Research Article

Solution of dynamic boundary value problems for an elastic disk with double porosity

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Abstract
The initial boundary value problems of dynamics are considered for the isotropic elastic body with double porosity. By the Laplace transform these problems are reduced to boundary value problems of pseudo-oscillations. Special representations are constructed for the general solution of pseudo-oscillation equations by means of metaharmonic functions. Such an approach facilitates the solution of problems and their solutions are written explicitly in form of absolutely and uniformly converging series. It is proved that inverse transforms yield solutions of initial dynamic problems. The question concerning the uniqueness of regular solutions of the considered problems is investigated.

Keywords Double porosity · Dynamic problems · Explicit solutions · Uniqueness theorems

1 Introduction

In recent years, interest has arisen in the study of problems of elasticity and thermoelasticity for porous solids. Theories of porous media are applied in many branches of engineering, technology [1, 2], geomechanics [3] and biomechanics [4].

The quasi-static theory for elastic materials with double porosity was developed by Aifantis and his coauthors in 1979-1986 [5–8]. This theory which the authors called consolidation theory combines the Barenblatt model [9] for a fluid flow in a medium of double porosity and the Biot model [10] for elastic materials with single porosity. In the quasi-static case the basic equations for water-saturated media with double porosity were obtained in [11, 12]. A fundamental solution of a system of equations of stationary oscillations in Aifantis's quasi-static theory of elasticity was constructed for solid bodies with double porosity in the paper [13].

However the above-mentioned theories of porous elasticity did not take into account the inertia term and the studies involved only static and quasi-static problems. But the inertia effect plays a key role in the investigation of various problems of oscillations and wave propagation in elastic media with double porosity. Hence it is important to study a complete dynamic model for materials with double porosity. A dynamic system for the description of deformation in media with single porosity was worked out by Biot [14, 15]. Flow and deformation processes in media with double porosity are considered with the inertia effect taken into account in the paper [16]. A complete dynamic case of the combined linear theory of flow and deformation of media with double porosity is treated in the paper [17]. Love and Rayleigh waves are respectively studied in [18, 19], respectively. The historical development of the mechanics of porous bodies, fundamental results and the sphere of their application are presented in detail in the monographs [20–23].

From the standpoint of application it is especially important to construct solutions in explicit form because such solutions allow one to perform effectively quantitative analyses of the considered problem. Questions related

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to this topic are considered, for example, in the works [24–28], where the explicit solutions of static boundary value problems of elasticity are constructed for the specific liquid-saturated media with double porosity. In [29, 30], the static two-dimensional boundary value problems are solved explicitly for a porous disk with voids.

In this article, two-dimensional boundary value problems of dynamics for porous materials are solved explicitly.

Methods for solving problems for private oscillations are not considered. These questions can be found in many papers, for example, [31–33].

stress) and those of mean pressures (or values normal derivatives pressures) of liquid in pores and cracks.

Basic equations of the motion of fluid-saturated media with double porosity contain the displacement vector in the neighborhood of some point and also the fluid pressure in cracks and the fluid pressure in pores in the neighborhood of the same point.

Let D be a finite plane region surrounded by a closed curve K. In what follows we assume that an isotropic and homogeneous porous elastic solid occupies a region of D.

A system of equations of the linear theory of elastic materials with double porosity has the form [8, 9]:

$$\begin{aligned} &\mu\,\Delta\mathbf{u}(\mathbf{x},t)+(\lambda+\mu)\boldsymbol{grad}\,\boldsymbol{div}\mathbf{u}(\mathbf{x},t)-\beta_1\boldsymbol{grad}\,p_1(\mathbf{x},t)-\beta_2\boldsymbol{grad}\,p_2(\mathbf{x},t)=\rho\partial_t^2\mathbf{u}(\mathbf{x},t)\\ &m_1\Delta p_1(\mathbf{x},t)-\alpha_1\partial_t p_1(\mathbf{x},t)+k(p_2(\mathbf{x},t)-p_1(\mathbf{x},t))-\beta_1\boldsymbol{div}\partial_t\mathbf{u}(\mathbf{x},t)=0\\ &m_2\Delta p_2(\mathbf{x},t)-\alpha_2\partial_t p_2(\mathbf{x},t)-k(p_2(\mathbf{x},t)-p_1(\mathbf{x},t))-\beta_2\boldsymbol{div}\partial_t\mathbf{u}(\mathbf{x},t)=0 \end{aligned} \tag{2.1}$$

In the present paper in Sect. 2 the basic equations of the motion of fluid-saturated media with double porosity are given, the basic the initial boundary value problems are formulated for an isotropic elastic body with double porosity. By the Laplace transform these problems are reduced to boundary value problems of pseudo-oscillations.

In Sect. 3 Green's identities are established and the uniqueness theorems are proved for solutions of both the initial problems and the corresponding problems of pseudo-oscillations.

In Sect. 4 the general solution of pseudo-oscillation equations are constructed by means of metaharmonic functions.

Examples of the application of these representations are given in Sect. 5. The problems of pseudo-oscillations for a specific elastic body - a porous disk - are solved. Solutions to these problems are obtained in the form of series. Conditions are provided that ensure the absolute and uniform convergence of these series and the use of the inverse Laplace theorem. It is proved that the inverse transforms provide a solution to the initial dynamic problems.

2 Formulation of boundary value problems

Let us consider the motion of a medium consisting of porous and permeable blocks separated from each other by a system of cracks (e.g. bone, granite). At every point of the medium are introduced two pressures: liquid pressure in pores and that in cracks. For such a body, called a medium with double porosity [7, 8], in the Aifantis theory of consolidation the problems of the theory of elasticity are formulated with the following boundary conditions: there are given the values of the displacement vector (or

where $\mathbf{x}=(x_1,x_2)\in D, t\in T, T\equiv[0,\infty)$ is time interval; λ, $\mu,\rho,\beta_1,\beta_2,m_i,\ \alpha_i,\ k$ are the known elastic and physical constants, $\mathbf{u}(\mathbf{x},t)=(u_1,u_2)$ is the displacement of the point $\mathbf{x}$; $p_1(\mathbf{x},t)$ and $p_2(\mathbf{x},t)$ are the average pressure values in the neighborhood of $\mathbf{x}$ in the cracks and pores, respectively.

Let us formulate the following initial boundary value problems.

Find in the $D\times T$ domain a regular solution $\mathbf{U}(\mathbf{x},t)=(\mathbf{u}(\mathbf{x},t),\ p_1(\mathbf{x},t),\ p_2(\mathbf{x},t))$ of system (2.1) that satisfies the initial conditions

$$\lim_{t\to 0}\mathbf{U}(\mathbf{x},t)=0,\ \lim_{t\to 0}\partial_t\mathbf{U}(\mathbf{x},t)=0, \tag{2.2}$$

and, on the boundary K, one of the conditions:

$$\lim_{x\to z}\mathrm{u}(\mathbf{x},t)=\mathbf{f}(\mathbf{z},t),\ \lim_{x\to z}p_\mathrm{i}(\mathbf{x},t)=f_{i+2}(\mathbf{z},t),\ i=1,2,\quad \text{in Problem I;} \tag{2.3}$$

$$\lim_{x\to z}\mathbf{R}(\partial_\mathrm{x},n)\mathbf{U}(\mathbf{x},t)=\mathbf{f}(\mathbf{z},t),\ \lim_{x\to z}\partial_\mathrm{n}p_i(\mathbf{x},t)=f_{i+2}(\mathbf{z},t),\ i=1,2,\ \text{in Problem II,} \tag{2.4}$$

where $\mathbf{z}=(z_1,z_2)\in S$, $\mathbf{n}(\mathbf{z})=(n_1(\mathbf{z}),n_2(\mathbf{z}))$ is the external normal to K; $\mathbf{f}=(f_1,f_2)$, $\mathbf{U}(\mathbf{x},t)\in C^1(\bar{D})\cap C^2(D)$, $\bar{D}=D\cup K$; $f_1,\ f_2,\ f_3,\ f_4$ are the given functions on S; $\partial_l=\frac{\partial}{\partial l}$, l is an arbitrary function. It is assumed [7] that $m_i>0, k>0, i=1,2$.

$$R(\partial_x,\mathbf{n})\mathbf{U}(\mathbf{x},t)=\mathbf{T}(\partial_x,\mathbf{n})\mathbf{u}(\mathbf{x},t)-\beta_1\mathbf{n}(\mathbf{x})p_1(\mathbf{x},t)-\beta_2\mathbf{n}(\mathbf{x})p_2(\mathbf{x},t) \tag{2.5}$$

is the stress vector in the porous medium and

$$\mathbf{T}(\partial_x,\mathbf{n})\mathbf{u}(\mathbf{x},t)=\mu\,\partial_n\mathbf{u}(\mathbf{x},t)+\lambda\,\mathbf{n}(\mathbf{x})\boldsymbol{div}\ \mathbf{u}(\mathbf{x},t)+\mu\sum_{i=1}^{2}n_i(\mathbf{x})\boldsymbol{grad}\,u_i(\mathbf{x},t)$$

is the stress vector in classical elasticity.

Let us assume that on the boundary K the so-called consistency conditions [31]

$$\partial_t^l f_j(\mathbf{z},0)=0,\ l=0,\ 1,\ 2,\ 3;\ j=1,2,3,4, \tag{2.6}$$

are fulfilled, while for large t the functions $\mathbf{U}$and f satisfy the conditions

$$\left|\partial_t^i \mathbf{U}(\mathbf{x},t)\right|<Me^{\xi_0 t},\ \left|\partial_t^q f_j(\mathbf{z},t)\right|<Me^{\xi_0 t} \tag{2.7}$$

uniformly with respect to $\mathbf{x}\in D$ and $\mathbf{z}\in K$,; $\xi_0\geq 0$, $M>0$ are constant values, $i=0,1,2, q=0,1,2,3,4; j=1,2,3,4$.

Consider the Laplace transform giving the originals $\mathbf{U}$ and f_j their images $\tilde{\mathbf{U}}$and $\tilde{f}_j$, respectively,

$$(\tilde{\mathbf{U}}(\mathbf{x},\tau),\tilde{f}_j(\mathbf{z},\tau))=\int_0^\infty e^{-t\tau}(\mathbf{U}(\mathbf{x},t),f_j(\mathbf{z},t))dt, \tag{2.8}$$

where $\tau=\xi+i\eta$, $\mathrm{Re}\tau=\xi>0$, $\mathrm{Im}\tau=\eta\in(-\infty;\infty)$, $\tilde{\mathbf{U}}=(\tilde{\mathbf{u}},\tilde{p}_1,\tilde{p}_2)$; $j=1,2,3,4$.

Given the conditions (2.2) and applying the rule of differentiation of the original, the dynamic problems I and II can be reduced to the following auxiliary so-called tasks of pseudo-oscillations:

$$\begin{aligned}&\mu\,\Delta\,\tilde{\mathbf{u}}(\mathbf{x},\tau)+(\lambda+\mu)\boldsymbol{grad\ div}\ \tilde{\mathbf{u}}(\mathbf{x},\tau)-\beta_1\boldsymbol{grad}\,\tilde{p}_1(\mathbf{x},\tau)-\beta_2\boldsymbol{grad}\,\tilde{p}_2(\mathbf{x},\tau)=\rho\tau^2\,\tilde{\mathbf{u}}(\mathbf{x},\tau)\\&m_1\Delta\,\tilde{p}_1(\mathbf{x},\tau)-\alpha_1\tau\,\tilde{p}_1(\mathbf{x},\tau)+k\,[\tilde{p}_2(\mathbf{x},\tau)-\tilde{p}_1(\mathbf{x},\tau)]-\beta_1\tau\,\boldsymbol{div}\ \tilde{\mathbf{u}}(\mathbf{x},\tau)=0\\&m_2\Delta\,\tilde{p}_2(\mathbf{x},\tau)-\alpha_2\tau\,\tilde{p}_2(\mathbf{x},\tau)-k\,[\tilde{p}_2(\mathbf{x},\tau)-\tilde{p}_1(\mathbf{x},\tau)]-\beta_2\tau\,\boldsymbol{div}\ \tilde{\mathbf{u}}(\mathbf{x},\tau)=0\end{aligned} \tag{2.9}$$

$$\tilde{\mathbf{u}}(\mathbf{z},\tau)=\tilde{\mathbf{f}}(\mathbf{z},\tau),\ \tilde{p}_i(\mathbf{z},\tau)=\tilde{f}_{i+2}(\mathbf{z},\tau),\ i=1,2\ \text{Problem I}_\tau; \tag{2.10}$$

$$\mathbf{R}(\partial_z,\mathbf{n})\,\tilde{\mathbf{U}}(\mathbf{z},\tau)=\tilde{\mathbf{f}}(\mathbf{z},\tau),\ \partial_{\mathrm{R}}\tilde{p}_i(\mathbf{z},\tau)=\tilde{f}_{i+2}(\mathbf{z},\tau)\ \ -\text{Problem II}_\tau \tag{2.11}$$

3 Uniqueness theorems

In this section Green's identities are established and the uniqueness theorems are proved for solutions of both the initial problems and the corresponding problems of pseudo-oscillations.

As is known [31], in the $D\times T$ domain, for arbitrary regular vectors $\mathbf{u}=(u_1,u_2)$

and $\partial_t\mathbf{u}=(\partial_t u_1,\partial_t u_2)$ and differential operator $A(\partial_x)=\mu\Delta+(\lambda+\mu)\boldsymbol{grad\ div}$ Green's formula has the form

$$\begin{aligned}&\int_D[\partial_t\mathbf{u}(\mathbf{x},t)\,A(\partial_x)\,\mathbf{u}(\mathbf{x},t)+E(\partial_t\mathbf{u}(\mathbf{x},t),\ \mathbf{u}(\mathbf{x},t))]dx\\&=\int_K\partial_t\mathbf{u}(\mathbf{y},t)\,\mathbf{T}(\partial_y,\mathbf{n})\,\mathbf{u}(\mathbf{y},t)\}d_yK,\end{aligned} \tag{3.1}$$

where $E(\partial_t\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t))$ is the nonnegative well-defined quadratic form. It is symmetric: $E(\partial_t\,\mathbf{u}(\mathbf{x},t),\ \mathbf{u}(\mathbf{x},t))=E(\mathbf{u}(\mathbf{x},t),\ \partial_t\mathbf{u}(\mathbf{x},t))$, and therefore

$$E(\partial_t\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t))=\frac{1}{2}\partial_t E(\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t)), \tag{3.2}$$

where

$$\begin{aligned}E(\mathbf{u},\mathbf{u})=&\frac{1}{3}(3\lambda+2\mu)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)^2\\&+\frac{\mu}{3}\left[\left(2\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+3\left(\frac{\partial u_2}{\partial x_2}\right)^2\right]\\&+\mu\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2\end{aligned}$$

is the potential deformation energy, $3\lambda+2\mu>0$, $\mu>0$. Using formulas (3.2) and the formulas

$$\begin{aligned}\partial_t\mathbf{u}\cdot\rho\,\partial_t^2\mathbf{u}=\frac{\rho}{2}\partial_t\left|\partial_t\mathbf{u}\right|^2,\ \ \partial_t\mathbf{u}\cdot\boldsymbol{grad}\,p_i=&\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial t}u_1p_i\right)\\&+\frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial t}u_2p_i\right)-p_i\,\boldsymbol{div}\frac{\partial}{\partial t}\mathbf{u},\end{aligned}$$

from (2.1) and (3.1) we obtain Green's formula for porous bodies in the case of dynamic problems:

$$\int_D \{\frac{\rho}{2}\partial_t|\partial_t\mathbf{u}(\mathbf{x},t)|^2+\frac{1}{2}\partial_t E(\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t))+m_1|\textbf{\textit{grad}}\,p_1(\mathbf{x},t)|^2+m_2|\textbf{\textit{grad}}\,p_2(\mathbf{x},t)|^2$$
$$+\frac{1}{2}\alpha_1\partial_t|p_1(\mathbf{x},t)|^2+\frac{1}{2}\alpha_2\partial_t|p_2(\mathbf{x},t)|^2+k[p_2(\mathbf{x},t)-p_1(\mathbf{x},t)]^2\}dx \tag{3.3}$$
$$=\int_K \{m_1p_1(\mathbf{y},t)\,\partial_n p_1(\mathbf{y},t)+m_2p_2(\mathbf{y},t)\,\partial_n p_2(\mathbf{y},t)+\partial_t\mathbf{u}(\mathbf{y},t)\mathbf{R}(\partial_y,\mathbf{n})\mathbf{U}(\mathbf{y},t)\}d_yK,$$

where the vector $\mathbf{R}(\partial_x,\mathbf{n})\mathbf{U}(\mathbf{x},t)$ is defined from (2.5).

Let $\mathbf{U}'$ and $\mathbf{U}''$ be arbitrary regular solutions of anyone of problems (2.1), (2.2), (2.3) or (2.1), (2.2), (2.4). Then the difference $\mathbf{U}(\mathbf{u},p_1,p_2)=\mathbf{U}'-\mathbf{U}''$is a regular solution of the corresponding homogeneous problems. In that case, the right-hand side of formula (3.3) is equal to zero and we can write

$$\frac{1}{2}\partial_t\int_D \left\{\rho|\partial_t\mathbf{u}(\mathbf{x},t)|^2+E(\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t))+\alpha_1\partial_t|p_1(\mathbf{x},t)|^2+\alpha_2\partial_t|p_2(\mathbf{x},t)|^2\right\}\mathrm{d}\mathbf{x}$$
$$=-\int_D \{m_1|\textbf{\textit{grad}}\,p_1(\mathbf{x},t)|^2+m_2|\textbf{\textit{grad}}\,p_2(\mathbf{x},t)|^2+k(p_2(\mathbf{x},t)-p_1(\mathbf{x},t))^2\}d\mathbf{x}\le 0$$

Therefore in the time interval $t\in[0;\infty)$ the value of the integral

$$\int_D \left\{\rho|\partial_t\mathbf{u}(\mathbf{x},t)|^2+E(\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t))+\alpha_1\partial_t|p_1(\mathbf{x},t)|^2+\alpha_2\partial_t|p_2(\mathbf{x},t)|^2\right\}\mathrm{d}\mathbf{x}$$

diminishes or remains constant and equal to zero. But since the integrand is nonnegative and equal to zero at the initial moment of time, each of the summands must be equal to zero too. Hence it follows that $\mathbf{u}=0$, $p_1=0$, $p_2=0$, $\mathbf{U}(\mathbf{u},p_1,p_2)$=0. So, the following statement is true.

Theorem 1 *A unique regular solution of homogeneous problems of dynamic is identically zero.Now let us prove the uniqueness theorems for pseudo-oscillatory problems. Let* $\tilde{\mathbf{U}}=(\tilde{\mathbf{u}},\tilde{p}_1,\tilde{p}_2)$ *be a solution of the homogeneous problem corresponding to one of problems* (2.9), (2.10) or (2.9), (2.11), *and* $\bar{\tilde{\mathbf{U}}}=(\bar{\tilde{\mathbf{u}}},\bar{\tilde{p}}_1,\bar{\tilde{p}}_2)$ *be the complex-conjugate vector.*

For arbitrary regular vectors $\tilde{\mathbf{u}}$ *and* $\bar{\tilde{\mathbf{u}}}$ *and differential operator* $A(\partial_x)=\mu\Delta+(\lambda+\mu)$***grad div*** *Green's formula has the form*

$$\int_D [\bar{\tilde{\mathbf{u}}}\,A(\partial_x)\,\tilde{\mathbf{u}}(\mathbf{x},\tau)+E(\tilde{\mathbf{u}}(\mathbf{x},\tau),\bar{\tilde{\mathbf{u}}}(\mathbf{x},\tau))]d\mathbf{x}$$
$$=\int_K \bar{\tilde{\mathbf{u}}}(\mathbf{y},\tau)\,\mathbf{T}(\partial_y,\mathbf{n})\,\tilde{\mathbf{u}}(\mathbf{y},\tau)\}d_yK, \tag{3.4}$$

where $E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}})=\frac{1}{3}(3\lambda+2\mu)\left|\frac{\partial\tilde{u}_1}{\partial x_1}+\frac{\partial\tilde{u}_2}{\partial x_2}\right|^2+\frac{\mu}{3}\left[\left|\frac{\partial\tilde{u}_1}{\partial x_1}-\frac{\partial\tilde{u}_2}{\partial x_2}\right|^2+3\left|\frac{\partial\tilde{u}_2}{\partial x_2}\right|^2\right]+\mu\left|\frac{\partial\tilde{u}_1}{\partial x_2}+\frac{\partial\tilde{u}_2}{\partial x_1}\right|^2.$

Since $3\lambda+2\mu>0$, μ>0 and $\tilde{\mathbf{u}}\cdot\bar{\tilde{\mathbf{u}}}=|\tilde{\mathbf{u}}|^2>0$, we have $E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}})$ =$E(\bar{\tilde{\mathbf{u}}},\tilde{\mathbf{u}})$,

$$\mathrm{Re}E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}})\equiv E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}})\ge 0,\ \ \mathrm{Im}E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}})=0. \tag{3.5}$$

Applying the equality $\int_D \tilde{p}_i\Delta\bar{\tilde{p}}_i d\mathbf{x}=\int_K \tilde{p}_i\partial_n\bar{\tilde{p}}_i dK-\int_D|\textbf{\textit{grad}}\,\tilde{p}_i|^2 d\mathbf{x}$, $i=1,2$ for system (2.9) we obtain Green's formula for the porous body

$$\int_D \{\rho\tau^2|\tilde{\mathbf{u}}|^2+E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}})+\frac{m_1}{\bar{\tau}}|\textbf{\textit{grad}}\,\tilde{p}_1|^2+\frac{m_2}{\bar{\tau}}|\textbf{\textit{grad}}\,\tilde{p}_2|^2+\frac{1}{\bar{\tau}}[(\alpha_1\bar{\tau}+k)|\tilde{p}_1|^2+$$
$$+(\alpha_2\bar{\tau}+k)|\tilde{p}_2|^2-k(\tilde{p}_1\bar{\tilde{p}}_2+\tilde{p}_2\bar{\tilde{p}}_1)]\}d\mathbf{x}= \tag{3.6}$$
$$=\int_K \{\frac{m_1}{\bar{\tau}}\tilde{p}_1(\mathbf{y},\tau)\partial_n\bar{\tilde{p}}_1(\mathbf{y},\tau)+\frac{m_2}{\bar{\tau}}\tilde{p}_2(\mathbf{y},\tau)\partial_n\bar{\tilde{p}}_2(\mathbf{y},\tau)+\bar{\tilde{\mathbf{u}}}(\mathbf{y},\tau)\mathbf{R}(\partial_y,\mathbf{n})\tilde{\mathbf{U}}(\mathbf{y},\tau)\}d_yK$$

For zero boundary conditions the right-hand side of equality (3.6) is equal to zero. Taking equality (3.5) into account and separating the real and the imaginary part in (3.6), we obtain

$$\int_D \{\rho(\xi^2-\eta^2)|\tilde{\mathbf{u}}|^2 + E(\tilde{\mathbf{u}},\bar{\tilde{\mathbf{u}}}) + \frac{m_1\xi}{|\tau|^2}|\boldsymbol{grad}\,\tilde{p}_1|^2 + \frac{m_2\xi}{|\tau|^2}|\boldsymbol{grad}\,\tilde{p}_2|^2 + \left|\sqrt{a_1}\tilde{p}_1 - \frac{a_{12}}{\sqrt{a_1}}\tilde{p}_2\right|^2 + \frac{a_1a_2-a_{12}^2}{a_1}|\tilde{p}_2|^2\}d\mathbf{x} = 0; \tag{3.7}$$

$$\eta\{\int_D \{2\xi|\tilde{u}|^2 + \frac{m_1}{|\tau|^2}|\boldsymbol{grad}\,\tilde{p}_1|^2 + \frac{m_2}{|\tau|^2}|\boldsymbol{grad}\,\tilde{p}_2|^2 + \frac{k}{|\tau|^2}|\tilde{p}_2-\tilde{p}_1|^2\}d\mathbf{x}\} = 0, \tag{3.8}$$

where $a_1 = \alpha_1 + \frac{k\xi}{|\tau|^2}, a_2 = \alpha_2 + \frac{k\xi}{|\tau|^2}, a_{12} = \frac{k\xi}{|\tau|^2}$.

Since $\xi,\ k,\ \alpha_1,\ \alpha_2 > 0$, we have $a_1a_2 - a_{12}^2 > 0$.

If $\eta = 0$, then from (3.7) we obtain $\mathbf{U}(\mathbf{u}, p_1, p_2) = 0$. If $\eta \neq 0$, then from (3.8) it follows that $\tilde{\mathbf{u}} = 0, \boldsymbol{grad}\,\tilde{p}_i = 0, \tilde{p}_1 = \tilde{p}_2 = \boldsymbol{const}$. Taking (2.9) into account, we finally obtain $\tilde{p}_1 = \tilde{p}_2 = 0$. The following statement is true.

Theorem 2 *Problems* I_τ *and* II_τ *have unique solutions.*

4 General representation of solution of a system of equations pseudo-oscillations

In this section for the general solution of system (2.9), special representation are constructed through elementary functions.

Applying the operator ***div*** to the first equation of system (2.9), we obtain a system of equations with respect to the sought value ***div*** $\tilde{\mathbf{u}}$,$\tilde{p}_1$ and $\tilde{p}_2$. The determinant of this system has the following form

$$\begin{aligned}\det(\Delta) =& \lambda_0 m_1 m_2 \Delta^3 + \{-\lambda_0(\alpha_1 m_2 + \alpha_2 m_1)\tau - \lambda_0 k(m_1+m_2) - (\beta_2^2 m_1 + \beta_1^2 m_2)\tau \\ & - m_1 m_2 \rho\tau^2\}\Delta^2 + \{\lambda_0 k(\alpha_1+\alpha_2)\tau + \rho\tau^3(\alpha_1 m_2 + \alpha_2 m_1) + \rho\tau^2 k(m_1+m_2) \\ & + k\tau(\beta_1^2+\beta_2^2) + \tau^2(\alpha_2\beta_1^2 + \alpha_1\beta_2^2) + 2\beta_1\beta_2 k\tau + \lambda_0\alpha_1\alpha_2\tau^2\}\Delta \\ & - \rho k\tau^3(\alpha_1+\alpha_2) - \rho\tau^4\alpha_1\alpha_2,\end{aligned} \tag{4.1}$$

where $\lambda_0 = \lambda + 2\mu > 0$. Expression (4.1) with respect to Δ is a third degree polynomial with complex coefficients, $\lambda_0 m_1 m_2 > 0$. Let us write it as the product

$$\det(\Delta) = \lambda_0 m_1 m_2 \prod_{j=1}^{3}(\Delta + \omega_j^2), \tag{4.2}$$

where the numbers $(-\omega_j^2)$ are the root of polynomial (4.1), $j = 1, 2, 3$. Relations between these roots are expressed by Vieta's formulas

$$\begin{aligned}\sum_{j=1}^{3}\omega_j^2 &= -\frac{1}{\lambda_0 m_1 m_2}[\lambda_0(\alpha_1 m_2 + \alpha_2 m_1)\tau + \lambda_0 k(m_1+m_2) \\ &\quad + (\beta_2^2 m_1 + \beta_1^2 m_2)\tau + m_1 m_2 \rho\,\tau^2] \\ \sum_{j=1}^{3}\omega_{j-1}^2\omega_j^2 &= \frac{\tau}{\lambda_0 m_1 m_2}[\lambda_0 k\,(\alpha_1+\alpha_2) + \rho\tau^2(\alpha_1 m_2 + \alpha_2 m_1) \\ &\quad + k\,\rho\,\tau(m_1+m_2) + k(\beta_1^2+\beta_2^2) + \tau(\alpha_2\beta_1^2 + \alpha_1\beta_2^2) \\ &\quad + 2\beta_1\beta_2 k + \lambda_0\alpha_1\alpha_2\tau], \\ \prod_{j=1}^{3}\omega_j^2 &= -\frac{\rho\tau^3}{\lambda_0 m_1 m_2}[k(\alpha_1+\alpha_2) - \tau\alpha_1\alpha_2].\end{aligned} \tag{4.3}$$

It is assumed that $\omega_0^2 \equiv \omega_3^2$.

Thus, using (4.2), with respect to ***div*** $\tilde{\mathbf{u}}$, $\tilde{p}_1$ and $\tilde{p}_2$ we obtain the new equations

$$\prod_{j=1}^{3}(\Delta + \omega_j^2)\begin{pmatrix}\boldsymbol{div}\ \tilde{\mathbf{u}} \\ \tilde{p}_1 \\ \tilde{p}_2\end{pmatrix} = 0. \tag{4.4}$$

Now, applying the operator ***rot*** $= -\partial_2 + \partial_1$ to the first equation of system (2.9), we obtain

$$(\Delta + \omega_4^2)\phi_4 = 0, \tag{4.5}$$

where $\omega_4^2 = -\frac{\rho\tau^2}{\mu}$, ***rot*** $\tilde{\mathbf{u}} = \phi_4$.

A solution of system (4.4) can be written in the form

$$\begin{aligned}&\tilde{\mathbf{u}} \equiv (\tilde{u}_1, \tilde{u}_2) = c_1\boldsymbol{grad}\,\phi_1 + c_2\boldsymbol{grad}\,\phi_2 + c_3\boldsymbol{grad}\,\phi_3 + c_4\boldsymbol{rot}\,\phi_4 \\ &\tilde{p}_1 = \varepsilon_1\phi_1 + \varepsilon_2\phi_2 + \varepsilon_3\phi_3, \\ &\tilde{p}_2 = \phi_1 + \phi_2 + \phi_3,\end{aligned} \tag{4.6}$$

where $\phi_l(\mathbf{x}, \tau)$ are scalar metaharmonic functions with a complex argument

$$(\Delta + \omega_l^2)\phi_l = 0,\ \ l = 1,\ 2,\ 3,\ 4 \tag{4.7}$$

while ω_l^2 are complex numbers which differ from one another in the half-plane $\mathrm{Re}\tau > 0, \omega_1^2 \neq \omega_2^2 \neq \omega_3^2$;

$$c_j = -\left\{ \left[\frac{\lambda_0}{\rho\,\beta_1\tau^3}(m_1\omega_j^2 + \alpha_1\tau + k) + \frac{\beta_1}{\rho\,\tau^2} \right]\varepsilon_j + \frac{\beta_1\,\beta_2\,\tau - \lambda_0 k}{\rho\,\beta_1\tau^3} \right\},\ c_4 = -\frac{\mu}{\rho\tau^2}, \tag{4.8}$$

where

$$\varepsilon_j = \frac{(\beta_1\beta_2\tau - \lambda_0 k)\,\omega_j^2 - \rho\,\tau^2 k}{(\lambda_0\omega_j^2 + \rho\tau^2)(m_1\omega_j^2 + \alpha_1\tau + k) + \beta_1^2\tau\,\omega_j^2},\quad j = 1, 2, 3. \tag{4.9}$$

By an immediate check we conclude that representations (4.6) satisfy equations (2.9).

Remark The check of the third equation of system (2.9) becomes easier if (4.9) in formula (4.8) is replaced by the following

$$\varepsilon_j' = \frac{\beta_2\,\omega_j^2(\beta_1\beta_2\,\tau - \lambda_0 k) + \beta_1\,\rho\,\tau^2(m_2\omega_j^2 + \alpha_2\tau + k)}{\beta_1\tau(k\,\rho\,\tau - \beta_1\beta_2\,\omega_j^2) - \lambda_0\,\beta_2\omega_j^2(m_1\omega_j^2 + \alpha_1\tau + k)}$$

It is proved that $\varepsilon_j = \varepsilon_j'$; $j = 1, 2, 3$.

5 Construction of solutions of boundary value problems of pseudo-oscillations for the porous disk

In this section, using representation (4.6), problems I_τ and II_τ are solved explicitly for a circle.

Let region D have the form of a disk bounded by a circumference K with radius R, the center of which coincides with the origin. Solutions of problems of pseudo-oscillation are obtained in the form of the series. The conditions are given which provide the absolute and uniform convergence of these series and the use of the Laplace inverse theorem. It is proved that inverse transforms yield solutions of initial dynamic problems.

Problem I_τ Regular solutions of equations (4.7) in the circle can be represented as follows [34]:

$$\phi_j(\mathbf{x},\tau) = \sum_{m=0}^{\infty} J_m(\omega_j r)(X_{mj}\cdot\boldsymbol{\nu}_m),\quad \phi_4(\mathbf{x},\tau) = \sum_{m=0}^{\infty} J_m(\omega_4 r)(X_{m4}\cdot\mathbf{s}_m), \tag{5.1}$$

where $\mathbf{x} = (r,\psi)$, $r^2 = x_1^2 + x_2^2$, J_m is the Bessel function of a complex variable with integer index $m X_{ml} = (\boldsymbol{X}_{ml}', \boldsymbol{X}_{ml}'')$ is the sought constant vector, $\boldsymbol{\nu}_m(\psi) = (\cos m\psi,\ \sin m\psi)$, $\mathbf{s}_m(\psi) = (-\sin m\psi,\ \cos m\psi)$, $l = 1, 2, 3, 4$.

Let us write the boundary conditions (2.10) of Problem I_τ in terms of normal and tangent components

$$\tilde{u}_n(\mathbf{z},\tau) = \tilde{f}_n(\mathbf{z},\tau),\ \tilde{u}_s(\mathbf{z},\tau) = \tilde{f}_s(\mathbf{z},\tau),\ \tilde{p}_i(\mathbf{z},\tau) = \tilde{f}_{i+2}(\mathbf{z},\tau),\ i = 1, 2, \tag{5.2}$$

where $\tilde{f}_n = n_1\tilde{f}_1 + n_2\tilde{f}_2, \tilde{f}_s = -n_2\tilde{f}_1 + n_1\tilde{f}_2$.

From representation (4.6) we write

$$\tilde{u}_n(\mathbf{x},\tau) = \partial_r\sum_{j=1}^{3} c_j\phi_j - \frac{c_4}{r}\partial_\psi\phi_4,\quad \tilde{u}_s(\mathbf{x},\tau) = \frac{1}{r}\partial_\psi\sum_{j=1}^{3} c_j\phi_j + c_4\partial_r\phi_4. \tag{5.3}$$

We expand the functions $\tilde{f}_n, \tilde{f}_s$ and $\tilde{f}_{i+2}$ into Fourier series

$$\begin{aligned}\tilde{f}_n(\mathbf{z},\tau) &= \sum_{m=0}^{\infty}(\boldsymbol{\alpha}_m(\tau)\cdot\boldsymbol{\nu}_m(\psi)),\\ \tilde{f}_s(\mathbf{z},\tau) &= \sum_{m=0}^{\infty}(\boldsymbol{\beta}_m(\tau)\cdot\mathbf{s}_m(\psi)),\\ \tilde{f}_{i+2}(\mathbf{z},\tau) &= \sum_{m=0}^{\infty}(\boldsymbol{\delta}_{mi}(\tau)\cdot\boldsymbol{\nu}_m(\psi)),\quad i = 1, 2,\end{aligned} \tag{5.4}$$

where $\boldsymbol{\alpha}_m(\tau) = (\alpha_{m1}, \alpha_{m2})$, $\boldsymbol{\beta}_m(\tau) = (\beta_{m1}, \beta_{m2})$ and $\boldsymbol{\delta}_{mi}(\tau) = (\delta_{mi1}, \delta_{mi2})$ are the Fourier coefficients of the respective functions $\tilde{f}_n, \tilde{f}_s$ and $\tilde{f}_{i+2}$, $i = 1, 2$.

Taking (4.6) into account, we substitute representations (5.1) into (5.3) and pass to the limit as $r \to R$. Then, taking (5.4) into account, from the boundary conditions (5.2), for each m we obtain a system of algebraic linear equations

$$\begin{aligned}&\sum_{j=1}^{3} c_j\omega_j J_m'(\omega_j R)X_{mj} + \frac{c_4}{R} m\,J_m(\omega_4 R)X_{m4} = \boldsymbol{\alpha}_m(\tau)\\ &\frac{1}{R}\sum_{j=1}^{3} c_j m\,J_m(\omega_j R)X_{mj} + \omega_4 J_m'(\omega_4 R)X_{m4} = \boldsymbol{\beta}_m(\tau)\\ &\sum_{j=1}^{3}\varepsilon_j J_m(\omega_j R)X_{mj} = \boldsymbol{\delta}_{m1}(\tau),\ \sum_{j=1}^{3} J_m(\omega_j R)X_{mj} = \boldsymbol{\delta}_{m2}(\tau),\ m = 0, 1, 2, \ldots\end{aligned} \tag{5.5}$$

where J_m' is the derivative of the function J_m with respect to its argument. We solve system (5.5) and substitute the obtained values of the vectors X_{mj} and X_{m4} into (5.1), and thereafter into (4.6). We obtain the solution of Problem I_τ.

Problem II_τ Passing in (2.5) to the normal and tangent components we obtain

$$\begin{aligned}\{\mathbf{R}(\partial_x,\mathbf{n})\tilde{\mathbf{U}}(\mathbf{x},\tau)\}_n &= \lambda_0\partial_r\tilde{u}_n(\mathbf{x},\tau) + \frac{\lambda}{r}\partial_\psi\tilde{u}_s(\mathbf{x},\tau) - \beta_1\tilde{p}_1(\mathbf{x},\tau) - \beta_2\tilde{p}_2(\mathbf{x},\tau),\\ \{\mathbf{R}(\partial_x,\mathbf{n})\tilde{\mathbf{U}}(\mathbf{x},\tau)\}_s &= \mu\left[\partial_r\tilde{u}_s(\mathbf{x},\tau) + \frac{1}{r}\partial_\psi\tilde{u}_n(\mathbf{x},\tau)\right]\end{aligned} \tag{5.6}$$

where $\tilde{u}_n(\mathbf{x},\tau)$ and $\tilde{u}_s(\mathbf{x},\tau)$ are defined by formulas (5.3). The boundary conditions (2.11) can be written in the form

$$\{\mathbf{R}(\partial_z,\mathbf{n})\tilde{\mathbf{U}}(\mathbf{z},\tau)\}_n=\tilde{f}_n(\mathbf{z},\tau),\quad \{\mathbf{R}(\partial_z,\mathbf{n})\tilde{\mathbf{U}}(\mathbf{z},\tau)\}_s=\tilde{f}_s(\mathbf{z},\tau),$$
$$\partial_R\tilde{p}_1(\mathbf{z},\tau)=\tilde{f}_3(\mathbf{z},\tau),\quad \partial_R\tilde{p}_2(\mathbf{z},\tau)=\tilde{f}_4(\mathbf{z},\tau), \tag{5.7}$$

where $\tilde{f}_n$, $\tilde{f}_s$ and $\tilde{f}_{i+2}$ are the known functions. It is assumed that they are expandable into Fourier series. Substitution (5.3) into (5.6), using (5.1) and (5.4) and passing to the limit as $r\to R$, from (5.7) we obtain for each m a system of algebraic linear equations

$$\begin{aligned}
&\lambda_0\left[\sum_{j=1}^{3}c_j\omega_j^2J''(\omega_jR)X_{mj}-\frac{c_4}{R^2}mJ_m(\omega_4R)X_{m4}+\frac{c_4\omega_4}{R}mJ'_m(\omega_4R)X_{m4}\right]-\\
&-\frac{\lambda}{R}\left[\frac{1}{R}\sum_{j=1}^{3}c_jm^2J_m(\omega_jR)X_{mj}+\omega_4mJ'_m(\omega_4R)X_{m4}\right]-\beta_1\sum_{j=1}^{3}\varepsilon_jJ_m(\omega_jR)X_{mj}-\\
&-\beta_2\sum_{j=1}^{3}J_m(\omega_jR)X_{mj}=\boldsymbol{\alpha}_m(\tau)\\
&\mu\left\{-\frac{1}{R^2}\sum_{j=1}^{3}c_jmJ_m(\omega_4R)X_{mj}+\omega_4^2J''_m(\omega_4R)X_{m4}+\frac{1}{R}\sum_{j=1}^{3}c_j\omega_jJ'_m(\omega_jR)X_{mj}+\right.\\
&\left.+\frac{c_4}{R}m^2J_m(\omega_4R)X_{m4}\right\}=\boldsymbol{\beta}_m(\tau)\\
&\sum_{j=1}^{3}\varepsilon_j\omega_jJ'_m(\omega_jR)X_{mj}=\boldsymbol{\delta}_{m1}(\mathbf{z},\tau),\quad \sum_{j=1}^{3}\omega_jJ'_m(\omega_jR)X_{mj}=\boldsymbol{\delta}_{m2}(\mathbf{z},\tau),\ m=0,\ 1,\ 2,\dots
\end{aligned} \tag{5.8}$$

Substituting the solution of system (5.8) into (5.1), and then into (4.6), we thus find the solutions $\tilde{\mathbf{u}}(\mathbf{x},\tau)$, $\tilde{p}_1(\mathbf{x},\tau)$ and $\tilde{p}_2(\mathbf{x},\tau)$ of Problem II_τ. From the uniqueness of the solutions of Problems I_τ and II_τ it follows that for each m the determinants of systems (5.5) and (5.8) are different from zero.

As is known, for $m\to\infty$ and $|\zeta|\le m$ we have the relation (see e.g. [35])

$$J_m(\zeta)\approx\left(\frac{\zeta}{2}\right)^m\frac{1}{m!}. \tag{5.9}$$

After substituting it into the solutions X_{mj} and X_{m4} of system (5.5), from representations (5.1) we can write

$$\left|\phi_j(\mathbf{x},\tau)\right|\le c\sum_{m=1}^{\infty}\frac{\alpha'_m}{m}\left(\frac{r}{R}\right)^m, \tag{5.10}$$

where $\alpha'_m=\max(|\alpha_m|,|\beta_m|,|\delta_{m1}|,|\delta_{m2}|)$, c is the constant not depending on m. As seen from (5.10) and (4.6), for $\tilde{\mathbf{u}}(\mathbf{x},\tau)$, $\tilde{p}_1(\mathbf{x},\tau)$ and $\tilde{p}_2(\mathbf{x},\tau)$, and also for their first order derivatives to be expanded into series which absolutely and uniformly converge on the boundary K, it is sufficient that the Fourier coefficients α'_m satisfy the condition $\left|\alpha'_m\right|\le\frac{c}{m^3}$, $m=1,2,\dots$ The latter estimate holds if $\tilde{f}\in C^2(K)$. For Problem II_τ, just like for I_τ, we can conclude that the condition $\tilde{f}\in C^1(K)$ ensures the uniform and absolute convergence of the series $\tilde{\mathbf{u}}(\mathbf{x},\tau)$, $\tilde{p}_1(\mathbf{x},\tau)$ and $\tilde{p}_2(\mathbf{x},\tau)$ and their first order derivatives when $\mathbf{x}\in\bar{D}$.

By analyzing the general form of solutions X_{ml} $(l=1,2,3,4)$ and applying conditions (2.6) and (2.7) as well as the asymptotic properties of the Bessel function for large ζ $(\mathrm{Re}\zeta>0)$ and relatively small m

$$J_m(\zeta)\approx\sqrt{\frac{2}{\pi\zeta}}\cos\left(\zeta-\frac{m\pi}{2}-\frac{\pi}{4}\right),$$

we can establish that the conditions $f_l(\mathbf{z},t)\in C^4(0\le t<\infty)$ give the following estimate for the solution of Problem I_τ

$$\left|\tilde{\mathbf{U}}(\mathbf{x},\tau)\right|\le\frac{c'}{|\tau|^4},\ c'=\boldsymbol{const} \tag{5.11}$$

for all τ uniformly for each $\mathbf{x}$, $\mathbf{x}\in\bar{D}$; $l=1,\ 2,\ 3,\ 4$. By an analogous reasoning we conclude that under the condition $f_l(\mathbf{z},t)\in C^3(0\le t<\infty)$ estimate (5.11) is fulfilled for the solution of Problem II_τ too. Since the Bessel functions are analytic in the complex half-plane $\mathrm{Re}\tau>0$, the values X_{mj} and X_{m4}, as well as ϕ_j, ϕ_4 $(j=1,2,3)$ and $\tilde{\mathbf{U}}=(\tilde{u}_1,\tilde{u}_2,\tilde{p}_1,\tilde{p}_2)$ are also analytic functions τ in the half-plane $\mathrm{Re}\tau>0$ for both problems. Under condition (5.11) the integral $\int_{-\infty}^{\infty}\left|\tilde{\mathbf{U}}(\mathbf{x},\ \xi+i\eta)\right|d\eta$ converges and $\tilde{\mathbf{U}}(\mathbf{x},\tau)\to 0$, as $|\tau|\to\infty$, uniformly with respect to $\mathbf{x}$, $\mathbf{x}\in\bar{D}$. By the Laplace inversion theorem (see e.g. [31]) vector $\tilde{\mathbf{U}}(\mathbf{x},\tau)$ is an image, and the original is defined by the integral

$$\mathbf{U}(\mathbf{x},t)=\frac{1}{2\pi i}\int\limits_{\xi-i\infty}^{\xi+i\infty}e^{t\tau}\tilde{\mathbf{U}}(\mathbf{x},\tau)d\tau, \tag{5.12}$$

where Re$\tau=\xi>\xi_0\geq 0$, ξ_0 is the growth exponent of the original. Condition (5.11) is sufficient for the existence of the derivatives $\partial_t\mathbf{U}(\mathbf{x},t)$ and $\partial_t^2\mathbf{u}(\mathbf{x},t)$.

Having in view expressions (2.8), (2.10) or (2.11), by direct computations we ascertain that the vector $\mathbf{U}(\mathbf{x},t)$ defined by formula (5.12) satisfies system (2.1), the initial conditions (2.2), and also the boundary conditions (2.3) or (2.4). In the formula (5.12) vector $\tilde{\mathbf{U}}(\mathbf{x},\tau)$ is solution pseudo-oscillations problems (2.9), (2.10) or (2.9), (2.11).

6 Conclusions

1. The novelty of the paper. We consider two-dimensional dynamic initial boundary value problems of elasticity for the body with double porosity. By the Laplace transform (2.8) these problems are reduced to the boundary value problems of pseudo- oscillations (2.9)–(2.11). For the general solution of system (2.9), special representations (4.6) are constructed through elementary (metaharmonic) functions. The properties of these functions are well known in mathematical physics. Such an approach facilitates the solution of two-dimensional problems of pseudo-oscillations. The solutions of this problems are obtained in the form of absolutely and uniformly convergent series. The conditions are written under which the inverse transforms exist and yield solutions of the initial problems. Green's formulas are derived and the uniqueness theorems are proved for solutions of both the initial problems and the corresponding problems of pseudo-oscillations.
2. Main result. Explicit solutions to the original problems are expressed by the formula (5.12), where vector $\tilde{\mathbf{U}}(\mathbf{x},\tau)$ is solution pseudo-oscillations problems (2.9), (2.10) or (2.9), (2.11).
3. The application of the considered method enables us to investigate a wide class of problems for systems of equations in modern linear theories of elasticity, thermoelasticity and porous elasticity for materials with double porosity; build explicit solutions basic boundary value problems not only for a circle, but also for a ring, a plane with a circular hole, etc.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interests.

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