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Holomorphy of adjoint L functions for quasisplit A_2

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Abstract

We study the poles of the twisted adjoint L function of a generic cuspidal automorphic representation of $GL(3)$ or a quasisplit unitary group using a method pioneered by Ginzburg and Jiang and based on the theory of integral representations.

1 Introduction

Let F be a number field.¹ Let G be a quasisplit F -group, isomorphic to SL_3 over \bar{F} . Thus, F is either SL_3 , or a quasisplit unitary group attached to some quadratic extension E/F . The finite Galois form of G 's L -group is then $PGL_3(\mathbb{C})$ in the split case, or the semidirect product of $PGL_3(\mathbb{C})$ with $\text{Gal}(E/F)$ in the nonsplit case. In either case we have an action of ${}^L G$ on $PGL_3(\mathbb{C})$ by conjugation, which may be regarded as an action on $GL_3(\mathbb{C})$ which fixes the center. This then induces an action on the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ which we denote Ad . Note that in the nonsplit case this does not coincide with the definition of Ad in [12]. Rather, the nontrivial element Fr of $\text{Gal}(E/F)$ will act by $X \mapsto -_t X$, as this is the differential of its action on $PGL_3(\mathbb{C})$. Let Ad' denote the representation of ${}^L G$ considered in [12]. Thus $\text{Ad}' = \text{Ad}$ in the split case, but in the nonsplit case it is the representation of ${}^L G$ where $GL_3(\mathbb{C})$ acts by conjugation and Fr acts by $X \mapsto -_t X$.

Let π be a globally generic irreducible cuspidal automorphic representation of G . Then we consider the adjoint L function $L(s, \pi, \text{Ad})$. One would like understand the poles of this L function. We discuss an attack based on the integral representation given in [9, 12] and a strengthening the results of [10]. Our proof also applies to certain twisted L functions.

We briefly review the local zeta integral for $L(s, \pi, \text{Ad})$ presented in [9, 12]. First, one fixes an embedding of G into the split exceptional group of type G_2 . Let P be the maximal standard parabolic subgroup of G_2 , whose Levi contains the root subgroup attached to the short root. Let f be a flat section of the family of induced representations attached to a family of characters of $P(F) \backslash P(\mathbb{A})$, and E_P be the corresponding Eisenstein series operator (these notions are reviewed in Sects. 2.2 and 2.6).

¹It seems that most of these results should also hold in the function field case.

One may then define

$$I(s, \varphi, f) := \int_{G(F)\backslash G(\mathbb{A})} \varphi(g)E_P f(g, s) dg \tag{1.1}$$

where $s \in \mathbb{C}$, and φ is a generic cusp form $G(F)\backslash G(\mathbb{A}) \rightarrow \mathbb{C}$. Both [9], and [12] present the argument under the assumption that the characters are unramified, but the extension to the general case is direct so we may regard the theorem as proved in the ramified case as well. The integral $I(s, \varphi, f)$ unfolds to a new integral $I(s, W_\varphi, f)$ where W_φ is the Whittaker function attached to φ . Assuming that W_φ and f are factorizable, $I(s, W_\varphi, f)$ then factors as a product of local zeta integrals $I(s, W_\nu, f_\nu)$. Here W_ν and f_ν are the local components at a place ν of W_φ and f respectively. Moreover, if W_ν, f_ν and χ_ν are all unramified,² and W_ν and f_ν are normalized, then (see note³)

$$I(s, W_\nu, f_\nu) = \frac{L(3s - 1, \pi_\nu, \text{Ad}' \otimes \chi_\nu)}{L(3s, \chi_\nu)L(6s - 2, \chi_\nu^2)L(9s - 3, \chi_\nu^3)}.$$

Hence,

$$I(s, \varphi, f) = \frac{L^S(3s - 1, \pi, \text{Ad}' \otimes \chi)}{L^S(3s, \chi)L^S(6s - 2, \chi^2)L^S(9s - 3, \chi^3)} \prod_{\nu \in S} I(s, W_\nu, f_\nu), \tag{1.2}$$

where S is a finite set of places, away from which W_ν, f_ν and χ_ν are unramified. Notice that $L(s, \pi, \text{Ad}) = L(s, \pi, \text{Ad}')$ if G is split and $L(s, \pi, \text{Ad}' \otimes \chi_{E/F})$, where $\chi_{E/F}$ is the quadratic character attached to the extension E/F by class field theory if G is not split.

One expects that in general the L function $L(s, \pi, \text{Ad} \otimes \chi)$ should be entire. Indeed, in the split case $L(s, \pi, \text{Ad} \times \chi) = L(s, \pi \otimes \chi \times \tilde{\pi})/L(s, \chi)$, and it follows that the possible poles are precisely the zeros of $L(s, \chi)$, unless χ is nontrivial and $\pi \otimes \chi \cong \tilde{\pi}$, in which case there are additional simple poles at $s = 0$ and $s = 1$. One expects that $L(s, \pi \otimes \chi \times \tilde{\pi})$ is divisible by $L(s, \chi)$ and hence that the only actual poles are the simple poles at $s = 0$ and $s = 1$ which occur when χ is nontrivial and $\pi \otimes \chi \cong \tilde{\pi}$. In the special case when $\pi = \otimes'_\nu \pi_\nu$ and at least one component π_ν is supercuspidal, this was proved by Flicker [5].

In the nonsplit case, one must replace $L(s, \pi \times \tilde{\pi})$ with the Asai L function of the stable base change lifting of π . One must also account for the image of certain theta liftings. Indeed, consider

$$\left\{ \begin{pmatrix} a & b \\ & t \\ c & d \end{pmatrix} \right\} \cong GL_2(\mathbb{C}) \times GL_1(\mathbb{C}) \subset GL_3(\mathbb{C}) \tag{1.3}$$

This subgroup is stable under the outer automorphism which realizes the action of the nontrivial Galois element of the L group. Thus we obtain a subgroup of the L group. This subgroup may be realized as the image of an L -homomorphism from the L group of a product of smaller unitary groups $U_{1,1} \times U_1$ (still attached to the same quadratic extension). See [20]. The above subgroup clearly stabilizes the one dimensional subspace of $\mathfrak{sl}_3(\mathbb{C})$ spanned by $\text{diag}(1, -2, 1)$, and so does the map $X \mapsto {}_tX := J {}^tXJ$, where $J = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Thus

²We say W_ν is unramified if it is the unramified vector in a Whittaker model attached to a character with trivial conductor.

³This corrects a typo which appears in [12]. In that paper, $\zeta(3s - 9)$ appears where $\zeta(9s - 3)$ would be correct.

we obtain a one dimensional space stable under the restriction of the representation Ad' to ${}^L(U_{1,1} \times U_1)$, so that $L(s, \pi, \text{Ad}')$ should have a pole whenever π is a lift from $U_{1,1} \times U_1$. (In the split case, this issue does not arise: the group (1.3) is the L group of a Levi subgroup, the corresponding “functorial lifting” is realized by formation of Eisenstein series, and an element of its image can never be cuspidal).

An approach to controlling poles of twisted adjoint L functions, which is based on the study of $I(s, \varphi, f)$ and does not depend on any property of a local component of π was pioneered in [10]. If $I(s, \varphi, f)$ has a pole at $s = s_0$ then each of its negative Laurent coefficients is a global integral, similar to $I(s, \varphi, f)$, but with the Eisenstein series replaced by its corresponding negative Laurent coefficient. Thus, it suffices to show that the negative Laurent coefficients of the Eisenstein series used in $I(s, \varphi, f)$, when restricted to the subgroup $G \hookrightarrow G_2$ are “orthogonal”⁴ to cusp forms. Following [10], this can be done by expressing such Laurent coefficients in terms of Eisenstein series induced from characters of the *other* maximal parabolic subgroup of G_2 , and then checking that the restrictions of these Eisenstein series are “orthogonal” to G -cuspforms. In [10], it is shown that the Eisenstein series appearing in the construction of $I(s, \varphi, f)$ for the case of trivial χ has only two poles in $\text{Re}(s) > \frac{1}{2}$, with one being simple and the other double. The residue of the simple pole is a constant function and thus obviously orthogonal to cusp forms. At the double pole, a “first term identity” is proved, which expresses the leading term of the Laurent expansion in terms of an Eisenstein series from the other parabolic. This rules out a double pole of the adjoint L function. In order to rule out a simple pole by this method, one would need a “second term identity.” In Sect. 3.3 we prove an identity of this type. It is my understanding that such an identity was first obtained by Jiang in unpublished work.

That being said, if the second pole of the Eisenstein series gave rise to a pole of the global adjoint L function $L(s, \pi, \text{Ad})$, this pole would occur at $s = 1$. Such a pole is impossible, because $L(s, \pi, \text{Ad}) = L(s, \pi \times \tilde{\pi})/\zeta(s)$, and $L(s, \pi \times \tilde{\pi})$ and $\zeta(s)$ both have simple poles at $s = 1$.

In this paper we pursue the approach pioneered by Ginzburg and Jiang. First, we analyze the poles of the Eisenstein series in the case of nontrivial χ . This allows us to deduce information about the poles of the local zeta integral. We also prove a key vanishing result needed to deduce holomorphy of $L(s, \pi, \text{Ad} \times \chi)$ at $\text{Re}(s) = \frac{1}{2}$ from (1.2). Then, we prove a weak result regarding local zeta integrals at ramified and Archimedean primes. While preparing this manuscript, I have learned that a stronger result—a local functional equation—has been obtained by Qing Zhang. The weaker result proved here suffices for our application, permitting us to deduce that each pole of the partial adjoint L function must be a pole of the global zeta integral for some choice of data.

Our main result is Theorem 6.1, which states that, in the split case, every pole of $L(s, \pi, \text{Ad} \times \chi)$ in the half plane $\text{Re}(s) \geq \frac{1}{2}$ is simultaneously a zero of $L(s, \chi)$ and a pole of the finite product over the ramified and Archimedean places. Using this result, together with knowledge of the form of the Gamma factor and the zeros of the Riemann zeta function, Buttcane and Zhou [3] were able to show holomorphy of the complete adjoint L function (and hence also all partial L functions) for an $SL(3, \mathbb{Z})$ Maass form with trivial central character (such a form generates a representation unramified at all finite places).

⁴“Orthogonal” is in quotes because the residues need not be L^2 . But one can extend the inner product on $L^2(G(F)\backslash G(\mathbb{A}))$ to allow pairing cusp forms with arbitrary smooth functions of moderate growth.

Since then, Qing Zhang has been able to strengthen the main result of this paper by treating ramified nonarchimedean places. Thus, one can take the finite product only over Archimedean places. This immediately gives an extension of the result of Buttcane and Zhou to Maass forms with trivial central character attached to congruence subgroups.

2 Induced representations, intertwining operators, and their poles

2.1 Characters and degenerate induced representations

Let F be a number field as before. If G is an F -group, write $X(G)$ for the group of rational characters of G and \mathbf{X}_G for the complex manifold of characters of $G(\mathbb{A})$ trivial on $G(F)$. These are groups and for the most part we write them additively. To reconcile with multiplicative notation for $G(\mathbb{A})$ and \mathbb{C}^\times , we use an exponential notation for the characters: the value of $\chi \in \mathbf{X}_G$ at $g \in G(\mathbb{A})$ is denoted g^χ . A similar notation is used for cocharacters. We identify $\chi \in X(G)$ with the character of $G(F) \backslash G(\mathbb{A})$ obtained by composing it with the absolute value on \mathbb{A}^\times . This extends to a mapping of $X(G) \otimes_{\mathbb{Z}} \mathbb{C}$ into \mathbf{X}_G . The image is the set of unramified characters, which we denote $\mathbf{X}_{G,\text{un}}$. Similarly we denote the complex manifold of all characters of $G(F_v)$ by $\mathbf{X}_{G,v}$ and the image of $X(G) \otimes_{\mathbb{Z}} \mathbb{C}$ in it by $\mathbf{X}_{G,v,\text{un}}$. We identify $\mathbf{X}_{GL_1,\text{un}}$ with \mathbb{C} using the map $s \rightarrow ||^s$. We denote the canonical pairing between characters and cocharacters by $\langle \cdot, \cdot \rangle$. If φ^\vee is a cocharacter, then $\langle \varphi^\vee, \chi \rangle \in \mathbf{X}_{GL_1}$.

We shall only require split connected reductive F -groups with simply connected derived groups. We always assume that each is equipped with a choice of split torus and Borel containing it. The torus is denoted T and the Borel B . Let G be such a group. For H a T -stable F -subgroup we write $\Phi(T, H)$ for the roots of T in H . The Weyl group is denoted W . It is realized as a quotient of the normalizer, $N_G(T)$, of T in G . We also assume G equipped with a realization, i.e. a family of isomorphisms $\{x_\alpha : \mathbb{G}_a \rightarrow U_\alpha\}_{\alpha \in \Phi(G, T)}$, such that $x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) \in N_G(T)$ for each α . This product is then a representative for the simple reflection attached to α and one may select representatives for other elements of the Weyl group using them.

Let M be a standard Levi. Then we may identify \mathbf{X}_M with $\{\chi \in \mathbf{X}_T : \langle \chi, \alpha^\vee \rangle = 0, \alpha \in \Phi(T, M)\}$. Likewise, we may identify $\mathbf{X}_{M,\text{un}}$ with $\{\chi \in \mathbf{X}_{T,\text{un}} : \langle \chi, \alpha^\vee \rangle = 0, \alpha \in \Phi(T, M)\}$. We would like to choose a complement $\mathbf{X}_{G,0}$ to $\mathbf{X}_{G,\text{un}}$ in \mathbf{X}_G . When $G = GL_1$ this is done by taking the normalized characters, i.e., those that are trivial on the multiplicative group of positive reals, embedded diagonally at all the infinite places. When G is a torus, it can be identified with several copies of GL_1 by choosing a \mathbb{Z} -basis for $X(G)$ and the subgroup $\mathbf{X}_{G,0}$ thus obtained is independent of the choice. If $G = G_{\text{der}}T$ where G_{der} is the derived group and T is a torus, then restriction gives an embedding $\mathbf{X}_G \hookrightarrow \mathbf{X}_T$, and we may apply the decomposition $\mathbf{X}_T = \mathbf{X}_{T,\text{un}} \oplus \mathbf{X}_{T,0}$ to obtain the corresponding decomposition of \mathbf{X}_G .

For archimedean local fields, we again define a character to be normalized if it is trivial on the positive reals. For nonarchimedean fields, we first choose a uniformizer and then say that a character is normalized if it is trivial on the uniformizer. This leads to similar decompositions $\mathbf{X}_{G,v} = \mathbf{X}_{G,v,\text{un}} \oplus \mathbf{X}_{G,v,0}$ into unramified characters and normalized characters. For any character χ , we define χ_{un} and χ_0 to be the components relative to this decomposition. If $\chi = s + \chi_0 \in \mathbf{X}_{GL_1}$ (resp. $\mathbf{X}_{GL_1,v}$) we define $L(\chi)$ to be the usual global (resp. local) L function $L(s, \chi_0)$ (keeping in mind that $\mathbf{X}_{GL_1,\text{un}}$ has been identified with \mathbb{C}). Note that $\mathbf{X}_{GL_1,0}$ is identified with the group of Hecke characters $F^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ and that this group is normally written additively. We shall occasionally break with the

practice of writing everything additively, and write $\mathbf{X}_{GL_1,0}$ multiplicatively, particularly when discussing L functions.

Take P a parabolic with Levi M and $\chi \in \mathbf{X}_M$. Write $\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi(P,T)} \alpha$. We define $I_P^G(\chi)$ to be the normalized K -finite induced representation of $G(\mathbb{A})$ and $\text{Ind}_P^G(\chi)$ to be the non-normalized version. For $\chi \in \mathbf{X}_{M,\nu}$ we define $I_P^G(\chi)$ to be the normalized K_ν -finite induced representation of $G(F_\nu)$ and $\text{Ind}_P^G(\chi)$ the non-normalized version. In either case, $I_P^G(\chi) = \text{Ind}_P^G(\chi + \rho_P)$, and is a subset of $I_B^G(\chi + \rho_P - \rho_B)$. In the important special case when the Levi of P is rank one with unique root α , this becomes $I_B^G(\chi - \frac{\alpha}{2})$.

2.2 Flat sections

Fix a reductive group G and standard Levi M , and a normalized character $\chi_0 \in \mathbf{X}_M$. We consider the family of induced representations $I_P^G(\chi)$ with χ in $\chi_0 + \mathbf{X}_{M,\text{un}}$. Denote the family as a whole by $\mathcal{I}_P^G(\chi_0)$. By a section we mean a function

$$\chi \mapsto f_\chi, \quad (\chi \in \chi_0 + \mathbf{X}_{M,\text{un}})$$

such that

- (1) $f_\chi \in I_P^G(\chi)$ for each $\chi \in \chi_0 + \mathbf{X}_{M,\text{un}}$,
- (2) f is smooth as a function $X_M \times G \rightarrow \mathbb{C}$.

We say that f is flat if $f(\chi_0 + s, k)$ is independent of $s \in \mathbf{X}_{M,\text{un}}$ for $k \in K$. (Here K is a fixed maximal compact subgroup.) The set of flat sections of $\mathcal{I}_P^G(\chi_0)$ is a complex vector space. Denote it $\text{Flat}(\chi_0)$.

2.3 Coordinates on \mathbf{X}_T in the case of G_2

Our main results deal with induced representations on the split exceptional group G_2 . I write α for the short root and β for the long root. *Unfortunately, this is the opposite of the notation used in [10]*. I write U_γ for the root subgroup attached to any root γ . I assume G_2 to be equipped with a choice of Borel and of maximal torus. These are B and T . I write $P = MU$ for the standard parabolic subgroup whose Levi contains U_α and $Q = LV$ for the one whose Levi contains U_β . For $\chi_1, \chi_2 \in \mathbf{X}_{GL_1}$ let $[\chi_1, \chi_2]$ denote the element of \mathbf{X}_T which satisfies

$$\langle [\chi_1, \chi_2], \alpha^\vee \rangle = \chi_1, \quad \langle [\chi_1, \chi_2], \beta^\vee \rangle = \chi_2.$$

Thus $\varpi_1 := [1, 0]$ and $\varpi_2 := [0, 1]$ are the two fundamental weights. Note that $[\chi_1, \chi_2] \in \mathbf{X}_M \iff \chi_1 = 0$ and $[\chi_1, \chi_2] \in \mathbf{X}_L \iff \chi_2 = 0$.

2.4 Normalization and poles of intertwining operators: GL_2 case

We study poles of intertwining operators. The theory is fairly uniform for split groups, and reduces to the special case of GL_2 . First we consider the case of GL_2 . Write B_{GL_2} for the standard Borel of GL_2 consisting of upper triangular matrices. Take $\chi = \prod_\nu \chi_\nu$ a character of $B_{GL_2}(\mathbb{A})$. Let w be the unique nontrivial element of the Weyl group, and α the unique positive root.

Lemma 2.1 *The normalized local intertwining operator*

$$M_\nu^*(w, \chi_\nu) := \frac{1}{L_\nu(\langle \chi_\nu, \alpha^\vee \rangle)} M_\nu(w, \chi)$$

extends analytically to all of \mathbf{X}_T . When $\langle \chi_v \circ \alpha^\vee \rangle$ is unramified $I_{BG L_2}^{GL_2}(\chi_v)$ has a normalized “spherical”⁵ vector which we denote $f_{\chi_v}^\circ$. Then

$$M_v(w, \chi_v) f_{\chi_v}^\circ = \frac{L(\langle \chi_v, \alpha^\vee \rangle)}{L(\langle \chi_v, \alpha^\vee \rangle + 1)} f_{w\chi_v}^\circ.$$

Proof In the nonarchimedean case, both assertions can be verified by fairly direct computations. Alternatively, the first assertion is a special case of a result of Winarsky, [23], and the second assertion is a special case of the result in Sect. 4 of Langlands, *Euler products* [17]. Over the reals, both assertions can be deduced from Proposition 2.6.3 of [2]. The second assertion is also the simplest case of the formula of Gindikin and Karpalevic [8], first proved for GL_n by Bhanu Murti [1]. Over the complex numbers, both assertions follow from Lemma 7.23 of [22]. See also [6] and additional references therein. The first assertion over either archimedean field also follows from the generalization found on p. 110 of [21]. \square

Consequently if $f = \prod_{v \notin S} f_{\chi_v}^\circ \prod_{v \in S} f_v$ is an element of the global induced space $I_{BG L_2}^{GL_2}(\chi)$, then

$$M(w, \chi) f = \frac{L(\langle \chi, \alpha^\vee \rangle)}{L(\langle \chi, \alpha^\vee \rangle + 1)} \prod_{v \notin S} f_{\chi_v}^\circ \prod_{v \in S} L_v(\langle \chi_v, \alpha^\vee \rangle + 1) M_v^*(w, \chi_v) f_v. \tag{2.2}$$

We deduce that poles of $M(w, \chi)$ come in three classes

- (1) Poles of $\prod_{\alpha > 0, w\alpha < 0} L(\langle \chi, \alpha^\vee \rangle)$, which are not cancelled by poles of $\prod_{\alpha > 0, w\alpha < 0} L(\langle \chi, \alpha^\vee \rangle + 1)$. These occur at $\langle \chi, \alpha^\vee \rangle = 1$, and in particular do not occur unless $\langle \chi, \alpha^\vee \rangle_0 = 0$.
- (2) Zeros of $L(\langle \chi, \alpha^\vee \rangle + 1)$. These are all in the strip $-1 < \text{Re} \langle \chi, \alpha^\vee \rangle_{\text{un}} < 0$.
- (3) Poles of $\prod_{v \in S} L_v(\langle \chi_v, \alpha^\vee \rangle + 1)$. These are all in the half plane $\text{Re} \langle \chi, \alpha^\vee \rangle_{\text{un}} \leq -1$.

2.5 Poles of intertwining operators on the principal series: general case

Now let G be a general split reductive group, B its Borel, $\chi = \prod_v \chi_v$ a character of $B(\mathbb{A})$, and w any element of the Weyl group. For each root α we have a map $SL_2 \rightarrow G$ and can decompose the standard intertwining operator $M(w, \chi)$ as a composite of intertwining operators indexed by $\{\alpha > 0 : w\alpha < 0\}$. Poles of the intertwining operator attached to a root α are of the same three types, along hyperplanes $\langle \alpha^\vee, \chi \rangle = c$ in the space $\mathbf{X}_{B, \text{un}}$ defined using the corresponding coroot.

2.6 Eisenstein series

Now suppose that G is a reductive group and P a parabolic subgroup. We fix a suitable maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ and let $\mathcal{A}(G)$ denote the space of automorphic forms (relative to K) $G(\mathbb{A}) \rightarrow \mathbb{C}$, that is, the space of smooth functions $\phi : G(F) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}$ of moderate growth which are finite under the action of K and the center, \mathfrak{z}_G of the universal enveloping algebra of the Lie algebra of $G(F_\infty)$.

Fix $\chi_0 \in \mathbf{X}_{M,0}$ and let $\text{Flat}(\chi_0)$ denote the space of flat sections of $\mathcal{I}_P^G(\chi_0)$. For $f \in \text{Flat}(\chi_0)$, we define the Eisenstein series

$$E_P f : G(\mathbb{A}) \times (\chi_0 + \mathbf{X}_{M, \text{un}}) \rightarrow \mathbb{C}$$

⁵By “spherical”, we mean fixed by the intersection of $SL_2(F_v)$ with the maximal compact subgroup.

by

$$E_P f(g, \chi) = \sum_{\gamma \in P(F) \backslash G(F)} f_\chi(\gamma g)$$

for values of χ such that this sum is convergent and by meromorphic continuation elsewhere. Outside the domain of convergence, one encounters poles of finite order along a locally finite set of root hyperplanes. For each χ away from the poles, $f \mapsto E_P f(\cdot, \chi)$ is an intertwining operator $I_P^G(\chi) \rightarrow \mathcal{A}(G)$. We denote it $E_P(\chi)$.

2.7 G_2 Eisenstein series

We briefly recall the Eisenstein series which appear in [9, 12] and their normalization. The Eisenstein series in question are attached to the parabolic $P = MU$ as in Sect. 3.2. In [9, 12], unramified Eisenstein series are considered. The space $\mathbf{X}_{M, \text{un}}$ is one dimensional and can conveniently be identified with \mathbb{C} using the mapping $s \mapsto \delta_P^s$. Here δ_P is the modular quasicharacter. In the notation of Sect. 3.2, $\delta_P^s = [0, 3s]$. Equivalently, the half-sum of the roots of P is $\rho_P = [0, \frac{3}{2}]$. Thus $\text{Ind}_P^G(\delta_P^s) = I_P^G([0, 3s - \frac{3}{2}]) \subset I_B^G[-1, 3s - 1]$. In order to generalize the construction of [9, 12] to get $L(s, \pi, \text{Ad}' \times \chi)$ for general χ , we would use $I_P^G([0, 3s - \frac{3}{2} + \chi])$,

2.8 Relevant intertwining operators for G_2 and their poles

We apply the material from Sects. 2.4 and 2.5 to the intertwining operators that appear in the constant term of our G_2 Eisenstein series. Write $c(u, \chi_0) = \frac{L(u, \chi_0)}{L(u+1, \chi_0)}$. For $\chi \in \mathbf{X}_T$ and $w \in W$ let $c(w, \chi) = \prod_{\alpha > 0, w\alpha < 0} c(\langle \alpha^\vee, \chi \rangle)$. Then decomposing $M(w, \chi)$ as a composite of operators attached simple reflections and letting $M^*(w, \chi)$ denote the corresponding composite of normalized operators yields the generalization of (2.2):

$$M(w, \chi).f = \prod_{\alpha > 0, w\alpha < 0} \left(\frac{L(\langle \chi, \alpha^\vee \rangle)}{L(\langle \chi, \alpha^\vee \rangle + 1)} \prod_{v \in S} L_v(\langle \chi_v, \alpha^\vee \rangle + 1) \right) \prod_{v \notin S} f_{\chi_v}^\circ \prod_{v \in S} M_v^*(w, \chi_v).f.$$

The constant term of our Eisenstein series may be expressed as a sum over $w \in W$ such that $w\alpha > 0$. Here α is the short simple root. There are six such w , one of each length from 0 to 5. We write w_i for the element of length i . If $\chi = [-1, 3s - 1 + \chi_0]$ and (see note⁶)

$$\mathcal{C} := (c(3s - 1, \chi_0), c(9s - 4, \chi_0^3), c(6s - 3, \chi_0^2), c(9s - 5, \chi_0^3), c(3s - 2, \chi_0)),$$

then $c(w_i, \chi) = \prod_{j=1}^i \mathcal{C}_j$. Recall that poles of the intertwining operators come in three classes. We analyze each class.

- (1) Poles which arise from the pole of the zeta function at 1 will occur at $2/3$ and 1 if χ_0 is trivial; at $5/9$ and $2/3$ if χ_0^3 is trivial, and at $2/3$ if χ_0^2 is trivial. If χ_0 is trivial then the pole at $2/3$ can be a triple pole. Otherwise it is simple. The other poles are always simple.
- (2) Poles which arise from zeros of a global L function are in the half plane $\text{Re}(s) < \frac{1}{2}$. For example, $c(9s - 5, \chi_0^3)$ could have poles as far right at $\text{Re}(s) = 5/9 - \varepsilon$, but it never occurs without $c(9s - 4, \chi_0^3)$ so the zeros of the L function in its denominator are always cancelled. We only get poles from the zeros of $L(9s - 3, \chi_0^3)$, and these are to the left of $s = 4/9$.

⁶In this section we deviate from the practice of writing $\mathbf{X}_{GL_1, \text{un}}$ additively, in order to be compatible with the usual notation for L functions.

(3) Poles which arise from local L functions are all in the half plane $\text{Re}(s) \leq 4/9$.

It follows that the only possible poles of our Eisenstein series in the half plane $\text{Re}(s) \geq 1/2$ are at $5/9, 2/3$ and 1 , and can occur only if χ_0 is trivial, quadratic, or cubic. Notice that the pole at $2/3$ would correspond to a pole of $L(s, \pi, \text{Ad}' \times \chi)$ at 1 . More detailed information regarding the poles of the intertwining operators in $\text{Re}(s) \geq \frac{1}{2}$ is recorded in the following table.

χ	Trivial	Nontrivial quadratic	Nontrivial cubic
w_0	Holomorphic	Holomorphic	Holomorphic
w_1	$2/3$	Holomorphic	Holomorphic
w_2	$2/3, 5/9$	Holomorphic	$5/9$
w_3	$2/3$ (double), $5/9$	$2/3$	$5/9$
w_4	$2/3$ (triple), $5/9$	$2/3$	$2/3, 5/9$
w_5	$2/3$ (triple), $5/9, 1$	$2/3$	$2/3, 5/9$

2.9 Application to the constant term of our G_2 Eisenstein series

Now, the poles of the Eisenstein series (and their orders) are the same as the poles of the constant term (and their orders), which is a sum of intertwining operators. Having determined the poles of the summands, and their orders, the next step is to account for the possibility of cancellation in the sum. In the unramified case this is done in [10].

Theorem 2.3 [Ginzburg–Jiang] *Assume that χ_0 is trivial. Then E_P has a simple pole at $s = 1$, and a double pole at $s = 2/3$. At $s = 5/9$ it is holomorphic.*

We sketch a proof which is slightly different than the one given in [10]. The technique is similar to [13] and will be worked out in detail for the ramified case below. We consider the two terms which have triple poles at $s = 2/3$. They correspond to the Weyl elements w_4 and w_5 . We may write $M(w_5, \chi) = M(s_\beta, w_4\chi)M(w_4, \chi)$. Here, s_β is the simple reflection in the Weyl group attached to the long root β (s_α is defined similarly). We check that when χ_0 is trivial, $(\beta^\vee, w_4\chi)$ vanishes at the point corresponding to $s = 2/3$. It follows from [15, Proposition 6.3] that $M(s_\beta, w_4\chi)$ is the scalar operator -1 at this point. Hence $M(w_4, \chi) + M(w_5, \chi)$ is equal to the composition of $M(w_4, \chi)$ and an operator which vanishes at $s = 2/3$. Similarly, the four terms which give poles at $5/9$ form two pairs such that the sum of each pair is holomorphic at $s = 5/9$.

Theorem 2.4 *Assume that χ_0 is nontrivial quadratic or cubic. Then E_P has a simple pole at $\frac{2}{3}$ and is otherwise holomorphic in $\text{Re}(s) \geq \frac{1}{2}$.*

Proof We have

$$M(w_5, \chi) = M(s_\beta, w_4\chi)M(s_\alpha, w_3\chi)M(s_\beta, w_2\chi)M(s_\alpha, w_1\chi)M(s_\beta, \chi)$$

The corresponding expression for w_i with $i < 5$ is obtained by taking only the rightmost operators in this composite.

We tabulate key data. First, the spaces that the six operators map into

$M(\dots, \chi)$	Maps to $I_B^{G_2}(\dots)$	$(s = 5/9, 3\chi_0 = 0)$	$(s=2/3, 3\chi_0 = 0)$	$(s = 2/3, 2\chi_0 = 0)$
w_0	$[-1, \chi_0 + 3s - 1]$	$[-1, \chi_0 + 2/3]$	$[-1, \chi_0 + 1]$	$[-1, \chi_0 + 1]$
w_1	$[3\chi_0 + 9s - 4, -\chi_0 - 3s + 1]$	$[1, -\chi_0 - 2/3]$	$[2, -\chi_0 - 1]$	$[\chi_0 + 2, \chi_0 - 1]$
w_2	$[-3\chi_0 - 9s + 4, 2\chi_0 + 6s - 3]$	$[-1, -\chi_0 + 1/3]$	$[-2, -\chi_0 + 1]$	$[\chi_0 - 2, 1]$
w_3	$[3\chi_0 + 9s - 5, -2\chi_0 - 6s + 3]$	$[0, \chi_0 - 1/3]$	$[1, \chi_0 - 1]$	$[\chi_0 + 1, -1]$
w_4	$[-3\chi_0 - 9s + 5, \chi_0 + 3s - 2]$	$[0, \chi_0 - 1/3]$	$[-1, \chi_0]$	$[\chi_0 - 1, \chi_0]$
w_5	$[-1, -\chi_0 - 3s + 2]$	$[-1, -\chi_0 + 1/3]$	$[-1, -\chi_0]$	$[-1, \chi_0]$

And next the elements of X_{GL_1} which will determine the poles of the rank one operators.

Pairing	General	$(s = 5/9, 3\chi_0 = 0)$	$(s=2/3, 3\chi_0 = 0)$	$(s = 2/3, 2\chi_0 = 0)$
$\langle \beta^\vee, \chi \rangle$	$\chi_0 + 3s - 1$	$\chi_0 + 2/3$	$\chi_0 + 1$	$\chi_0 + 1$
$\langle \alpha^\vee, w_1\chi \rangle$	$3\chi_0 + 9s - 4$	1	2	$\chi_0 + 2$
$\langle \beta^\vee, w_2\chi \rangle$	$2\chi_0 + 6s - 3$	$-\chi_0 + 1/3$	$-\chi_0 + 1$	1
$\langle \alpha^\vee, w_3\chi \rangle$	$3\chi_0 + 9s - 5$	0	1	$\chi_0 + 1$
$\langle \beta^\vee, w_4\chi \rangle$	$\chi_0 + 3s - 2$	$\chi_0 - 1/3$	χ_0	χ_0

The key facts are the following: when the pairing is 1 the corresponding rank-one operator has a pole. When it is zero, the corresponding rank-one operator is the scalar operator -1 . When it is -1 the corresponding rank-one operator has a kernel. Otherwise, the rank-one operator is an isomorphism. From this we can see that $M(w_i, [-1, \chi_0 + 3s - 1])$ has a pole at $s = 2/3$ if $2\chi_0 = 0$ and $i \geq 3$ or if $3\chi_0 = 0$ and $i \geq 4$.⁷ In either case, the operators $M(w_i, [-1, \chi_0 + 1])$ all land in different spaces. Hence there is no possibility of cancellation among them and the poles of the individual intertwining operators are inherited by their sum (i.e., the constant term) and then by the Eisenstein series itself.

We also see that $M(w_i, [-1, \chi_0 + 3s - 1])$ has a simple pole at $s = 5/9$ if $i \geq 2$ and $3\chi_0 = 0$. In this case we have two pairs of operators which land in the same space. To study them, set $u = 3s - 5/3 = 3(s - 5/9)$ so that $3s - 1 = u + 2/3$. Thus u is a convenient local coordinate in a neighborhood of $s = 5/9$. The first key point is that

$$\langle \alpha^\vee, w_3[-1, \chi_0 + 2/3] \rangle = 0 \implies M(s_\alpha, w_3[-1, \chi_0 + u + 2/3]) = -1 + O(u).$$

Hence

$$\begin{aligned} &M(w_3, [-1, \chi_0 + u + 2/3]) + M(w_4, [-1, \chi_0 + u + 2/3]) \\ &= (1 + M(s_\alpha, w_3[-1, \chi_0 + u + 2/3])) \circ M(w_3, [-1, \chi_0 + u + 2/3]) \end{aligned}$$

is the composite of an operator with a simple pole at $u = 0$ and an operator which vanishes at $u = 0$. Thus, it is holomorphic at $u = 0$.

The second key point is that $M(s_\beta, -s_\beta\chi) = M(s_\beta, \chi)^{-1}$. (This is an identity of meromorphic functions.) Hence

$$\begin{aligned} H(u) &:= M(s_\beta, [-3u, \chi_0 + u - 1/3])M(s_\alpha, [3u, \chi_0 - 2u - 1/3]) \\ &\quad \times M(s_\beta, [-3u - 1, 2u + 1/3 - \chi_0]) \end{aligned}$$

⁷We have reverted to writing $X_{GL_1, \text{un}}$ additively: $2\chi_0$ instead of χ_0^2 and $3\chi_0$ instead of χ_0^3 .

$$\begin{aligned} &= (M(s_\beta, [0, \chi_0 - 1/3]) + O(u))(-1 + O(u)) \times (M(s_\beta, [-1, 1/3 - \chi_0]) + O(u)) \\ &= -1 + O(u). \end{aligned}$$

But

$$\begin{aligned} &M(w_2, [-1, \chi_0 + u + 2/3]) + M(w_5, [-1, \chi_0 + u + 2/3])s \\ &= (1 + H(u)) \circ M(w_2, [-1, \chi_0 + u + 2/3]). \end{aligned}$$

Once again we have an operator with a simple pole composed with an operator that has a zero. This completes the proof. \square

Remark 2.5 The existence of the pole of E_P at $2/3$ in the cubic case can also be deduced from the existence of a pole of the adjoint L function: cuspidal representations of $GL_3(\mathbb{A})$ satisfying $\pi \cong \pi \otimes \chi$ exist by Theorem 2.4(iv) of [4]. Thanks to David Loeffler for explaining this to me. For such a representation $L(s, \pi, \text{Ad} \otimes \chi) = L(s, \pi \times \tilde{\pi} \times \chi) / L(s, \chi)$ will have a pole at $s = 1$. As local L functions are nonvanishing this pole will be inherited by the partial L function, and then, by Theorem 5.1 below by the global zeta integral. In the case when χ is nontrivial quadratic the existence of a pole at $s = 2/3$ can also be deduced from Theorem 3.4 below.

2.10 Key vanishing property of the Eisenstein series

Proposition 2.6 *Assume that $2\chi_0 = 0$. Then $E_P([-1, 3s - 1 + \chi_0])$ vanishes at $s = \frac{1}{2}$.*

Proof The proof is similar to that of Theorem 2.4. Note that $w_5[-1, \frac{1}{2} + \chi_0] = [-1, \frac{1}{2} + \chi_0]$, $w_4[-1, \frac{1}{2} + \chi_0] = w_1[-1, \frac{1}{2} + \chi_0]$ and $w_3[-1, \frac{1}{2} + \chi_0] = w_2[-1, \frac{1}{2} + \chi_0]$. Write $w_4 = w_{4,1}w_1$ and $w_3 = w_{3,2}w_2$. As in the proof of Theorem 2.4 one readily checks that $M(w_5, [-1, 3s - 1 + \chi_0])$, $M(w_{4,1}, w_1[-1, 3s - 1 + \chi_0])$ and $M(w_{3,2}, w_2[-1, 3s - 1 + \chi_0])$ are all $-1 + O(s - \frac{1}{2})$ at $s = \frac{1}{2}$. This time none of the intertwining operators has a pole, so the sum vanishes at $s = \frac{1}{2}$.

Alternatively, having established that $M(w_5, [-1, 3s - 1 + \chi_0]) = -1 + O(s - \frac{1}{2})$, We can deduce vanishing of the Eisenstein series from the fact that it is holomorphic and satisfies the functional equation

$$E_P([-1, 3s - 1 + \chi_0]) = E_P([-1, 2 - 3s + \chi_0]) \circ M(w_5, [-1, 3s - 1 + \chi_0]).$$

\square

3 Siegel–Weil type identities

In this section we prove identities relating degenerate Eisenstein series induced from the two different parabolic subgroups of G_2 . Such identities are sometimes called Siegel–Weil n th term identities. A conceptual explanation for their existence comes from embeddings of degenerate induced representations into principal series representations induced from the Borel, together with the symmetry of the principal series (see [14]). We first prove a technical result which extends this philosophy to flat sections and Eisenstein series.

3.1 Surjectivity property of intertwining operators

In the next few sections we consider an alternate normalization of the intertwining operator, which is different than the one considered in Sect. 2.

Take $\chi \in \mathbf{X}_{M,v} \subset \mathbf{X}_{T,v}$. Then the standard intertwining operator $M(w_\alpha, \chi + \frac{\alpha}{2})$ maps $I_B^G(\chi + \frac{\alpha}{2})$ to $I_P^G(\chi) \subset I_B^G(\chi - \frac{\alpha}{2})$. We sketch a proof that the map is surjective. Take any

$f \in I_P^G(\chi)$ and let f_s° denote the spherical vector in $\text{Ind}_B^G(s + \frac{1}{2})\alpha$. Then let $\tilde{f} := f \cdot f_s^\circ$, which lies in the space $I_B^G(\chi + s\alpha)$. It follows immediately from the integral formula for the standard intertwining operator that

$$M(w_\alpha, \chi + s\alpha)\tilde{f} = fM(w_\alpha, s\alpha) \cdot f_s^\circ = f \frac{\zeta(2s)}{\zeta(2s + 1)} f_{-s}^\circ.$$

But $f_{-1/2}^\circ$ is just the constant function 1, so when $s = 1/2$ we obtain a nonzero scalar multiple of f .

In the global setting, this fails because $\zeta(2s)$ has a pole at $s = 1/2$. Let

$$M^*(w_\alpha, \chi) = (\langle \alpha^\vee, \chi \rangle_{\text{un}} - 1)M(w_\alpha, \chi).$$

(Recall that $\langle \alpha^\vee, \chi \rangle_{\text{un}} \in \mathbf{X}_{GL_1, \text{un}}$ which has been identified with \mathbb{C} .) Then $M^*(w_\alpha, \chi)$ has no poles in $\text{Re}(s) \geq 0$, and for $\langle \chi, \alpha^\vee \rangle = 0$ maps $I_B^G(\chi + \frac{\alpha}{2})$ (by which we now mean the global version) surjectively onto $I_P^G(\chi) \subset I_B^G(\chi - \alpha/2)$.

Proposition 3.1 *If $\Phi(T, M) = \{\pm\alpha\}$ and f is a flat section of $\mathcal{I}_P^G(\chi_0)$, then there exists a flat section \tilde{f} of $\mathcal{I}_B^G(\chi_0)$ such that $M^*(w_\alpha, \chi + \frac{\alpha}{2})\tilde{f}_{\chi + \frac{\alpha}{2}} = f_\chi$ for all $\chi \in \chi_0 + \mathbf{X}_{M, \text{un}}$.*

Proof As before, we construct \tilde{f} by taking the product of f and the normalized spherical vector in $\text{Ind}_B^G(s + \frac{1}{2})\alpha$. (Note that $\mathbf{X}_{T, \text{un}} = \mathbf{X}_{M, \text{un}} + \mathbb{C}\alpha$.) Then for $\chi \in \mathbf{X}_M$ and $s \in \mathbb{C}$

$$M^*(w_\alpha, \chi + s\alpha)\tilde{f}_{\chi + s\alpha} = f_\chi \cdot (2s - 1) \cdot \frac{\zeta(2s)}{\zeta(2s + 1)} \cdot f_{-s}.$$

Again, $f_{-1/2}$ is the constant function 1, so we get a nonzero scalar multiple of f_χ .

Remark 3.2 For each fixed χ , the operator $M^*(w_\alpha, \chi + \frac{\alpha}{2})$ maps $I_B^G(\chi + \frac{\alpha}{2})$ onto $I_P^G(\chi) \subset I_B^G(\chi - \frac{\alpha}{2})$. The key point is that this extends to a map from flat sections of $\mathcal{I}_B^G(\chi_0)$ to flat sections of $\mathcal{I}_P^G(\chi_0)$.

Proposition 3.3 *Take $P = MU$ with $\Phi(T, M) = \{\pm\alpha\}$ and f a flat section of $\mathcal{I}_P^G(\chi_0)$. Assume $\langle \alpha^\vee, \chi_0 \rangle = 0$. Choose \tilde{f} a flat section of $\mathcal{I}_B^G(\chi_0)$ such that $M^*(w_\alpha, \chi + \frac{\alpha}{2})\tilde{f}|_{\chi_0 + \mathbf{X}_{M, \text{un}}} = f$. Define $E_B^{\alpha*}\tilde{f}(g, \chi) = (\langle \chi, \alpha^\vee \rangle_{\text{un}} - 1)E_B\tilde{f}(g, \chi + \frac{\alpha}{2})$. Then*

$$E_B^{\alpha*}\tilde{f}(g, \chi) = E_P f(g, \chi), \quad (\forall \chi \in \chi_0 + \mathbf{X}_{M, \text{un}}, g \in G(\mathbb{A})).$$

Proof The two sides have the same constant term, namely

$$\sum_{w\alpha > 0} M\left(w, \chi - \frac{\alpha}{2}\right) \cdot M^*\left(w_\alpha, \chi + \frac{\alpha}{2}\right) \cdot \tilde{f} = \sum_{w\alpha > 0} M\left(w, \chi - \frac{\alpha}{2}\right) \cdot f.$$

Hence their difference is both a cusp form and a linear combination of Eisenstein series. As such, it is zero.

3.2 An identity of ramified Eisenstein series on G_2

Theorem 3.4 *Take $\eta \in \mathbf{X}_{GL_1}$, nontrivial quadratic, and \tilde{f} a flat section of $\mathcal{I}_B^G[0, \eta]$. Let f be the flat section of $\mathcal{I}_P^G[0, \eta]$ determined by \tilde{f} as in Proposition 3.1. The standard intertwining operator $M(w_\alpha w_\beta, [1, \eta - 1])$ is an isomorphism. Let \tilde{h} be the flat section of $\mathcal{I}_B^G[\eta, 0]$ satisfying $\tilde{h}|_{[\eta, 1]} = M(w_\alpha w_\beta, [1, \eta - 1])\tilde{f}|_{[1, \eta - 1]}$. Let h be the flat section of $\mathcal{I}_Q^G[\eta, 0]$ determined by \tilde{h} as in Proposition 3.1. Then $E_P f$ has a pole at $[0, \eta + \frac{1}{2}]$, $E_Q h$ has a pole at $[\eta + \frac{1}{2}, 0]$, and the two residues are the same.*

Proof Note that $\alpha = [2, -3]$ and $\beta = [-1, 2]$. Note also that

$$w_\alpha w_\beta [0, 1] = [-1, 2] = \beta \equiv \varpi_1 \pmod{2}, \quad w_\beta w_\alpha \alpha = [0, 1].$$

It follows that

$$w_\alpha w_\beta [0, s + \eta] + u \frac{\alpha}{2} = [u + \eta, 0] + s \frac{\beta}{2}.$$

Now,

$$E_P [0, s + \eta] \cdot f = \left(u - \frac{1}{2}\right) E_B([0, s + \eta] + u \frac{\alpha}{2}) \tilde{f}|_{u=\frac{1}{2}},$$

$$E_Q [u + \eta, 0] \cdot h = \left(s - \frac{1}{2}\right) E_B([u + \eta, 0] + s \frac{\alpha}{2}) \tilde{h}|_{s=\frac{1}{2}},$$

and

$$E_B \left([0, s + \eta] + u \frac{\alpha}{2}\right) \tilde{f} = E_B \left([u + \eta, 0] + s \frac{\alpha}{2}\right) M \left(w_\alpha w_\beta, [0, s + \eta] + u \frac{\alpha}{2}\right) \tilde{f}$$

$$= E_B \left([u + \eta, 0] + s \frac{\alpha}{2}\right) \left(\tilde{h} + \text{higher order terms.}\right)$$

It follows that the residue of $E_P \cdot f$ at $s = \frac{1}{2}$ and that of $E_Q \cdot h$ at $u = \frac{1}{2}$ are two different expressions for the value of the meromorphic continuation of $(s - \frac{1}{2})(u - \frac{1}{2})E_B([0, s + \eta] + u \frac{\alpha}{2}) \cdot \tilde{f}$ to $s = u = \frac{1}{2}$. \square

3.3 An identity of unramified Eisenstein series on G_2

In this section we prove an intriguing identity between unramified G_2 Eisenstein series. This result was previously obtained by D. Jiang in unpublished work.

Define $[a, b]$ as before and write $f_{[a,b]}^\circ$ for the spherical vector in the corresponding induced representation. Let $c(s) = \frac{\zeta(s)}{\zeta(s+1)}$. Identify $X_{GL_1, \text{un}}$ with \mathbb{C} as usual. Recall that

$$M(w, [a, b]) \cdot f_{[a,b]}^\circ = \prod_{\alpha > 0, w\alpha < 0} c([a, b] \circ \alpha^\vee) \cdot f_{w \cdot [a,b]}^\circ$$

Also, $c(s)$ has a simple zero at $s = -1$ a simple pole at $s = 1$ and is holomorphic and nonvanishing at each other integer. Write $c_{i,j}$ for the i th Laurent coefficient at j , so that

$$c(j + s) = \sum_{i=\text{ord}_j(c)}^{\infty} c_{i,j} s^i, \quad \text{ord}_j(c) = \begin{cases} 1, & j = -1, \\ -1, & j = 1, \\ 0, & j \in \mathbb{Z} \setminus \{\pm 1\}. \end{cases}$$

Also $c(s)c(-s) = 1$ and $c(0) = -1$, which implies that

$$c_{1,-1} = -c_{-1,1} \quad c_{2,-1} = -\frac{c_{0,1}}{c_{-1,1}^2} \quad c_{3,-1} = \frac{c_{1,1}}{c_{-1,1}^2} - \frac{c_{0,1}^2}{c_{-1,1}^3}$$

$$c_{0,0} = -1, \quad c_{2,0} = -\frac{1}{2}c_{1,0}^2.$$

Theorem 3.5 The meromorphic function

$$E_Q f_{[3u+2,-1]}^\circ - \frac{1}{3}c(u)c(3u-1)E_P f_{[-1,u+1]}^\circ$$

vanishes identically at $u = 0$.

Remark 3.6 It would be interesting to make sense of this identity in terms of the functional equations of the Eisenstein series $E_B \cdot f^\circ$, which are parametrized by the Weyl group of G_2 . However, this is not so trivial, as $[3u + 2, -1]$ is not in the same Weyl orbit as $[-1, u + 1]$ for $u \neq 0$.

Proof It suffices to prove that the constant term vanishes. We compute the constant terms of $E_Q \cdot f_{[3u+2,-1]}^\circ$ and $E_P \cdot f_{[-1,u+1]}^\circ$. We have

$$\begin{aligned} (E_Q \cdot f_{[3u+2,-1]}^\circ)_B &= f_{[3u+2,-1]}^\circ + c(3u+2)f_{[-2-3u,3u+1]}^\circ \\ &\quad + c(3u+1)c(3u+2) \left(f_{[6u+1,-1-3u]}^\circ + c(3u)c(3u-1)c(6u+1)f_{[1-3u,-1]}^\circ \right) \\ &\quad + c(3u+1)c(3u+2)c(6u+1) \left(f_{[-6u-1,3u]}^\circ + c(3u)f_{[3u-1,-3u]}^\circ \right) \\ (E_P \cdot f_{[-1,u+1]}^\circ)_B &= f_{[-1,u+1]}^\circ + c(u+1)f_{[3u+2,-1-u]}^\circ \\ &\quad + c(u+1)c(3u+2)f_{[-3u-2,2u+1]}^\circ \\ &\quad + c(u+1)c(3u+2)c(2u+1)f_{[3u+1,-2u-1]}^\circ \\ &\quad + c(u+1)c(3u+2)c(2u+1)c(3u+1) \left(f_{[-3u-1,u]}^\circ + c(u)f_{[-1,-u]}^\circ \right) \end{aligned}$$

The terms are grouped according to which T -eigenspace they reside in when $u = 0$. The proof is by direct computation. For each T -eigenspace we consider the terms in the Laurent expansion up to $O(u)$. For example take the character $[-1, 1]$ of T . In $(E_Q \cdot f_{[3u+2,-1]}^\circ)_B$ it does not appear. In $-\frac{1}{3}c(u)c(3u-1)(E_P \cdot f_{[-1,u+1]}^\circ)_B$ the contribution is $-\frac{1}{3}c(u)c(3u-1)f_{[3u+2,-1]}^\circ$, which nontrivial, but vanishes at $u = 0$ because $c(3u-1)$ vanishes at $u = 0$ and none of the other terms has a pole at $u = 0$. This is a fairly simple example.

We consider one additional example, which is a little more complex. Recall that

$$f_{[a,b]}^\circ(ntk) = |t^{\varpi_1}|^{a+1}|t^{\varpi_2}|^{b+1}, \quad (n \in N(\mathbb{A}), t \in T(\mathbb{A}), k \in K),$$

where N is the unipotent radical of B . Hence

$$f_{[a,b]+u[c,d]}^\circ(ntk) = f_{[a,b]}^\circ(ntk) \cdot \left(\sum_{m=0}^\infty \frac{1}{m!} [(\log |t^{\varpi_1}|c + \log |t^{\varpi_2}|d) \cdot u]^m \right).$$

Write $\langle c, d \rangle$ for the mapping

$$(ntk) \mapsto \log |t^{\varpi_1}|c + \log |t^{\varpi_2}|d, \quad (c, d \in \mathbb{Z}, n \in N(\mathbb{A}), t \in T(\mathbb{A}), k \in K).$$

Then we have

$$f_{[a,b]+u[c,d]}^\circ = f_{[a,b]}^\circ \cdot \sum_{m=0}^\infty \frac{\langle c, d \rangle^m}{m!} u^m.$$

Note that $\langle c_1, d_1 \rangle + \langle c_2, d_2 \rangle = \langle c_1 + c_2, d_1 + d_2 \rangle$. We compare the contributions to the constant terms of $E_Q \cdot f_{[3u+2,-1]}^\circ$ and $E_P \cdot f_{[-1,u+1]}^\circ$ which lie in the $[1, -1]$ eigenspace when $u = 0$. The relevant portion of $\frac{1}{3}c(u)c(3u-1)(E_P \cdot f_{[-1,u+1]}^\circ)_B$ is

$$\frac{1}{3}c(u)c(3u-1)c(u+1)c(3u+2)c(2u+1)f_{[3u+1,-2u-1]}^\circ \tag{[1,-1],P}$$

while the relevant portion of $E_Q \cdot f_{[s+2,-1]}^\circ$ is

$$c(s+1)c(s+2) \left(f_{[2s+1,-1-s]}^\circ + c(s)c(s-1)c(2s+1)f_{[1-s,-1]}^\circ \right). \tag{[1,-1],Q}$$

It makes sense to simplify $([1,-1],Q)$ before substituting $s = 3u$. However, notice that once this substitution is made, the factor of $c(3u+2)$ appears in both $([1,-1],P)$ and $([1,-1],Q)$. So, we may omit it from both. We expand the remainder of $([1,-1],P)$.

$$\begin{aligned}
 & \frac{1}{3}c(u)c(3u-1)c(u+1)c(2u+1)f_{[3u+1,-2u-1]}^\circ \\
 &= \frac{1}{3}(-1+c_{1,0}u+O(u^2))\left(-\frac{3}{c_{-1,1}}u-9\frac{c_{0,1}}{c_{-1,1}^2}u^2+O(u^3)\right)\left(\frac{c_{-1,1}}{u}+c_{0,1}+O(u)\right) \\
 & \quad \times \left(\frac{c_{-1,1}}{2u}+c_{0,1}+O(u)\right)f_{[1,-1]}^\circ(1+\langle 3,-2\rangle u+O(u^2)) \\
 &= f_{[1,-1]}^\circ\left(\frac{c_{-1,1}}{2u}-\frac{c_{-1,1}c_{1,0}}{2}+3c_{0,1}+\frac{c_{-1,1}}{2}\langle 3,-2\rangle+O(u)\right) \tag{3.7}
 \end{aligned}$$

We simplify the expression in brackets in $([1,-1],Q)$

$$\begin{aligned}
 & f_{[2s+1,-1-s]}^\circ+c(s)c(s-1)c(2s+1)f_{[1-s,-1]}^\circ \\
 &= f_{[1,-1]}^\circ\left(1+s\langle 2,-1\rangle+(-1+c_{1,0}s)\left(-\frac{s}{c_{-1,1}}-\frac{c_{0,1}s^2}{c_{-1,1}^2}\right)\right) \\
 & \quad \times \left(\frac{c_{-1,1}}{2s}+c_{0,1}\right)(1+\langle -1,0\rangle s+O(s^2)) \\
 &= f_{[1,-1]}^\circ\left(1+s\langle 2,-1\rangle+\frac{1}{2}\left(1+s[-c_{1,0}+\frac{3c_{0,1}}{c_{-1,1}}+\langle -1,0\rangle]\right)\right) \\
 &= f_{[1,-1]}^\circ\left(\frac{3}{2}+\frac{s}{2}\left(\langle 3,-2\rangle-c_{1,0}+\frac{3c_{0,1}}{c_{-1,1}}\right)\right)
 \end{aligned}$$

Multiplying by $c(s+1) = (c_{-1,1}/s + c_{0,1} + O(s))$ yields

$$\begin{aligned}
 & f_{[1,-1]}^\circ\left(\frac{3c_{-1,1}}{2s}+\frac{c_{-1,1}}{2}\left(\langle 3,-2\rangle-c_{1,0}+\frac{3c_{0,1}}{c_{-1,1}}\right)+\frac{3}{2}c_{0,1}+O(s)\right) \\
 &= f_{[1,-1]}^\circ\left(\frac{3c_{-1,1}}{2s}+\frac{c_{-1,1}}{2}\langle 3,-2\rangle-\frac{c_{1,0}c_{-1,1}}{2}+3c_{0,1}+O(s)\right),
 \end{aligned}$$

and now substituting $s = 3u$ yields (3.7). The other eigenspaces are treated similarly. \square

4 Global zeta integral involving a Q-Eisenstein series

In this section we consider the degenerate Eisenstein series $E_Q \cdot h$ on $G_2(\mathbb{A})$, where $Q = LV$ is the parabolic subgroup whose Levi contains the root subgroup attached to the long simple root. We let H_ρ be a quasisplit subgroup of G_2 of type A_2 defined as in [12]. Thus $\rho \in F^\times$ and H_ρ is isomorphic to SL_3 if ρ is a square, and a quasisplit unitary group attached to the corresponding quadratic extension if F is ρ is a nonsquare. For any normalized character χ_0 of $L(F)\backslash L(\mathbb{A})$ and any flat section h of $I = \mathcal{I}_Q^{G_2}(\chi_0)$ the restriction of $E_\rho \cdot h$ to $H_\rho(\mathbb{A})$ is a smooth function of moderate growth on $H_\rho(F)\backslash H_\rho(\mathbb{A})$, so it may be integrated against a cuspform on H_ρ . In the split case, this integral is identically zero for all χ_0, h , as shown in Proposition 4.6 of [10]. We extend the same idea to the general case.

First we need to analyze $Q(F)\backslash G_2(F)/H_\rho(F)$. Following [12], we regard G_2 as a subgroup of (split) SO_8 preserving the quadratic form $\underline{x} \mapsto {}_t \underline{x} \cdot \underline{x}$. Here ${}_t$ is the ‘‘other transpose’’ (as in [12]). Then G_2 is contained in the group SO_7 preserving the subspace $V_0 := \{ \underline{x} = {}_t [x_1 \dots x_8] : x_4 = x_5 \}$. The group H_ρ is the stablizer of $\nu_\rho := {}_t [0 \ 0 \ 1 \ 0 \ 0 \ \rho \ 0 \ 0]$ in G_2 . Thus $G_2(F)/H_\rho(F)$ may be identified with the orbit $O_\rho := G_2(F) \cdot \nu_\rho$, and $Q(F)\backslash G_2(F)/H_\rho(F)$ can be identified with the set of $Q(F)$ -orbits in O_ρ .

Lemma 4.1 *The orbit O_ρ is equal to $\{ \underline{x}_- \in V_0 : {}_t \underline{x}_- \cdot \underline{x}_- = 2\rho \}$. If ρ is not square it is a union of two $Q(F)$ orbits namely*

$$\left\{ \underline{x}_- = [x_1 \dots x_8] \in O_\rho : x_8 \neq 0 \right\}, \quad \left\{ \underline{x}_- = [x_1 \dots x_8] \in O_\rho : x_8 = 0 \right\}.$$

If $\rho = a^2$ then $\{x = [x_1 \dots x_8] \in O_\rho : x_8 \neq 0\}$ is still a single $Q(F)$ -orbit, while

$$\{x = [x_1 \dots x_8] \in O_\rho : x_8 = 0\}$$

is a union of three orbits, viz.

$$\begin{aligned} & \{x = [x_1 \dots x_8] \in O_\rho : x_8 = 0, \{x_6, x_7\} \neq \{0\}\}, \\ & \left\{ [x_1 \ x_2 \ x_3 \ a \ a \ 0 \ 0 \ 0] \right\}, \quad \text{and} \quad \left\{ [x_1 \ x_2 \ x_3 \ -a \ -a \ 0 \ 0 \ 0] \right\}. \end{aligned}$$

Proof It's clear that each element x of $G_2(F) \cdot v_\rho$ satisfies $t_x \cdot x = 2\rho$. Let $v'_\rho = {}^t[\rho \ 0 \dots 0 \ 1]$. Then a suitable representative for $s_\alpha s_\beta s_\alpha$ maps v_ρ to v'_ρ . One checks that the second, third, fourth, sixth, and seventh elements of the last column of an element of the standard maximal unipotent subgroup of $G_2(F)$ can be chosen arbitrarily and it follows that the $Q(F)$ orbit of v'_ρ (and also of v_ρ) contains every element of $\{x = [x_1 \dots x_8] \in O_\rho : x_8 \neq 0\}$. On the other hand, this subset is clearly $Q(F)$ -stable, so it is the $Q(F)$ orbit of v'_ρ . Likewise, a suitable representative of $s_\beta s_\alpha$ maps v_ρ to $v''_\rho = {}^t[0 \ \rho \ 0 \dots 0 \ 1 \ 0]$ and the orbit of v''_ρ under the Borel of $G_2(F)$ is readily seen to be all x with $t_x \cdot x = 2\rho$, $x_8 = 0$, $x_7 \neq 0$. Clearly, each vector with x_6, x_7 not both zero is $Q(F)$ -equivalent to one with $x_7 \neq 0$. It follows that $\{x : t_x \cdot x = 2\rho, x_8 = 0, \{x_6, x_7\} \neq \{0\}\}$ is the $Q(F)$ orbit of v''_ρ . Recall that $x \in V_0$ forces $x_4 = x_5$. If $x_6 = x_7 = x_8 = 0$ we get $t_x \cdot x = 2x_4^2$, so that $t_x \cdot x = 2\rho \implies \rho = x_4^2$. If ρ is not a square this is impossible. It follows that the two $Q(F)$ orbits already described exhaust all vectors with $t_x \cdot x = 2\rho$. If $\rho = a^2$ we have the two additional subsets described. Each is readily seen to be a $Q(F)$ -orbit. Let $X_{-\alpha}$ be the matrix with a one at positions (2, 1), (4, 3) and (5, 3), a -1 at positions (8, 7), (6, 5) and (6, 4), and zeros everywhere else. It spans the root subspace for the root $-\alpha$. Let $x_{-\alpha}(r) = \exp(rX_{-\alpha})$. (This is a polynomial formula, because $X_{-\alpha}$ is nilpotent.) Then

$$x_{-\alpha}(\pm a) \cdot v_{a^2} = {}^t[0 \ 0 \ 1 \ \pm a \ \pm a \ 0 \ 0 \ 0],$$

which proves that our two additional $Q(F)$ -orbits are still in the $G_2(F)$ -orbit of v_{a^2} . Clearly, the four taken together exhaust $\{x \in V_0 : t_x \cdot x = 2\rho\}$. This completes the proof. \square

Corollary 4.2 *If ρ is not a square, then $\{e, s_\alpha s_\beta\}$ is a set of representatives for $Q(F) \backslash G_2(F) / H_\rho(F)$. If $\rho = a^2$, then $\{e, s_\alpha s_\beta, x_{-10}(a), x_{-10}(-a)\}$ is a set of representatives.*

Proof Let each element act on v_ρ and consider the explicit description of the $Q(F)$ -orbits in O_ρ .

4.1 A certain period

Let $\varphi_{32} : SL_2 \rightarrow G_2$ be the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & & & & & & b \\ & a & & & & & & b \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ c & & & & & & d & \\ & c & & & & & & d \end{pmatrix}.$$

The image is contained in H_ρ for every ρ . (Cf. (2) on p. 195 of [12].) If we embed $H_\rho(F)$ into $GL_3(E)$ using the identification E^3 with the six dimensional subspace of F^8 stabilized by H_ρ as in [12] remarks 2, then φ_{32} corresponds to the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \frac{b}{2\tau} \\ & 1 \\ 2\tau c & d \end{pmatrix},$$

where $\tau \in E$ satisfies $\tau^2 = \rho$. From this point of view, the image of φ_{32} is a smaller special unitary group, and we denote in H'_ρ . We consider the period

$$\mathcal{P}(\varphi) = \int_{H'_\rho(F)\backslash H'_\rho(\mathbb{A})} \varphi(h) dh. \tag{4.3}$$

We say that an automorphic representation π is H'_ρ -distinguished if \mathcal{P} does not vanish identically on it.

Proposition 4.4 *Let h be a flat section of $\mathcal{I}_Q^{G_2}(\chi_0)$ and φ a cuspform defined on $H_\rho(\mathbb{A})$. Assume that $s \in \mathbf{X}_{Q,\text{un}}$ is not a pole of E_Q . Then for a suitable measure on the space $H'_\rho(\mathbb{A})\backslash H_\rho(\mathbb{A})$ we have*

$$\int_{H_\rho(F)\backslash H_\rho(\mathbb{A})} \varphi(g)E_Q \cdot h(g, s) dg = \int_{H'_\rho(\mathbb{A})\backslash H_\rho(\mathbb{A})} h(s_\alpha s_\beta g, s) \mathcal{P}(R(g) \cdot \varphi) dg. \tag{4.5}$$

(Here, R denotes right translation.) In particular, the integral vanishes identically on any cuspidal automorphic representation which is not H'_ρ -distinguished. If H_ρ is split, then no cuspidal automorphic representation is H'_ρ -distinguished, and the integral (4.5) is always zero.

Proof For s in the domain of convergence for E_Q , our integral is equal to

$$\sum_{\gamma \in Q(F)\backslash G_2(F)/H_\rho(F)} \int_{H_\rho(F)\cap \gamma^{-1}Q(F)\gamma\backslash H_\rho(\mathbb{A})} h(\gamma g, s)\varphi(g) dg.$$

If γ is the identity, or if $\rho = a^2$ and $\gamma = x_{-a}(\pm a)$ then $H_\rho \cap \gamma^{-1}Q\gamma$ contains the unipotent radical of the standard Borel subgroup of H_ρ . Hence the function $g \mapsto h(\gamma g, s)$ is invariant by this subgroup on the left, and the integral over $H_\rho(F) \cap \gamma^{-1}Q(F)\gamma\backslash H_\rho(\mathbb{A})$ factors through the mapping that sends φ to its constant term– which is zero. If $\gamma = s_\alpha s_\beta$, then $H_\rho \cap \gamma^{-1}Q\gamma = H'_\rho$, and $\gamma H'_\rho \gamma^{-1}$ is contained in the derived group of the Levi L of Q . Hence the function $g \mapsto f(s_\alpha s_\beta g, s)$ is invariant by $H'_\rho(\mathbb{A})$ on the right. We are reduced to showing that

$$\int_{H'_\rho(F)\backslash H_\rho(\mathbb{A})} h(s_\alpha s_\beta g, s)\varphi(g) dg = \int_{H'_\rho(\mathbb{A})\backslash H_\rho(\mathbb{A})} h(s_\alpha s_\beta g_1, s) \int_{H'_\rho(F)\backslash H'_\rho(\mathbb{A})} \varphi(h'_\rho g_1) dh'_\rho dg_1$$

for a suitable measure dg_1 . Here, dg and dh'_ρ are the Haar measures on the respective groups. The existence of a suitable invariant measure on each of the corresponding local homogeneous spaces $H'_\rho(F_v)\backslash H_\rho(F_v)$ follows from Proposition 4.3.5 of [2]. The existence of dg_1 then follows easily from expressing dg and dh'_ρ in terms of the Haar measures on the local groups.

If H_ρ is split, then H'_ρ is a nonstandard Levi subgroup and is conjugate to a standard Levi subgroup. It suffices to show that the period along the standard Levi subgroup vanishes. As noted in [10] this follows easily from cuspidality. \square

Remark 4.6 We may regard H'_ρ as $SU_{1,1} \subset U_{1,1} \subset U_{2,1}$ and one could enlarge our period \mathcal{P} , to an integral over $U_{1,1}(F)\backslash U_{1,1}(\mathbb{A})$ against a character. Such periods are considered in [7], where they are used to characterize the image of the theta lifting. Recall that the Ad' L function of a representation in the image of this lifting should have a pole at $s = 1$. So, the emergence of the H'_ρ period as an obstruction to proving holomorphy in general makes perfect sense.

4.2 Conclusions regarding poles of global zeta integrals

Theorem 4.7 *Suppose that φ generates a cuspidal automorphic representation which is not H'_ρ distinguished. Then the global zeta integral $I(s, \varphi, f)$ defined in (1.1) has no poles in the half plane $\text{Re}(s) \geq \frac{1}{2}$, except for a simple pole at $s = \frac{2}{3}$ which can occur only when χ_0 is cubic. Moreover, if χ_0 is quadratic, then $I(\frac{1}{2}, \varphi, f) = 0$ for all φ and f .*

Remark 4.8 In the split case, no cuspidal representation is H'_ρ distinguished. For these purposes the trivial character is both cubic and quadratic.

Proof In the case when χ is unramified this is a slight refinement of Proposition 4.6 of [10]. We know from Theorems 2.3 and 2.4 that the only possible poles in $\text{Re}(s) \geq \frac{1}{2}$ occur at $s = \frac{2}{3}$ when χ_0 is trivial, quadratic, or cubic, and at $s = 1$ when χ_0 is trivial. When H_ρ is split and χ_0 is trivial, the possibility of a pole at $s = 1$ or a double pole at $s = 2/3$ are ruled out by [10]. When χ_0 is quadratic, the possibility of a pole at $\frac{2}{3}$ is ruled out by the same argument, using Theorem 3.4 and Proposition 4.4. The vanishing at $\frac{1}{2}$ when χ_0 is quadratic follows from Proposition 2.6. \square

5 Nonvanishing of local zeta integrals

In this section we prove that for any fixed $s_0 \in \mathbb{C}$ there is a choice of data such that the local zeta integral for the adjoint L function is nonzero at s_0 . It then follows that any pole of the partial adjoint L function would give rise to a pole of the global zeta integral, except possibly at poles of the “normalizing factor” $L^S(3s, \chi)L^S(6s - 2, \chi^2)L^S(9s - 3, \chi)$. We take up some notations from [12]: for fixed $\rho \in F$, H_ρ is defined as in the previous section. We equip it with a choice of Borel subgroup $B = TN$ where T is a maximal torus and N is a maximal unipotent. (This departs from our previous usage of B, T and N for corresponding subgroups of G_2 .) Also w_2 is the second simple reflection in G_2 (attached to the long simple root; this departs from the usage of w_2 in Sect. 2.8), and N_2 is a two dimensional unipotent subgroup, with the property that $H_\rho \cap w_2^{-1}Pw_2 = N_2T_{\text{sp}}$, where T_{sp} denotes the one dimensional maximal F -split torus contained in the standard Borel of H_ρ . Finally, ψ_N is a certain generic character of N . Details are found in [12]. It is convenient to identify $\mathbf{X}_{M, \text{un}}$ with \mathbb{C} via the map $s \mapsto \delta_p^s$.

For $\rho \in F$, π an irreducible admissible ψ_N -generic representation of H_ρ , W in the ψ_N -Whittaker model $\mathcal{W}_{\psi_N}(\pi)$ of π , and $f \in \text{Flat}(\chi_0)$ we define

$$I(W, f; s) = \int_{N_2 \backslash H_\rho} W(g)f_s(w_2g) dg.$$

Theorem 5.1 For any $\rho \in F^\times$, any irreducible admissible generic representation π of H_ρ , any $\chi_0 \in \mathbf{X}_{M,0}$, and any fixed $s_0 \in \mathbb{C}$, there exist $W \in \mathcal{W}_{\psi_N}(\pi)$ and $f \in \text{Flat}(\chi_0)$ such that $I(W, f; s_0) \neq 0$.

Proof Expressing the Haar measure on H_ρ as a suitable product measure on the open Bruhat cell yields

$$I(W, f; s) = \int_{N_2 \backslash N} \int_T \int_{N^-} W(ntn^-) f_s(wntn^-) \delta_B^{-1}(t) dn^- dt dn.$$

We may identify $N_2 \backslash N$ with the complementary subgroup

$$N'_2 := \left\{ \begin{pmatrix} 1 & r & -\frac{r^2}{2} \\ & 1 & -r \\ & & 1 \end{pmatrix} : r \in F \right\},$$

which is T_{sp} -stable, and fix a fundamental domain $[T_{\text{sp}} \backslash T]$ for T_{sp} in T , to express $I(W, f; s)$ as

$$\begin{aligned} & \int_{N'_2} \int_{T_{\text{sp}}} \int_{[T_{\text{sp}} \backslash T]} \int_{N^-} W(nt_{\text{sp}}t'n^-) f_s(w_2nt_{\text{sp}}t'n^-) \delta_B^{-1}(t_{\text{sp}}t') dn^- dt' dt_{\text{sp}} dn \\ &= \int_{T_{\text{sp}}} t_{\text{sp}}^{w_2\chi_0 + w_2s - 3\gamma} I(W, f; t_{\text{sp}}, s) dt_{\text{sp}}, \end{aligned}$$

where γ is the common restriction of the two simple roots of H_ρ to T_{sp} (so that $\delta_B^{-1}(t_{\text{sp}}) = t_{\text{sp}}^{-4\gamma}$) and

$$I(W, f; t_{\text{sp}}, s) := \int_{N'_2} \int_{[T_{\text{sp}} \backslash T]} \int_{N^-} W(t_{\text{sp}}nt'n^-) f_s(w_2nt'n^-) \delta_B^{-1}(t') dn^- dt' dn.$$

Note that the function $(n, t', n^-) \mapsto Pw_2nt'n^-$ is a continuous injection of $N'_2 \times [T_{\text{sp}} \backslash T] \times N^-$ into $P \backslash G_2$. Now fix s_0 , and let $\phi_f(n, t', n^-) := \delta_B(t')^{-1} f_{s_0}(w_2nt'n^-)$, which we view as a “test function” on $N'_2 \times [T_{\text{sp}} \backslash T] \times N^-$. First assume that F is nonarchimedean. Then for any smooth function ϕ_1 of compact support defined on $N'_2 \times [T_{\text{sp}} \backslash T] \times N^-$ we can choose f so that $\phi_f = \phi_1$. But then

$$I(W, f; s_0) = I'(\phi_f * W; s_0),$$

where $*$ denotes the action by convolution and $I'(W, s)$ is defined for $W \in \mathcal{W}_{\psi_N}(\pi)$ and $s \in \mathbb{C}$ by

$$I'(W, s) := \int_{T_{\text{sp}}} W(t_{\text{sp}}) t_{\text{sp}}^{w_2\chi_0 + w_2s - 3\gamma} dt_{\text{sp}}.$$

Hence $I(W, f; s_0)$ vanishes for all $f \in \text{Flat}(\chi_0)$, $W \in \mathcal{W}_{\psi_N}(\pi)$ if and only if $I'(W; s_0)$ vanishes for all $W \in \mathcal{W}_{\psi_N}(\pi)$.

But now let ϕ_2 be a Schwartz function on F and $x : F \rightarrow H_\rho$ an embedding into N chosen so that $t_{\text{sp}}x(a)t_{\text{sp}}^{-1} = x(t_{\text{sp}}^\gamma a)$ and $\psi_N(x(a)) = \psi(a)$. Then

$$\begin{aligned} I'((\phi_2 \circ x) * W, s) &= \int_{T_{\text{sp}}} \int_F W(t_{\text{sp}}x(a)) t_{\text{sp}}^{w_2\chi_0 + w_2s - 3\gamma} \phi_2(a) da dt_{\text{sp}} \\ &= \int_{T_{\text{sp}}} \int_F W(t_{\text{sp}}) t_{\text{sp}}^{w_2\chi_0 + w_2s - 3\gamma} \psi(t_{\text{sp}}^\gamma a) \phi_2(a) da dt_{\text{sp}} \\ &= \int_{T_{\text{sp}}} W(t_{\text{sp}}) t_{\text{sp}}^{w_2\chi_0 + w_2s - 3\gamma} \widehat{\phi}_2(t_{\text{sp}}^\gamma) dt_{\text{sp}}. \end{aligned}$$

Notice that $\widehat{\phi}_2$ is a Schwartz function which can be chosen arbitrarily. Clearly, we can now choose $W \in \mathcal{W}_{\psi_N}(\pi)$ which does not vanish identically on T_{sp} and then choose ϕ_2 so that $I'((\phi_2 \circ x) * W, s_0) \neq 0$.

This completes the proof in the nonarchimedean case. In the archimedean case, the same argument shows that the mapping $(W, f) \mapsto I(W, f; s_0)$ does not vanish identically on $\mathcal{W}_{\psi_N}(\pi) \times \text{SI}_P^G(\chi_0 + s_0)$, where SI denotes smooth induction, as opposed to K -finite induction. But since the space of K -finite vectors is dense in the smooth induced representation, it then follows that $(W, f) \mapsto I(W, f; s_0)$ can not vanish identically on $\mathcal{W}_{\psi_N}(\pi) \times \text{Ind}_P^G(\chi_0 + s_0)$ either, completing the proof in this case. \square

6 Application to poles of the adjoint L function

Theorem 6.1 *Assume that G is split, and let S be a finite set of places, including all archimedean places and all places where either π_v or χ_v is ramified. Then the partial twisted adjoint L function $L^S(s, \pi, \text{Ad} \otimes \chi)$ has no poles in the half-plane $\text{Re}(s) \geq \frac{1}{2}$, except possibly for a simple pole at $\text{Re}(s) = 1$ when χ is nontrivial and $\pi \cong \pi \otimes \chi$ (which forces χ to be cubic). If this pole is present, then it is inherited by the complete L function $L(s, \pi, \text{Ad} \times \chi)$. Every other pole of $L(s, \pi, \text{Ad} \times \chi)$ in $\text{Re}(s) \geq \frac{1}{2}$ is a zero of the Hecke L function $L(s, \chi)$, and a pole of $\prod_{v \in S} L_v(s, \pi_v, \text{Ad} \times \chi_v)$.*

Proof According to the results which we have proved so far, the global zeta integral $I(s, \varphi, f)$ has no poles in $\text{Re}(s) \geq \frac{1}{2}$ except possibly for a simple pole at $\text{Re}(s) = \frac{2}{3}$ which can occur only when χ is nontrivial cubic. By Theorem 5.1, these properties are inherited by the ratio of partial L functions⁸

$$\frac{L^S(3s - 1, \pi, \text{Ad}' \otimes \chi)}{L^S(3s, \chi)L^S(6s - 2, \chi^2)L^S(9s - 3, \chi^3)},$$

and then, since local L functions are nonvanishing meromorphic functions, by

$$\frac{L^S(3s - 1, \pi, \text{Ad}' \otimes \chi)}{L(3s, \chi)L(6s - 2, \chi^2)L(9s - 3, \chi^3)}.$$

The product $L(3s, \chi)L(6s - 2, \chi^2)L(9s - 3, \chi^3)$ has no poles in $\text{Re}(s) \geq \frac{1}{2}$ except for the simple pole of $L(6s - 2, \chi^2)$ at $s = \frac{1}{2}$ which occurs only if χ^2 is trivial. But we have seen that $I(\frac{1}{2}, \varphi, f) = 0$ when χ is quadratic.

This completes the proof of our assertions regarding the partial L function. Since local L functions are meromorphic but nonvanishing, passing from the partial to the completed L function may introduce additional poles, but will not cancel the pole at 1 in the case when it occurs. On the other hand, it follows immediately from the definitions that $L(s, \pi, \text{Ad} \times \chi) = L(s, (\pi \otimes \chi) \times \widetilde{\pi})/L(s, \chi)$. By a result of Mœglin and Waldspurger [19, Corollaire, p. 667], the numerator has at most two simple poles, which occur at 0 and 1

⁸The particularly careful reader may have noticed that the equality

$$I(s, W_v, f_v) = \frac{L_v(3s - 1, \pi, \text{Ad}' \otimes \chi)}{L_v(3s, \chi)L_v(6s - 2, \chi^2)L_v(9s - 3, \chi^3)} \tag{6.2}$$

is only attained by taking normalized spherical vectors if $\psi_{N,v}$ is unramified, in addition to π_v and χ_v . We remark briefly on the places where π_v and χ_v are unramified and $\psi_{N,v}$ is not. The form of the global character ψ_N emerges from the unfolding in [9, 12] ensures that when $\psi_{N,v}$ is ramified, its orbit under the maximal F_v -split torus contains an element which is unramified. Then $W_v(g) = W'_v(tg)$, where W'_v is in the Whittaker model attached to this unramified additive character, and t is a suitable element of the torus. Making a change of variables in the integral we find that (6.2) holds up to an exponential factor.

and occur if and only if $\pi \cong \pi \otimes \chi$. Thus, any other poles which appear must be zeros of the denominator. \square

Remark 6.3 The expression $L(s, \pi, \text{Ad}) = L(s, \pi \times \tilde{\pi})/\zeta(s)$ also gives us a shorter proof that $L(s, \pi, \text{Ad})$ is holomorphic and nonvanishing at $s = 1$, without appealing to Theorem 3.5 and Proposition 4.4. The functional equations of the global Rankin–Selberg L -function and the global Dedekind zeta function give a functional equation of $L(s, \pi, \text{Ad})$. By the functional equation, the set of poles of $L(s, \pi, \text{Ad})$ is symmetric as $s \mapsto 1 - s$.

In the nonsplit case, the same argument gives the following result.

Theorem 6.4 *Assume that G is nonsplit, and let S be a finite set of places, including all archimedean places such that π_v and χ_v are unramified for $v \notin S$. Then the partial twisted L function $L^S(s, \pi, \text{Ad}' \otimes \chi)$ has no poles in the half-plane $\text{Re}(s) \geq \frac{1}{2}$, except possibly at $s = 1$. The pole at $s = 1$ can occur only when χ is trivial, quadratic, or cubic. If a pole occurs when χ is trivial or quadratic, then π is H'_ρ -distinguished. The pole at $s = 1$ is at most a double pole when χ is trivial, and at most a simple pole when χ is nontrivial quadratic or cubic.*

Remark 6.5 In the nonsplit case we have $L(s, \pi, \text{Ad}' \times \chi) = L(s, \text{sbc}(\pi), \text{Asai} \times \chi)/L(s, \chi)$, where sbc denotes the stable base change lifting from $U_{2,1}$ to $\text{Res}_{E/F} GL_3$, constructed in [16], and Asai is the Asai representation. In the important special case $\chi = \chi_{E/F}$ this becomes $L(s, \pi, \text{Ad}) = L(s, \text{sbc}(\pi), \text{Asai} \times \chi_{E/F})/L(s, \chi_{E/F})$. It is proved in [16], that $L(s, \text{sbc}(\pi), \text{Asai})$ will have a simple pole at $s = 1$ if $\text{sbc}(\pi)$ is cuspidal, but also that $\text{sbc}(\pi)$ need not be cuspidal. Arguing as on p. 22 of [11], we may deduce that $L(s, \text{sbc}(\pi), \text{Asai} \times \chi_{E/F})$ is holomorphic and nonvanishing at $s = 1$, and deduce that $L(s, \pi, \text{Ad})$ has the same property, still provided that $\text{sbc}(\pi)$ is cuspidal. A more thorough analysis of the nonsplit case is left to future work.

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