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Characterizations of continuous operators on $C_b(X)$ with the strict topology

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Abstract
Let X be a completely regular Hausdorff space and $C_b(X)$ be the space of all bounded continuous functions on X, equipped with the strict topology β. We study some important classes of $(\beta, \|\cdot\|_E)$-continuous linear operators from $C_b(X)$ to a Banach space $(E, \|\cdot\|_E)$: β-absolutely summing operators, compact operators and β-nuclear operators. We characterize compact operators and β-nuclear operators in terms of their representing measures. It is shown that dominated operators and β-absolutely summing operators $T : C_b(X) \to E$ coincide and if, in particular, E has the Radon–Nikodym property, then β-absolutely summing operators and β-nuclear operators coincide. We generalize the classical theorems of Pietsch, Tong and Uhl concerning the relationships between absolutely summing, dominated, nuclear and compact operators on the Banach space $C(X)$, where X is a compact Hausdorff space.

Keywords Spaces of bounded continuous functions · k-spaces · Radon vector measures · Strict topologies · Absolutely summing operators · Dominated operators · Nuclear operators · Compact operators · Generalized DF-spaces · Projective tensor product

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1 Introduction and preliminaries

The Riesz representation theorem plays a crucial role in the study of operators on the Banach space $C(X)$ of continuous functions on a compact Hausdorff space X. Due to this theorem, different classes of operators on $C(X)$ have been characterized in terms of their representing Radon vector measures.

Absolutely summing operators between Banach spaces have been the object of several studies (see [1, pp. 209–233] and [5, 8, 11, 27, 28, 31, 34]). It originates in the fundamental paper of Grothendieck [17] from 1953. Grothendieck's inequality has equivalent formulation using the theory of absolutely summing operators (see [1, Theorem 8.3.1] and [4, 22]). In the multilinear case, it is also connected with the Bohnenblust–Hille and the Hardy–Littlewood inequalities (see [2]). There is a vast literature on absolutely summing operators from the Banach space $C(X)$ to a Banach space E (see [1], [9, Chap. VI], [11, 34, 43]).

The concept of nuclearity in Banach spaces is due to Grothendieck [17, 18] and Ruston [33] and has the origin in Schwartz's kernel theorem [18]. Many authors have studied nuclear operators between locally convex spaces (see [21, §17.3], [37, Chap. 3, §7], [46, p. 289]) and Banach spaces (see [9, Chap. VI], [11, 16] [46, p. 279]). If F is a Banach space, nuclear operators from the Banach space $C(X, F)$ of F-valued continuous functions on a compact Hausdorff space X to E have been studied intensively by Popa [29], Saab [35], Saab and Smith [36]. In particular, a characterization of nuclear operators from $C(X)$ to E in terms of their representing measures can be found in [9, Theorem 4, pp. 173–174], [34, Proposition 5.30], [43, Proposition 1.2].

The interplay between absolutely summing operators, dominated operators of Dinculeanu (see [12, §19], [13, §1]) and nuclear operators $T : C(X) \to E$ has been an interesting issue in operator theory. Pietsch [27, 2.3.4, Proposition, p. 41] proved that dominated operators and absolutely summing operators on the Banach space $C(X)$ coincide. It is known that if in particular, E has the Radon–Nikodym property, then absolutely summing and nuclear operators $T : C(X) \to E$ coincide (see [9, Corollary 5, p. 174]). Moreover, Uhl [44, Theorem 1] showed that if, E has the Radon–Nikodym property, then every dominated operator $T : C(X) \to E$ is compact.

The aim of this paper is to extend these classical results to the setting, where X is a completely regular Hausdorff k-space.

Throughout the paper, we assume that $(X, \mathcal{T})$ is a completely regular Hausdorff space. By $\mathcal{K}$ we denote the family of all compact sets in X. Let $\mathcal{B}o$ denote the σ-algebra of Borel sets in X.

Let $C_b(X)$ (resp. $B(\mathcal{B}o)$) denote the Banach space of all bounded continuous (resp. bounded $\mathcal{B}o$-measurable) scalar functions on X, equipped with the topology τ_u of the uniform norm $\|\cdot\|_\infty$. By $\mathcal{S}(\mathcal{B}o)$ we denote the space of all $\mathcal{B}o$-simple scalar functions on X. Let $C_b(X)'$ stand for the Banach dual of $C_b(X)$.

Following [15, 37] and [45, Definition 10.4, p. 137] the *strict topology* β on $C_b(X)$ is the locally convex topology determined by the seminorms

$$p_w(u) := \sup_{t\in X} w(t)|u(t)| \quad \text{for} \quad u \in C_b(X),$$

where w runs over the family $\mathcal{W}$ of all bounded functions $w : X \to [0, \infty)$ which vanish at infinity, that is, for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that $\sup_{t\in X\setminus K} w(t) \le \varepsilon$. Let $\mathcal{W}_1 := \{w \in \mathcal{W} : 0 \le w \le \mathbb{1}_X\}$. For $w \in \mathcal{W}_1$ and $\eta > 0$ let

$$U_w(\eta) := \{u \in C_b(X) : p_w(u) \le \eta\}.$$

Note that the family $\{U_w(\eta) : w \in \mathcal{W}_1, \eta > 0\}$ is a local base at 0 for β.

The strict topology β on $C_b(X)$ has been studied intensively (see [15, 20, 38, 41, 45]). Note that β can be characterized as the finest locally convex Hausdorff topology on $C_b(X)$ that coincides with the compact-open topology τ_c on τ_u -bounded sets (see [41, Theorem 2.4]). The topologies β and τ_u have the same bounded sets. This means that $(C_b(X), \beta)$ is a generalized DF-space (see [38, Corollary]), and it follows that $(C_b(X), \beta)$ is quasinormable (see [32, p. 422]). If, in particular, X is locally compact (resp. compact), then β coincides with the original strict topology of Buck [6] (resp. $\beta = \tau_u$).

Recall that a countably additive scalar measure μ on $\mathcal{B}o$ is said to be a *Radon measure* if its variation $|\mu|$ is regular, that is, for every $A \in \mathcal{B}o$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ with $K \subset A \subset O$ such that $|\mu|(O\setminus K) \le \varepsilon$. Let $M(X)$ denote the Banach space of all scalar Radon measures, equipped with the total variation norm $\|\mu\| := |\mu|(X)$.

The following characterization of the topological dual of $(C_b(X), \beta)$ will be of importance (see [15, Lemma 4.5]), [20, Theorem 2].

Theorem 1.1 *For a linear functional Φ on $C_b(X)$ the following statements are equivalent:*

(i) *Φ is β-continuous.*
(ii) *There exists a unique $\mu \in M(X)$ such that*

$$\Phi(u) = \Phi_\mu(u) = \int_X u\, d\mu \quad \text{for} \quad u \in C_b(X)$$

and $\|\Phi_\mu\|' = |\mu|(X)$ for $\mu \in M(X)$ (here $\|\cdot\|'$ denotes the conjugate norm in $C_b(X)'$).

The following result will be useful (see [41, Theorem 5.1]).

Theorem 1.2 *For a subset $\mathcal{M}$ of $M(X)$ the following statements are equivalent:*

(i) $\sup_{\mu\in\mathcal{M}} |\mu|(X) < \infty$ *and $\mathcal{M}$ is uniformly tight, that is, for each $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that* $\sup_{\mu\in\mathcal{M}} |\mu|(X \setminus K) \le \varepsilon$.
(ii) *The family $\{\Phi_\mu : \mu \in \mathcal{M}\}$ is β-equicontinuous.*

Recall that a completely regular Hausdorff space $(X, \mathcal{T})$ is a *k-space* if any subset A of X is closed whenever $A \cap K$ is compact for all compact sets K in X. In particular, every locally compact Hausdorff space, every metrizable space and every space satisfying the first countability axiom is a k-space (see [14, Chap. 3, § 3]).

From now on, we will assume that $(X, \mathcal{T})$ is a k-space. Then, the space $(C_b(X), \beta)$ is complete (see [15, Theorem 2.4]).

We assume that $(E, \|\cdot\|_E)$ is a Banach space. Let $B_{E'}$ stand for the closed unit ball in the Banach dual E' of E.

Recall that a bounded linear operator $T : C_b(X) \to E$ is said to be *absolutely summing* if there exists a constant $c > 0$ such that for any finite set $\{u_1, \dots, u_n\}$ in $C_b(X)$,

$$\sum_{i=1}^{n} \|T(u_i)\|_E \leq c \sup \left\{ \sum_{i=1}^{n} |\Phi(u_i)| : \Phi \in B_{C_b(X)'} \right\}. \tag{1.1}$$

The infimum of number of $c > 0$ satisfying (1.1) denoted by $\|T\|_{\mathrm{as}}$ is called an *absolutely summing norm* of T.

It is known that a bounded linear operator $T : C_b(X) \to E$ is absolutely summing if and only if T maps unconditionally convergent series in $C_b(X)$ into absolutely convergent series in E (see [9, Definition 1, p. 161 and Proposition 2, p. 162]).

For $t \in X$, let δ_t stand for the point mass measure, that is, $\delta_t(A) := \mathbb{1}_A(t)$ for $A \in \mathcal{B}o$. Then $\delta_t \in M^+(X)$ and $\int_X u \, \mathrm{d}\delta_t = u(t)$ for $u \in C_b(X)$. Clearly, $\|\delta_t\| = \delta_t(X) = 1$.

Lemma 1.3 *For a bounded linear operator $T : C_b(X) \to E$, the following statements are equivalent*:

(i) *T is absolutely summing*.
(ii) *There exists $c > 0$ such that for any set $\{u_1, \dots, u_n\}$ in $C_b(X)$*,

$$\sum_{i=1}^{n} \|T(u_i)\|_E \leq c \sup \left\{ \sum_{i=1}^{n} \left| \int_X u_i \, \mathrm{d}\mu \right| : \mu \in M(X), |\mu|(X) \leq 1 \right\}.$$

Proof (i)⇒(ii) There exists $c > 0$ such that for any set $\{u_1, \dots, u_n\}$ in $C_b(X)$,

$$\sum_{i=1}^{n} \|T(u_i)\|_E \leq c \sup \left\{ \sum_{i=1}^{n} |\Phi(u_i)| : \Phi \in B_{C_b(X)'} \right\}.$$

Note that we have (see [1, p. 205]),

$$\sup \left\{ \sum_{i=1}^{n} |\Phi(u_i)| : \Phi \in B_{C_b(X)'} \right\} = \sup \left\{ \left\| \sum_{i=1}^{n} \varepsilon_i u_i \right\|_\infty : (\varepsilon_i) \in \{-1, 1\}^n \right\}.$$

Hence, we get,

$$\begin{aligned}\sum_{i=1}^{n}\|T(u_i)\|_E &\le c\sup\left\{\left\|\sum_{i=1}^{n}\varepsilon_i u_i\right\|_\infty : (\varepsilon_i)\in\{-1,1\}^n\right\}\\ &= c\sup\left\{\left|\sum_{i=1}^{n}\varepsilon_i u_i(t)\right| : (\varepsilon_i)\in\{-1,1\}^n, t\in X\right\}\\ &\le c\sup\left\{\sum_{i=1}^{n}|u_i(t)| : t\in X\right\} = c\sup\left\{\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\delta_t\right| : t\in X\right\}\\ &\le c\sup\left\{\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\mu\right| : \mu\in M(X), |\mu|(X)\le 1\right\}.\end{aligned}$$

(ii)⇒(i) This is obvious. □

The general theory of absolutely summing operators between locally convex spaces was developed by Pietsch [27].

Following [27, 1.2, pp. 23–24], we say that a sequence (u_n) in $C_b(X)$ is β *-weakly summable* if $\sum_{n=1}^{\infty}|\int_X u_n\,\mathrm{d}\mu|<\infty$ for every $\mu\in M(X)$. By $\ell^1_w(C_b(X),\beta)$, we denote the linear space of all β-weakly summable sequences in $C_b(X)$.

Let $(u_n)\in\ell^1_w(C_b(X),\beta)$. Then, in view of [27, 1.2.3, pp. 23–24] for each $w\in\mathcal{W}_1$ and $\eta>0$ there exists $\varrho_{w,\eta}>0$ such that

$$\mathcal{E}_{w,\eta}((u_n)) := \sup\left\{\sum_{n=1}^{\infty}\left|\int_X u_n\,\mathrm{d}\mu\right| : \mu\in U_w(\eta)^0\right\}\le\varrho_{w,\eta},$$

where $U_w(\eta)^0$ stands for the polar of $U_w(\eta)$ with respect to the pairing $\langle C_b(X), M(X)\rangle$. Then, $\mathcal{E}_{w,\eta}$ is a seminorm on $\ell^1_w(C_b(X),\beta)$ and the family $\{\mathcal{E}_{w,\eta} : w\in\mathcal{W}_1, \eta>0\}$ generates the so-called $\mathcal{E}$ *-topology* on $\ell^1_w(C_b(X),\beta)$ (see [27, 1.2.3]).

Let $\mathcal{F}(\mathbb{N})$ denote the family of all finite sets in $\mathbb{N}$, the set of all natural numbers. By $\ell^1_s(C_b(X),\beta)$ we denote the $\mathcal{E}$-closed subspace of $\ell^1_w(C_b(X),\beta)$ consisting of all β *-summable sequences* in $C_b(X)$ (see [27, 1.3]). In view of [27, Theorem 1.3.6] a sequence $(u_n)\in\ell^1_s(C_b(X),\beta)$ if and only if the net $(s_M)_{M\in\mathcal{F}(\mathbb{N})}$ of partial sums $s_M := \sum_{i\in M}u_i$ forms a β-Cauchy sequence in $C_b(X)$, where $\mathcal{F}(\mathbb{N})$ is directed by inclusion.

Let $\ell^1(E)$ stand for the linear space of all *absolutely summable* sequences in E, i.e., $(e_n)\in\ell^1(E)$ if $\sum_{n=1}^{\infty}\|e_n\|_E<\infty$. Then, $\ell^1(E)$ can be equipped with the norm $\pi_E((e_n)) := \sum_{n=1}^{\infty}\|e_n\|_E$ (see [27, 1.4]).

According to [27, 2.1], we have

Definition 1.4 A $(\beta,\|\cdot\|_E)$-continuous linear operator $T : C_b(X)\to E$ is said to be β *-absolutely summing* if $\sum_{n=1}^{\infty}\|T(u_n)\|_E<\infty$ whenever $(u_n)\in\ell^1_s(C_b(X),\beta)$.

Recall that a linear operator $T : C_b(X)\to E$ is said to be β *-compact* (resp. β *-weakly compact*) if there exists a β-neighborhood V of 0 such that $T(V)$ is a relatively norm compact (resp. relatively weakly compact) subset of E.

We will say that an operator $T : C_b(X) \to E$ is compact (resp. weakly compact) if T is τ_u-compact (resp. τ_u-weakly compact).

Proposition 1.5 *Let $T : C_b(X) \to E$ be a $(\beta, \|\cdot\|_E)$-continuous linear operator. Then, the following statements are equivalent:*

(i) *T is weakly compact (resp. compact).*
(ii) *T is β-weakly compact (resp. β-compact).*

Proof (i)⇒(ii) Assume that (i) holds. Topologies β and τ_u have the same bounded sets in $C_b(X)$, so T maps β-bounded sets onto relatively weakly compact (resp. norm compact) sets in E. Since the space $(C_b(X), \beta)$ is quasinormable, by the Grothendieck classical result (see [32, p. 429]), we obtain that T is β-weakly compact (resp. β -compact).

(ii)⇒(i) This is obvious because $\beta \subset \tau_u$. □

Following [12, § 19, Section 3], [13, § 1, Section H] one can distinguish an important class of linear operators on $C_b(X)$.

Definition 1.6 A linear operator $T : C_b(X) \to E$ is said to be *dominated* if there exists $\mu \in M^+(X)$ such that

$$\|T(u)\|_E \leq \int_X |u| \, d\mu \quad \text{for} \quad u \in C_b(X).$$

Then, we say that T is *dominated* by μ.

According to [25, Proposition 3.1] we have.

Proposition 1.7 *Every dominated operator $T : C_b(X) \to E$ is $(\beta, \|\cdot\|_E)$-continuous and weakly compact.*

Following [37, Chap. 3, §7] (see also [21, §17.3, p. 376]) and using Theorem 1.2 we have the following definition.

Definition 1.8 A linear operator $T : C_b(X) \to E$ is said to be β *-nuclear*, if there exist a uniformly bounded and uniformly tight sequence (μ_n) in $M(X)$, a bounded sequence (e_n) in E and a sequence $(\lambda_n) \in \ell^1$ such that

$$T(u) = \sum_{n=1}^{\infty} \lambda_n \Big(\int_X u \, d\mu_n \Big) e_n \quad \text{for} \quad u \in C_b(X). \tag{1.2}$$

If $T : C_b(X) \to E$ is β-nuclear operator, let us put

$$\|T\|_{\beta\text{- nuc}} := \inf\left\{ \sum_{n=1}^{\infty} |\lambda_n||\mu_n|(X)\|e_n\|_E \right\},$$

where the infimum is taken over all sequences (μ_n) in $M(X)$, (e_n) in E and $(\lambda_n) \in \ell^1$ such that T admits a representation (1.2).

Every β-nuclear operator $T : C_b(X) \to E$ is $(\beta, \|\cdot\|_E)$-continuous and β-compact (see [37, Chap. 3, §7, Corollary 1]).

In [24], the theory of integral representation of continuous operators on $C_b(X)$, equipped with the strict topology β has been developed. Making use of the results of [24], we study β-absolutely summing operators, compact operators and β-nuclear operators $T : C_b(X) \to E$. We characterize compact operators and β-nuclear operators $T : C_b(X) \to E$ in terms of their representing measures (see Theorems 4.1 and 5.1 below). It is shown that dominated operators and β-absolutely summing operators $T : C_b(X) \to E$ coincide (see Corollary 3.4) and if, in particular, E has the Radon–Nikodym property, then β-absolutely summing and β-nuclear operators $T : C_b(X) \to E$ coincide (see Corollary 5.2). We prove that a natural kernel operator $T : C_b(X) \to C(K)$ is β-nuclear (see Theorem 6.3).

2 Integral representation

In this section, we collect basic concepts and facts concerning integral representation of operators on $C_b(X)$ that will be useful (see [24] for notation and more details).

Let $m : \mathcal{B}o \to E$ be a finitely additive measure. By $|m|(A)$ (resp. $\|m\|(A)$), we denote the variation (resp. the semivariation) of m on $A \in \mathcal{B}o$ (see [9, Definition 4, p. 2]). Then, $\|m\|(A) \le |m|(A)$ for $A \in \mathcal{B}o$.

For $e' \in E'$, let

$$m_{e'}(A) := e'(m(A)) \ \text{ for } \ A \in \mathcal{B}o.$$

Then,

$$\|m\|(A) = \sup_{e' \in B_{E'}} |m_{e'}|(A),$$

where $|m_{e'}|(A)$ stands for the variation of $m_{e'}$ on $A \in \mathcal{B}o$.

Recall that a countably additive measure $m : \mathcal{B}o \to E$ is called a *Radon measure* if its semivariation $\|m\|$ is regular, i.e., for each $A \in \mathcal{B}o$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ with $K \subset A \subset O$ such that $\|m\|(O \backslash K) \le \varepsilon$ (see [24, Definition 3.3]).

We will need the following result (see [12, §15.6, Proposition 19]).

Lemma 2.1 *Assume that $m : \mathcal{B}o \to E$ is a Radon measure and $|m|(X) < \infty$. Then, $|m| \in M^+(X)$.*

Assume that $m : \mathcal{B}o \to E$ is a finitely additive measure with $\|m\|(X) < \infty$. Then, for every $v \in B(\mathcal{B}o)$, one can define the so-called *immediate integral* $\int_X v \,\mathrm{d}m \in E$ by

$$\int_X v\,dm := \lim \int_X s_n\,\mathrm{d}m, \tag{2.1}$$

where (s_n) is a sequence in $\mathcal{S}(\mathcal{B}o)$ such that $\|s_n - v\|_\infty \to 0$ (see [9, p. 5], [13, § 1, Section G]). Then, for $v \in B(\mathcal{B}o)$,

$$\left\| \int_X v\,\mathrm{d}m \right\|_E \le \|v\|_\infty \, \|m\|(X).$$

For $e' \in E'$, we have

$$e'\left(\int_X v\,\mathrm{d}m \right) = \int_X v\,\mathrm{d}m_{e'} \quad \text{for} \quad v \in B(\mathcal{B}o). \tag{2.2}$$

Let $ca(\mathcal{B}o)$ denote the Banach space of all countably additive scalar measures on $\mathcal{B}o$, equipped with the total variation norm $\|\mu\| := |\mu|(X)$. For $\mu \in ca(\mathcal{B}o)^+$, let $\mathcal{L}^1(\mu)$ denote the space of all μ-integrable scalar functions on X, equipped with the seminorm $\|v\|_1 := \int_X |v|\,\mathrm{d}\mu$ for $v \in \mathcal{L}^1(\mu)$. Then

$$C_b(X) \subset B(\mathcal{B}o) \subset \mathcal{L}^1(\mu).$$

Assume that $m : \mathcal{B}o \to E$ is a countably additive measure of finite variation $|m|$, i.e., $|m|(X) < \infty$. Then $|m| \in ca(\mathcal{B}o)^+$ (see [9, Proposition 9, p. 3]). Since $\mathcal{S}(\mathcal{B}o)$ is $\|\cdot\|_1$ -dense in $\mathcal{L}^1(|m|)$, for every

$$\int_X v\,\mathrm{d}m := \lim \int_X s_n\,\mathrm{d}m, \tag{2.3}$$

where (s_n) is a sequence in $\mathcal{S}(\mathcal{B}o)$ such that $\|s_n - v\|_1 \to 0$ (see [13, § 2, Sect. D]).

Note that for $v \in B(\mathcal{B}o) \subset \mathcal{L}^1(|m|)$, the integral $\int_X v\,dm$ defined in (2.3) coincides with the immediate integral defined in (2.1). We have

$$\left\| \int_X v\,\mathrm{d}m \right\|_E \le \int_X |v|\,\mathrm{d}|m| \quad \text{for} \quad v \in \mathcal{L}^1(|m|). \tag{2.4}$$

Hence, the corresponding integration operator $T_m : \mathcal{L}^1(|m|) \to E$ given by

$$T_m(v) := \int_X v\,\mathrm{d}m \quad \text{for} \quad v \in \mathcal{L}^1(|m|)$$

is $(\|\cdot\|_1, \|\cdot\|_E)$-continuous.

Let $C_b(X)'_\beta$ and $C_b(X)''_\beta$ denote the dual and the bidual of $(C_b(X), \beta)$. Since β -bounded subsets of $C_b(X)$ are τ_u-bounded, the strong topology $\beta(C_b(X)'_\beta, C_b(X))$ in $C_b(X)'_\beta$ coincides with the $\|\cdot\|'$-norm topology in $C_b(X)'$ restricted to $C_b(X)'_\beta$. Hence, we have $C_b(X)''_\beta = (C_b(X)'_\beta, \|\cdot\|')'$ and we get $\Psi \in C_b(X)''_\beta$ $\|\Psi\|'' = \sup\{|\Psi(\Phi)| : \Phi \in C_b(X)'_\beta, \|\Phi\|' \le 1\}$. Then, one can embed isometrically $B(\mathcal{B}o)$ into $C_b(X)''_\beta$ by the mapping $\pi : B(\mathcal{B}o) \to C_b(X)''_\beta$, where for $v \in B(\mathcal{B}o)$,

$$\pi(v)(\Phi_\mu) := \int_X v \, \mathrm{d}\mu \quad \text{for} \quad \mu \in M(X).$$

Note that $C_b(X)'_\beta$ is a closed subspace of $(C_b(X)', \|\cdot\|')$ (see [24, p. 847]).

Let $i_E : E \to E''$ stand for the canonical injection, that is, $i_E(e)(e') := e'(e)$ for $e \in E$, $e' \in E'$. Let $j_E : i_E(E) \to E$ denote the left inverse of i_E, i.e., $j_E(i_E(e)) := e$ for $e \in E$.

Assume that $T : C_b(X) \to E$ is a $(\beta, \|\cdot\|_E)$-continuous linear operator. Then we can define the biconjugate mapping

$$T'' : C_b(X)''_\beta \to E''$$

by putting $T''(\Psi)(e') := \Psi(e' \circ T)$ for $\Psi \in C_b(X)''_\beta$ and $e' \in E'$. Then T'' is $(\|\cdot\|'', \|\cdot\|_{E''})$-continuous. Let

$$\hat{T} := T'' \circ \pi : B(\mathcal{B}o) \to E''.$$

Then, $\hat{T}$ is $(\|\cdot\|_\infty, \|\cdot\|_{E''})$-continuous.

For $A \in \mathcal{B}o$, let

$$\hat{m}(A) := \hat{T}(\mathbb{1}_A).$$

Hence, $\hat{m} : \mathcal{B}o \to E''$ is a finitely additive bounded measure (i.e., $\|\hat{m}\|(X) < \infty$) and is called a *representing measure* of T. For every $e' \in E'$, let

$$\hat{m}_{e'}(A) := \hat{m}(A)(e') \quad \text{for} \quad A \in \mathcal{B}o.$$

Then for every $v \in B(\mathcal{B}o)$, we have (see [24, Theorem 3.1])

$$\hat{T}(v) = \int_X v \, \mathrm{d}\hat{m} \quad \text{and} \quad \hat{T}(v)(e') = \int_X v \, \mathrm{d}\hat{m}_{e'} \quad \text{for every} \quad e' \in E',$$

where $\hat{m}_{e'} \in M(X)$ for every $e' \in E'$. From the general properties of the operator T'' it follows that $\hat{T}(C_b(X)) \subset i_E(E)$ and

$$T(u) = j_E\big(\hat{T}(u)\big) = j_E\left(\int_X u \, \mathrm{d}\hat{m}\right) \quad \text{for} \quad u \in C_b(X). \tag{2.5}$$

According to [24, Theorem 4.2], we have the following characterization of $(\beta, \|\cdot\|_E)$ -continuous weakly compact operators $T : C_b(X) \to E$.

Theorem 2.2 *Let $T : C_b(X) \to E$ be a $(\beta, \|\cdot\|_E)$-continuous linear operator and $\hat{m} : \mathcal{B}o \to E''$ be its representing measure. Then the following statements are equivalent:*

(i) *T is weakly compact.*
(ii) *$\hat{m}(A) \in i_E(E)$ for every $A \in \mathcal{B}o$.*
(iii) *$\hat{m} : \mathcal{B}o \to E''$ is a Radon measure.*
(iv) *$\hat{m} : \mathcal{B}o \to E''$ is countably additive.*

(v) $T(u_n) \to 0$ *whenever* (u_n) *is a uniformly bounded sequence in* $C_b(X)$ *such that* $u_n(t) \to 0$ *for every* $t \in X$.

(vi) $T(u_n) \to 0$ *whenever* (u_n) *is a uniformly bounded sequence in* $C_b(X)$ *such that* $\operatorname{supp} u_k \cap \operatorname{supp} u_n = \emptyset$ *for* $n \neq k$.

The following result will be useful.

Theorem 2.3 *Let* $T : C_b(X) \to E$ *be a* $(\beta, \|\cdot\|_E)$*-continuous linear operator and* $\hat{m} : \mathcal{B}o \to E''$ *be its representing measure. Then the following statements hold:*

(i) *If* T *is weakly compact, then* $m := j_E \circ \hat{m} : \mathcal{B}o \to E$ *is a Radon measure and*

$$T(u) = \int_X u\, dm \quad \text{for} \quad u \in C_b(X).$$

(ii) *If* $|\hat{m}|(X) < \infty$*, then* T *is weakly compact and* $\hat{m}$ *is a Radon measure with* $|\hat{m}| \in M^+(X)$.

Proof (i) See [24, Theorem 3.5] and Theorem 2.2.

(ii) Assume that $|\hat{m}|(X) < \infty$. Then $\hat{m}$ is strongly additive (see [9, Proposition 15, p. 7]) and hence the operator $\hat{T} : B(\mathcal{B}o) \to E''$ is weakly compact (see [9, Theorem 1, p. 148]). Therefore, in view of (2.5), the operator $T : C_b(X) \to E$ is weakly compact and by Theorem 2.2, $\hat{m}$ is a Radon measure. Using Lemma 2.1, we get $|\hat{m}| \in M^+(X)$. □

3 Absolutely summing operators

In this section, we characterize β-absolutely summing operators $T : C_b(X) \to E$ and show that β-absolutely summing operators and dominated operators on $C_b(X)$ coincide.

We will need the following lemma.

Lemma 3.1 *For a sequence* (u_n) *in* $C_b(X)$*, the following statements are equivalent*:

(i) $\sup \left\{ \| \sum_{i \in M} \varepsilon_i\, u_i \|_\infty : \varepsilon_i = \pm 1, M \in \mathcal{F}(\mathbb{N}) \right\} < \infty$.

(ii) $\sum_{n=1}^\infty |\Phi(u_n)| < \infty$ *for all* $\Phi \in C_b(X)'$.

(iii) $\sum_{n=1}^\infty \left| \int_X u_n\, d\mu \right| < \infty$ *for all* $\mu \in M(X)$.

Proof (i)⇔(ii) It is well known (see [10, Chap. 5, Theorem 6, p. 44]).

(ii)⇒(iii) This follows from Theorem 1.1 because $\beta \subset \tau_u$.

(iii)⇒(i) Assume that (iii) holds. Then, for $\varepsilon_i = \pm 1$, $M \in \mathcal{F}(\mathbb{N})$ and $\mu \in M(X)$, we have

$$\left|\int_X \left(\sum_{i\in M} \varepsilon_i\, u_i\right) \mathrm{d}\mu\right| = \left|\sum_{i\in M} \int_X \varepsilon_i\, u_i\, \mathrm{d}\mu\right| \le \sum_{i\in M} \left|\int_X u_i\, \mathrm{d}\mu\right|$$
$$\le \sum_{n=1}^{\infty} \left|\int_X u_n\, \mathrm{d}\mu\right| < \infty.$$

This means that $\{\sum_{i\in M} \varepsilon_i\, u_i : \varepsilon_i = \pm 1,\ M \in \mathcal{F}(\mathbb{N})\}$ is $\sigma(C_b(X), M(X))$-bounded, and hence it is β-bounded. It follows that $\sup\{\|\sum_{i\in M} \varepsilon_i\, u_i\|_\infty : \varepsilon_i = \pm 1,$ $M \in \mathcal{F}(\mathbb{N})\} < \infty$ because τ_u and β have the same bounded sets. □

The following theorem characterizes β-absolutely summing operators $T : C_b(X) \to E$ (see [9, Proposition 2, p. 162], [22, Proposition 3.1] if X is compact).

Theorem 3.2 *Let $T : C_b(X) \to E$ be a $(\beta, \|\cdot\|_E)$-continuous linear operator. Then the following statements are equivalent:*

(i) *There exists $c > 0$ such that for any finite set $\{u_1, \dots, u_n\}$ in $C_b(X)$,*

$$\sum_{i=1}^{n} \|T(u_i)\|_E \le c\ \sup\left\{\sum_{i=1}^{n} \left|\int_X u_i\, d\mu\right| : \mu \in M(X),\ |\mu|(X) \le 1\right\}.$$

(ii) $\sum_{n=1}^{\infty} \|T(u_n)\|_E < \infty$ *if* $\sum_{n=1}^{\infty} |\int_X u_n\, d\mu| < \infty$ *for every* $\mu \in M(X)$.

(iii) $\sum_{n=1}^{\infty} \|T(u_n)\|_E < \infty$ *if* $\sum_{n=1}^{\infty} u_n$ *is unconditionally β-convergent.*

(iv) *T is β-absolutely summing.*

Proof (i)⇒(ii) Assume that (i) holds. Let (u_n) be a sequence in $C_b(X)$ such that $\sum_{n=1}^{\infty} |\int_X u_n\, \mathrm{d}\mu| < \infty$ for every $\mu \in M(X)$. Then, by Lemma 3.1, we have $\sum_{n=1}^{\infty} |\Phi(u_n)| < \infty$ for all $\Phi \in C_b(X)'$. Hence, by [27, 1.2.3, pp. 23–24], we get

$$\|(u_n)\|_1^w := \sup\left\{\sum_{n=1}^{\infty} |\Phi(u_n)| : \Phi \in C_b(X)', \|\Phi\|' \le 1\right\} < \infty.$$

Hence, for every $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} \|T(u_i)\|_E \le c\ \sup\left\{\sum_{i=1}^{n} |\Phi(u_i)| : \Phi \in C_b(X)', \|\Phi\|' \le 1\right\} \le c\, \|(u_n)\|_1^w,$$

and it follows that $\sum_{n=1}^{\infty} \|T(u_n)\|_E < \infty$, as desired.

(ii)⇒(iii) Assume that (ii) holds and the series $\sum_{n=1}^{\infty} u_n$ is unconditionally β-convergent in $C_b(X)$. Then $\sum_{n=1}^{\infty} |\int_X u_n\, \mathrm{d}\mu| < \infty$ for every $\mu \in M(X)$ and it follows that $\sum_{n=1}^{\infty} \|T(u_n)\|_E < \infty$.

(iii)⇒(iv) Assume that (iii) holds and $(u_n) \in \ell_s^1(C_b(X), \beta)$. Then a net $(s_M)_{M\in\mathcal{F}(\mathbb{N})}$ is a β-Cauchy sequence, where $s_M := \sum_{i\in M} u_i$ for $M \in \mathcal{F}(\mathbb{N})$. Let σ be a permutation of $\mathbb{N}$. Let $w \in \mathcal{W}_1$ and $\varepsilon > 0$ be given. Then, there exists $M \in \mathcal{F}(\mathbb{N})$ such that $p_w(\sum_{j\in L} u_j) \leq \varepsilon$ for every $L \in \mathcal{F}(\mathbb{N})$ with $L \cap M = \emptyset$. Choose $k \in \mathbb{N}$ such that $M \subset \{\sigma(i) : 1 \leq i \leq k\}$. Then for $n, m \in \mathbb{N}$ with $m > n > k$, we have $p_w\big(\sum_{i=n}^{m} u_{\sigma(i)}\big) \leq \varepsilon$. This means that the partial sums $\sum_{i=1}^{n} u_{\sigma(i)}$ form a β-Cauchy sequence in $C_b(X)$. Since the space $(C_b(X), \beta)$ is complete, we obtain that the series $\sum_{n=1}^{\infty} u_n$ is unconditionally β-convergent in $C_b(X)$. Hence, we get $\sum_{n=1}^{\infty} \|T(u_n)\|_E < \infty$.

(iv)⇒(i) Assume that (iv) holds. Let $w \in \mathcal{W}_1$. Then in view of [27, Theorem 2.1.2] there exists $c_w > 0$ such that $\pi_E((T(v_n))) = \sum_{n=1}^{\infty} \|T(v_n)\|_E \leq c_w$ whenever $(v_n) \in \ell_w^1(C_b(X), \beta)$ with $\mathcal{E}_{w,1}((v_n)) \leq 1$. Hence for $(v_n) \in \ell_w^1(C_b(X), \beta)$, we have

$$\pi_E((T(v_n))) = \sum_{n=1}^{\infty} \|T(v_n)\|_E \leq c_w\, \mathcal{E}_{w,1}((v_n)).$$

Let $u_i \in C_b(X)$ for $i = 1, \ldots, n$. Define $v_i = u_i$ for $i = 1, \ldots, n$ and $v_i = 0$ for $i > n$. Then

$$\sum_{i=1}^{n} \|T(u_i)\|_E \leq c_w \sup \left\{ \sum_{i=1}^{n} \left| \int_X u_i \, d\mu \right| : \mu \in U_w(1)^0 \right\}. \tag{3.1}$$

Note that $B_\infty(1) := \{u \in C_b(X) : \|u\|_\infty \leq 1\} \subset U_w(1)$. Hence, $U_w(1)^0 \subset B_\infty(1)^0$, where the polars are taken with respect to the pairing $\langle C_b(X), M(X)\rangle$. In view of Theorem 1.1 for $\mu \in M(X)$, we have

$$|\mu|(X) = \sup \left\{ \left| \int_X u \, \mathrm{d}\mu \right| : u \in C_b(X), \|u\|_\infty \leq 1 \right\}.$$

It follows that $B_\infty(1)^0 = \{\mu \in M(X) : |\mu|(X) \leq 1\}$. By (3.1) we get

$$\sum_{i=1}^{n} \|T(u_i)\|_E \leq c_w \sup \left\{ \sum_{i=1}^{n} \left| \int_X u_i \, \mathrm{d}\mu \right| : \mu \in M(X), |\mu|\,(X) \leq 1 \right\}.$$

Thus (i) holds. □

We show that dominated operators and β-absolutely summing operators on $C_b(X)$ coincide (see [27, 2.3.4, Proposition, p. 41]).

We will need the following lemma.

Lemma 3.3 *Assume that $\mu \in M(X)$. Then for $O \in \mathcal{T}$, we have*

$$|\mu|(O) = \sup \left\{ \left| \int_X u \, d\mu \right| : u \in C_b(X), \|u\|_\infty = 1 \text{ and } \operatorname{supp} u \subset O \right\}. \tag{3.2}$$

Proof For $u \in C_b(X)$ with $\|u\|_\infty = 1$ and $\operatorname{supp} u \subset O$, we have

$$\left| \int_O u \, \mathrm{d}\mu \right| \leq \|u\|_\infty \, |\mu|(O) \leq |\mu|(O).$$

Now let $\varepsilon > 0$ be given. Then there exists a $\mathcal{B}o$-partition $(A_i)_{i=1}^n$ of O such that

$$|\mu|(O) - \frac{\varepsilon}{3} \le \Big| \sum_{i=1}^n \mu(A_i) \Big|.$$

For $i = 1, \dots, n$ choose $K_i \in \mathcal{K}$ with $K_i \subset A_i$ such that $|\mu|(A_i \backslash K_i) \le \frac{\varepsilon}{3n}$ for $i = 1, \dots, n$. Choose pairwise disjoint $O_i \in \mathcal{T}$ with $K_i \subset O_i$ for $i = 1, \dots, n$ such that $|\mu|(O_i \backslash K_i) \le \frac{\varepsilon}{3n}$. For $i = 1, \dots, n$ choose $u_i \in C_b(X)$ with $0 \le u_i \le \mathbb{1}_X$, $u_i|_{K_i} \equiv 1$ and $u_i|_{X\backslash(O_i \cap O)} \equiv 0$. Let $u := \sum_{i=1}^n u_i$. Then $\|u\|_\infty = 1$ with $\operatorname{supp} u \subset O$ and

$$\int_O u \, \mathrm{d}\mu = \sum_{i=1}^n \int_O u_i \, \mathrm{d}\mu = \sum_{i=1}^n \int_{O_i \cap O} u_i \, \mathrm{d}\mu.$$

Then

$$\begin{aligned} |\mu|(O) - \tfrac{\varepsilon}{3} &\le \Big| \sum_{i=1}^n \mu(A_i) - \sum_{i=1}^n \mu(K_i) \Big| \\ &\quad + \Big| \sum_{i=1}^n \int_{K_i} u_i \, \mathrm{d}\mu - \sum_{i=1}^n \int_{O_i \cap O} u_i \, \mathrm{d}\mu \Big| + \Big| \int_O u \, \mathrm{d}\mu \Big| \\ &\le \sum_{i=1}^n |\mu| \, (A_i \backslash K_i) + \sum_{i=1}^n |\mu| \, ((O_i \cap O)\backslash K_i) + \Big| \int_O u \, \mathrm{d}\mu \Big| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \Big| \int_O u \, \mathrm{d}\mu \Big|, \end{aligned}$$

that is, $|\mu|(O) \le |\int_O u \, \mathrm{d}\mu| + \varepsilon$. Thus (3.2) holds. □

Now we can state our main result (see [27, 2.3.4, Proposition, p. 41]).

Corollary 3.4 *Assume that $T : C_b(X) \to E$ is a $(\beta, \| \cdot \|_E)$-continuous linear operator and $\hat{m} : \mathcal{B}o \to E''$ is its representing measure. Then the following statements are equivalent*:

(i) $|\hat{m}|(X) < \infty$.
(ii) *T is dominated.*
(iii) *T is β-absolutely summing.*
(iv) *T is absolutely summing.*

In this case, $\|T\|_{as} = |\hat{m}|(X)$.

Proof (i)⇔(ii) This follows from [25, Theorem 3.1].

(ii)⇒(iii) Assume that (ii) holds. Then T is dominated by $|\hat{m}|$, so

$$\|T(u)\|_E \le \int_X |u| \, \mathrm{d}\, |\hat{m}| \quad \text{for} \quad u \in C_b(X).$$

Let $u_1, \dots, u_n \in C_b(X)$. Then we have

$$\begin{aligned}\sum_{i=1}^{n}\|T(u_i)\|_E &\le \sum_{i=1}^{n}\int_X |u_i|\,\mathrm{d}\,|\hat{m}| \le \int_X \left(\sum_{i=1}^{n}|u_i|\right)\mathrm{d}\,|\hat{m}| \\ &\le \sup_{t\in X}\left(\sum_{i=1}^{n}|u_i(t)|\right)|\hat{m}|(X) = \sup_{t\in X}\left(\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\delta_t\right|\right)|\hat{m}|(X) \\ &\le \sup\left\{\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\mu\right| : \mu\in M(X),\ |\mu|(X)\le 1\right\}|\hat{m}|(X).\end{aligned}$$

In view of Theorem 3.2 T is β-absolutely summing and $\|T\|_{\mathrm{as}} \le |\hat{m}|(X)$.

(iii)⇒(i) Assume that (iii) holds. Then in view of Theorem 3.2, there exists $c > 0$ such that for every $u_1, \ldots, u_n \in C_b(X)$, we have

$$\sum_{i=1}^{n}\|T(u_i)\|_E \le c\,\sup\left\{\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\mu\right| : \mu\in M(X), |\mu|(X)\le 1\right\}.$$

Let (u_n) be a sequence in $C_b(X)$ such that $\sup_n \|u_n\|_\infty = a < \infty$ and $\operatorname{supp} u_n \cap \operatorname{supp} u_k = \emptyset$ if $n \ne k$. Then, for $\mu \in M(X)$ with $|\mu|(X) \le 1$, we have

$$\begin{aligned}\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\mu\right| &\le \sum_{i=1}^{n}\|u_i\|_\infty\,|\mu|(\operatorname{supp} u_i) \le a\sum_{i=1}^{n}|\mu|\,(\operatorname{supp} u_i) \\ &= a\,|\mu|\left(\bigcup_{i=1}^{n}\operatorname{supp} u_i\right) \le a\,|\mu|(X) \le a.\end{aligned}$$

Then $\sum_{n=1}^{\infty}\|T(u_n)\|_E \le ca < \infty$, so $\|T(u_n)\|_E \to 0$ and according to Theorem 2.2 T is weakly compact. Hence by Theorem 2.3 $m := j_E \circ \hat{m} : \mathcal{B}o \to E$ is a Radon measure and

$$T(u) = \int_X u\,\mathrm{d}m \quad \text{for} \quad u \in C_b(X).$$

Now, we shall show that $|m|(X) = |\hat{m}|(X) < \infty$. In fact, let $(A_i)_{i=1}^n$ be a $\mathcal{B}o$-partition of X and $\varepsilon > 0$ be given. Choose $e'_1, \ldots, e'_n \in B_{E'}$ such that $\|m\|(A_i) \le |m_{e'_i}|(A_i) + \frac{\varepsilon}{4n}$ for $i = 1, \ldots, n$. Hence

$$\sum_{i=1}^{n}\|m(A_i)\|_E \le \sum_{i=1}^{n}\|m\|(A_i) \le \sum_{i=1}^{n}|m_{e'_i}|(A_i) + \frac{\varepsilon}{4}. \tag{3.3}$$

For each $i = 1, \ldots, n$ one can choose $K_i \in \mathcal{K}$ with $K_i \subset A_i$ such that $|m_{e'}|(A_i \backslash K_i) \le \frac{\varepsilon}{4n}$. Hence $|m_{e'}|(A_i) \le |m_{e'_i}|(K_i) + \frac{\varepsilon}{4n}$ for $i = 1, \ldots, n$. Then we can choose pairwise disjoint open sets O_i with $K_i \subset O_i$ for $i = 1, \ldots, n$. According to Lemma 3.3 for each $i = 1, \ldots, n$ there exists $u_i \in C_b(X)$ with $\|u_i\|_\infty = 1$ and $\operatorname{supp} u_i \subset O_i$ such that

$$|m_{e'_i}|(O_i) \le \left|\int_X u_i\,\mathrm{d}m_{e'_i}\right| + \frac{\varepsilon}{2n}. \tag{3.4}$$

Hence, by (2.2) and Lemma 3.3, we have

$$\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}m_{e_i'}\right| = \sum_{i=1}^{n}|e_i'(T(u_i))| \le \sum_{i=1}^{n}\|T(u_i)\|_E$$
$$\le c\ \sup\left\{\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}\mu\right| : \mu\in M(X), |\mu|(X)\le 1\right\}$$
$$\le c\ \sup\left\{\sum_{i=1}^{n}|\mu|(O_i) : \mu\in M(X), |\mu|(X)\le 1\right\}\le c.$$

Hence using (3.3) and (3.4), we have

$$\sum_{i=1}^{n}\|m(A_i)\|_E \le \sum_{i=1}^{n}|m_{e_i'}|(A_i)+\frac{\varepsilon}{4}\le\sum_{i=1}^{n}\left(|m_{e_i'}|(K_i)+\frac{\varepsilon}{4n}\right)+\frac{\varepsilon}{4}$$
$$\le\sum_{i=1}^{n}|m_{e_i'}|(O_i)+\frac{\varepsilon}{2}\le\sum_{i=1}^{n}\left|\int_X u_i\,\mathrm{d}m_{e_i'}\right|+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\le c+\varepsilon.$$

It follows that $\sum_{i=1}^{n}\|m(A_i)\|_E\le c$, so $|m|(X)\le c$. Thus, $|\hat{m}|(X)\le c$ and hence $|\hat{m}|(X)\le\|T\|_{\mathrm{as}}$.

(iii)⇔(iv) This follows from Lemma 1.3 and Theorem 3.2. □

Let $\varphi\in L^1(\mu)$, where $\mu\in M^+(X)$. We define the multiplication operator $M_\varphi : C_b(X)\to L^1(\mu)$ by $M_\varphi(u) := \varphi u$ for $u\in C_b(X)$. For $A\in\mathcal{B}o$, let $m_\varphi(A) := \varphi\mathbb{1}_A$.

Proposition 3.5 *Assume that $\varphi\in L^1(\mu)$, where $\mu\in M^+(X)$. Then the following statements hold*:

(i) $|m_\varphi|(A)=\int_A|\varphi|\,d\mu$ *for* $A\in\mathcal{B}o$ *and* $|m_\varphi|\in M^+(X)$.

(ii) $\|M_\varphi(u)\|_1=\int_X|u|\,d\,|m_\varphi|$ *for* $u\in C_b(X)$, *that is*, M_φ *is dominated by* $|m_\varphi|$.

(iii) $m_\varphi : \mathcal{B}o\to L^1(\mu)$ *is a Radon measure and*

$$M_\varphi(u)=\int_X u\,dm_\varphi\ \text{ for } u\in C_b(X).$$

(iv) M_φ *is β-absolutely summing*.

Proof (i) Let $A\in\mathcal{B}o$ and $(A_i)_{i=1}^n$ be a finite $\mathcal{B}o$-partition of A. Then

$$\sum_{i=1}^{n}\|m_\varphi(A_i)\|_1=\sum_{i=1}^{n}\int_X|\varphi|\,\mathbb{1}_{A_i}\mathrm{d}\mu=\int_A|\varphi|\,\mathrm{d}\mu.$$

Hence, $|m_\varphi|(A)=\int_A|\varphi|\,\mathrm{d}\mu$ and it follows that $|m_\varphi|$ is countably additive. Since $|m_\varphi|\ll\mu$ and $\mu\in M^+(X)$, we obtain that $|m_\varphi|\in M^+(X)$.

(ii) From (i) it follows that $|\varphi|=\frac{\mathrm{d}|m_\varphi|}{\mathrm{d}\mu}$ (= the Radon–Nikodym derivative of $|m_\varphi|$ with respect to μ). Since $C_b(X)\subset L^1(\mu)$, in view of [7, Theorem C.8, p. 380] for $u\in C_b(X)$, we get

$$\|M_\varphi(u)\|_1 = \int_X |\varphi\, u|\, \mathrm{d}\mu = \int_X |u|\, \mathrm{d}|m_\varphi|.$$

(iii) Since $\|m_\varphi\|(A) \leq |m_\varphi|(A)$ for $A \in \mathcal{B}o$ and $|m_\varphi| \in M^+(X)$, we obtain that m_φ is a Radon measure. Note that for $s \in \mathcal{S}(\mathcal{B}o)$, $\int_X s\, \mathrm{d}m_\varphi = \varphi\, s$.

Let $u \in C_b(X)$ and choose a sequence (s_n) in $\mathcal{S}(\mathcal{B}o)$ such that $\|u - s_n\|_\infty \to 0$. Hence

$$\|M_\varphi(u) - \varphi s_n\|_1 = \int_X |\varphi u - \varphi s_n|\, \mathrm{d}\mu \leq \int_X |\varphi|\, \mathrm{d}\mu\, \|u - s_n\|_\infty.$$

This means that $M_\varphi(u) = \int_X u\, \mathrm{d}m_\varphi$.

(iv) In view of (ii) and Proposition 1.7 M_φ is $(\beta, \|\cdot\|_1)$-continuous. Hence, by Corollary 3.4 M_φ is β-absolutely summing. □

The next result shows that every β-absolutely summing operator $T : C_b(X) \to E$ admits a factorization through L^1-space (see [9, Corollary 7, pp. 164–165], [11, Corollary 2.5], [43, Theorem 1.8] if X is compact).

Corollary 3.6 *Let $T : C_b(X) \to E$ be a β-absolutely summing operator and $\hat{m} : \mathcal{B}o \to E''$ be its representing measure. Then, $m := j_E \circ \hat{m} : \mathcal{B}o \to E$ is a Radon measure with $|m| \in M^+(X)$ and the following statements hold*:

(i) *The inclusion map $I : C_b(X) \to L^1(|m|)$ is a β-absolutely summing operator with $\|I\|_{as} = |m|(X)$.*

(ii) *The integration operator $S : L^1(|m|) \to E$ defined by*

$$S(v) := \int_X v\, dm \quad \text{for all} \quad v \in L^1(|m|)$$

is bounded with $\|S\| \leq 1$ and $T = S \circ I$.

Proof In view of Theorem 2.3 $m := j_E \circ \hat{m} : \mathcal{B}o \to E$ is a Radon measure with $|m| \in M^+(X)$.

(i) Since $|m| \in M^+(X)$ in view of Proposition 3.5, I is β-absolutely summing and $\|I\|_{\mathrm{as}} = \int_X \mathbb{1}_X\, \mathrm{d}|\hat{m}| = |\hat{m}|(X) = |m|(X)$.

(ii) In view of Theorem 2.3 we have that $T(u) = \int_X u\, \mathrm{d}m$ for $u \in C_b(X)$.

Thus, we get $T = S \circ I$, where by (2.4) $\|S\| \leq 1$. □

4 Compact operators

The tensor product $ca(\mathcal{B}o) \otimes E$ consists of all measures $m : \mathcal{B}o \to E$ of the form $m = \sum_{i=1}^n (\mu_i \otimes e_i)$, where $\mu_i \in ca(\mathcal{B}o)$ and $e_i \in E$ for $i = 1, \dots, n$. Then $m(A) = \sum_{i=1}^n \mu_i(A) e_i$ for $A \in \mathcal{B}o$.

Now, we can state a characterization of β-compact operators $T : C_b(X) \to E$ in terms of their representing measures $\hat{m} : \mathcal{B}o \to E''$ (see [9, Theorem 18, p. 161], [34, Theorem 5.27] if X is compact).

Theorem 4.1 *Let $T : C_b(X) \to E$ be a $(\beta, \|\cdot\|_E)$-continuous linear operator and $\hat{m} : \mathcal{B}o \to E''$ be its representing measure. Then the following statements are equivalent:*

(i) *T is β-compact.*
(ii) *$\hat{m}$ has a relatively norm compact range in E''.*

Proof (i)⇒(ii) Assume that (i) holds. Then $T'' : C_b(X)'' \to E''$ is compact and hence $\hat{T} := T'' \circ \pi : B(\mathcal{B}o) \to E''$ is compact. Since

$$\left\{\hat{m}(A) : A \in \mathcal{B}o\right\} = \left\{\hat{T}(\mathbb{1}_A) : A \in \mathcal{B}o\right\} \subset \left\{\hat{T}(v) : v \in B(\mathcal{B}o), \|v\|_\infty \leq 1\right\},$$

we obtain that $\hat{m}(\mathcal{B}o)$ is relatively norm compact in E''.

(ii)⇒(i) Assume that (ii) holds. Since $\hat{m}(\mathcal{B}o)$ is weakly compact, the corresponding integration operator $\hat{T} : B(\mathcal{B}o) \to E''$ is weakly compact (see [19, Theorem 7]). Then, in view of (2.5), T is weakly compact, and by Theorem 2.3 $m := j_E \circ \hat{m} : \mathcal{B}o \to E$ is countably additive and $m(\mathcal{B}o)$ is relatively norm compact in E. According to the proof of [34, Theorem 5.18], there exists a sequence (m_k) in $ca(\mathcal{B}o) \otimes E$ such that $\|m - m_k\| \to 0$.

For each $k \in \mathbb{N}$, let $T_k : C_b(X) \to E$ be the finite rank operator defined by $T_k(u) := \int_X u \, \mathrm{d}m_k$. For $u \in C_b(X)$, we have

$$\|T_k(u) - T(u)\|_E = \left\| \int_X u \, \mathrm{d}(m_k - m) \right\|_E \leq \|u\|_\infty \, \|m_k - m\|(X),$$

and it follows that $\|T_k - T\| \to 0$. Hence, T is a compact operator and using Proposition 1.5 we have that T is β-compact. ☐

5 Nuclear operators

We state our main result that characterizes β-nuclear operators $T : C_b(X) \to E$ in terms of their representing measures (see [9, Theorem 4, p. 179], [34, Proposition 5.30], [43, Proposition 1.2] if X is a compact Hausdorff space).

Let (Ω, Σ, μ) be a finite measure space. Recall that a bounded linear operator $S : L^1(\mu) \to E$ is said to be *representable* if there exists an essentially bounded μ-Bochner integrable function $f : \Omega \to E$ such that $S(v) = \int_\Omega v(\omega) f(\omega) \, \mathrm{d}\mu$ for all $v \in L^1(\mu)$.

Theorem 5.1 *Let $T : C_b(X) \to E$ be a $(\beta, \|\cdot\|_E)$-continuous linear operator and $\hat{m} : \mathcal{B}o \to E''$ be its representing measure. Then the following statements are equivalent:*

(i) *T is β-nuclear.*
(ii) *$|\hat{m}|(X) < \infty$ and m has a $|m|$-Bochner integrable derivative.*
(iii) *$|\hat{m}|(X) < \infty$ and there exists a representable operator $S : L^1(|m|) \to E$ such that $T = S \circ I$, where $I : C_b(X) \to L^1(|m|)$ denotes the inclusion map.*

In this case, $\|T\|_{\beta\text{- nuc}} = |\hat{m}|(X) = |m|(X)$.

Proof (i)⇒(ii) This follows from [26, Theorem 3.1].

(ii)⇒(i) Assume that (ii) holds, that is, $|\hat{m}|(X) < \infty$ and there exists a function $f \in L^1(|m|, E)$ such that $m(A) = \int_A f(t)\,\mathrm{d}|m|$ for $A \in \mathcal{B}o$. Then, $|m|(X) = \|f\|_1$. Hence, we easily obtain that

$$T(u) = \int_X u(t)f(t)\,\mathrm{d}|m| \quad \text{for} \quad u \in C_b(X).$$

Let $L^1(|m|)\hat{\otimes}E$ denote the projective tensor product of $L^1(|m|)$ and E, equipped with the norm γ defined for $w \in L^1(|m|)\hat{\otimes}E$ by

$$\gamma(w) := \inf\Big\{ \sum_{n=1}^{\infty} |\lambda_n| \|v_n\|_1 \|e_n\|_E \Big\},$$

where the infimum is taken over all sequences (v_n) in $L^1(|m|)$ and (e_n) in E with $\lim_n \|v_n\|_1 = 0 = \lim_n \|e_n\|_E$ and $(\lambda_n) \in \ell^1$ such that $w = \sum_{n=1}^{\infty} \lambda_n (v_n \otimes e_n)$ (see [34, Proposition 2.8, pp. 21–22]). It is known that $L^1(|m|)\hat{\otimes}E$ is isometrically isomorphic to the Banach space $(L^1(|m|, E), \|\cdot\|_1)$ throughout the isometry J, where

$$J(v \otimes e) := v(\cdot) \otimes e \quad \text{for} \quad v \in L^1(|m|),\ e \in E,$$

(see [9, Example 10, p. 228], [34, Example 2.19, p. 29]).

Let $\varepsilon > 0$ be given. Then, there exist sequences (v_n) in $L^1(|m|)$ and (e_n) in E with $\lim_n \|v_n\|_1 = 0 = \lim_n \|e_n\|_E$ and $(\lambda_n) \in \ell^1$ such that

$$J^{-1}(f) = \sum_{n=1}^{\infty} \lambda_n (v_n \otimes e_n) \quad \text{in} \quad (L^1(|m|)\hat{\otimes}E, \gamma)$$

and

$$\sum_{n=1}^{\infty} |\lambda_n| \|v_n\|_1 \|e_n\|_E \le \gamma(J^{-1}(f)) + \varepsilon = \|f\|_1 + \varepsilon. \tag{5.1}$$

Hence

$$f = J\Big(\sum_{n=1}^{\infty} \lambda_n (v_n \otimes e_n) \Big) = \sum_{n=1}^{\infty} \lambda_n (v_n \otimes e_n) \quad \text{in} \quad (L^1(|m|, E), \|\cdot\|_1)$$

and we obtain that

$$T(u) = \sum_{n=1}^{\infty} \lambda_n \Big(\int_X u\, v_n \, \mathrm{d}|m| \Big) e_n \quad \text{for} \quad u \in C_b(X).$$

For $n \in \mathbb{N}$, let

$$\mu_n(A) := \int_A v_n \, \mathrm{d}|m| \quad \text{for} \quad A \in \mathcal{B}o.$$

Note that $\mu_n \in M(X)$ and $|\mu_n|(X) = \|v_n\|_1$. Then we have $\sup_n |\mu_n|(X) = \sup_n \|v_n\|_1 < \infty$. To show that the family $\{\mu_n : n \in \mathbb{N}\}$ is uniformly tight, let $\varepsilon > 0$ be given. Since $\|v_n\|_1 \to 0$, we can choose $n_\varepsilon \in \mathbb{N}$ such that $|\mu_n|(X) \le \varepsilon$ for $n > n_\varepsilon$. For $n = 1, \dots, n_\varepsilon$ choose $K_n \in \mathcal{K}$ such that $|\mu_n|(X \setminus K_n) \le \varepsilon$. Denote $K := \bigcup_{n=1}^{n_\varepsilon} K_n$. Then, $|\mu_n|(X \setminus K) \le \varepsilon$ for every $n \in \mathbb{N}$, as desired.

Clearly for $n \in \mathbb{N}$, we have (see [7, Theorem C.8]),

$$\int_X u\, v_n \, \mathrm{d}|m| = \int_X u \, \mathrm{d}\mu_n \quad \text{for} \quad u \in C_b(X).$$

Hence, we have

$$T(u) = \sum_{n=1}^{\infty} \lambda_n \Big(\int_X u \, \mathrm{d}\mu_n \Big) e_n \quad \text{for} \quad u \in C_b(X),$$

and this means that T is β-nuclear. By (5.1) we get

$$\|T\|_{\beta\text{- nuc}} \le \|f\|_1 = |m|(X). \tag{5.2}$$

(ii)⇒(iii) Assume that (ii) holds, that is, $|\hat{m}|(X) < \infty$ and there exists a $|m|$-Bochner integrable function $f : X \to E$ such that $m(A) = \int_A f(t) \, \mathrm{d}|m|$ for $A \in \mathcal{B}o$. Let

$$S(v) := \int_X v \, \mathrm{d}m \quad \text{for all} \quad v \in L^1(|m|).$$

Then, $S(u) = T(u)$ for $u \in C_b(X)$ and $m(A) = S(\mathbb{1}_A)$ for $A \in \mathcal{B}o$. Hence, by [9, Lemma 4, p. 62] f is essentially bounded and

$$S(v) = \int_X v(t) f(t) \, \mathrm{d}|m| \quad \text{for all} \quad v \in L^1(|m|).$$

(iii)⇒(ii) This is obvious.

Thus, (i)⇔(ii)⇔(iii) hold. Moreover, if T is β-nuclear and $\varepsilon > 0$ is given, then there exist a uniformly bounded and uniformly tight sequence (μ_n) in $M(X)$, a bounded sequence (e_n) in E and a sequence $(\lambda_n) \in \ell^1$ such that

$$T(u) = \sum_{n=1}^{\infty} \lambda_n \Big(\int_X u \, \mathrm{d}\mu_n \Big) e_n \quad \text{for} \quad u \in C_b(X)$$

and

$$\sum_{n=1}^{\infty} |\lambda_n||\mu_n|(X)\|e_n\|_E \leq \|T\|_{\beta\text{- nuc}} + \varepsilon. \tag{5.3}$$

Following the proof of [26, Theorem 3.1], we have

$$m(A) = \sum_{i=1}^{\infty} \lambda_n \, \mu_n(A) \, e_n \quad \text{for} \quad A \in \mathcal{B}o.$$

Now, if Π is a finite $\mathcal{B}o$-partition of X, then

$$\begin{aligned}\sum_{A\in\Pi} \|m(A)\|_E &= \sum_{A\in\Pi} \Big\| \sum_{n=1}^{\infty} \lambda_n \, \mu_n(A) \, e_n \Big\|_E \leq \sum_{A\in\Pi} \sum_{n=1}^{\infty} |\lambda_n||\mu_n(A)|\|e_n\|_E \\ &= \sum_{n=1}^{\infty} |\lambda_n| \Big(\sum_{A\in\Pi} |\mu_n(A)| \Big) \|e_n\|_E \leq \sum_{n=1}^{\infty} |\lambda_n||\mu_n|(X)\|e_n\|_E.\end{aligned}$$

Thus, in view of (5.3), we get

$$|m|(X) \leq \sum_{n=1}^{\infty} |\lambda_n||\mu_n|(X)\|e_n\|_E \leq \|T\|_{\beta\text{- nuc}} + \varepsilon.$$

Hence using (5.2) we have $\|T\|_{\beta\text{- nuc}} = |m|(X) = |\hat{m}|(X)$. Thus the proof is complete. □

In view of Theorem 5.1 and Corollary 3.4, we get (see [9, Corollary 5, p. 174]).

Corollary 5.2 *Assume that $T : C_b(X) \to E$ is a $(\beta, \|\cdot\|_E)$-continuous linear operator.*

(i) *If the operator T is β-nuclear, then T is β-absolutely summing and $\|T\|_{as} = \|T\|_{\beta\text{- nuc}}$.*
(ii) *If E has the Radon–Nikodym property, then T is β-absolutely summing if and only if T is β-nuclear.*

As a consequence of Corollarys 3.4 and 5.2, we have

Corollary 5.3 *Let $T : C_b(X) \to E$ be a dominated operator. If E has the Radon–Nikodym property, then T is β-compact.*

Remark 5.4 If X is a compact Hausdorff space, the related result to Corollary 5.3 was obtained in the different way by Uhl [44, Theorem 1].

Remark 5.5 A relationship between vector measures $m : \Sigma \to E$ with a μ-Bochner integrable derivatives (with respect to a finite measure μ) and the nuclearity of the corresponding integration operators $T_m : L^\infty(\mu) \to E$ has been studied by Swartz [42] and Popa [30].

6 Nuclearity of kernel operators

It is well known that if K is a compact Hausdorff space, $\mu \in M^+(K)$ and $k(\cdot,\cdot) \in C(K \times K)$, then the corresponding kernel operator $T : C(K) \to C(K)$ between Banach spaces, defined by

$$T(u)(t) = \int_X u(s)k(t,s)\,\mathrm{d}\mu(s) \quad \text{for} \quad u \in C(K), t \in K,$$

is nuclear (see [16, Theorem V.22, p. 99] if $X = [a,b]$).

Now as an application of Theorem 5.1, we extend this result to the setting, where X is a k-space and the kernel operator $T : C_b(X) \to C(K)$ is acting from the space $(C_b(X), \beta)$ to a Banach space $(C(K), \|\cdot\|_\infty)$, where K is a compact Hausdorff space.

From now on we assume that $\mu \in M^+(X)$ and $k(\cdot,\cdot) \in C_b(K \times X)$ with $\sup_{t \in K} |k(t,s)| \geq c$ for every $s \in X$ and some $c > 0$.

We start with the following lemma.

Lemma 6.1 *For every $v \in B(\mathcal{B}o)$, the mapping $\Psi_v : X \ni s \mapsto v(s)k(\cdot,s) \in C(K)$ is $(\mathcal{T}, \|\cdot\|_\infty)$-continuous.*

Proof Let $s_0 \in X$ and $\varepsilon > 0$ be given. Then for every $t \in K$ there exist a neighborhood V_t of t and a neighborhood W_t of s_0 such that

$$|k(z,s) - k(t,s_0)| \leq \frac{\varepsilon}{\|v\|_\infty} \quad \text{for all} \quad z \in V_t, s \in W_t.$$

Hence there exist $t_1, \dots, t_n \in K$ such that $K = \bigcup_{i=1}^n V_{t_i}$. Let us put $W := \bigcap_{i=1}^n W_{t_i}$. Let $t \in K$ and choose i_0 with $1 \leq i_0 \leq n$ such that $t \in V_{t_{i_0}}$. Then for $s \in W$, we have $|k(t,s) - k(t,s_0)| \leq \frac{\varepsilon}{\|v\|_\infty}$. Hence

$$\|\Psi_v(s) - \Psi_v(s_0)\|_\infty \leq \|v\|_\infty \sup_{t \in K} |k(t,s) - k(t,s_0)| \leq \varepsilon.$$

This means that Ψ_v is $(\mathcal{T}, \|\cdot\|_\infty)$-continuous. □

Let $L^1(\mu, C(K))$ stand for the Banach space of μ-Bochner integrable functions on X with values in $C(K)$. In view of [23, Theorem 5.1] we have

$$C_b(X, C(K)) \subset L^1(\mu, C(K)).$$

Hence, in view of Lemma 6.1, we can define the kernel operator $S : B(\mathcal{B}o) \to C(K)$ by

$$S(v) := \int_X \Psi_v(s)\,\mathrm{d}\mu(s) = \int_X v(s)k(\cdot,s)\,\mathrm{d}\mu(s) \quad \text{for all} \quad v \in B(\mathcal{B}o).$$

For $t \in K$, let $\phi_t(w) := w(t)$ for $w \in C(K)$. Then $\phi_t \in C(K)'$ and using Hille's theorem (see [13, §1, Section J, Theorem 36]), we get

$$S(v)(t) = \int_X v(s)k(t,s)\,\mathrm{d}\mu(s) \quad \text{for all} \quad v \in B(\mathcal{B}o), t \in K.$$

Then for $v \in B(\mathcal{B}o)$,

$$\begin{aligned}\|S(v)\|_\infty &= \sup_{t\in K} |S(v)(t)| \le \sup_{t\in K} \int_X |v(s)||k(t,s)|\,\mathrm{d}\mu(s)\\ &\le \int_X |v(s)| \sup_{t\in K} |k(t,s)|\,\mathrm{d}\mu(s) \le \|v\|_\infty \sup_{t\in K, s\in X} |k(t,s)|\,\mu(X),\end{aligned}$$

that is, S is a $(\|\cdot\|_\infty, \|\cdot\|_\infty)$-bounded operator.

Define a measure $m_k : \mathcal{B}o \to C(K)$ by

$$m_k(A) := S(\mathbb{1}_A) = \int_A k(\cdot,s)\,\mathrm{d}\mu(s) \quad \text{for} \quad A \in \mathcal{B}o.$$

Then,

$$S(v) = \int v\,\mathrm{d}m_k \quad \text{for all} \quad v \in B(\mathcal{B}o)$$

and for $A \in \mathcal{B}o, t \in K$, we have

$$m_k(A)(t) = \int_A k(t,s)\,\mathrm{d}\mu(s).$$

Proposition 6.2 *The measure m_k has the following properties*:

(i) *m_k is of bounded variation and for every $A \in \mathcal{B}o$,*

$$|m_k|(A) = \int_A \sup_{t\in K} |k(t,s)|\,\mathrm{d}\mu(s).$$

(ii) *$|m_k| \in M^+(X)$ and m_k is a Radon measure and*

$$m_k(A) = \int_A \frac{k(\cdot,s)}{\sup_{t\in K} |k(t,s)|}\,\mathrm{d}|m_k|(s) \quad \text{for all} \quad A \in \mathcal{B}o,$$

where the function $X \ni s \mapsto \frac{k(\cdot,s)}{\sup_{t\in K}|k(t,s)|} \in C(K)$ belongs to $L^1(|m_k|, C(K))$.

Proof (i) See [9, Theorem 4, p. 46].

(ii) Note that $|m_k|(A) \le \mu(A) \sup\{|k(t,s)| : t \in K, s \in X\}$ for $A \in \mathcal{B}o$. Then $|m_k| \in M^+(X)$ and hence m_k is a Radon measure. From (i) it follows that $\sup_{t\in K} |k(t,\cdot)| = \frac{\mathrm{d}|m_k|}{\mathrm{d}\mu}$ (= the Radon–Nikodym derivative of $|m_k|$ with respect to μ). Since $|m_k| \in M^+(X)$, using [23, Theorem 5.1] we get $C_b(X, C(K)) \subset L^1(|m_k|, C(K))$.

Let $v(s) := \frac{1}{\sup_{t\in K}|k(t,s)|}$ for $s \in X$. Then $v \in B(\mathcal{B}o)$ and by Lemma 6.1 the function

$$X \ni s \mapsto \frac{k(\cdot, s)}{\sup_{t\in K} |k(t,s)|} \in C(K)$$

belongs to $L^1(|m_k|, C(K))$. Hence we can define the measure $m_0 : \mathcal{B}o \to C(K)$ by

$$m_0(A) := \int_A \frac{k(\cdot, s)}{\sup_{t\in K} |k(t,s)|} \,\mathrm{d}|m_k|(s) \quad \text{for} \quad A \in \mathcal{B}o.$$

Using Hille's theorem and [7, Theorem C.8] for $A \in \mathcal{B}o$ and each $\tau \in K$, we get

$$\begin{aligned} m_0(A)(\tau) = \phi_\tau(m_0(A)) &= \phi_\tau\left(\int_X \frac{\mathbb{1}_A(s)k(\cdot, s)}{\sup_{t\in K} |k(t,s)|} \,\mathrm{d}|m_k|(s)\right) \\ &= \int_X \frac{\mathbb{1}_A(s)k(\tau, s)}{\sup_{t\in K} |k(t,s)|} \,\mathrm{d}|m_k|(s) \\ &= \int_X \frac{\mathbb{1}_A(s)k(\tau, s)}{\sup_{t\in K} |k(t,s)|} \sup_{t\in K} |k(t,s)| \,\mathrm{d}\mu(s) \\ &= \int_A k(\tau, s)\,\mathrm{d}\mu(s) = m_k(A)(\tau). \end{aligned}$$

Thus $m_k(A) = m_0(A) := \int_A \frac{k(\cdot,s)}{\sup_{t\in K} |k(t,s)|} \,\mathrm{d}|m_k|(s)$ for every $A \in \mathcal{B}o$. □

Define the kernel operator $T : C_b(X) \to C(K)$ by

$$T(u) := \int_X u(s)k(\cdot, s)\,\mathrm{d}\mu(s) \quad \text{for all} \quad u \in C_b(X).$$

Let us consider the mapping $\lambda : K \ni t \mapsto \mu_t \in M(X)$, where for $t \in K$,

$$\mu_t(A) := \int_A k(t,s)\,\mathrm{d}\mu(s) \quad \text{for all} \quad A \in \mathcal{B}o.$$

Then

$$T(u)(t) = \int_X u(s)\,\mathrm{d}\mu_t(s) \quad \text{for all} \quad u \in C_b(X), t \in K,$$

that is, T is a kernel operator in the sense of Sentilles (see [39, 40]) with the kernel λ and $T(u)(t) = \lambda(u)(t)$ for $u \in C_b(X)$, $t \in K$.

Now, we are ready to state our desire result.

Theorem 6.3 *The kernel operator $T : C_b(X) \to C(K)$ is β-nuclear and*

$$\|T\|_{\beta\text{- }nuc} = \int_X \sup_{t\in K} |k(t,s)|\,\mathrm{d}\mu(s).$$

Proof For every $u \in C_b(X)$, using Proposition 6.2, we get

$$\|T(u)\|_\infty = \sup_{t\in K} |T(u)(t)| = \sup_{t\in K} \left| \int_X u(s)k(t,s)\,\mathrm{d}\mu(s) \right|$$
$$\leq \int_X |u(s)| \sup_{t\in K} |k(t,s)|\,\mathrm{d}\mu(s) = \int_X |u(s)|\,\mathrm{d}|m_k|(s).$$

Hence, T is dominated, and by Proposition 1.7 T is $(\beta, \|\cdot\|_\infty)$-continuous and weakly compact. In view of Theorem 2.3,

$$T(u) = \int_X u\,\mathrm{d}m \quad \text{for all} \quad u \in C_b(X),$$

where $m := j_{C(K)} \circ \hat{m}$ and $\hat{m}$ is the representing measure of T.

On the other hand,

$$T(u) = S(u) = \int_X u\,\mathrm{d}m_k \quad \text{for all} \quad u \in C_b(X),$$

and since m_k and m are Radon measures, we derive that $m_k = m$. In view of Proposition 6.2 and Theorem 5.1, we obtain that T is a β-nuclear operator and

$$\|T\|_{\beta\text{- nuc}} = |m|(X) = |m_k|(X) = \int_X \sup_{t\in K} |k(t,s)|\,\mathrm{d}\mu(s).$$

□

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