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ORIGINAL RESEARCH PAPER

Schur convexity of Bonferroni harmonic mean

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Abstract
In this paper, we research the Schur convexity, Schur geometric convexity and Schur harmonic convexity of the Bonferroni harmonic mean. Some inequalities identified with the Bonferroni harmonic mean are set up to represent the utilizations of the acquired outcomes.

Keywords Bonferroni harmonic mean · Schur's condition · Majorization relationship · Inequality

Mathematics Subject Classification Prime 4600

1 Introduction

Arithmetic, Geometric and Harmonic means are three important means, which have been extensively used in the information aggregation [5, 6, 7, 11, 12, 17, 18, 19, 35, 36]. For a collection of real numbers $a_i(i = 1, 2, \ldots n)$, the Arithmetic mean (AM), the Geometric mean (GM) and the Harmonic mean (HM) are defined by:

$$\mathrm{AM}\, a_i(i = 1, 2 \ldots n) = \frac{1}{n}\sum_{i=1}^{n} a_i$$

$$\mathrm{BM}\, a_i(i = 1, 2 \ldots n) = \frac{1}{n}\prod_{i=1}^{n} a_i^n$$

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$$\mathrm{HM}\, a_i(i = 1, 2 \dots n) = n \Big/ \sum_{i=1}^{n} \frac{1}{a_i}$$

respectively. The fundamental characteristic of arithmetic mean is that it focuses on the group opinions, while geometric mean gives more importance to the individual opinions and harmonic mean is the reciprocal of arithmetic mean, which is a conservative average to be used to provide for aggregation lying between the max and min operators and is widely used as a tool to aggregate central tendency data [30].

In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data information. However, in many situations the input arguments take the form of triangular fuzzy numbers because of time pressure, lack of knowledge, and peoples limited expertise related with problem domain. Therefore, "how to aggregate fuzzy data by using the harmonic mean?" is an interesting research topic and is worth paying attention too. So Xu [30] developed the fuzzy harmonic mean operators such as fuzzy weighted harmonic mean (FWHM) operator, fuzzy ordered weighted harmonic mean (FOWHM) operator and fuzzy hybrid harmonic mean (FHHM) operator and applied them to MAGDM. Wei [25] developed fuzzy induced ordered weighted harmonic mean (FIOWHM) operator and then based on the FWHM and FIOWHM operators, presented the approach to MAGDM. H. Sun and M. Sun [23] further applied the BM operator to fuzzy environment, introduced the fuzzy Bonferroni harmonic mean (FBHM) operator and the fuzzy ordered Bonferroni harmonic mean (FOBHM) operator and applied the FOBHM operator to multiple attribute decision making.

The Bonferroni mean operator was initially proposed by Bonferroni [2] and was also investigated intensively by Yager [32].

Definition 1.1 [2] Let p, $q > 0, p + q \neq 0$ and let $a_i(i = 1, 2, \dots n)$ be a collection of non-negative numbers. If

$$\mathrm{BM}^{p,q}\left(a_1, a_2, \dots a_n\right) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} x_i^p x_j^q \right)^{\frac{1}{p+q}} \tag{1.1}$$

then $\mathrm{BM}^{p,q}$ is called the Bonferroni mean (BM) operator. It has important application in multi criteria decision-making [1, 13, 14, 29, 31, 32].

Beliakov et al. [1] further extended the BM operator by considering the correlations of any three aggregated arguments instead of any two.

Definition 1.2 [1] Let p, $q, r > 0, p + q + r \neq 0$ and let $a_i(i = 1, 2, \dots n)$ be a collection of non-negative numbers. If

$$\mathrm{GBM}^{p,q,r}\left(a_1, a_2, \dots a_n\right) = \left(\frac{1}{n(n-1)(n-2)} \sum_{i,j,k=1, i \neq j \neq k}^{n} x_i^p x_j^q x_k^r \right)^{\frac{1}{p+q+r}} \tag{1.2}$$

then $\mathrm{GBM}^{p,q,r}$ is called the generalized Bonferroni mean (GBM) operator. In particular, if $r=0$, then the GBM operator reduces to the BM operator. However, it is noted that both BM operator and the GBM operator do not consider the situation that $i=j$ or $j=k$ or $i=k$, and the weight vector of the aggregated arguments is not also considered. To overcome this drawback, Xia et al. [29]. defined the weighted version of the GBM operator.

Definition 1.3 [29] Let $p, q, r > 0, p+q+r \neq 0$ and let $a_i (i = 1, 2, \dots n)$ be a collection of non-negative numbers with the weight vector $w = \left(w_1, w_{2,} \dots w_n\right)^T$ such that $w_i \geq 0, i = 1, 2, \dots n$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{GWBM}^{p,q,r}\left(a_1, a_2, \dots a_n\right) = \left(\sum_{i,j,k=1, i\neq j\neq k}^{n} w_i w_j w_k x_i^p x_j^q x_k^r\right)^{\frac{1}{p+q+r}}$$

then $\mathrm{GWBM}^{p,q,r}$ is called the generalized weighted Bonferroni mean (GWBM) operator.

Some special cases can be obtained as the change of the parameters as follows:

Case 1 If $r = 0$ then the GWBM operator reduces to the following:

$$\mathrm{GWBM}^{p,q,0}\left(a_1, a_2, \dots a_n\right) = \left(\sum_{i,j,k=1, i\neq j\neq k}^{n} w_i w_j w_k x_i^p x_j^q\right)^{\frac{1}{p+q}} = \left(\sum_{i,j=1, i\neq j}^{n} w_i w_j x_i^p x_j^q \sum_{k=1}^{n} w_k\right)^{\frac{1}{p+q}}$$

$$\mathrm{GWBM}^{p,q,0}\left(a_1, a_2, \dots a_n\right) = \left(\sum_{i=1}^{n} w_i w_j x_i^p x_j^q\right)^{\frac{1}{p+q}}$$

which is the weighted Bonferroni mean (WBM) operator.

Case 2 If $q = 0$ and $r = 0$, then the GWBM operator reduces to the following:

$$\mathrm{GWBM}^{p,0,0}\left(a_1, a_2, \dots a_n\right) = \left(\sum_{i,j,k=1, i\neq j\neq k}^{n} w_i w_j w_k x_i^p x_j^q\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} w_i x_i^p \sum_{j=1}^{n} w_j \sum_{k=1}^{n} w_k\right)^{\frac{1}{p}}$$

$$\mathrm{GWBM}^{p,0,0}\left(a_1, a_2, \ldots a_n\right) = \left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}}$$

which is the generalized weighted averaging operator. Further in this case, let us look at the GWBM operator for some special cases of p.

1. If $p = 1$, the GWBM operator reduces to the weighted averaging (WA) operator.
2. If $p \to 0$, then the GWBM operator reduces to the weighted geometric (WG) operator.
3. If $p \to +\infty$, then the GWBM operator reduces to the max operator.

To aggregate the triangular fuzzy correlated information, based on the BM and weighted harmonic mean operators, H. Sun and M. Sun [23] developed the fuzzy Bonferroni harmonic mean operator. Since this operator considers the weight vector of the aggregated arguments, we redefine this operator as fuzzy weighted Bonferroni harmonic mean operator.

Definition 1.4 (*42*). Let $\hat{a}_i = \left[a_i^L, a_i^M, a_i^U\right](i = 1, 2, \ldots n)$ be a collection of triangular fuzzy numbers, let $\left(w_1, w_2, \ldots w_n\right)^T$ be the weight vector $a_i(i = 1, 2, \ldots n)$ where $w_i > 0, i = 1, 2, \ldots n$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{FWBHM}^{\mathrm{p,q}}\left(\hat{\mathrm{a}}_1, \hat{\mathrm{a}}_2, \ldots \hat{\mathrm{a}}_\mathrm{n}\right) = \frac{1}{\left(\sum_{i,j=1}^{n} (w_i w_j)/\hat{\mathrm{a}}_i^p \hat{\mathrm{a}}_j^q\right)^{\frac{1}{p+q}}}$$

$$= \left[\frac{1}{\left(\sum_{i,j=1}^{n} (w_i w_j)/(\hat{a}_i^L)^p \left(\hat{a}_j^L\right)^q\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} (w_i w_j)/(\hat{a}_i^M)^p \left(\hat{a}_j^M\right)^q\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} (w_i w_j)/(\hat{a}_i^U)^p \left(\hat{a}_j^U\right)^q\right)^{\frac{1}{p+q}}}\right]$$

where $p;q \geq 0$, then $\mathrm{FWBHM}^{p,q}$ is called the fuzzy weighted Bonferroni harmonic mean (FWBHM) operator [10, 3].

In particular, considering the triangular fuzzy numbers. Let $\hat{a}_i = \left[a_i^L, a_i^M, a_i^U\right]$ $(i = 1, 2, \ldots n)$ reduce to the interval numbers $\hat{a}_i = \left[a_i^L, a_i^M\right](i = 1, 2, \ldots n)$ then the FWBHM operator (10) reduces to the uncertain weighted Bonferroni harmonic mean (UWBHM) operator as follows:

$$\mathrm{UWBHM}^{\mathrm{p,q}}\left(\hat{a}_1, \hat{a}_2, \ldots \hat{a}_n\right) = \frac{1}{\left(\sum_{i,j=1}^{n} (w_i w_j)/\hat{a}_i^p \hat{a}_j^q\right)^{\frac{1}{p+q}}}$$

$$=\left[\frac{1}{\left(\sum_{i,j=1}^{n}\left(w_iw_j\right)/\left(\hat{a}_i^L\right)^p\left(\hat{a}_j^L\right)^q\right)^{\frac{1}{p+q}}},\frac{1}{\left(\sum_{i,j=1}^{n}\left(w_iw_j\right)/\left(\hat{a}_i^U\right)^p\left(\hat{a}_j^U\right)^q\right)^{\frac{1}{p+q}}}\right]$$

If $w=\left({}^1/_n,{}^1/_n,\cdots{}^1/_n\right)^T$ then the UWBHM operator reduces to the uncertain Bonferroni harmonic mean (UBHM) operator as follows:

$$\mathrm{UBHM}^{\mathrm{p,q}}\left(\mathrm{a}_1,\mathrm{a}_2,\ldots\mathrm{a}_\mathrm{n}\right)=\frac{1}{\left(1/n^2\sum_{i,j=1}^{n}(w_iw_j)/\hat{a}_i^p\hat{a}_j^q\right)^{\frac{1}{p+q}}}$$

$$=\left[\frac{1}{\left(1/n^2\sum_{i,j=1}^{n}\left(w_iw_j\right)/\left(\hat{a}_i^L\right)^p\left(\hat{a}_j^L\right)^q\right)^{\frac{1}{p+q}}},\frac{1}{\left(1/n^2\sum_{i,j=1}^{n}\left(w_iw_j\right)/\left(\hat{a}_i^U\right)^p\left(\hat{a}_j^U\right)^q\right)^{\frac{1}{p+q}}}\right]$$

If $a_i^L=a_i^U=a_i$ for all, then the UBHM operator reduces to the weighted Bonferroni harmonic mean (WBHM) operator as follows:

$$\mathrm{WBHM}^{\mathrm{p,q}}\left(\mathrm{a}_1,\mathrm{a}_2,\ldots\mathrm{a}_\mathrm{n}\right)=\frac{1}{\left(\sum_{i,j=1}^{n}(w_iw_j)/a_i^pa_j^q\right)^{\frac{1}{p+q}}}$$

In the Case $w=\left({}^1/_n,{}^1/_n,\cdots{}^1/_n\right)^T$ then the WBHM operator reduces to the Bonferroni harmonic mean (BHM) operator as follows:

$$\mathrm{BHM}^{\mathrm{p,q}}\left(\mathrm{a}_1,\mathrm{a}_2,\ldots\mathrm{a}_\mathrm{n}\right)=\frac{1}{\left(1/n^2\sum_{i,j=1}^{n}(1/a_i^pa_j^q\right)^{\frac{1}{p+q}}} \tag{1.4}$$

In recent years, the Schur convexity of functions relating to special means is a very significant research subject and has attracted the interest of many mathematicians. As supplements to the Schur convexity of functions, the Schur geometrically convex functions and Schur harmonically convex functions were investigated [8, 21, 26, 27].

In [9]. the authors discussed the Schur convexity, Schur geometric convexity, Schur harmonic convexity and Schur m-power convexity of the geometric Bonferroni mean.

This motivated us to determine the Schur convexity, Schur geometric convexity, Schur harmonic convexity and Schur m-power convexity of the Bonferroni harmonic mean.

Our main results are as follows.

Theorem 1.1 *For fixed non-negative real numbers* p, q *with* $p+q \neq 0$, *if* $x = (x_1, x_2, \dots x_n)$ *then* $\mathrm{BHM}^{p,q}(x)$ *is Schur concave, Schur geometric convex and Schur harmonic convex on* $R^n_{++} := (0, +\infty)^n$.

Theorem 1.2 *For fixed non-negative real numbers* p, q *with* $p+q \neq 0$, *if* $x = (x_1, x_2, \dots x_n)$ *then (x) is Schur m - power convexity on* R^n_{++}.

1. *If* $m < 0$, *then* $\mathrm{BHM}^{p,q,r}(X)$ *is Schur m-power convex*;
2. *If* $m > 0$, *then* $\mathrm{BHM}^{p,q,r}(X)$ *is Schur m-power concave*;
3. *If* $m = 0$, *then* $\mathrm{BHM}^{p,q,r}(X)$ *is Schur m-power convex (concave);*

2 Preliminaries

We begin with recalling some basic concepts and notations in the theory of majorization. For more details, we refer the reader to [2, 32].

Definition 2.1 Let $x = (x_1, x_2, x_3 \dots, x_n)$. and $y = \left(y_1, y_2, y_3 \dots, y_n\right) \in R^n$.

1. x is said to be majorized by y (in symbols $x \prec y$.),$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, 3, \dots, n-1$. and $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} y_i$ where $x_{[1]} \geq, \dots, \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangement of x and y and y in a descending order.
2. $\Omega \subseteq R^n$is called a convex set, if $\left(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n\right) \in \Omega$, for any x and $y \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.
3. Let $\Omega \subseteq R^n$, the function $\varphi : \Omega \to R^n$ is said to be schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur concave function on Ω, if and only if $-\varphi$ is Schur convex function.

Definition 2.2 [22] Let $x = (x_1, x_2, x_3 \dots, x_n)$. and $y = \left(y_1, y_2, y_3, \dots, y_n\right) \in R^n_+$.

1. $\Omega \subseteq R^n$ is called geometrically convex set, if $(x_1^{\alpha} y_1^{\beta}, \dots, x_n^{\alpha} y_n^{\beta}) \in R^n$ for any $\boldsymbol{x}$ and $y \in \Omega$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.
2. Let $\Omega \subseteq R^n_+$ the function $\varphi : \Omega \to R_+$ is said to be schur geometrically convex function on Ω if $(\log x_1, \log x_2 \dots \log x_n)(\log y_1, \log y_2 \dots \log y_n)$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex function.

Definition 2.3 [4] Let $x = (x_1, x_2, x_3 \ldots, x_n)$ and $y = \left(y_1, y_2, y_3, \ldots, y_n\right) \in R^n_+$.

1. A set $\Omega \subseteq R^n$. is said to be a harmonically convex set, if

$$\left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda) y_2}, \ldots, \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n}\right) \in \Omega$$

for any $\boldsymbol{x}$ and $y \in \Omega$, and $\lambda \in [0, 1]$.

2. A function $\varphi : \Omega \to R_+$. Is said to be a Schur-harmonically convex function on Ω, if $\frac{1}{x} \prec \frac{1}{y}$ implies $\varphi(x) \le \varphi(y)$. φ is said to be a Schur harmonically concave function on Ω. If and only if $-\varphi$ is a Schur-harmonically convex function.

Lemma 2.1 *Let $\Omega \subseteq R^n$ be symmetric with non empty interior convex set and let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schur convex (concave) if.*

$$(x_1 - x_2)\left(\frac{\partial\phi(X)}{\partial x_1} - \frac{\partial\phi(X)}{\partial x_2}\right) \ge 0 (\le 0).$$

holds for any $x = \left(x_1, x_2, x_3 \ldots, x_n\right) \in \Omega^0$.

Lemma 2.2 *Let $\Omega \subseteq R^n$ be a symmetric geometrically convex set with non empty interior Ω^0. Let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schur gemetrically convex (concave) function $x = \left(x_1, x_2, x_3 \ldots, x_n\right) \in \Omega^0$ if and only if ϕ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1\frac{\partial\phi(X)}{\partial x_1} - x_2\frac{\partial\phi(X)}{\partial x_2}\right) \ge 0 (\le 0).$$

Lemma 2.3 *Let $\Omega \subseteq R^n$ be symmetric harmonically convex set with non empty interior Ω^0 Let $\varphi : \Omega \to R_+$ be continuous on Ω. and differentiable on Ω^0 Then φ. is Schur harmonically convex (concave) function $x = \left(x_1, x_2, x_3 \ldots, x_n\right) \in \Omega^0$. if and only if ϕ is symmetric on Ω and.*

$$(x_1 - x_2)\left(x_1^2\frac{\partial\phi(X)}{\partial x_1} - x_2^2\frac{\partial\phi(X)}{\partial x_2}\right) \ge 0 (\le 0).$$

holds for any $x = \left(x_1, x_2, \ldots, x_n\right) \in \Omega^0$.

Lemma 2.4 [20, 29] *Let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schur m-power convex on function $x = \left(x_1, x_2, \dots, x_n\right) \in \Omega^0$ if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{m-1}\frac{\partial\varphi(x)}{\partial x_1} - x_2^{m-1}\frac{\partial\varphi(x)}{\partial x_2}\right] \geq 0 \, if \, m \neq 0 \tag{2.4}$$

and

$$\left(\log x_1 - \log x_2\right)\left[x_1^m\frac{\partial\varphi(x)}{\partial x_1} - x_2^m\frac{\partial\varphi(x)}{\partial x_2}\right] \geq 0 \, if \, m = 0 \tag{2.4}$$

Lemma 2.5 *Let$\left(x_1, x_2, \dots x_n,\right) \in R_n^+$ and $A_n(x) = \frac{1}{n}\sum_{i=1}^n x_i$. Then*

$$u = \underbrace{A_n(x), A_n(x), \dots \dots A_n(x)}_{n} \prec (x_1, x_2, ..x_n) = x$$

Lemma 2.6 *If $x_i > 0, i = 1, 2, \dots n,$, for any non negative constant c satisfying $0 \leq c < \frac{1}{n}\sum_{i=1}^n x_i$ one has*

$$\left(\frac{x_1}{\sum_{i=1}^n x_i}, \dots \frac{x_n}{\sum_{i=1}^n x_i}\right) \prec \left(\frac{x_1 - c}{\sum_{i=1}^n (x_i - c)}, \dots \frac{x_n - c}{\sum_{i=1}^n (x_i - c)}\right).$$

3 Proof of main results

The Bonferroni harmonic mean (BHM) is defined by

$$\mathrm{BHM}^{\mathrm{p,q}}(x) = \frac{1}{\left(1/n^2 \sum_{i,j=1}^n (1/x_i^p x_j^q\right)^{\frac{1}{p+q}}}$$

Taking the natural logarithm gives

$$\log \mathrm{BHM}^{\mathrm{p,q}}(x) = \log 1 - \log\left(1/n^2 \sum_{i,j=1}^n (1/x_i^p x_j^q\right)^{\frac{1}{p+q}}$$

$$\log \mathrm{BHM}^{\mathrm{p,q}}(x) = -\frac{1}{\mathrm{p+q}}(\log n^2 + \mathrm{Q}) \tag{3.1}$$

Where

$$Q=\sum_{J=3}^{n}\left[log\left(\frac{1}{x_1^p x_j^q}\right)+log\left(\frac{1}{x_2^p x_j^q}\right)\right]+\sum_{i=3}^{n}\left[log\left(\frac{1}{x_i^p x_1^q}\right)+log\left(\frac{1}{x_i^p x_2^q}\right)\right]$$
$$+\left[log\left(\frac{1}{x_1^p x_2^q}\right)+log\left(\frac{1}{x_2^p x_1^q}\right)\right]+\sum_{i,j=3,i\neq j}^{n}\left[log\left(\frac{1}{x_i^p x_j^q}\right)\right].$$

Partially differentiating the Eq. (3.1) with respect to x_1, we have

$$\frac{\partial \mathrm{BHM}^{\mathrm{p,q}}(x)}{\partial x_1}=\frac{\mathrm{BHM}^{\mathrm{p,q}}(x)}{p+q}\frac{\partial}{\partial x_1}\sum_{J=3}^{n}\left[log\left(\frac{1}{x_1^p x_j^q}\right)+log\left(\frac{1}{x_2^p x_j^q}\right)\right]$$
$$+\sum_{i=3}^{n}\left[log\left(\frac{1}{x_i^p x_1^q}\right)+log\left(\frac{1}{x_i^p x_2^q}\right)\right]$$
$$+\left[log\left(\frac{1}{x_1^p x_2^q}\right)+log\left(\frac{1}{x_2^p x_1^q}\right)\right]$$

$$\frac{\partial \mathrm{BHM}^{\mathrm{p,q}}(x)}{\partial x_1}=\frac{(n-1)\mathrm{BHM}^{p,q}(x)}{p+q}\frac{\partial}{\partial x_1}\left[logx_1^p+logx_1^q\right]$$

$$\frac{\partial \mathrm{BHM}^{\mathrm{p,q}}(x)}{\partial x_1}=\frac{(n-1)\mathrm{BHM}^{p,q}(x)}{p+q}\frac{p+q}{x_1},$$

$$\frac{\partial \mathrm{BHM}^{\mathrm{p,q}}(x)}{\partial x_1}=\frac{(n-1)\mathrm{BHM}^{p,q}(x)}{x_1}.$$

Partially differentiating the Eq. (3.1) with respect to x_2, we have

$$\frac{\partial \mathrm{BHM}^{\mathrm{p,q}}(x)}{\partial x_2}=\frac{\mathrm{BHM}^{p,q}(x)}{p+q}\frac{\partial}{\partial x_2}\sum_{J=3}^{n}\left[log\left(\frac{1}{x_1^p x_j^q}\right)+log\left(\frac{1}{x_2^p x_j^q}\right)\right]$$
$$+\sum_{i=3}^{n}\left[log\left(\frac{1}{x_i^p x_1^q}\right)+log\left(\frac{1}{x_i^p x_2^q}\right)\right]$$
$$+\left[log\left(\frac{1}{x_1^p x_2^q}\right)+log\left(\frac{1}{x_2^p x_1^q}\right)\right]$$

$$\frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2} = \frac{(n-1)\mathrm{BHM}^{p,q}(x)}{p+q} \frac{\partial}{\partial x_2} \left[log x_2^p + log x_2^q \right]$$

$$\frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2} = \frac{(n-1)\mathrm{BHM}^{p,q}(x)}{p+q} \frac{p+q}{x_2},$$

$$\frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2} = \frac{(n-1)\mathrm{BHM}^{p,q}(x)}{x_2}.$$

Proof of Theorem 1.1 By Lemma 2.1, direct computation gives

$$\Delta_1 = \left(x_1 - x_2\right)\left(\frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_1} - \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2}\right) = \frac{-(n-1)\mathrm{BHM}^{p,q}\left(x_1 - x_2\right)^2}{x_1 x_2} \leq 0$$

This implies that $\Delta_1 \leq 0$ for $x \in R^n$ By Lemma 2.1, we conclude that BHM ^{p,q}(x) is Schur concave on R^n_{++}.

In view of the discrimination criterion of Schur geometric convexity, we start with the following calculations:

$$\Delta_2 = \left(log x_1 - log x_2\right)\left(x_1 \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_1} - x_2 \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2}\right) = 0.$$

This implies that $\Delta_2 = 0$ for $x \in R^n$

By Lemma 2.2, we conclude that BHM ^{p,q}(x) is neither Schur geometrically convexity nor Schur geometrically concave on R^n_{++}.

Finally, we discuss the Schur harmonic convexity of BHM ^{p,q}(x).

A direct computation gives

$$\Delta_3 = \left(x_1 - x_2\right)\left(x_1^2 \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_1} - x_2^2 \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2}\right) = (n-1)\mathrm{BHM}^{p,q}\left(x_1 - x_2\right)^2 \geq 0.$$

This implies that $\Delta_3 \geq 0$ for $x \in R^n_n$. By Lemma 2.3, we conclude that BHM ^{p,q}(x) is Schur harmonically convex on R^n_{++}.

This completes proof of Theorem 1.1.

Proof of Theorem 1.2 Now we discuss the Schur m-power convexity of BHM ^{p,q}(x).

It is easy to see that BHM ^{p,q}(x) is symmetric on R^n_{++}. Without loss of generality, we may assume that $x_1 \geq x_2$

A direct computation gives

$$\Delta = \left(\frac{x_1^m - x_2^m}{m}\right)\left(x_1^{1-m} \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_1} - x_2^{1-m} \frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2}\right)$$

$$= \frac{\left(x_1^m - x_2^m\right)(n-1)\mathrm{BHM}}{m}\left[\frac{1}{x_1^m} - \frac{1}{x_2^m}\right]$$

$$= \frac{-\left(x_1^m - x_2^m\right)^2(n-1)\mathrm{BHM}}{m x_1^m x_2^m}.$$

If $m<0$, then $\Delta \geq 0$. From Lemma 2.4, it follows that BHM ^{p,q}(x) is Schur m—power convex for $x \in R^n_{++}$.

If m > 0, then $\Delta \leq 0$. From Lemma 2.4, it follows that BHM ^{p,q}(x) is Schur m—power convex for $x \in R^n_{++}$.

If m = 0, then by direct computation gives,

$$\Delta = \left(logx_1 - logx_2\right)\left(x_1\frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_1} - x_2\frac{\partial \mathrm{BHM}^{p,q}(x)}{\partial x_2}\right) = 0$$

From Lemma 2.4, it follows that BHM ^{p,q}(x) is neither Schur geometrically convexity nor Schur geometrically concave for $x \in R^n_{++}$.

The proof of Theorem 1.2 is completed.

4 Applications

Theorem 4.1 *For fixed non-negative real numbers p, q with $p + q \neq 0$, then for arbitrary $x \in R^n_{++}$.*

$$A_n(x) \geq \mathrm{BHM}^{p,q}(x) \tag{4.1}$$

Proof From Theorem 1.2 BHM^{p,q}(x) is Schur concave on R^n_{++}.

Using Lemma 2.5, one has

$$\underbrace{A_n(x), A_n(x), \ldots \ldots A_n(x)}_{n} \prec (x_1, x_2, ..x_n)$$

Thus, we deduce from Definition 2.1 that $\mathrm{BHM}^{p,q}(x)(A_n(x), A_n(x), \ldots \ldots A_n(x)) \geq \mathrm{BHM}^{p,q}(x)(x_1, x_2, ..x_n)$

Which implies that

$$A_n(x) \geq \mathrm{BHM}^{p,q}(x)$$

Theorem 4.1 is proved.

Theorem 4.2 *For fixed non-negative real numbers* p, q *with* $p + q \neq 0$, *and let* c *be a constant satisfying* $0 \leq c < A_n(x), (X - c) = (x_1 - c, x_2 - c, \dots x_n - c)$ *then for arbitrary* $x \in R^n_{++}$.

$$\mathrm{BHM}^{p,q}(X - c) \leq \left(1 - \frac{c}{A_n(x)}\right)\mathrm{BHM}^{p,q}(x)$$

Proof By the majorization relationship given in Lemma (2.6),

$$\left(\frac{x_1}{\sum_{i=1}^n x_i}, \dots \frac{x_n}{\sum_{i=1}^n x_n}\right) \prec \left(\frac{x_1 - c}{\sum_{i=1}^n (x_i - c)}, \dots \frac{x_n - c}{\sum_{i=1}^n (x_n - c)}\right),$$

From Theorem (1.1)

$$\mathrm{BHM}^{p,q}\left(\frac{x_1}{\sum_{i=1}^n x_i}, \dots \frac{x_n}{\sum_{i=1}^n x_n}\right) \geq \mathrm{BHM}^{p,q}\left(\frac{x_1 - c}{\sum_{i=1}^n (x_i - c)}, \dots \frac{x_n - c}{\sum_{i=1}^n (x_n - c)}\right)$$

i.e.,

$$\frac{\mathrm{BHM}^{p,q}(x_1, x_{2,} \dots x_n)}{\sum_{i=1}^n x_i} \geq \frac{\mathrm{BHM}^{p,q}(x_1 - c, x_2 - \mathrm{c} \dots x_n - c)}{\sum_{i=1}^n x_i - nc}$$

which implies that

$$\mathrm{BHM}^{p,q}(X - c) \leq (1 - \frac{c}{A_n(x)})\mathrm{BHM}^{p,q}(x)$$

Theorem 4.2 is proved.

5 Conclusion

We prove the Bonferroni mean $\mathrm{BHM}^{p,q}$ by introducing non-negative parameters p, q under the condition of Schur concave, Schur geometric convex and Schur harmonic convex on R^n_{++}.

As an application of the Schur convexity, we establish two inequalities for generalized geometric Bonferroni mean $\mathrm{BHM}^{p,q}$. For details, we refer the interested reader to [15, 16, 24, 28, 33, 34] and the references therein

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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