A as above. Also, $\mathcal{V}$ is canonically oriented, by (1), since every choice of bases $(\phi^\pm, \chi^\pm)$ with (8) leads to an isomorphism $\mathcal{V} \to \mathbf{C}^2$, and such $(\phi^\pm, \chi^\pm)$ form an orbit of the connected Lie subgroup Spin$^c(4)$ of the direct product $U(2) \times U(2)$ consisting of all $(\mathfrak{A}, \mathfrak{B})$ with $\det \mathfrak{A} = \det \mathfrak{B}$.

LEMMA 5.3. *For any spin*$^c(4)$*-geometry* $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ *there exist norm-preserving real-linear isomorphic identifications* $\mathcal{S}^\pm = \mathbf{H}$ *and* $\mathcal{V} = \mathbf{H}$, *which are also complex-linear for* $\mathcal{S}^\pm$ *and orientation-preserving for* $\mathcal{V}$, *with* (6), *and which make the* Clifford multiplication $\mathcal{V} \times \mathcal{S}^+ \to \mathcal{S}^-$, *that is, the evaluation pairing given by* $(A, \phi) \mapsto A\phi$, *appear as the quaternion multiplication. Specifically, such identifications are provided by declaring* $\phi^\pm$ *equal to* $1 \in \mathbf{H}$ *and* $\chi^\pm$ *equal to* $\mathbf{j} \in \mathbf{H}$, *for any fixed* $(\phi^\pm, \chi^\pm)$ *with* (8).

This is clear from (7) and Remark 5.2.

Insisting in Lemma 5.3 that the identification $\mathcal{V} = \mathbf{H}$ be orientation-preserving might seem pedantic—after all, the orientation of $\mathcal{V}$ can always be reversed. The point is, however, that given a spin^c-structure on a 4-manifold M, the canonical orientation of $\mathcal{V}$ leads to an orientation of M (see Section 6). When the spin^c-structure comes from an almost complex structure (along with a compatible metric, cf. Section 7), the latter orientation agrees with another canonical orientation of M, provided by (1). Without this agreement, the generalization of formula (4), obtained in Section 14, might well read $\nu_f = T\Sigma$ instead of $\nu_f = \overline{T\Sigma}$

6. Spin^c-Structures over 4-Manifolds

By a *spin*^{c}*-structure* over a 4-manifold M we mean a triple $(\sigma^+, \sigma^-, \kappa)$ of complex vector bundles of ranks 2, 2 and 1 over M, all endowed with Hermitian fibre metrics, whose fibres form, at every point $y \in M$, a spin$^c(4)$-geometry (Definition 5.1), varying with y, and such that the associated space $\mathcal{V} = \mathcal{V}_y$ with (9) has a fixed isomorphic identification with T_yM depending, along with the spin^c (4)-geometry itself, C^∞-differentiably on y. (See [6], [8].) In other words, we then have fixed norm-preserving identifications $\kappa = [\sigma^+]^{\wedge 2} = [\sigma^-]^{\wedge 2}$ and the *Clifford multiplication* $TM \otimes \sigma^+ \to \sigma^-$, which is a C^∞ morphism of real vector bundles acting, at every $y \in M$, as a pairing $T_yM \times \sigma_y^+ \ni (v, \phi) \mapsto v\phi \in \sigma_y^-$ such that the operator $\phi \mapsto v\phi$ equals, for any given v, a scalar $|v| \geq 0$ times a norm-preserving *unimodular* complex isomorphism $\sigma_y^+ \to \sigma_y^-$. (Unimodularity makes sense here, since both $\sigma^\pm$ have the same determinant bundle κ.) This defines a Euclidean norm $v \mapsto |v|$ in T_yM, and hence a Riemannian metric g on M. Also, M is canonically oriented (since so is the space $\mathcal{V}$ with (9)). In other words, a spin^c-structure really lives over an *oriented Riemannian* 4-manifold.

A special case is a *spin structure* (σ^+, σ^-) for a 4-manifold M, that is, a spin^c-structure $(\sigma^+, \sigma^-, \kappa)$ over M in which κ is the product line bundle $M \times \mathbf{C}$ with the standard (constant) fibre metric. See [7], [6].

7. Almost Complex Surfaces as Spin^c-Manifolds

An *almost Hermitian structure* (J, g) on a manifold M consists of an almost complex structure J on M and a Riemannian metric g on M compatible with J. Such (J, g) may be viewed as a special case of a spin^c-structure (see [6, 8, 9]), as described below in real dimension 4.

More precisely, almost Hermitian structures on any 4-manifold M are in a natural bijective correspondence with pairs formed by a spin^c-structure $(\sigma^+, \sigma^-, \kappa)$ over M and a global unit C^∞ section ψ of σ^+.

In fact, given such $(\sigma^+, \sigma^-, \kappa)$ and ψ, the Clifford multiplication by ψ identifies TM with σ^-, thus introducing an almost complex structure J on M, compatible with the Riemannian metric g on M obtained as the real part of the Hermitian fibre metric $\langle,\rangle$ in σ^- (which, since $|\psi| = 1$, is the same g as in Section 6). In addition, we have a natural isomorphic identification $\sigma^+ = \iota \oplus \kappa$ for the trivial complex line bundle $\iota = M \times \mathbf{C}$, where the subbundles $\mathrm{Span}_{\mathbf{C}}\psi$ and $\psi^\perp$ of σ^+ are identified with ι and κ via the norm-preserving isomorphisms $\phi \mapsto \langle\psi, \phi\rangle$ and $\phi \mapsto \psi \wedge \phi$.

Conversely, any almost Hermitian structure (J, g) on M arises in this manner from $(\sigma^+, \sigma^-, \kappa)$, where $\sigma^- = TM$ with the Hermitian fibre metric $\langle,\rangle$ whose real part is g, while $\kappa = \det_{\mathbf{C}} TM$ and $\sigma^+ = \iota \oplus \kappa$ for $\iota = M \times \mathbf{C}$ as above. This identifies κ with both $[\sigma^\pm]^{\wedge 2}$, as required. Also, σ^+ is naturally a real vector subbundle of $\eta = \mathrm{Hom}_{\mathrm{R}}(TM, TM)$. Namely, the sections of ι provided by the constant functions $1, i : M \to \mathbf{C}$ are to be identified with the sections Id and J of η, while $\kappa = \det_{\mathbf{C}} TM$ becomes a subbundle of η if one lets $u \wedge v \in \kappa_y$, for any $y \in M$, operate on vectors $w \in T_yM$ via $w \mapsto \langle v, w\rangle u - \langle u, w\rangle v$. The Clifford multiplication by $w \in T_yM$, for $y \in M$, then is the evaluation operator $\sigma_y^+ \ni A \mapsto Aw \in \sigma_y^+$. (That $|Aw| = |A||w|$ is easily verified if one writes $A = (a, bu \wedge v)$ with $a, b, \in \mathbf{C}$ and $\langle,\rangle$-orthonormal vectors $u, v \in T_yM$.) The distinguished unit section ψ of σ^+ is the constant function 1 identified as above with a section of $\iota \subset \sigma^+$, so that the Clifford multiplication by ψ is a $\mathbf{C}$-linear bundle isomorphism $TM \to TM = \sigma^-$. Therefore, these $(\sigma^+, \sigma^-, \kappa)$ and ψ in turn lead, via the construction of the preceding paragraph, to the original (J, g).

Remark 7.1. For $n = 2$, the Lie-group embedding $\mathrm{U}(n) \to \mathrm{Spin}^c(2n)$ mentioned in Section 1 is given by $\mathfrak{A} \mapsto (\mathrm{diag}(1, \det \mathfrak{A}), \mathfrak{A})$, in the notation of Remark 5.2. (See [8, p. 53].) In fact, given an almost complex surface M and $y \in M$, it is this homomorphism that renders equivariant the mapping which sends any orthonormal basis (ϕ^-, χ^-) of $\sigma_y^- = T_yM$ to the basis $(\phi^+, \chi^+, \phi^-, \chi^-)$ of $\sigma_y^+ \oplus \sigma_y^-$ with (8), defined by $\phi^+ = 1$ and $\chi^+ = \phi^- \wedge \chi^-$, where $\sigma_y^+ = \mathbf{C} \oplus \kappa_y$, and so $\phi^+ \in \mathbf{C} \subset \sigma_y^+$, while $\chi^+ \in \kappa_y \subset \sigma_y^+$.

8. An Explicit Diffeomorphism $S^2 \times S^2 \approx G_2^+(\mathbf{R}^4)$

A spin^c(4)-geometry (see Definition 5.1) gives rise to a specific diffeomorphic identification $S^2 \times S^2 \approx G_2^+(\mathbf{R}^4)$ with $G_2^+()$ as in Section 3.

Namely, given a spin^c(4)-geometry $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ and any complex lines $\mathcal{L} \subset \mathcal{S}^+$ and $\mathcal{L}' \subset \mathcal{S}^-$ (cf. Section 2), let us denote by $\mathcal{T} = \Psi(\mathcal{L}, \mathcal{L}')$ the real subspace $\{A \in \mathcal{V} : A\mathcal{L} \subset \mathcal{L}'\}$ of the oriented real 4-space $\mathcal{V}$ given by (9). Then

(a) $\mathcal{T}$ is naturally real-isomorphic to the complex line Hom$(\mathcal{L}, \mathcal{L}')$. Consequently, dim $\mathcal{T} = 2$ and $\mathcal{T}$ is canonically oriented, that is, $\mathcal{T} \in G_2^+(\mathcal{V})$.

(b) $\overline{\mathcal{T}}^\perp = \Psi(\mathcal{L}^\perp, \mathcal{L}')$, where $\overline{\mathcal{T}}^\perp \subset \mathcal{V}$ is the crthogonal complement of the oriented plane $\mathcal{T}$, endowed with the *nonstandard* orientation (cf. Section 2).

(c) Given $\psi \in \mathcal{S}^+\backslash\{0\}$, we have $\psi \in \mathcal{L}$ (or, $\psi \in \mathcal{L}^\perp$) if and only if the operator $\mathcal{T} \ni A \mapsto A\psi \in S^-$ is an orientation preserving (or, reversing) real-linear isomorphism onto a complex line in $\mathcal{S}^-$.

Finally, for $\Psi(\mathcal{L}, \mathcal{L}') = \mathcal{T}$ depending as above on complex lines $\mathcal{L}, \mathcal{L}'$,

$$\Psi : P(\mathcal{S}^+) \times P(\mathcal{S}^-) \to G_2^+(\mathcal{V}) \text{ is a diffeomorphism,} \tag{10}$$

with $P(W)$ defined as in Section 2 for $W = S^\pm$.

The remainder of this section is devoted to proving (a)–(c) and (10). We set $\mathcal{S}^\pm = \mathcal{V} = \mathbf{H}$, as in Lemma 5.3, and choose $p, q \in \mathbf{H}\backslash\{0\}$ with $\mathcal{L} = p\mathbf{C}$, $\mathcal{L}' = q\mathbf{C}$. Then $\mathcal{T} = \{x \in \mathbf{H} : xp \in q\mathbf{C}\} = \mathrm{q}\mathbf{C}p^{-1}$, with $\mathbf{C} = \mathrm{Span}_\mathbf{R}\,(1, \mathbf{i}) \subset \mathbf{H}$, so that (a) follows, and $(qp^{-1}, q\mathbf{i}p^{-1})$ is a positive-oriented basis of $\mathcal{T}$.

Since the quaternion norm is multiplicative, the left or right multiplication by a nonzero quaternion is a homothety. Given $p, q \in \mathbf{H}\backslash\{0\}$, both $(p, p\mathbf{i}, p\mathbf{j}, p\mathbf{k})$ and $(qp^{-1}, q\mathbf{i}p^{-1}, q\mathbf{j}p^{-1}, q\mathbf{k}p^{-1})$ thus are real-orthogonal bases of $\mathbf{H}$. For $\mathcal{L} = p\mathbf{C}$, we now have $\mathcal{L}^\perp = p\mathbf{j}\mathbf{C}$ (as one sees using the first basis: $(p, p\mathbf{i})$ is an $\mathbf{R}$-basis of $\mathcal{L}$, and so $p\mathbf{j}$ and $-p\mathbf{k} = p\mathbf{j}\mathbf{i}$ must form an $\mathbf{R}$-basis of $\mathcal{L}^\perp$). Also, since $\mathcal{T} = \mathrm{Span}_\mathbf{R}\,(qp^{-1}, q\mathbf{i}p^{-1})$, orthogonality of the *second* basis implies that $\overline{\mathcal{T}}^\perp = \mathrm{Span}_\mathbf{R}\,(q\mathbf{j}p^{-1}, q\mathbf{k}p^{-1})$, that is, $\overline{\mathcal{T}}^\perp = q\mathbf{C}(p\mathbf{j}^{-1}) = \psi(\mathcal{L}^\perp, \mathcal{L}')$, which proves (b). (We write $\overline{\mathcal{T}}^\perp$ rather than $\mathcal{T}^\perp$, since the positive-oriented basis $(qp^{-1}, q\mathbf{i}p^{-1})$ of $\mathcal{T}$ and its analogue $(q(p\mathbf{j})^{-1}, q\mathbf{i}(p\mathbf{j})^{-1}) = (-q\mathbf{j}p^{-1}, -q\mathbf{k}p^{-1})$ for the orthogonal complement together form a *negative-oriented* basis of $\mathbf{H} = \mathcal{V}$, due to (6) and connectedness of $\mathbf{H} \setminus \{0\}$.)

Furthermore, the $\mathbf{C}$-linear operator $\mathbf{H} \to \mathbf{H}$ of left multiplication by any fixed nonreal quaternion, being a matrix operator of the form (7), must have a pair of conjugate nonreal eigenvalues. By Lemma 5.3, this applies to $\Omega = A^{-1}B : \mathcal{S}^+ \to \mathcal{S}^+$ (or, $\Omega = -AB^{-1}$: $\mathcal{S}^- \to \mathcal{S}^-$) whenever $A, B \in \mathcal{V}$ are linearly independent over $\mathbf{R}$. For any $\mathcal{T} \in G_2^+(\mathcal{V})$, let us now define $\Psi'(\mathcal{T})$ to be the pair $(\mathcal{L}, \mathcal{L}') \in P(\mathcal{S}^+) \times P(\mathcal{S}^-)$ obtained by fixing any positive-oriented basis (A, B) of $\mathcal{T}$ and choosing $\mathcal{L}$ (or, $\mathcal{L}'$) to be the eigenspace of $A^{-1}B$ (or, $-AB^{-1}$) for the unique eigenvalue $z \in \mathbf{C}$ with Im $z > 0$. That Ψ' does not depend on the choice of the basis (A, B) is easily seen, for $\mathcal{L}$, if we replace (A, B) with $(B, -A)$ or a basis of the form (A, B'), and, for $\mathcal{L}'$, if instead of (A, B) we use $(B, -A)$ or (A', B). It is now immediate that $A\mathcal{L} \subset \mathcal{L}'$ and $B\mathcal{L} \subset \mathcal{L}'$, i.e., $\Psi(\mathcal{L}, \mathcal{L}') = \mathcal{T}$, which shows that $\Psi \circ \Psi' = \mathrm{Id}$. Moreover, using Lemma 5.3 as before, we get $\Psi' \circ \Psi = \mathrm{Id}$. Namely, $\mathcal{T} = q\mathbf{C}p^{-1}$ has the positive-oriented basis $(A, B) = (qap^{-1}, qazp^{-1})$, with $a, z \in \mathbf{C}$, $a \neq 0$ and $\operatorname{Im} z > 0$. This makes $A^{-1}B$ (or, $-AB^{-1}$) appear as the left quaternion multiplications by pzp^{-1} (or, $-qz^{-1}q^{-1}$), which has the eigenspace $\mathcal{L} = p\mathbf{C}$ (or, $\mathcal{L}' = q\mathbf{C}$) with the eigenvalue z (or, z^{-1}). Now (10) follows, with $\Psi' = \Psi^{-1}$.

Finally, according to the above description of $\Psi' = \Psi^{-1}$ in terms of a positive-oriented basis (A, B) of $\mathcal{T}$, condition $\psi \in \mathcal{L}$ (or, $\psi \in \mathcal{L}^\perp$) is equivalent to $B\psi = zA\psi$ (or, $B\psi = \bar{z}A\psi$) for some $z \in \mathbf{C}$ with $\operatorname{Im} z > 0$. (The case of $\mathcal{L}^\perp$ follows from that of $\mathcal{L}$ since $\Omega = A^{-1}B$ is a homothety, cf. (9), and so $\mathcal{L}^\perp$ must be Ω-invariant as long as $\mathcal{L}$ is.) In other words, $\psi \in \mathcal{L}$ (or, $\psi \in \mathcal{L}^\perp$) if and only if $(A\psi, B\psi)$ is a positive (or, negative) oriented real basis of a complex line. This yields (c).

9. Line Bundles Associated with a Spin$^{\mathrm{c}}$-Structure

Given a spin$^{\mathrm{c}}$-structure $(\sigma^+, \sigma^-, \kappa)$ over a 4-manifold M (see Section 6), let λ^+ and λ^- be the complex line bundles over the Grassmannian manifold G_2^+M (defined in Section 3), whose fibres at any $(y, \mathcal{T})$ are the lines $\mathcal{L}, \mathcal{L}'$ in σ_y^+, σ_y^- with $\Psi(\mathcal{L}, \mathcal{L}') = \mathcal{T}$ for Ψ as in (10), where $(\mathcal{S}^+, \mathcal{S}^-, \kappa) = (\sigma_y^+, \sigma_y^-, \kappa_y)$ and $\mathcal{V}$, given by (9), is identified with T_yM. We then have

$$\pi^*\sigma^\pm = \lambda^\pm \oplus \mu^\pm, \tag{11a}$$

$$\pi^*\kappa = \lambda^\pm \otimes \mu^\pm, \tag{11b}$$

for the bundle projection $\pi : G_2^+M \to M$ and the line bundles $\mu^\pm$ which are the orthogonal complements of $\lambda^\pm$ in $\pi^*\sigma^\pm$. (The natural isomorphic identification in (11b) is obvious from (11a) as $\kappa = [\sigma^\pm]^{\wedge 2}$, cf. Section 6.)

We wish to emphasize that even when the spin$^{\mathrm{c}}$-structure $(\sigma^+, \sigma^-, \kappa)$ in question arises from an almost Hermitian structure (J, g), as in Section 7, J and g do not seem to provide any shortcuts for defining $\lambda^\pm$ and $\mu^\pm$. If anything, they are rather an impediment, unless one simply ignores them; this amounts to ignoring both the decomposition $\sigma^+ = \iota \oplus \kappa$ and the presence of a distinguished unit section ψ of σ^+, so that, in effect, one treats $(\sigma^+, \sigma^-, \kappa)$ as if it were just any spin$^{\mathrm{c}}$-structure, with no additional features.

For any oriented 4-manifold M and G_2^+M, π as above, the tangent bundle $T[G_2^+M]$ admits specific oriented real-plane subbundles τ, ν such that $\pi^*TM = \tau \oplus \nu \subset T[G_2^+M]$. Namely, τ is the *tautological bundle* with the fibre $\mathcal{T} \subset T_yM$ over any $(y, \mathcal{T})$. The other summand ν, which might be called the *tautonormal bundle*, is obtained by choosing a Riemannian metric on M and setting $\nu = \tau^\perp \subset \pi^*TM$.

If M now happens to be the canonically-oriented base manifold of a spin$^{\mathrm{c}}$-structure $(\sigma^+, \sigma^-, \kappa)$, the definitions of $\lambda^\pm, \mu^\pm$ and (a), (b) in Section 8 give the following natural isomorphic identifications of oriented real-plane bundles:

$$\tau = \mathrm{Hom}_{\mathbf{C}}(\lambda^+, \lambda^-), \qquad \bar{\nu} = \mathrm{Hom}_{\mathbf{C}}(\mu^+, \lambda^-), \tag{12}$$

with $\bar{\nu}$ standing for ν with the reversed orientation. We may treat τ, ν as complex line bundles, using their orientations and the fibre metrics on them induced by a fixed Riemannian metric on M, along with Remark 2.1 (or, just (12)). Then $\bar{\nu} = \nu^*$ and $\mathrm{Hom}_{\mathbf{C}}(\bar{\nu}, \tau) = \nu \otimes \tau$. Hence, by (12), $\nu \otimes \tau = \mu^+ \otimes \overline{\lambda^-} \otimes \lambda^- \otimes \overline{\lambda^+} = \mu^+ \otimes \overline{\lambda^+}$, so that a natural isomorphism also exists between $\mathrm{Hom}_{\mathbf{C}}(\bar{\nu}, \tau)$ and $\mathrm{Hom}_{\mathbf{C}}(\lambda^+, \mu^+)$.

Since $T\Sigma = F^*\tau$ and $\nu_f = F^*\nu$, where ν_f and $F : \Sigma \to G_2^+ M$ are the normal bundle and the Gauss mapping of any given immersion $f : \Sigma \to M$ of an oriented real surface Σ (see Section 3), this leads to an *adjunction formula*, which equates $\mathrm{Hom}_{\mathbf{C}}(\overline{\nu_f}, T\Sigma)$ with the F-pullback of $\mathrm{Hom}_{\mathbf{C}}(\lambda^+, \mu^+)$.

10. The Mapping (10) and Antilinearity

Let $\mathcal{V}$ be the real 4-space with (9) for a spin^c (4)-geometry $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$, and let Ψ be the diffeomorphism appearing in (10). Then, for any $\mathcal{L}' \in P(\mathcal{S}^-)$,

$$\text{the differential of } P(\mathcal{S}^+) \ni \mathcal{L} \mapsto \Psi(\mathcal{L}, \mathcal{L}') \in G_2^+(\mathcal{V}) \text{ is antilinear} \tag{13}$$

at every point of $P(\mathcal{S}^+)$, antilinearity referring to the standard complex structure of the complex projective line $P(\mathcal{S}^+)$ and the almost complex structure on $G_2^+(\mathcal{V})$ associated, as in Section 3, with the Euclidean inner product in $\mathcal{V}$ mentioned in Remark 5.2.

In fact, let $\mathcal{S}^\pm = \mathcal{V} = \mathbf{H}$ as in Lemma 5.3. As stated in the two paragraphs following (10), for any given $\mathcal{L} \in P(\mathcal{S}^+)$ and $\mathcal{L}' \in P(\mathcal{S}^-)$ we then have $\mathcal{L} = p\mathbf{C}$, $\mathcal{L}^\perp = p\mathbf{j}\mathbf{C}$ and $\mathcal{L}' = q\mathbf{C}$ with some unit quaternions p, q, and so $(q\bar{p}, q\mathbf{i}\bar{p})$ is a positive-oriented basis of the plane $\mathcal{T} = \Psi\,(\mathcal{L}, \mathcal{L}')$, that is, $\mathcal{T} = q\mathbf{C}\bar{p}$, while $\mathcal{T}^\perp = q\mathbf{C}\mathbf{j}\bar{p}$. (Note that $p^{-1} = \bar{p}$ since $|p| = 1$.) As usual, $\mathbf{C} = \mathrm{Span}_{\mathbf{R}}(1, \mathbf{i}) \subset \mathbf{H}$. Thus, we may write $\mathcal{T} = \mathbf{C}$, identifying $z \in \mathbf{C}$ with $qz\overline{p}$. The above basis of $\mathcal{T}$ then becomes $(1, i)$, so that $\mathbf{C}$ with its usual structure is precisely the complex line formed by the oriented Euclidean plane $\mathcal{T}$ (see Remark 2.1).

By (2), $T_{\mathcal{L}}[P(\mathcal{S}^+)] = \mathrm{Hom}_{\mathbf{C}}(\mathcal{L}, \mathcal{L}^\perp) = \mathrm{Hom}_{\mathbf{C}}(p\mathbf{C}, p\mathbf{j}\mathbf{C}) = \mathbf{C}$, where each $c \in \mathbf{C}$ is identified with the operator $p\mathbf{C} \to p\mathbf{j}\mathbf{C}$ sending p to $p\mathbf{j}c$. Similarly, the description of $T_L[G_m(W)]$ in Section 2 identifies the tangent space of $G_2^+(\mathcal{V})$ at $\mathcal{T}$ with $\mathrm{Hom}_{\mathbf{R}}\,(\mathcal{T}, \mathcal{V}/\mathcal{T}) = \mathrm{Hom}_{\mathbf{R}}(\mathcal{T}, \mathcal{T}^\perp)$. We will now show that the differential of the mapping (13) at $\mathcal{L}$ sends $c \in \mathbf{C} = T_{\mathcal{L}}[P(\mathcal{S}^+)]$ to the operator $A_c : \mathbf{C} \to \mathcal{T}^\perp$ given by $A_c z = -qz\bar{c}\mathbf{j}\bar{p}$ (with $\mathcal{T} = \mathbf{C}$ as before). Since $A_{ic}z = -A_c(iz)$, (13) will follow.

To this end, let us replace p by a C^1 function of a real parameter t, valued in unit quaternions, equal at $t = 0$ to the original p, and such that dp/dt at $t = 0$ equals $p\mathbf{j}c$. Then $d\mathbf{w}/dt$ at $t = 0$ for the t-dependent basis $\mathbf{w} = (q\bar{p}, q\mathbf{i}\bar{p})$ of the t-dependent plane $\mathcal{T} = q\mathbf{C}\bar{p}$ equals $\mathbf{v} = (A_c 1, A_c i)$, with A_c defined above, which (cf. Remark 2.2) proves our claim about A_c.

Remark 10.1. A virtually identical argument shows that the differential of $P(\mathcal{S}^-) \ni \mathcal{L}' \mapsto \Psi(\mathcal{L}, \mathcal{L}') \in G_2^+(\mathcal{V})$ is *complex-linear* at every point of $P(\mathcal{S}^-)$, for any fixed $\mathcal{L}' \in P(\mathcal{S}^-)$. (Note that in $\mathcal{T} = q\mathbf{C}\bar{p}$ conjugation is applied to p, but not to q.) Along with (13) this implies that the almost complex structure of $G_2^+(\mathcal{V})$ is integrable and (10) is a biholomorphism, provided that the factor $P(\mathcal{S}^+)$ in $P(\mathcal{S}^+) \times P(\mathcal{S}^-)$ carries the *conjugate* of its standard complex structure.

Note the well-known fact that, for a Euclidean space W of any dimension, the almost complex structure on $G_2^+(W)$ described in Section 3 is integrable.

11. Transversality

Given a C^1 section ϕ of a vector bundle η over a manifold N and a point $\xi \in N$ with $\phi(\xi) = 0$,

(a) ϕ is transverse at ξ to the zero section of η if and only if, for the fibre-valued function $\phi' : U \to \mathbf{F}$ representing ϕ in some, or any, local trivialization of η over a neighborhood U of ξ, the differential $d\phi'_\xi\colon T_\xi N \to \mathbf{F}$ is surjective.

(b) ϕ is transverse at ξ to the zero section of η whenever this is the case for η, ϕ restricted to a submanifold of N containing ξ.

This is a trivial exercise; (b) easily follows from (a).

Let $P(W)$ now be a real, complex or quaternion projective line obtained as in Section 2 from a plane W over a field $\mathbf{K}$ (one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$), and let η be the tautological $\mathbf{K}$-line bundle over $P(W)$, with the fibre L at any $L \in P(W)$. Choosing, in addition, a $\mathbf{K}$-valued sesquilinear inner product $\langle,\rangle$ in W and a nonzero vector $u \in W$, we now also define a $\mathbf{K}$-line bundle ζ over $P(W)$ along with C^∞ sections ϕ in η and χ in ζ. Namely, $\zeta = \eta^\perp$ is the orthogonal complement of η treated as a subbundle of the product bundle $\theta = P(W) \times W$, while ϕ, χ are the η and ζ components of u, which is a constant section of θ, relative to the decomposition $\theta = \eta \oplus \zeta$. Then

(i) Either of ϕ, χ has just one zero, at which it is transverse to the respective zero section in η or ζ.

(ii) If $\mathbf{K} = \mathbf{C}$, then, for some $\phi', \chi' : U \to \mathbf{C}$ representing ϕ, χ as in (a) above, the differential of ϕ' (or, χ') at the zero in question is an antilinear (or, complex-linear) isomorphism.

In fact, we may fix $v \in W$ with $\langle u, v\rangle = 0$ and $|v| = |u|$. The unique zero of ϕ (or, χ) is $u^\perp = \mathbf{K}v \in P(W)$ (or, $v^\perp = \mathbf{K}u \in P(W)$); let us agree to identify its neighborhood $M' = P(W) \setminus \{v^\perp\}$ (or, $M' = P(W) \setminus \{u^\perp\}$) with $\mathbf{K}$, using the diffeomorphism $\mathbf{K} \to M'$ given by $z \mapsto \mathbf{K}(zu + v)$ (or $z \mapsto \mathbf{K}(zv + u)$), cf. Remark 2.3. Under this identification, a local trivializing section of η (or, ζ) is defined on $M' = \mathbf{K}$ by $z \mapsto zu + v$ (or, $z \mapsto \bar{z}u + v$) and, evaluating $\langle,\rangle$-orthogonal projections of u onto the directions of $zu + v$ and $\bar{z}u + v$ in W, we see that the $\mathbf{K}$-valued functions ϕ', χ' corresponding to ϕ, χ as in (a) above are $z \mapsto \bar{z}$ and $z \mapsto z$ divided, in both cases, by $(|z|^2 + 1)\,|u|^2$. Their differentials at the respective zeros (that is, at $z = 0$) are $z \mapsto \bar{z}/|u|^2$ and $z \mapsto z/|u|^2$. This gives (ii) and, combined with (a) above, proves the transversality claim in (i).

Remark 11.1. Suppose that K, K' are two-dimensional submanifolds of a real 4-manifold M, both containing some given point $\xi \in N$, and K is the set of zeros of a C^∞ section ϕ of a complex line bundle λ over N, transverse to the zero section. Let α, β, γ be the differentials at ξ of the inclusion mappings $K \to N$, $K' \to N$ and, respectively, of the restriction to $K' \cap U$ of a function $\phi' : U \to \mathbf{C}$ that represents ϕ in some local trivialization of λ having a domain U with $\xi \in U$. Finally, let N, K and K' all carry some fixed almost complex structures, so that they are all oriented via (1).

If γ is injective and β is antilinear, while α, γ are both complex-linear or both antilinear, then the orientation of K defined by (1) is the opposite of the orientation that $K \subset N$ acquires by being the zero set of ϕ, cf. Section 2.

In fact, as $\phi' = 0$ on $K \cap U$, injectivity of γ gives $T_\xi N = T_\xi K \oplus T_\xi K'$. The complex-plane orientation of $T_\xi N$ (given by (1)) is the direct sum of the complex-line orientation in $T_\xi K$ and the opposite of the orientation in $T_\xi K'$ pulled back by γ from $\mathbf{C}$. In fact, our assumptions state that the complex structure prescribed in $T_\xi K$, and the one in $T_\xi K'$ making γ antilinear, are either both identical with, or both conjugates of, the complex structures that $T_\xi K$, $T_\xi K'$ inherit from $T_\xi N$ by being its complex subspaces.

12. Poincaré Duals of Q and $\bar{Q}$ in G_2^+M

Let K be a compact, oriented, not necessarily connected, codimension-two submanifold of an oriented even-dimensional real manifold N, and let λ be a complex line bundle over N. We will say that λ is *Poincaré-dual* to K if K is the canonically oriented set of zeros (cf. Section 2) of some C^∞ section of λ transverse to the zero section.

This implies that $c_1(\lambda) \in H^2(N, \mathbf{Z})$ corresponds to $[K] \in H_{n-2}(N, \mathbf{Z})$ under the Poincaré duality ($n = \dim M$), as $\int_\Delta c_1(\lambda) = [K] \cdot [\Delta]$ for any 2-cycle Δ in N, due to the Poincaré index formula for λ restricted to Δ.

THEOREM 12.1. *Let M be an almost Hermitian surface treated as a* 4*-manifold along with a spin^c-structure $(\sigma^+, \sigma^-, \kappa)$ and a fixed global unit C^∞ section ψ of σ^+, cf. Section* 7*. The line bundles λ^+ and μ^+ over G_2^+M, introduced in Section* 9*, then are Poincaré-dual to the oriented six-dimensional submanifolds $\bar{Q}$ and Q of G_2^+M, defined at the end of Section* 3.

More precisely, the λ^+ and μ^+ components ϕ, χ of $\pi^*\psi$ relative to the decomposition (11a) *are transverse to the zero sections in λ^+ and μ^+, and their respective oriented manifolds of zeros are $\bar{Q}$ and Q.*

Proof. Let us fix any $y \in M$ and consider the spin^c(4)-geometry $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K}) = (\sigma_y^+, \sigma_y^-, \kappa)$ with the identification $\mathcal{V} = T_yM$ (see Section 6). The fibre $G_2^+(T_yM)$ of G_2^+M over y thus is identified with $G_2^+(\mathcal{V})$. That the respective *sets of* zeros of ϕ, χ in G_2^+M are $\bar{Q}$ and Q is now clear from (c) in Section 8 along with the fact that the Clifford multiplication by ψ is an isomorphism $TM \to \sigma^-$ of *complex* vector bundles (see Section 7), and the definitions of λ^+, μ^+ in Section 9.

The same definitions clearly imply that the pullbacks of λ^+, μ^+ under (13), for any fixed $\mathcal{L}'$, are the bundles η, ζ over $P(W)$, for $W = \mathcal{S}^+$, described in Section 11, while ϕ and χ, pulled back to $P(\mathcal{S}^+)$ via (13), coincide with the sections ϕ and χ defined in Section 11 for $u = \psi(y)$. Our transversality claim thus is obvious from (i) in Section 11 combined with (b) in Section 11 for the submanifold of $N = G_2^+M$ obtained as the image of (13).

What remains to be shown is that the orientations of $\bar{Q}$ and Q defined at the end of Section 3 coincide with the orientations which they acquire by being the zero sets of transverse sections in complex line bundles over the oriented eight-dimensional manifold G_2^+M (cf. Section 2).

However, $\bar{Q}$, Q and G_2^+M are bundle spaces over M, and their orientations described in Section 3 are opposite to the direct sums of the orientations of the base M and those of the fibres $\bar{P}(\mathcal{V})$, $P(\mathcal{V})$ or $G_2^+(\mathcal{V})$. All four orientations are induced by almost complex structures via (1). It, therefore, suffices to establish agreement between the standard orientations of the complex projective lines $\bar{P}(\mathcal{V})$, $P(\mathcal{V})$, and the orientations of $\bar{P}(\mathcal{V})$, $P(\mathcal{V})$ as submanifolds of the canonically oriented almost complex surface $G_2^+(\mathcal{V})$ which are the zero sets of ϕ, χ restricted to the fibre $G_2^+(\mathcal{V})$.

Such agreement is in turn immediate from Remark 11.1 applied to $N = G_2^+(\mathcal{V})$, $\xi = \mathcal{T}$ for any fixed $\mathcal{T} \in \bar{P}(\mathcal{V})$ (or, $\mathcal{T} \in P(\mathcal{V})$), K' which is the image of (13) with $\mathcal{L}'$ chosen so that $\mathcal{T} = \Psi(\mathcal{L}, \mathcal{L}')$ for some $\mathcal{L}$, along with $K = \bar{P}(\mathcal{V})$ (or $K = P(\mathcal{V})$) and with ϕ which is the restriction to the fibre $G_2^+(\mathcal{V})$ of our ϕ (or, of our χ). That the assumptions listed in Remark 11.1 are all satisfied is clear from (ii) in Section 11, (10), (13) and Remark 3.1. This completes the proof. □

COROLLARY 12.2. *For any almost Hermitian structure* (J, g) *on a* 4*-manifold* M*, the line bundle* $\pi^*\kappa$ *over* G_2^+M*, with* $\kappa = \det_{\mathbf{C}}\mathrm{TM}$ *defined as in Section* 4*, is Poincaré-dual to the union* $Q \cup \bar{Q} \subset G_2^+M$*. Thus,* (5) *holds for every immersion* $f : \Sigma \to M$ *of a closed oriented real surface* Σ*.*

In fact, for ϕ, χ as in Theorem 12.1, $\phi \otimes \chi$ is transverse to the zero section in $\pi^*\kappa = \lambda^+ \otimes \mu^+$ (cf. (11b)), by (a) in Section 11 with $Q \cap \bar{Q} = \emptyset$.

One could also prove Corollary 12.2 without mentioning $(\sigma^+, \sigma^-, \kappa)$ and ψ at all. Instead, one might use the section of $\pi^*\kappa$, for $\kappa = \det_{\mathbf{C}}TM$, which assigns to $(y, \mathcal{T}) \in G_2^+M$ the complex exterior product $v \wedge w$ for any positive-oriented g-orthonormal basis (v, w) of the plane $\mathcal{T} \subset T_yM$. This section is transverse to the zero section in $\pi^*\kappa$, as it equals $-2i\phi \otimes \chi$ (by Remark 14.1 below);

however, having to establish its transversality directly would make such a proof quite tedious.

13. Framed Spin^c-Structures on 4-Manifolds

By a *framed spin*^{c}*-structure* over a 4-manifold M we mean a quadruple $(\sigma^+, \sigma^-, \kappa, \psi)$ consisting of a spin^c-structure $(\sigma^+, \sigma^-, \kappa)$ over M and a C^∞ section ψ of σ^+ defined only up to multiplication by positive C^∞ functions on M and transverse to the zero section. In the special case where $(\sigma^+, \sigma^-, \kappa)$ is a spin structure (with $\kappa = M \times \mathbf{C}$, as at the end of Section 6), we will refer to the triple $(\sigma^+, \sigma^-, \psi)$ as a *framed spin structure* over M.

According to the second paragraph of Section 7, an almost Hermitian structure on a 4-manifold M is nothing else than a framed spin^c-structure $(\sigma^+, \sigma^-, \kappa, \psi)$ over M such that $\psi \neq 0$ everywhere. On the other hand, every spin^c-structure $(\sigma^+, \sigma^-, \kappa)$ over a compact 4-manifold is a part of a framed spin^c-structure, with ψ that may be chosen arbitrarily C^1-close to any given C^∞ section of σ^+.

Remark 13.1. For any framed spin^c-structure $(\sigma^+, \sigma^-, \kappa, \psi)$ over a 4-manifold M, let $Z \subset M$ be the discrete set of all zeros of ψ. The open submanifold $M' = M \setminus Z$ then carries the *residual* almost complex structure J, determined as in Section 7 by $(\sigma^+, \sigma^-, \kappa)$ restricted to M' and $\psi/|\psi|$.

LEMMA 13.2. *Given a framed spin*^{c}*-structure* $(\sigma^+, \sigma^-, \kappa, \psi)$ *over a* 4*-manifold* M*, let* ϕ, χ *be the* λ^+ *and* μ^+ *components of the section* $\pi^*\psi$ *of* $\pi^*\sigma^+$ *relative to the decomposition* (11a). *Then* ϕ *and* χ *are transverse to the zero sections in* λ^+ *and* μ^+*, respectively.*

Proof. Transversality at points of the open dense set $\pi^{-1}(M') \subset G_2^+ M$, with $M' = M \setminus Z$ as in Remark 13.1, is obvious from the final clause of Theorem 12.1, applied to $(\sigma^+, \sigma^-, \kappa)$ (and ψ) restricted to M'. Note that we may assume that $|\psi| = 1$ on M$'$ by suitably rescaling both fibre metrics of $\sigma^\pm$, namely, dividing them by $|\psi|^2$.

Given fibre-valued functions ϕ', χ' that represent ϕ, χ in local trivializations of λ^+ and μ^+, it is clear that (ϕ', χ') will similarly represent $\pi^*\psi = \phi + \chi$ in the corresponding direct-sum local trivialization of $\pi^*\sigma^+ = \lambda^+ \oplus \mu^+$. Our transversality assertion at points of $\pi^{-1}(Z)$ is now immediate from (a) in Section 11, which completes the proof. □

At the end of Section 3 we associated with any almost complex surface M a pair $Q, \bar{Q}$ of oriented six-dimensional submanifolds of the Grassmannian manifold $G_2^+ M$. Lemma 13.2 provides a natural generalization of that construction to the case where M is a 4-manifold carrying a framed spin^c-structure $(\sigma^+, \sigma^-, \kappa, \psi)$. Namely, we declare $\bar{Q}$ and Q to be the zero sets in $G_2^+ M$ of the sections ϕ, χ of λ^+ and μ^+ obtained as the components of $\pi^*\psi$ relative to the decomposition

(11a). In view of Lemma 13.2, Q, $\bar{Q}$ are, again, not-necessarily connected, oriented six-dimensional submanifolds of G_2^+M. The conclusion of Theorem 12.1 also remains valid, although now it is nothing else than the definition of $\bar{Q}$ and Q.

Given M, $(\sigma^+, \sigma^-, \kappa, \psi)$ as above and an immersion $f\colon \Sigma \mapsto M$ of an oriented real surface Σ, we call $x \in \Sigma$ a *complex point* of f if it is a *Q-complex point* or a *$\bar{Q}$-complex point*, in the sense that $F(x) \in Q$ or, respectively, $F(x) \in \bar{Q}$, where $F\colon \Sigma \mapsto G_2^+M$ is the Gauss mapping of f (see the end of Section 3). This generalizes the definitions given in Section 4, and will in turn allow us to define, in Sections 14 and 15, what it means for such an immersion f to be totally real or pseudoholomorphic (which, again, is a straightforward extension from the almost-complex case).

Remark 13.3. For M, $(\sigma^+, \sigma^-, \kappa, \psi)$ and Z as in Remark 13.1, and for any immersion $f : \Sigma \mapsto M$ of an oriented real surface Σ, every point $x \in \Sigma$ with $f(x) \in Z$ is a complex point of f. More precisely, every oriented plane in $T_{f(x)}M$ then lies in both Q and $\bar{Q}$.

14. Totally Real Immersions of Oriented Surfaces

Given a framed spin^c-structure $(\sigma^+, \sigma^-, \kappa, \psi)$ over a 4-manifold M (see Section 13), we will say that an immersion $f\colon \Sigma \mapsto M$ of an oriented real surface Σ is *totally real* if f has no complex points (defined as in Section 13), that is, if we have $(Q \cup \bar{Q}) \cap F(\Sigma) = \emptyset$ for its Gauss mapping F.

Relations (3) and (4) remain valid in this case. In fact, the pullbacks $F^*\lambda^+$ and $F^*\mu^+$ are canonically trivialized by the λ^+ and μ^+ components of $\pi^*\psi$, so that (11b) yields (3), while (12) now gives $F^*\tau = F^*\lambda^- = F^*\bar{\nu}$, and hence (4). (Note that $T\Sigma = F^*\tau$ and $\nu_f = F^*\nu$, cf. the end of Section 9).

Immersions of oriented real surfaces in 4-manifolds M with framed spin^c-structures $(\sigma^+, \sigma^-, \kappa, \psi)$ obviously include, as a special case, their immersions in almost complex surfaces (see Section 13). However, for totally real immersions one can also turn this relation around and view the former as a special case of the latter. Namely, the image of such an immersion $f : \Sigma \to M$ must lie in the open submanifold $M \setminus Z$ on which $\psi \neq 0$ (cf. Remark 13.3) and, by Theorem 12.1, $f\colon \Sigma \to M \setminus Z$ then is a totally real immersion in the almost complex surface formed by $M \setminus Z$ with the residual almost complex structure J described in Remark 13.1.

Let $(\sigma^+, \sigma^-, \kappa, \psi)$ again be a framed spin^c-structure over a 4-manifold M, and let E^+ denote the total space of the unit-circle bundle of κ. Every totally real immersion $f : \Sigma \to M$ of an oriented closed real surface Σ gives rise to a mapping $\Xi : \Sigma \to E^+$ with $\Xi(x) = (y, \rho/|\rho|)$, where, for any $x \in \Sigma$, we set $y = f(x)$ and $\rho = \phi \otimes \chi$ with ϕ, χ standing for the λ^+ and μ^+ components of $\pi^*\psi$ at $F(x)$. (Thus, $\phi \otimes \chi \in (\pi^*\kappa)_{F(x)} = \kappa_y$ by (11b).) The homotopy class of this mapping $\Sigma \to E^+$ may be called the *oriented Maslov invariant* of f. (See [2] and Remark 14.1 below.)

Here $\pi\colon G_2^+ M \to M$ is, as usual, the bundle projection of the Grassmannian manifold, and F is the Gauss mapping of f, cf. the end of Section 3.

According to Gromov [4, p. 192], the h-principle holds for totally real immersions of closed real surfaces in almost complex surfaces. Therefore, the oriented Maslov invariant classifies such immersions up to the equivalence relation of being homotopic through totally real immersions. (See [2].)

The last conclusion remains valid, more generally, for totally real immersions of oriented closed real surfaces in a 4-manifold M with a framed spin^c-structure $(\sigma^+, \sigma^-, \kappa, \psi)$. Namely, Z in Remark 13.1 is a discrete subset of M. Hence, for dimensional reasons, the set of homotopy classes of mappings $\Sigma \to E^+$ remains unchanged when E^+ is replaced by its portion lying over $M \setminus Z$, while the set of equivalence classes of totally real immersions $\Sigma \to M$ is the same as for totally real immersions $\Sigma \to M \setminus Z$.

Remark 14.1. As explained in Remark 14.2 below, $v \wedge w = -2i\phi \otimes \chi$ whenever (v, w) is a positive-oriented g-orthonormal basis of a real plane $\mathcal{T} \subset T_y M$ at a point y of a 4-manifold M with a Hermitian structure (J, g), and ϕ, χ denote the λ^+ and μ^+ components of $\pi^*\psi$ at $(y, \mathcal{T})$ relative to the decomposition (11a) for the spin^c-structure $(\sigma^+, \sigma^-, \kappa)$ and the unit section ψ of σ^+ associated with (J, g) as in Section 7. Here $v \wedge w$ is the *complex* exterior product, so that $v \wedge w \in \kappa_y$ with $\kappa = \det_{\mathbf{C}} TM$ (see Section 7); also, $\phi \otimes \chi \in (\pi^*\kappa)_{(y,\mathcal{T})} = \kappa_y$ in view of (11b).

Consequently, the above definition of the Maslov invariant is equivalent to that in [2], which uses $v \wedge w/|v \wedge w|$ instead of our $\rho/|\rho|$. (The $-i$ factor, representing a rotation, leaves the homotopy class unaffected.)

Remark 14.2. Relation $v \wedge w = 2i\phi \otimes \chi$, under the assumptions listed in Remark 14.1, amounts to $A\psi \wedge B\psi = -2i\phi \wedge \chi$ for any positive-oriented orthonormal basis (A, B) of $\mathcal{T} \in G_2^+(\mathcal{V})$, where $\mathcal{V}$ is the space (9) for a fixed spin^c(4)-geometry $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$, and ϕ, χ are the $\mathcal{L}$ and $\mathcal{L}^\perp$ components of $\psi \in \mathcal{S}^+$, with complex lines $\mathcal{L} \in P(\mathcal{S}^+)$ and $\mathcal{L}' \in P(\mathcal{S}^-)$ such that $\mathcal{T} = \Psi(\mathcal{L}, \mathcal{L}')$ for Ψ as in (10). Note that it is the Clifford multiplication by $\psi(y)$, for $y \in M$, that turns $\mathcal{V} = T_y M$ into a *complex* vector space by identifying it with σ_y^- (see Section 7).

Equality $A\psi \wedge B\psi = -2i\phi \wedge \chi$ follows in turn since, due to the matrix form of any $A \in \mathcal{V}$ (see Remark 5.2), A^*A equals $|A|^2$ times the identity of $\mathcal{S}^+$. The Euclidean inner product $\langle , \rangle$ of $\mathcal{V}$ thus is given by $4\langle A, B\rangle = \mathrm{Trace}_{\mathbf{R}} A^*B$ which, if $A \neq 0$, coincides with $|A|^{-2}\mathrm{Trace}_{\mathbf{R}}\, A^{-1}B$. However, $A^{-1}B$ has the eigenvalues $z, \bar{z}$ with $\mathrm{Im}\, z > 0$, realized by eigenvectors in $\mathcal{L}$ and, respectively, $\mathcal{L}^\perp$. (See the end of Section 8.) As $\langle A, B\rangle = 0$, that is, $\mathrm{Trace}_{\mathbf{R}} A^{-1}B = 0$, we then have Re $z = 0$. Also, A, B are linear isometries (being homotheties of norm 1, cf. Remark 5.2). Hence $|z| = 1$, and so $z = i$. Now $B\phi = iA\phi$, $B\chi = -iA\chi$. Thus, from

$\psi = \phi + \chi$, we get $iA\psi \wedge B\psi = 2A\phi \wedge A\chi = 2(\det A)\phi \wedge \chi$ with $\det A = 1$ (see Remark 5.2).

15. Pseudoholomorphic Immersions

Let $(\sigma^+, \sigma^-, \kappa, \psi)$ be a framed spin^c-structure over a 4-manifold M (see Section 13). We say that an immersion $f : \Sigma \to M$ of an oriented real surface Σ is *pseudoholomorphic* if the image $F(\Sigma) \subset G_2^+ M$ of the Gauss mapping of f (cf. the end of Section 3) is contained in the set Q described in Section 13, that is, if every point of Σ is a Q-complex point of f.

In terms of the set Z of zeros of ψ, an immersion $f : \Sigma \to M$ of an oriented real surface is pseudoholomorphic if and only if its restriction to $\Sigma \setminus f^{-1}(Z)$ is pseudoholomorphic, in the sense of Section 4, relative to the residual almost complex structure J on $M' = M \setminus Z$ described in Remark 13.1. (In fact, by Remark 13.3, all points of $f^{-1}(Z)$ are Q-complex points of f.) Such an immersion, therefore, induces an almost complex structure on $\Sigma \setminus f^{-1}(Z)$, which is always integrable as $\dim \Sigma = 2$, and has an extension to a complex structure on Σ (by Remark 15.1 below, as Z is discrete).

Remark 15.1. Let g be a Riemannian metric on a manifold M and let J be an almost complex structure on $M' = M \setminus \{y\}$, compatible with g, for some given point $y \in M$. If $\Sigma \subset M$ is a two-dimensional real submanifold such that $y \in \Sigma$ and the inclusion mapping $\Sigma' \to M'$ of $\Sigma' = \Sigma \setminus \{y\}$ is pseudoholomorphic relative to J, then the (integrable) almost complex structure which J induces on Σ' has a C^∞ extension to Σ.

This is clear as J restricted to $T\Sigma'$ is uniquely determined, as in Remark 2.1, by the metric on Σ' induced by g and the orientation of Σ' induced by J via (1), both of which admit obvious extensions to Σ.

16. A Projection of $\mathbf{C}\mathrm{P}^n$ onto S^{2n}

Throughout this section, V is a fixed complex vector space of complex dimension $n < \infty$ with a Hermitian inner product $\langle , \rangle$ and the corresponding norm $|\ |$, while S, P stand for its natural compactifications: namely, $S \approx S^{2n}$ is obtained by adding to V a new point at infinity, that is, $S = V \cup \{\infty\}$, while $P \approx \mathbf{C}\mathrm{P}^n$ is the set of all complex lines through zero in the direct-product complex vector space $V \times \mathbf{C}$, and so $P = P(V \times \mathbf{C})$ (cf. Section 2). We use the homogeneous-coordinate notation, with $[y, z] \in P$ denoting the line spanned by $(y, z) \in V \times \mathbf{C}$ when $y \in V$, $z \in \mathbf{C}$ and $|y| + |z| > 0$. Thus, $P = V \cup H$, where we treat V as a subset of P, identifying $y \in V$ with $[y, 1]$, and $H \approx \mathbf{C}\mathrm{P}^{n-1}$ is the hyperplane of all $[y, 0]$ with $y \in V \setminus \{0\}$. For $y \in V \setminus \{0\}$ we now have $[y, 0] = L \times \{0\} \in H$, with $L = \mathbf{C}y \in P(V)$.

Finally, Φ, pr denote the *inversion* Φ: $S \to S$ with $\Phi(y) = y/|y|^2$ for any $y \in V \setminus \{0\}$, $\Phi(0) = \infty$, $\Phi(\infty) = 0$, and the *projection* pr: $P \to S$ with pr $([y, z]) = y/z$ if $z \neq 0$ and pr $([y, 0]) = \infty$.

The C^∞-manifold structure of S is introduced by an atlas of two V-valued charts, formed by Id: $S \setminus \{\infty\} = V \to V$ and $\Phi : S \setminus \{0\} \to V$. As one easily verifies in both charts, Φ is a diffeomorphism $S \to S$, while pr: $P \to S$ is always of class C^∞ and, if $n = 1$, it is a diffeomorphism as well. (A function of $[y, z]$ is differentiable if and only if it is differentiable in (y, z).) Thus, pr: $V \cup H \to V \cup \{\infty\}$ is the result of combining Id: $V \to V$ with the constant mapping $H \to \{\infty\}$. In other words, pr identifies S with the quotient of P obtained by collapsing H to the single point ∞.

For any $L \in P(V)$, that is, any complex line L through 0 in V,

$$L \text{ is the image of the differential of } \Phi \circ \text{pr at the point } L \times \{0\} \in P, \tag{14}$$

with L treated as a real vector subspace of $V = T_0V = T_0S$. In fact, since pr is constant on H, its real rank at $L \times \{0\}$ is at most 2, and so (14) will follow if we show that the image in question contains L. To this end, note that, as we just saw, pr and Φ send the complex projective line $P' = P(L \times \mathbf{C}) \subset P$ and the 2-sphere $S' = L \cup \{\infty\} \subset S$ diffeomorphically onto S'. Thus, Φ o pr: $P' \to S'$ is a diffeomorphism; hence, the image of its differential at $L \times \{0\}$ is $T_0S' = T_0L = L$. Next, in terms of the decomposition $V = \mathbf{C}y \oplus y^\perp$ of $V = T_yV = T_yS$ for any $y \in V \setminus \{0\}$,

$$\text{the differential } d\Phi_y \text{ is complex-linear on } y^\perp \text{ and antilinear on } \mathbf{C}y. \tag{15}$$

The $y^\perp$ part is obvious since $d\Phi_y$ acts on the Re$\langle,\rangle$-orthogonal complement of y, which includes $y^\perp$, via multiplication by $|y|^{-2}$, as one sees applying Φ to curves on which the norm $|\ |$ is constant. For the $\mathbf{C}y$ part, note that $L = \mathbf{C}y$ is tangent at y to the Φ-invariant submanifold $L \setminus \{0\} \subset V$, which we may identify with $\mathbf{C} \setminus \{0\}$ using the diffeomorphism of $\mathbf{C} \setminus \{0\}$ onto $L \setminus \{0\}$ sending z to $zy/|y|$. This makes Φ restricted to $V \setminus \{0\}$ appear as the standard antiholomorphic inversion $z \mapsto z/|z|^2 = 1/\bar{z}$.

17. Standard Framed Spin Structures on the 4-Sphere

We define a specific spin structure (σ^+, σ^-) over the sphere S^4 (see the end of Section 6) by first identifying S^4 with the quaternion projective line $P(W) \approx \mathbf{H}\mathrm{P}^1$ for a given quaternion plane W (cf. Section 2 and Lemma 17.1(a) below), and then choosing σ^+, σ^- to be, respectively, the tautological $\mathbf{H}$-line bundle over $P(W)$, with the fibre L at any $L \in P(W)$, and the quotient $\sigma^- = \theta/\sigma^+$, where $\theta = P(W) \times W$ is the product bundle over $P(W)$ with the fibre W. A fixed quaternion inner product $\langle,\rangle$ in W now turns both $\sigma^\pm$ into normed quaternion line bundles, with the structure group SU(2) (cf. the matrix in (7)), which also makes them Hermitian plane bundles (see the lines following (1)), and yields the norm-preserving identifications $\kappa = [\sigma^+]^{\wedge 2} = [\sigma^-]^{\wedge 2}$, required in Section 6, for $\kappa = P(W) \times \mathbf{C}$. Finally, equality $T[P(W)] = \mathrm{Hom}_{\mathbf{H}}(\sigma^+, \sigma^-)$, due to (2), provides the Clifford multiplication by any $A \in T_L[P(W)]$, $L \in P(W)$. As $A : L \to W/L$ is $\mathbf{H}$-linear and $\dim_{\mathbf{H}} L = \dim_{\mathbf{H}} W/L = 1$, the properties of A named in (9) follow easily.

Every nonzero vector $u \in W$ now gives rise to a section ψ of σ^+ with $\psi(L) = \mathrm{pr}_L u \in L$ for $L \in P(W)$, where pr_L is the $\langle,\rangle$-orthogonal projection onto L. Thus, $\psi(L) = \langle u, w\rangle w/|w|^2$ if $L = \mathbf{H}w$. For any $u \in W \setminus \{0\}$ the resulting triple $(\sigma^+, \sigma^-, \psi)$ is a framed spin structure, cf. Section 13, and will be called a *standard framed spin structure* over $P(W)$. In fact, ψ has just one zero, at $u^\perp \in P(W)$, and is transverse to the zero section by (i) in Section 11.

LEMMA 17.1. *Let $(\sigma^+, \sigma^-, \psi)$ be the standard framed spin structure over $P(W)$ corresponding to a nonzero vector u in a quaternion inner-product plane W, and let the quaternion line $V = u^\perp \in P(W)$ be treated as a complex Hermitian plane, cf. Section* 2. *If $S = V \cup \{\infty\}$ is the* 4*-sphere described in Section* 16 *and $\Theta : S \to P(W)$ is the extension, with $\Theta(\infty) = V$, of $\Theta : V \to P(W)$ defined in Remark* 2.3, *then*

(*a*) *$\Theta : S \to P(W)$ is a C^∞ diffeomorphism.*
(*b*) *The residual almost complex structure J on $P(W) \setminus \{V\}$, associated with ψ as in Remark* 13.1*, corresponds under Θ to the obvious structure on the complex vector space $V = S \setminus \{\infty\}$.*

Proof. Both Θ and Θ^{-1} appear as real-rational mappings in standard projective coordinates for $P(W)$ (cf. Remark 2.3) and the two charts for S (see Section 16), which proves (a). Given $y \in V$ and $v \in V = T_yV = T_yS$, the operator $A : L \to W/L$ with $L = \mathbf{H}w$ for $w = y + u$, corresponding to $d\Theta_y v$ under (2), sends $w = y + u$ to the coset $v + L$ (see Remark 2.3), so that, by $\mathbf{H}$-linearity of A, the A-image of $\psi(L) = \mathrm{pr}_L u = \langle u, w\rangle w/|w|^2$ is the coset of L in W containing $\langle u, w\rangle v/|w|^2$, i.e., v times the *real* scalar $|u|^2/(|y|^2 + |u|^2)$. (Note that $\langle u, w\rangle = |u|^2$ as $y \in V = u^\perp$.) The dependence of that coset on v thus is $\mathbf{C}$-linear, relative to the obvious complex structure of V. This yields (b), completing the proof. □

Remark 17.2. Combined with the second paragraph of Section 15, Lemma 17.1 shows that an immersion f of an oriented real surface Σ in the 4-sphere $S = V \cup \{\infty\}$ is pseudoholomorphic relative to some standard framed spin structure $(\sigma^+, \sigma^-, \psi)$ over S if and only if, restricted to $\Sigma \setminus f^{-1}(\infty)$, it is pseudoholomorphic as an immersion into the complex plane V.

18. Smoothness at Infinity for Holomorphic Curves in $\mathbf{C}^n$

As a step towards a classification proof in Section 20, we will now show that any smooth real surface in the sphere $S^{2n} = \mathbf{C}^n \cup \{\infty\}$, obtained by adding the point ∞ to a holomorphic curve in $\mathbf{C}^n$, is the image of a holomorphic curve in $\mathbf{CP}^n = \mathbf{C}^n \cup \mathbf{CP}^{n-1}$ under a natural projection (see Section 16) which restricted to $\mathbf{C}^n$ is the identity, and sends the hyperplane $\mathbf{CP}^{n-1}$ onto ∞.

LEMMA 18.1. *Given V, $\langle,\rangle$, S, ∞, P, H, Φ,* pr *as in Section* 16, *let Σ be a two-dimensional real submanifold of S such that $\infty \in \Sigma$ and $\Sigma \setminus \{\infty\}$ is a complex* *submanifold of $V = S \setminus \{\infty\}$. Furthermore, let $L = d\Phi_\infty(T_\infty\Sigma)$ be the real vector subspace of $V = T_0V = T_0S$ obtained as the image of $T_\infty\Sigma$ under the differential of Φ at ∞. Then*

(i) *L is a complex line in V.*
(ii) *Σ is the* pr*-image of a one-dimensional complex submanifold $\hat\Sigma$ of P which intersects H, transversally, at the single point $L \times \{0\}$.*

A proof of Lemma 18.1 is given in the next section.

Remark 18.2. Given a complex inner-product space V of any finite dimension n, we can now describe all two-dimensional real submanifolds Σ of the $2n$-sphere $S = V \cup \{\infty\}$ for which $\infty \in \Sigma$ and $\Sigma \setminus \{\infty\}$ is a complex submanifold of V. Namely, they coincide with the pr-images of the one-dimensional complex submanifolds $\hat\Sigma$ of the projective space $P = V \cup H$ that intersect the hyperplane $H \approx \mathbf{C}P^{n-1}$, transversally, at a single point. In fact, every such Σ is of this form by Lemma 18.1(ii), while the converse statement is immediate as pr: $\hat\Sigma \to S$ is an immersion: by (14), its real rank at the intersection point of $\hat\Sigma$ with H equals 2.

19. Proof of Lemma 18.1

Denoting by Γ the unit circle about 0 in L, let us choose $\varepsilon \in (0, \infty)$ and a C^∞ mapping $[0, \varepsilon] \times \Gamma \ni (t, u) \mapsto y(t, u) \in N \cap V$, where $N = \Phi(\Sigma) \subset S = V \cup \{\infty\}$, such that $y(0, u) = 0$, $\dot y(0, u) = u$ with $\dot y(t, u) = d[y(t, u)]/dt$, and $y(t, u) \neq 0$ unless $t = 0$, for every $u \in \Gamma$. For instance, we might set $y(t, u) = \exp_0 tu$, using the exponential mapping at $0 \in N$ of any Riemannian metric on N.

To prove (i), let us fix $u \in \Gamma$ and, for any given $t \in (0, \varepsilon]$, write $x = y(t, u)$, $v = \dot y(t, u)$. Thus, $d\Phi_x v \in T_{\Phi(x)}\Sigma$, as $v \in T_xN$ and $N = \Phi(\Sigma)$. Since $T_{\Phi(x)}\Sigma$ is closed under multiplication by $-i$, combining this with (15) we obtain $iv^{\mathrm{rad}} - iv^{\mathrm{tng}} \in T_xN$, where v^{rad}, v^{tng} are the components of v relative to the decomposition $V = \mathbf{C}x \oplus x^\perp$. In the last relation, v^{rad}, v^{tng} and x all depend on t, and taking its limit as $t \to 0$ we get $iu \in T_0N$, since $v^{\mathrm{rad}} \to u$, $v^{\mathrm{tng}} \to 0$ as $t \to 0$ (see the next paragraph) and, clearly, $x \to x(0) = 0$. Therefore, $u, iu \in L = T_0N$, which yields (i).

To obtain relations $v^{\mathrm{rad}} \to u$ and $v^{\mathrm{tng}} \to 0$ as $t \to 0$, note that $v \to u$ by continuity of $\dot y(t, u)$ in (t, u). (In all limits, $t \to 0$.) Next, $x/t \to u$ (since $\dot y(0, u) = u$), and so $x/|x| \to u$, as $x/|x| = (x/t)/|x/t|$. Thus, $v^{\mathrm{rad}} = \langle v, x/|x|\rangle x/|x| \to \langle u, u\rangle u = u$, and $v^{\mathrm{tng}} = v - v^{\mathrm{rad}} \to u - u = 0$.

Since $[y, z] = [t^{-1}y, t^{-1}z]$ for $[y, z] \in P$ and $t \in \mathbf{R}\setminus\{0\}$, the definitions of Φ and pr give $\mathrm{pr}^{-1}(\Phi(y(t, u))) = [t^{-1}y(t, u), t^{-1}\,|y(t, u)|^2]$, while $t^{-1}y(t, u) \to u$ and $t^{-1}\,|y(t, u)|^2 \to 0$ as $t \to 0$, uniformly in $u\ \varepsilon\ \Gamma$.

In fact, for $\omega : [0, \varepsilon] \times \Gamma \to V$ given by $\omega(t, u) = \int_0^1 (1 - s)\,\ddot y(st, u)ds$, repeated integration by parts leads to the Taylor formula $y(t, u) = t[u + t\omega(t, u)]$, so that convergence follows from continuity of ω, and is uniform due to compactness of Γ.

In other words, the restriction to $\Sigma \setminus \{\infty\} = \Phi(N \setminus \{0\})$ of the biholomorphism pr^{-1}: $S \setminus \{\infty\} \to P \setminus H$ (that is, of the identity mapping of V) has a limit at ∞ equal to $L \times \{0\} \in H$, that is, to $[u,0]$ for any $u \in \Gamma$. Setting $\hat{f} = \mathrm{pr}^{-1}$ on the complement $\Sigma \setminus \{\infty\}$ and $\hat{f}(\infty) = L \times \{0\}$ we now obtain a continuous mapping $\hat{f} : \Sigma \to P$, which is holomorphic on $\Sigma \setminus \{\infty\}$. Also, the complex structure of $\Sigma \setminus \{\infty\}$ has an extension to Σ.

Such an extension exists in view of Remark 15.1 applied to $M = S\setminus\{0\}$, $y = \infty$, and J which is the standard integrable almost complex structure on $M' = V \setminus \{0\} = S \setminus \{0, \infty\}$, along with the metric g obtained as the pull-back under Φ of the standard Euclidean metric $\mathrm{Re}\langle,\rangle$ on V. Note that, since Φ is conformal, g equals a positive function times $\mathrm{Re}\langle,\rangle$, and so is still compatible with J.

Some neighborhood of ∞ in Σ is therefore biholomorphic to a disk in $\mathbf{C}$, so that the mapping $\hat{f} : \Sigma \to P$, continuous on Σ and holomorphic on $\Sigma \setminus \{\infty\}$, must be holomorphic on Σ. Also, by continuity, $\mathrm{pr} \circ \hat{f}$ coincides with the inclusion mapping $\Sigma \to S$. Thus, $\hat{f}$ is a C^∞ embedding, and hence a holomorphic embedding. Setting $\hat{\Sigma} = \hat{f}(\Sigma)$ we thus obtain a biholomorphism $\hat{f} : \Sigma \to \hat{\Sigma}$, the inverse of which is pr restricted to $\hat{\Sigma}$ (and valued in Σ). Since pr is constant on H, transversality of $\hat{\Sigma}$ and H follows, completing the proof of Lemma 18.1.

20. Real Surfaces, Pseudoholomorphically Immersed in S^4

For the 4-sphere $S = V \cup \{\infty\}$ obtained as in Section 16 (with $n = 2$) from a Hermitian plane V, an immersion $f : \Sigma \to S$ of an oriented real surface Σ will be called *pseudoholomorphic* if its restriction to $\Sigma \setminus f^{-1}(\infty)$ is a pseudoholomorphic immersion in the complex plane V. According to Remark 17.2, this amounts to its being pseudoholomorphic for a certain standard framed spin structure $(\sigma^+, \sigma^-, \psi)$ over S defined as in Section 17.

In this section we classify such immersions of *closed* surfaces Σ, beginning with the special case of embeddings.

THEOREM 20.1. *Let* $P \approx \mathbf{CP}^2$, $S \approx S^4$ *and* pr: $P \to S$ *be defined as in Section* 16 *with* $n = 2$, *for some two-dimensional complex inner-product space* V, *so that* $P = V \cup H$ *and* $S = V \cup \{\infty\}$, *where* $H \approx \mathbf{CP}^1$ *is a projective line in* P, *while* pr *sends* H *to* ∞ *and coincides with the identity on* V.

The oriented closed real surfaces pseudoholomorphically embedded in S *then are all diffeomorphic to* S^2, *and coincide with the* pr-*images of complex projective lines other than* H *in the projective space* P.

In fact, $\infty \in \Sigma$ for any pseudoholomorphically embedded, oriented closed real surface $\Sigma \subset S$ (as Σ cannot lie entirely in $V = S\setminus\{\infty\}$), and so Remark 18.2 with

$n = 2$ gives $\Sigma = \mathrm{pr}(\hat{\Sigma})$ for a one-dimensional compact complex submanifold $\hat{\Sigma}$ which intersects the projective line H, transversally, at a single point. By Chow's theorem [3], $\hat{\Sigma}$ is algebraic, and hence of degree one, as required. The converse statement is clear from Remark 18.2.

THEOREM 20.2. *Under the hypotheses of Theorem* 20.1, *the pseudoholomorphic immersions $f : \Sigma \to S$ of any oriented real surface Σ are nothing else than the composites* $\mathrm{pr} \circ \hat{f}$, *where $\hat{f}$ runs through all pseudoholomorphic immersions $\Sigma \to P$ transverse to H.*

This is immediate if one replaces Σ by the f-image of a suitable neighborhood in Σ of any given point of $f^{-1}(\infty)$ and, again, applies Remark 18.2.

Unlike Theorem 20.1, the last result does not even assume closedness of Σ. However, when the surface Σ *is* closed, Chow's theorem [3, p. 167] implies that the image $\hat{f}(\Sigma)$ is an algebraic curve of some degree d. In the simplest case, where some finite number δ of ordinary double points constitute both the only singularities of $\hat{f}(\Sigma)$ and the only self-intersections of f, the genus of Σ is $[(d-1)(d-2)-2\delta]/2$, by Plücker's formula [3, p. 280].

21. Blow-Ups, Blow-Downs and Framed Spin^c-Structures

The one-to-one correspondence, in Theorem 20.2, between pseudoholomorphic immersions from Σ into S, and those from Σ into P, is a special case of two mutually inverse constructions. They involve an embedded $\mathbf{C}\mathrm{P}^1$ with the self-intersection number $+1$ (not -1), and hence should not be confused with their obvious analogues in the complex category.

The first construction is *blow-down*. It may be applied to a complex surface $\hat{M}$ along with a fixed complex submanifold $H \subset \hat{M}$, biholomorphic to $\mathbf{C}\mathrm{P}^1$, such that some neighborhood $\hat{U}$ of H in $\hat{M}$ has a biholomorphic identification with a neighborhood of a projective line $\mathbf{C}\mathrm{P}^1$ in $\mathbf{C}\mathrm{P}^2$, under which $H = \mathbf{C}\mathrm{P}^1$. Contracting $H = \mathbf{C}\mathrm{P}^1$ to a single point, denoted by ∞, we now transform $\hat{U}$ into a neighborhood U of ∞ in the four-sphere $S^4 = \mathbf{C}^2 \cup \{\infty\}$. (This is the $n = 2$ case of the construction in Section 16.) The replacement of H by ∞ in $\hat{M}$ thus leads to a new 4-manifold M along with an H-collapsing projection pr: $\hat{M} \to M$. Smoothness of M follows since M is the result of gluing $\hat{M} \setminus H$ and U together using the diffeomorphism pr: $\hat{U} \setminus H \to U \setminus \{\infty\}$ of Section 16. In addition, any Hermitian metric g on $\hat{M}$ can obviously be modified so as to coincide, near H, with the standard Fubini-Study metric on $\mathbf{C}\mathrm{P}^2$. Our gluing procedure then may clearly be extended to the framed spin^c-structures over $\hat{U} \setminus H \subset \mathbf{C}\mathrm{P}^2$ and $U \setminus \{\infty\} \subset S^4$, of which the former corresponds to the (almost) Hermitian structure, cf. Section 13, and the latter involves a standard framed spin structure $(\sigma^+, \sigma^-, \psi)$ over S^4, with ψ having a zero at ∞ (see Section 17). The result is a framed spin^c-structure over M with the analogue of ψ vanishing at ∞.

The opposite construction is *blow-up*, for which the starting point is a framed spin^c-structure $(\sigma^+, \sigma^-, \kappa, \psi)$ over a 4-manifold M with ψ that vanishes at some given point $\infty \in M$. An additional assumption is that on some neighborhood U of ∞ in M the framed spin^c-structure may be identified, via a suitable bundle isomorphism, with a standard framed spin structure on S^4, restricted to an open set (also denoted by U) and having a zero at a point $\infty \in U$. As in Section 16, U is obtained from an open set $\hat{U} \subset \mathbf{C}\mathrm{P}^2$ by collapsing a projective line $H = \mathbf{C}\mathrm{P}^1 \subset \hat{U}$ to the point ∞. Let $\hat{M}$ now be the set obtained from M by replacing $\{\infty\}$ with H. For reasons similar to those in the preceding paragraph, $\hat{M}$ is a smooth 4-manifold with a C^∞ projection mapping pr: $\hat{M} \to M$ sending H onto $\{\infty\}$ and diffeomorphic (equal to the identity) on $\hat{M} \setminus H = M \setminus \{\infty\}$. Also, the original framed spin^c-structure has an extension from $M \setminus \{\infty\} = \hat{M} \setminus H$ to $\hat{M}$ such that, on $\hat{U}$, it is the one associated with the complex structure and the Fubini-Study metric of $\mathbf{C}\mathrm{P}^2$.

Note that both constructions can be performed repeatedly, as long as the choices of H (or, ∞) that we use are pairwise disjoint (or, distinct). In addition, one may also apply the standard *complex* blow-down or blow-up to a copy of $\mathbf{C}\mathrm{P}^1$ biholomorphically embedded in $\hat{M}$ with the self-intersection number -1 or, respectively, to a point in M at which ψ is nonzero and in a neighborhood of which the residual almost complex structure (see Remark 13.1) is integrable.

Finally, in both cases described above, for any oriented real surface Σ, relation $f = \mathrm{pr} \circ \hat{f}$ defines a bijective correspondence between pseudoholomorphic immersions $\hat{f} : \Sigma \to \hat{M}$ which are transverse to H, and pseudoholomorphic immersions $f : \Sigma \to M$. This conclusion, obtained from Remark 18.2 exactly as in our two-line "proof" of Theorem 20.2, includes Theorem 20.2 as a special case with $M = S$ and $\hat{M} = P$.

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