European Journal of Mathematics (2019) 5:798–827
https://doi.org/10.1007/s40879-018-0298-x

RESEARCH ARTICLE

G-birational superrigidity of Del Pezzo surfaces of degree 2 and 3

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Received: 26 April 2018 / Revised: 14 August 2018 / Accepted: 4 September 2018 /
Published online: 31 October 2018

Abstract
Any minimal Del Pezzo G-surface S of degree smaller than 3 is G-birationally rigid. We classify those which are G-birationally superrigid, and for those which fail to be so, we describe the equations of a set of generators for the infinite group $\mathrm{Bir}^G(S)$ of G-birational automorphisms.

Keywords Birational rigidity · Del Pezzo surfaces · Cubic surfaces · Bertini involution · Geiser involution

Mathematics Subject Classification 14E07

1 Introduction

The group of birational automorphisms of $\mathbb{P}^2(\mathbb{C})$ is classically known as Cremona group, denoted $\mathrm{Cr}_2(\mathbb{C})$. The classification of its finite subgroups up to conjugacy rose the interest of many classical authors and it has been completed in [5]. In this paper, we refine the description of the conjugacy class of some special finite subgroups.

The first named author was supported by the Brazilian National Council for Scientific and Technological Development (CNPq). The second named author was supported by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (the London School of Geometry and Number Theory), University College London and Imperial College, London.

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The key reduction step in the classification consists in associating to any finite subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ a group of automorphisms of a rational surface, isomorphic to G, see [5, Section 3.4]. Via a G-equivariant version of Mori theory, one can suppose that the surface is minimal with respect to the G-action. Here, we concentrate our attention to those finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ which act minimally by automorphisms on Del Pezzo surfaces S of degree 2 and 3. In particular, when the normaliser of G is not generated by automorphisms of the Del Pezzo surface, *i.e.*, the surface S is not G-birationally superrigid, we describe explicitly the generators of the normaliser.

In order to formulate our main results, we recall the definition of minimal G-surface. Let (S, ρ) be a G-surface, *i.e.*, a nonsingular surface S defined over $\mathbb{C}$, endowed with the action of a finite group of automorphisms $\rho\colon G \to \mathrm{Aut}(S)$. Given two G-surfaces (S, ρ) and (S', ρ'), we say that a rational map $\varphi\colon S \dashrightarrow S'$ is *G-rational* if for any $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} S & \overset{\varphi}{\dashrightarrow} & S' \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \rho'(g')} \\ S & \underset{\varphi}{\dashrightarrow} & S' \end{array}$$

for some $g' \in G$. Then, a *minimal G-surface* S is a G-surface with the property that any G-birational morphism $S \to S'$ is an isomorphism. Equivalently, it is the output of a G-equivariant minimal model program, and as in the non-equivariant case, if S is rational, it is either a Del Pezzo surface with $\mathrm{Pic}^G(S) \simeq \mathbb{Z}$, *i.e.*, $-K_S$ is ample, or a conic bundle with $\mathrm{Pic}^G(S) \simeq \mathbb{Z}^2$ (*cf.* [5, Theorem 3.8]).

The main properties investigated in this paper are described in the following definitions.

Definition 1.1 Let (S, ρ) be a minimal Del Pezzo G-surface. Then (S, ρ) is *G-birationally rigid* if there is no G-birational map from S to any other minimal G-surface. Equivalently, if S' is any minimal G-surface and $\varphi\colon S \dashrightarrow S'$ is any G-birational map, then S is G-isomorphic to S', not necessarily via φ. More precisely, there exists a G-birational automorphism $\sigma: S \dashrightarrow S$ such that $\varphi \circ \sigma$ is a G-biregular map.

Definition 1.2 The minimal Del Pezzo G-surface (S, ρ) is *G-birationally superrigid* if it is G-birationally rigid and in addition, in the notation of Definition 1.1, any G-birational map $\varphi\colon S \dashrightarrow S'$ is biregular. In particular, the group of G-biregular automorphisms coincides with the group of G-birational automorphisms, *i.e.*, $\mathrm{Aut}^G(S) = \mathrm{Bir}^G(S)$.

A classical theorem by Segre [9] and Manin [8] establishes that nonsingular cubic surfaces of Picard number 1 defined over a non-algebraically closed field are birationally rigid. In analogy with this arithmetic case, Dolgachev and Iskoviskikh showed in [5, Section 7.3] that minimal Del Pezzo G-surfaces of degree smaller than 3 are G-birationally rigid. In this paper we determine which minimal Del Pezzo G-surfaces of degree 2 and 3 are G-birationally superrigid. When the G-surface is not G-birationally superrigid, we describe the generators of the group of G-birational automorphisms $\mathrm{Bir}^G(S)$, or equivalently the normaliser of the corresponding subgroup G in $\mathrm{Cr}_2(\mathbb{C})$. Here, we collect our main results, adopting the notation of [5]:

Theorem 1.3 *Let G be a non-cyclic group and S be a minimal Del Pezzo G-surface of degree 3. Then S is G-birationally superrigid, unless G is isomorphic to the symmetric group S_3 and S is not the Fermat cubic surface.*

In this case, the group $\mathrm{Bir}^G(S)$ is generated by two or three Geiser involutions whose base points lie on the unique G-fixed line and by a subgroup of $\mathrm{Aut}(S)$ isomorphic to:

(i) *S_3 if S is of type V, VIII;*
(ii) *$S_3 \times 2$ if S is of type VI;*
(iii) *$S_3 \times 3$ if S is of type III, IV.*

The group $\mathrm{Bir}^G(S)$ of the very general non-G-birationally superrigid minimal Del Pezzo G-surface of degree 3 with $G \simeq S_3$ is not finite.

For the proof, see Sect. 4.1.

Theorem 1.4 *Let G be a cyclic group and S be a minimal Del Pezzo G-surface of degree 3. Then S is G-birationally superrigid if and only if G is of order 6 of type $A_5 + A_1$. More precisely, if S is not G-birationally superrigid, then G is isomorphic to one of the following:*

(i) *a cyclic group of order 3 of type $3A_2$. The group $\mathrm{Bir}^G(S)$ is (infinitely) generated by the Geiser involutions whose base points lie on the unique G-fixed nonsingular cubic curve and by a subgroup of $\mathrm{Aut}(S)$ isomorphic to $3^3 \rtimes S_3$, if S is the Fermat cubic surface, or by $\mathrm{Aut}(S)$ itself otherwise.*
(ii) *a cyclic group of order 6 of type $E_6(a_2)$. The group $\mathrm{Bir}^G(S)$ is (infinitely) generated by three Geiser involutions, the Bertini involutions whose base points lie on a G-invariant nonsingular cubic curve C and by a subgroup of $\mathrm{Aut}(S)$ isomorphic to $3^3 \times 2$, if S is the Fermat cubic surface, or by $\mathrm{Aut}(S)$ itself otherwise.*
(iii) *a cyclic group of order 9 of type $E_6(a_1)$. The group $\mathrm{Bir}^G(S)$ is finitely generated by three Geiser involutions whose base loci are coplanar and by a subgroup of $\mathrm{Aut}(S)$ isomorphic to the dihedral group D_{18}.*
(iv) *a cyclic group of order 12 of type E_6. The group $\mathrm{Bir}^G(S)$ is finitely generated by G, by a Bertini involution and by a Geiser involution whose base loci are aligned.*

For the proof, see Sect. 4.2.

Theorem 1.5 *Let G be a non-cyclic group and S be a minimal Del Pezzo G-surface of degree 2. Then S is G-birationally superrigid.*

For the proof, see Sect. 5.1.

Theorem 1.6 *Let G be a cyclic group and S be a minimal Del Pezzo G-surface of degree 2. Then S is G-birationally superrigid if and only if G is one of the following:*

(i) *a group of order 2 of type A_1^7;*
(ii) *a group of order 6 of types $E_7(a_4)$, $A_5 + A_1$, $D_6(a_2) + A_1$;*

(iii) *a group of order* 14 *of type* $E_7(a_1)$*;*
(iv) *a group of order* 18 *of type* E_7*.*

Moreover, if S is not G-birationally superrigid, then G is isomorphic to one of the following:

(v) *a cyclic group of order* 4 *of type* $2A_3 + A_1$*. The group* $\mathrm{Bir}^G(S)$ *is generated by infinitely many Bertini involutions whose base loci lie in the unique G-fixed nonsingular curve of genus one and by a subgroup of* $\mathrm{Aut}(S)$ *isomorphic to* $2 \times 4^2 \rtimes 2$*, if S is of type* II*, or by* $\mathrm{Aut}(S)$ *itself otherwise.*
(vi) *a cyclic group of order* 12 *of type* $E_7(a_2)$*. The group* $\mathrm{Bir}^G(S)$ *is generated by two Bertini involutions and by a subgroup of* $\mathrm{Aut}(S)$ *isomorphic to* 2×12*.*

For the proof, see Sect. 5.2.

Corollary 1.7 *Let G be a cyclic group and S be a minimal Del Pezzo G-surface of degree smaller than* 3*. Then, S is G-birationally superrigid if and only if the group* $\mathrm{Bir}^G(S)$ *of G-birational automorphisms is finite.*

Proof It is an immediate corollary of Theorems 1.4 and 1.6. In particular, see Lemmas 4.9, 4.11 and 5.3. The authors are not aware of a proof that does not rely on the above classification. □

In the paper we also provide explicit equations for the listed Del Pezzo surfaces S and the generators of the group $\mathrm{Bir}^G(S)$, unless it coincides with $\mathrm{Aut}^G(S)$. The types of the G-surfaces appearing in Theorems 1.3, 1.4 and 1.6 are described in full detail in Lemma 4.5, Propositions 4.7 and 5.2. For convenience, we summarise the contents of Theorems 1.3, 1.4 and 1.6 in Tables 1 and 2.

The structure of the paper is as follows: in Sect. 3 we rewrite in full detail the proof of the G-equivariant version of the above-mentioned Segre–Manin theorem, see Theorem 3.1. Note that the statement is essentially proved in [5, Corollary 7.11]. Building on this result, we classify the minimal Del Pezzo G-surfaces of degree 3 and 2 which are not G-birationally superrigid in Sects. 4 and 5 respectively.

2 Preliminaries

Let S be a nonsingular surface. A linear system $\mathcal{M}$ on S is *mobile* if its fixed locus does not contain any divisorial component. The pair $(S, D+\mathcal{M})$ is the datum of a nonsingular surface S, a $\mathbb{Q}$-divisor D whose coefficients are at most 1 and a mobile linear system $\mathcal{M}$, or equivalently one of its general members. Let $\alpha\colon \widetilde{S} \to S$ be a birational morphism. For each prime divisor E_i of $\widetilde{S}$ there exists a coefficient $a(E_i, S, D + \mathcal{M})$, called *discrepancy*, such that the following relation holds:

$$K_{\widetilde{S}} + \alpha_*^{-1}(D) + \alpha_*^{-1}(\mathcal{M}) \sim_{\mathbb{Q}} \alpha^*(K_S + D + \mathcal{M}) + \sum_i a(E_i, S, D + \mathcal{M}) E_i.$$

In particular, observe that the multiplicity $\mathrm{mult}_p(\mathcal{M})$ of $\mathcal{M}$ at a point $p \in S$ equals $1 - a(E, S, \mathcal{M})$, where E is the exceptional divisor of the blow-up of S at p.

Table 1 Minimal Del Pezzo G-surfaces of degree 3 which are not G-birationally superrigid

Type of G	G	Type of S	Equation of S	$\mathrm{Aut}^G(S)$	Geiser invol.	Bertini invol.
$3A_2$	3	I	$t_0^3+t_1^3+t_2^3+t_3^3$	$3^3 \rtimes S_3$	∞	0
$3A_2$	3	III	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$H_3(3) \rtimes 4$	∞	0
			$20a^3+8a^6=1$			
$3A_2$	3	IV	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$H_3(3) \rtimes 2$	∞	0
			$20a^3+8a^6 \neq 1,\ 8a^3 \neq 1$			
			$a-a^4 \neq 1$			
$E_6(a_2)$	6	I	$t_0^3+t_1^3+t_2^3+t_3^3$	$3^2 \times 2$	3	∞
$E_6(a_2)$	6	III	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$H_3(3) \rtimes 4$	3	∞
			$20a^3+8a^6=1$			
$E_6(a_2)$	6	IV	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$H_3(3) \rtimes 2$	3	∞
			$20a^3+8a^6 \neq 1,\ 8a^3 \neq 1$			
			$a-a^4 \neq 1$			
$E_6(a_1)$	9	I	$t_3^2t_1+t_1^2t_2+t_2^2t_3+t_0^3$	D_{18}	3	0
E_6	12	III	$t_3^2t_1+t_2^2t_3+t_0^3+t_1^3$	12	1	1
	S_3	VI	$t_0^3+t_1^3+t_2^3+t_3^3+at_0t_1(t_2+t_3)$	$S_3 \times 2$	2	0
			$a \neq 0$			
	S_3	III–IV	$t_0^3+t_1^3+t_2^3+t_3^3+t_0t_1(at_2+bt_3)$	$S_3 \times 3$	3	0
		V–VIII	$a^3 \neq b^3 \neq 0,$	S_3		

Table 2 Minimal Del Pezzo G-surfaces of degree 2 which are not G-birationally superrigid

Type of G	G	Type of S	Equation of S	$\mathrm{Aut}^G(S)$	Bertini invol.
$2A_3 + A_1$	4	II	$t_3^2 + t_2^4 + t_0^4 + t_1^4$	$2 \times 4^2 \rtimes 2$	∞
$2A_3 + A_1$	4	III	$t_3^2 + t_2^4 + t_0^4 + 2\sqrt{3}it_0^2t_1^2 + t_1^4$	$2 \times 4A_4$	∞
$2A_3 + A_1$	4	V	$t_3^2 + t_2^4 + t_0^4 + at_0^2t_1^2 + t_1^4$	$2 \times AS_{16}$	∞
			$a^2 \neq 0, -12, 4, 36$		
$E_7(a_2)$	12	III	$t_3^2 + t_0^4 + t_1^4 + t_0t_2^3$	2×12	2

A pair $(S, D+\mathcal{M})$ is *canonical* if $a(E, S, D+\mathcal{M}) \geqslant 0$ for any exceptional divisor E and for any $f : \widetilde{S} \to S$ birational morphism. A pair $(S, D+\mathcal{M})$ is called *log Calabi–Yau* if $K_S + D + \mathcal{M} \sim_{\mathbb{Q}} 0$.

Let G be a finite group of automorphisms acting effectively on a surface S. In the introduction we have already recalled the definition of a G-rational map. This concept must not be confused with that of a *G-equivariant map*, *i.e.*, a birational map which makes the following diagram commute:

$$\begin{array}{ccc} S & \overset{\varphi}{\dashrightarrow} & S' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ S & \underset{\varphi}{\dashrightarrow} & S' \end{array}$$

for every $g \in G$.

The *degree d* of a Del Pezzo surface S is defined to be the self-intersection number of the canonical class K_S, in symbols $d := K_S^2$. We briefly recall some properties of Del Pezzo surfaces of degree $\leqslant 3$, see for instance [6, Chapter III, Theorem 3.5].

- A Del Pezzo surface S of degree 1 is a nonsingular hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$, embedded via the third pluricanonical linear system $|-3K_S|$. Via the linear system $|-2K_S|$, S can be realised as a double cover of the singular quadric $\mathbb{P}(1, 1, 2)$ branched along a nonsingular sextic curve. In particular, since the double cover is canonical, its deck transformation τ is a central element in the group of automorphisms $\mathrm{Aut}(S)$, see also [5, Section 6.7].

$$\begin{array}{ccc} S & \xhookrightarrow{\varphi_{|-3K_S|}} & \mathbb{P}(1,1,2,3) \\ {\scriptstyle \varphi_{|-2K_S|}}\downarrow{\scriptstyle 2:1} & & \\ \mathbb{P}(1,1,2). & & \end{array}$$

- A Del Pezzo surface S of degree 2 is a nonsingular hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, embedded via the second pluricanonical linear system $|-2K_S|$. Via the canonical map, S can be realised as a double cover of $\mathbb{P}^2$ branched along a nonsingular quartic curve. In particular, since the double cover is canonical, its deck transformation τ is a central element in the group of automorphisms $\mathrm{Aut}(S)$, see also [5, Section 6.6].

$$\begin{array}{ccc} S & \xhookrightarrow{\varphi_{|-2K_S|}} & \mathbb{P}(1,1,1,2) \\ {\scriptstyle \varphi_{|-K_S|}}\big\downarrow{\scriptstyle 2:1} & & \\ \mathbb{P}^2. & & \end{array}$$

- A Del Pezzo surface S of degree 3 is a nonsingular hypersurface of degree 3 in the projective space $\mathbb{P}^3$, embedded via the anticanonical linear system $|-K_S|$.

$$S \xhookrightarrow{\varphi_{|-K_S|}} \mathbb{P}^3.$$

3 *G*-equivariant Segre–Manin theorem

In this section we present the proof, essentially due to Dolgachev and Iskoviskikh, of the following G-equivariant version of a classical arithmetic theorem by Segre [9] and Manin [8] (*cf.* also [2, Section 1.5]).

Theorem 3.1 (G-equivariant Segre–Manin theorem [5, Section 7.3]) *Every minimal Del Pezzo G-surface S of degree $d \leqslant 3$ is G-birationally rigid.*

The main ingredients of the proof are Noether–Fano inequalities, which in modern language recast the failure of birational superrigidity in terms of the existence of a non-canonical log Calabi–Yau pair.

Theorem 3.2 (Noether–Fano inequalities [3, Theorem 3.2.1 (ii), Theorem 3.2.6 (ii)]) *Let G be a finite group, S and S' be two minimal G-surfaces and $\varphi\colon S \dashrightarrow S'$ be a G-birational map. Suppose S is a minimal Del Pezzo surface and let $\mathcal{M}$ be a G-invariant mobile linear system on S defined in the following way:*

- *if S' is a Del Pezzo G-surface, then $\mathcal{M} := \varphi_*^{-1}(|H|)$ is the strict transform via φ^{-1} of the linear system $|H|$, where H is a very ample multiple of $-K_{S'}$;*
- *if $\psi\colon S' \to C$ is a G-conic bundle and H is a very ample G-invariant divisor of C, then $\mathcal{M} := \varphi_*^{-1}(|\psi^*(H)|)$ is the strict transform via φ^{-1} of the linear system $|\psi^*(H)|$.*

Then, there exists a positive rational number λ such that

$$K_S + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$$

and the following statements hold:

(i) *If S' is a Del Pezzo surface and $(S, \lambda\mathcal{M})$ is canonical, then φ is biregular.*
(ii) *If S' is a conic bundle, then $(S, \lambda\mathcal{M})$ is not canonical.*

Proof of Theorem 3.1 Let $\varphi\colon S \dashrightarrow S'$ be a G-birational non-biregular map to a minimal G-surface S'. In order to prove that S is G-birationally rigid we need to exhibit a G-birational map $\sigma : S \dashrightarrow S$ such that $\varphi \circ \sigma$ is a G-biregular map.

Step 1 (non-canonical log Calabi–Yau pair). By Theorem 3.2, the existence of φ is equivalent to the existence of a mobile G-invariant linear system $\mathcal{M}$ on S such that

- (log Calabi–Yau) $K_S + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$;
- (not canonical singularities) the pair $(S, \lambda\mathcal{M})$ is not canonical.

Since K_S generates $\operatorname{Pic}^G(S)$ in degree $\leqslant 3$, we can suppose $\lambda = \frac{1}{n}$ for some $n \in \mathbb{N}$.

Step 2 (orbit of length $\leqslant 3$). The proof of Lemma 3.4 implies that there exists a G-orbit O contained in the non-canonical locus of the log Calabi–Yau pair $\left(S, \frac{1}{n}\mathcal{M}\right)$ such that

$$m := \operatorname{mult}_p(\mathcal{M}) > n \quad \text{for all points } p \in O.$$

Lemma 3.5 grants that the length of O is strictly less than the degree d of S.

Step 3 (Geiser and Bertini involution). By hypothesis, the degree of S is at most 3 and we are left with few possibilities:

Case 1. *O consists of a single G-fixed point p and the degree of S is either* 2 *or* 3. Let $\pi : \widetilde{S} \to S$ be the blow-up of S at p with exceptional divisor E. Then, the surface $\widetilde{S}$ is a Del Pezzo surface of degree 1 or 2 if S has degree 2 or 3 respectively (*cf.* Lemma 3.7), and it is endowed with a G-action via pullback of the G-action on S. These surfaces are endowed with a central G-invariant biregular involution τ, which descends to a G-birational non-biregular involution σ_1 on S, named Bertini or Geiser involution respectively. The defined G-birational maps are collected in the following diagram:

$$\begin{array}{ccccc} \widetilde{S} & \xrightarrow{\ \tau\ } & \widetilde{S} & & \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\ S & \overset{\sigma_1}{\dashrightarrow} & S & \overset{\varphi}{\dashrightarrow} & S'. \end{array}$$

Let a, b, c, d be integers such that $\tau^*(H) \sim aH + bE$ and $\tau^*(E) \sim cH + dE$, where $H := -\pi^* K_S$ is the pullback of the ample generator of $\operatorname{Pic}^G(S)$. Then, we obtain

$$\sigma_1^{-1}(\mathcal{M}) = \pi_*\tau^*\pi_*^{-1}\mathcal{M} \sim_{\mathbb{Q}} \pi_*\tau^*(nH - mE) \sim_{\mathbb{Q}} -(an - cm)K_S.$$

Note in particular that $c > 0$, because E is not τ-invariant and so $\tau^* E$ is not contracted by π: by the ampleness of $-K_S$, we obtain

$$0 < (-K_S.\pi_*\tau^*E) = (H.\tau^*E) = cH^2.$$

Since τ preserves the canonical class $K_{\widetilde{S}} \sim -H + E$, we obtain also that $a - c = 1$, so that

$$\sigma_1^{-1}(\mathcal{M}) \sim_{\mathbb{Q}} -(an - cm)K_S = -(n - c(m - n))K_S < -nK_S.$$

Case 2. *O consists of two points p_1 and p_2 and the degree of S is* 3. Analogously, the blow-up of S at p_1 and p_2 is a Del Pezzo surface of degree 1 endowed with a G-equivariant involution which descends to a non-biregular Bertini involution on S, denoted σ_1.

In all the cases, the Noether–Fano inequalities (*cf.* Lemma 3.4) force $\sigma_1^{-1}(\mathcal{M}) \sim_{\mathbb{Q}} -k_1 K_S$ with $k_1 < n$.

Step 4 (inductive step). By Theorem 3.2, either $\varphi \circ \sigma_1$ is G-biregular or the pair $\left(S, \frac{1}{k_1}\sigma_1^{-1}(\mathcal{M})\right)$ is not canonical. In the latter case, we can repeat the above arguments and construct a sequence of Bertini or Geiser G-involutions $\sigma_1, \ldots, \sigma_s$ on S such that $\varphi_s := \varphi \circ \sigma_1 \circ \cdots \circ \sigma_s$ is non-biregular and again, by Theorem 3.2, the mobile pair $\left(S, \frac{1}{k_s}\varphi_s^{-1}(\mathcal{M})\right)$, with $k_s < k_{s-1}$, is not canonical. However, if $s > n$, then the mobile linear system $\varphi_s^{-1}(\mathcal{M})$ would not be $\mathbb{Q}$-linearly equivalent to an effective divisor, which is a contradiction. Hence, there exists an integer s such that φ_s is G-biregular. We conclude that S is G-birationally rigid. □

Corollary 3.3 *Let S be a minimal Del Pezzo G-surface of degree* 3 (*resp.* 2). *Then, every G-birational map is a composition of a G-biregular map, Geiser and/or Bertini involutions* (*resp. a G-biregular map and Bertini involutions*).

We now prove the lemmas used in the proof of Theorem 3.1.

Lemma 3.4 *Let S be a G-surface and* $(S, \mathcal{M})$ *be a G-pair, i.e., $\mathcal{M}$ is a G-invariant mobile linear system. If* $(S, \mathcal{M})$ *is not canonical, then there exists a G-orbit O in S such that*

$$\mathrm{mult}_O(\mathcal{M}) > 1,$$

i.e., the multiplicity of each point of O on $\mathcal{M}$ is greater than 1.

Proof Let $\alpha : \widetilde{S} \to S$ be a G-equivariant log resolution of the pair $(S, \mathcal{M})$. This means that α is a G-equivariant birational morphism such that the fixed locus of the pullback linear system $\alpha^*(\mathcal{M})$ has simple normal crossing. We prove the statement by induction on the number s of G-equivariant blow-ups through which α factors. If α is the blow-up of S at a single G-orbit O with exceptional divisor E, then

$$K_{\widetilde{S}} + \alpha_*^{-1}\mathcal{M} \sim_{\mathbb{Q}} \alpha^*(K_S + \mathcal{M}) + (1 - \mathrm{mult}_O(\mathcal{M}))E.$$

Since the pair $(S, \mathcal{M})$ is not canonical, by definition $a(E, S, \mathcal{M}) := 1 - \mathrm{mult}_O(\mathcal{M}) < 0$. Suppose now that $\alpha = \alpha_{s-1} \circ \alpha_1$, where α_i are a composition of i G-equivariant blow-ups:

$$\alpha : \widetilde{S} \xrightarrow{\alpha_{s-1}} S_1 \xrightarrow{\alpha_1} S.$$

Let O' be the centre of the blow-up α_1 with exceptional divisor E_1. Then, either $\mathrm{mult}_{O'}(\mathcal{M}) > 1$, or $a(E_1, S, \mathcal{M}) \geqslant 0$. In the latter case, since the pair $(S_1, (\alpha_1)_*^{-1}\mathcal{M} - a(E_1, S, \mathcal{M})E_1)$ is not canonical, a fortiori the pair $(S_1, \mathcal{M}_1 := (\alpha_1)_*^{-1}\mathcal{M})$ is not canonical, but by induction hypothesis there exists $O_1 \subseteq S_1$ such that $\operatorname{mult}_{O_1}(\mathcal{M}_1) > 1$. This implies that $\operatorname{mult}_O(\mathcal{M}) > 1$ for $O := \alpha_1(O_1)$. □

Lemma 3.5 *Let S be a minimal Del Pezzo G-surface of degree d. If $\varphi\colon S \dashrightarrow S'$ is a non-biregular G-birational map, then the G-orbit O defined in Lemma 3.4 has length $|O|$ strictly smaller than d.*

Proof Let $\mathcal{M}$ be the linear system defined in Theorem 3.2. Consider C_1 and C_2 two general G-invariant $\mathbb{Q}$-divisors of $\mathcal{M} \sim_{\mathbb{Q}} -nK_S$. For any G-orbit O defined in Lemma 3.4, the following sequence of inequalities holds:

$$dn^2 = C_1 \cdot C_2 \geqslant \sum_{p \in O} \operatorname{mult}_p(C_1)\operatorname{mult}_p(C_2) > |O|n^2,$$

which implies $d > |O|$. □

Remark 3.6 Lemma 3.5 implies immediately that any Del Pezzo G-surface of degree 1 is G-birationally superrigid, see also [5, Corollary 7.11].

Lemma 3.7 *Let S be a minimal Del Pezzo G-surface of degree d and $\mathcal{M}$ be a mobile linear system on S such that $K_S + \mathcal{M} \sim_{\mathbb{Q}} 0$. Let $\pi\colon S' \to S$ be a G-equivariant blow-up of S at a G-orbit O defined in Lemma 3.4. Then, S' is a Del Pezzo surface, i.e. $-K_{S'}$ is ample.*

Proof By the Nakai–Moishezon criterion for amplitude [7, Theorem 1.2.23], it is enough to check that

- $(-K_{S'}.C) > 0$ for any curve $C \subset S'$;
- $K_{S'}^2 > 0$.

Note that

$$K_{S'} + \pi_*^{-1}\mathcal{M} = \pi^*(K_S + \mathcal{M}) + (1 - \operatorname{mult}_O(\mathcal{M}))E \sim_{\mathbb{Q}} (1 - \operatorname{mult}_O(\mathcal{M}))E.$$

In particular, we obtain that for any curve $C \subset S$ different from E

$$(-K_{S'}.C) = \big((\pi_*^{-1}\mathcal{M} + (\operatorname{mult}_O(\mathcal{M}) - 1)E).C\big) > 0,$$

since $\mathcal{M}$ is an ample linear system and because of Lemma 3.4. If $C = E$, then

$$(-K_{S'}.E) = ((-\pi^*K_S - E).E) = -E^2 > 0.$$

Finally, $K_{S'}^2 = K_S^2 - |O| > 0$, by Lemma 3.5. □

4 *G*-birational superrigidity of cubic surfaces

Let G be a finite group of automorphisms acting effectively on a minimal Del Pezzo surface of degree 3. It is well known that any Del Pezzo surface of degree 3 is a nonsingular cubic surface embedded in $\mathbb{P}^3 = \mathbb{P}(V)$ via the canonical embedding and every automorphism of S lifts to an automorphism of $\mathbb{P}^3$. The 4-dimensional vector space V is a G-representation, unique up to scaling by a character of G.

The content of this section is the proof of Theorem 1.4. Proposition 4.1 is one of the main ingredients of the proof.

Proposition 4.1 *A minimal Del Pezzo G-surface S of degree 3 is not G-birationally superrigid if and only if it admits either G-equivariant Geiser or Bertini involutions. This is equivalent to the existence on S of a G-fixed point, not lying on a line, or a G-orbit of length 2, not lying on a line or a conic in S and such that no tangent space of one point contains the other.*

Proof This is a corollary of Theorem 3.1 and Corollary 3.3. The second statement follows from Lemma 4.2. □

Lemma 4.2 *Let S be a nonsingular cubic surface.*

(i) *A point p in S is the base locus of a Geiser involution if and only if no line contained in S passes through p.*
(ii) *The points $\{p_1, p_2\}$ in S are the base locus of a Bertini involution if and only if*
   (a) *there is no line in S passing through p_1 or p_2;*
   (b) *there is no conic contained in S passing through p_1 and p_2;*
   (c) *p_i is not contained in the tangent space of S at p_j, $i \neq j$.*

Proof Let $f\colon \widetilde{S} \to S$ be the blow-up of S at p or at the pair $\{p_1, p_2\}$ respectively. By construction of Geiser and Bertini involution, we just need to check that $\widetilde{S}$ is a Del Pezzo surface. Recall that a Del Pezzo surface is the blow-up of $\mathbb{P}^2$ at most at eight points in general position, namely if

- no three of them lie on a line;
- no six of them lie on a conic;
- no eight of them lie on a nodal or cuspidal cubic with one of them at the singular point.

See for instance [1, Exercise V.21.(1)]. Let $g\colon S \to \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ at six points $q_1, \dots, q_6$ in general position. The point p in S is the base locus of a Geiser involution if and only if:

- p does not lie in the exceptional locus of g;
- the strict transform $\widetilde{l}$ of the line l passing through q_i and q_j does not contain p;
- the strict transform $\tilde{c}$ of a conic c passing through five of the points q_i does not contain p.

Equivalently, we require that no (-1)-curve contains p. Indeed, the g-exceptional lines and the curves $\widetilde{l}$ and $\tilde{c}$ are all the 27 (-1)-curves in S.

In order to construct a Bertini involution, we need to check in addition that the strict transform $\tilde{s}$ of a singular cubic curve s containing all the points q_i does not contain both p_1 and p_2. Suppose on the contrary that such a curve $\tilde{s}$ exists. We distinguish two cases: either q_i is a singular point of s or one of the p_i, say p_1, is a singular point of $\tilde{s}$. In the former case, $\tilde{s}$ is a conic. Indeed, it is a nonsingular rational curve with

$$(\mathcal{O}_{\mathbb{P}^3}(1).\tilde{s}) = \left(-K_S.\left(g^*\mathcal{O}_{\mathbb{P}^2}(3) - 2E_i - \sum_{j=1}^{5} E_j\right)\right) = 2.$$

Vice versa, if the points $\{p_1, p_2\}$ lie on a nonsingular conic c in S, then

$$(K_{\tilde{S}}.\tilde{c}) = \big((f^*K_S + F_1 + F_2).\tilde{c}\big) = (K_S.c) + ((F_1 + F_2).c) = 0,$$

where $\tilde{c}$ is the strict transform of c via f, and F_1 and F_2 are the f-exceptional divisors. Thus, $\widetilde{S}$ is not a Del Pezzo surface. In the latter case, $\tilde{s}$ is an anticanonical divisor, hence a hyperplane section singular at p_i. In particular, the tangent plane at p_1 contains both the points p_1 and p_2. □

In the following Lemma 4.3, we show that orbits of length 2 lie on invariant lines passing through a fixed point for the action of G on S.

Lemma 4.3 *Let S be a minimal cubic G-surface admitting an orbit of length 2, then G fixes a point in S.*

Proof Denote by q_1 and q_2 the points in the orbit of length 2 and by $l_{q_1q_2}$ the line passing through those points in $\mathbb{P}^3$. Note that the line $l_{q_1q_2}$ is G-invariant and it is not contained in S. Differently, it could be contracted, violating the minimality of G.

Moreover, the line $l_{q_1q_2}$ intersects S with multiplicity 1 at q_1 and q_2. Otherwise, if the multiplicity at one of the two points is $\geqslant 2$, then so it is at the other point due to the group action. However, this is a contradiction, since $l_{q_1q_2}$ would intersect S with multiplicity at least 4, while S has degree 3. This implies that the invariant line $l_{q_1q_2}$ intersects S in a third point, thus fixed by the action of G. □

Our strategy to show that a nonsingular cubic surface is G-birationally superrigid is the following:

- find G-fixed points and orbits of length 2 aligned with them, see Lemma 4.3;
- if the conditions of Lemma 4.2 do not hold for these G-orbits, then S is G-birationally superrigid.

In view of the latter, recall that a point of intersection of three lines on a cubic surface is called *Eckardt point*. It is just the case to mention that a point p is an Eckardt point if and only if the intersection of its tangent space to S and S itself is the union of three lines passing through p.

Remark 4.4 Notice that if $\{p_1, p_2\}$ is a G-orbit then condition (c) in Lemma 4.2 always holds, since otherwise the line between p_1 and p_2 is bitangent to S.

4.1 *G*-birational superrigidity for non-cyclic groups

Suppose now that G is non-cyclic. Minimal non-cyclic finite groups acting effectively by automorphisms on cubic surfaces and fixing a point have been classified by Dolgachev and Duncan [4]. Any cubic surface endowed with an action of such a group is projectively equivalent to a surface S_{ab} defined by

$$F_{ab} = t_0^3 + t_1^3 + t_2^3 + t_3^3 + t_0 t_1 (a t_2 + b t_3), \tag{1}$$

where a and b are parameters, and the fixed point is $p_0 = (0:0:1:-1)$, see [4, Theorem 8.1]. In particular, G is a subgroup of the stabiliser of the point p_0. Since F_{ab} specialises to the Fermat cubic equation F_{00}, G is a subgroup of the stabiliser of the point p_0 in $\mathrm{Aut}(S_{00})$, see the proof of [4, Theorem 8.1]. The automorphism group of the Fermat cubic surface is $3^3 \rtimes S_4$, where S_4 is the group of permutations of the variables and 3^3 is the 3-torsion group of $\mathrm{PGL}(4, \mathbb{C})$ generated for instance by the following automorphisms:

$$\begin{aligned}\sigma(t_0:t_1:t_2:t_3) &= (\epsilon_3 t_0:t_1:t_2:t_3),\\ \rho(t_0:t_1:t_2:t_3) &= (t_0:\epsilon_3 t_1:t_2:t_3),\\ \theta(t_0:t_1:t_2:t_3) &= (t_0:t_1:\epsilon_3 t_2:t_3),\end{aligned}$$

where ϵ_3 is a primitive third root of unity. The stabiliser of the point p_0 is $3^2 \rtimes K_4 \simeq 6 \times S_3$, where K_4 is the non-normal Klein subgroup of S_4 generated by (12) and (34) and 3^2 is generated by σ and ρ.

In particular, the skew lines $l_1 = \{t_0 = t_1 = 0\}$ and $l_2 = \{t_2 = t_3 = 0\}$ are G-invariant, since they are invariant under the action of $3^2 \rtimes K_4$. The intersection $l_1 \cap S_{ab}$ consists of three points p_0, $p_1 := (0:0:1:-\epsilon_3)$ and $p_2 := (0:0:1:-\epsilon_3^2)$. In particular,

$$T_{p_i} S_{ab} \cap S_{ab} = \left\{\epsilon_3^i t_2 + t_3 = t_0^3 + t_1^3 + (a - \epsilon_3^i b)\, t_0 t_1 t_2 = 0\right\}.$$

As the values of the parameters (a, b) vary, we have the following cases.

Type $a = b = 0$. The surface S_{00} is the Fermat cubic surface. The points p_i are Eckardt points. No orbit can be the base locus of a Geiser or a Bertini involution. By Theorem 3.1, we conclude that S_{00} is G-birationally superrigid.

Type $a^3 = b^3 \neq 0$. Up to a linear change of coordinates, we can suppose that $a = b \neq 0$. The group G is isomorphic to $2 \times S_3$ or S_3, where S_3 is generated by $\sigma\rho^2$ and (12), and 2 is generated by (34), see [4, Theorem 8.1, Case 3.2.]. Hence, the only fixed point is the Eckardt point p_0 and the only invariant line through p_0 is l_1. Note that the surface S_{ab} is of type VI in the sense of [5, Table 4] and the automorphism group of $\mathrm{Aut}(S_{ab})$ is isomorphic to $2 \times S_3$. We consider the cases $G \simeq 2 \times S_3$ and $G \simeq S_3$ separately.

- $G \simeq 2 \times S_3$. The conic $C = \{t_0 + t_1 = t_2^2 - t_2 t_3 + t_3^2 + a t_0 t_1 = 0\}$ passes through the length-two orbit $\{p_1, p_2\}$. By Lemma 4.2 and Theorem 3.1, we conclude that S_{ab} is G-birationally superrigid.

- $G \simeq S_3$. The fixed points p_1 and p_2 are not Eckardt points. Therefore, S_{ab} is not G-birationally superrigid and the group $\mathrm{Bir}^G(S_{ab})$ is generated by $\mathrm{Aut}(S_{ab})$ and the two Geiser involutions with base locus p_1 and p_2 respectively. The equations of these Geiser involutions and the infinitude of the group $\mathrm{Bir}^G(S_{ab})$ for the very general surface S_{ab} are discussed in the following paragraphs.

Type $a^3 \neq b^3$. The group G is isomorphic to S_3, see [4, Theorem 8.1, Case 3.1]. The only fixed points are p_0, p_1, p_2. None of them is an Eckardt point and the only invariant line through p_i is l_1. Therefore, S_{ab} is not G-birationally superrigid and the group $\mathrm{Bir}^G(S_{ab})$ is generated by G-biregular automorphisms of S_{ab} and three Geiser involutions with base loci contained in $l_1 \cap S_{ab}$.

The Geiser involutions are given by the equations

$$\varphi_{p_0}(t_0{:}t_1{:}t_2{:}t_3) = \left(t_0{:}t_1{:}t_3 - \frac{(a-b)\,t_0t_1}{3(t_2+t_3)}{:}t_2 + \frac{(a-b)\,t_0t_1}{3(t_2+t_3)}\right),$$
$$\varphi_{p_1}(t_0{:}t_1{:}t_2{:}t_3) = \left(t_0{:}t_1{:}\epsilon_3^2t_3 - \frac{(a-\epsilon_3 b)\,t_0t_1}{3(t_2+\epsilon_3^2t_3)}{:}\epsilon_3t_2 + \frac{(\epsilon_3a-\epsilon_3^2b)\,t_0t_1}{3(t_2+\epsilon_3^2t_3)}\right),$$
$$\varphi_{p_2}(t_0{:}t_1{:}t_2{:}t_3) = \left(t_0{:}t_1{:}\epsilon_3t_3 - \frac{(a-\epsilon_3^2 b)\,t_0t_1}{3(t_2+\epsilon_3t_3)}{:}\epsilon_3^2t_2 + \frac{(\epsilon_3^2a-\epsilon_3b)\,t_0t_1}{3(t_2+\epsilon_3t_3)}\right).$$

We complete the list of generators, computing the normaliser $N_{\mathrm{Aut}(S_{ab})}(G)$ of G in $\mathrm{Aut}(S_{ab})$. We adopt the surface type convention of [5].

Lemma 4.5 *The normaliser of G in $\mathrm{Aut}(S_{ab})$, denoted $N_{\mathrm{Aut}(S_{ab})}(G)$, is isomorphic to $S_3 \times 3$, if S_{ab} is of type* III *or* IV, *or to S_3, if S_{ab} is of type* V *or* VIII.

Proof Due to [5, Theorem 6.14], the group $\mathrm{Aut}(S_{ab})$ is one of the following.

- Type III. $\mathrm{Aut}(S_{ab}) \simeq H_3(3) \rtimes 4$, where $H_3(3)$ is the Heisenberg group of unipotent 3×3-matrices over the finite field $\mathbb{F}_3$, see Sect. 4.2, Type E_6, for explicit generators. The generator of 4 conjugates the non-conjugate subgroups of type S_3 in $H_3(3) \rtimes 2$, see [5, Theorem 6.14, Type III]. We conclude that $N_{H_3(3)\rtimes 4}(S_3) = N_{H_3(3)\rtimes 2}(S_3)$.
- Type IV. $\mathrm{Aut}(S_{ab}) \simeq H_3(3) \rtimes 2$. It contains two non-conjugate subgroups isomorphic to S_3, normalized by the subgroups isomorphic to $S_3 \times 3$ obtained from the previous ones by adding the central element, see [5, Theorem 6.14, Type III].
- Type V. $\mathrm{Aut}(S_{ab}) \simeq S_4$. Any subgroup isomorphic to S_3 is a non-normal maximal subgroup of S_4.
- Type VIII. $\mathrm{Aut}(S_{ab}) \simeq S_3$. □

Let G be again the group of biregular automorphisms acting minimally on S_{ab} with a fixed point p_0 and isomorphic to S_3. The following lemma establishes the infinitude of the group of G-birational automorphisms $\mathrm{Bir}^G(S_{ab})$ for the very general surface S_{ab}.

Let $\mathcal{S} \subset \mathbb{P}^3_{(t_0:t_1:t_2:t_3)} \times \mathbb{C}^2_{(a,b)}$ be the hypersurface given by the equation $\{F_{ab} = 0\}$, see equation (1), and $\mathcal{S}'$ be the divisor $\{a = b\}$ in $\mathcal{S}$ (equivalently $\{a = \epsilon_3^i b\}$). Denote by

$f\colon \mathcal{S} \to \mathbb{C}^2_{(a,b)}$ the family of cubic surfaces S_{ab} and by $f'\colon \mathcal{S}' \to \mathbb{C}_{(a)}$ that of surfaces S_{ab} with the property that $a = b$ (equivalently $a = \epsilon_3^i b$). The Geiser involutions φ_{p_i} on S_{ab} glue together to birational involutions of $\mathcal{S}$ and $\mathcal{S}'$ respectively, as their equations are polynomial in (a, b).

Lemma 4.6 *The group* $\mathrm{Bir}^G(S_{ab})$ *is not a finite group for the very general surface* S_{ab} *in* $\mathcal{S}$ *and in* $\mathcal{S}'$.

Proof Let Δ be the diagonal in $\mathcal{S} \times_f \mathcal{S}$ and $\Gamma_{(\varphi_{p_2} \circ \varphi_{p_1})^n}$ be the graph of the composition $(\varphi_{p_2} \circ \varphi_{p_1})^n$ in $\mathcal{S} \times_f \mathcal{S}$. There is an induced projection morphism

$$\mathrm{pr}\colon \Gamma_{(\varphi_{p_2} \circ \varphi_{p_1})^n} \cap \Delta \subseteq \Delta \to \mathbb{C}^2_{(a,b)}.$$

Define the (closed) algebraic subset

$$\begin{aligned} C_n &= \left\{(a, b) \in \mathbb{C}^2_{(a,b)} \mid (\varphi_{p_2} \circ \varphi_{p_1})^n = \mathrm{id}|_{S_{(a,b)}}\right\} \\ &= \left\{(a, b) \in \mathbb{C}^2_{(a,b)} \mid \dim \mathrm{pr}^{-1}(a, b) = 2\right\}. \end{aligned}$$

Note that the locus of surfaces S_{ab} with infinite $\mathrm{Bir}^G(S_{ab})$ contains $\mathbb{C}^2_{(a,b)} \setminus \bigcup_n C_n$. Therefore, if there exists $(a_0, b_0) \in \mathbb{C}^2_{(a,b)}$ such that $\mathrm{Bir}^G(S_{a_0b_0})$ is not finite, then C_n is a proper closed subset of $\mathbb{C}^2_{(a,b)}$ and the lemma holds.

We claim that $\mathrm{Bir}^G(S_{11})$ is not finite, *i.e.*, we can choose (a_0, b_0) equal to $(1, 1)$. To this aim, recall the following facts:

- for any $p \in S_{ab}$ the point $\varphi_{p_i}(p)$ is aligned with p_i and p;
- the involutions φ_{p_1} and φ_{p_2} fix the pencil of cubic curves

$$C_{(\lambda:\mu)} = \left\{\lambda t_0 - \mu t_1 = t_0^3 + t_1^3 + t_2^3 + t_3^3 + t_0 t_1 (t_2 + t_3) = 0\right\}.$$

Fix $(\lambda : \mu) \in \mathbb{P}^1_{(\lambda:\mu)}$ such that $C_{(\lambda:\mu)}$ is nonsingular. Observe that the point p_0 is an inflection point of $C_{(\lambda:\mu)}$. Due to the previous facts, the following relations for the elliptic curve $(C_{(\lambda:\mu)}, p_0)$ hold:

$$\begin{aligned} p_1 + p_2 &= 0; \\ p_1 + p + \varphi_{p_1}(p) &= 0; \\ p_2 + \varphi_{p_1}(p) + \varphi_{p_2} \circ \varphi_{p_1}(p) &= 0. \end{aligned}$$

In particular,

$$\varphi_{p_2} \circ \varphi_{p_1}(p) = p + 2p_1.$$

One can check (use MAGMA) that for a suitable choice of $(\lambda : \mu)$ (*e.g.* $(1 : 1)$), the point p_1 is not a torsion point. This implies that $\varphi_{p_2} \circ \varphi_{p_1}$ has infinite order in S_{11}.

The same proof holds for $\mathcal{S}'$ since $S_{11} \subset \mathcal{S}'$. □

Open Question *Is the group* $\mathrm{Bir}^G(S_{ab})$ *not finite for any* $(a, b) \neq (0, 0)$*?*

4.2 *G*-birational superrigidity for cyclic groups

In this section, we discuss the birational superrigidity of minimal cubic surfaces endowed with the action of a finite cyclic group G. Dolgachev and Iskovskikh classified these groups in [5]. For the convenience of the reader, we recall their result.

Here and in the following we denote by ϵ_n a primitive n-th root of unity.

Proposition 4.7 ([5, Corollary 6.11]) *Let $S = V(F)$ be a nonsingular cubic surface, endowed with a minimal action of a cyclic group G of automorphisms, generated by g. Then, one can choose coordinates in such a way that g and F are given in the following list:*

(i) $3A_2$, *order 3*, $g(t_0:t_1:t_2:t_3) = (t_0:t_1:t_2:\epsilon_3 t_3)$,
$$F = t_0^3 + t_1^3 + t_2^3 + t_3^3 + \alpha t_0 t_1 t_2;$$

(ii) $E_6(a_2)$, *order* 6, $g(t_0:t_1:t_2:t_3) = (t_0:t_1:-t_2:\epsilon_3 t_3)$,
$$F = t_0^3 + t_1^3 + t_3^3 + t_2^2(\alpha t_0 + t_1);$$

(iii) $A_5 + A_1$, *order* 6, $g(t_0:t_1:t_2:t_3) = (t_0:\epsilon_3^2 t_1:\epsilon_3 t_2:\epsilon_6 t_3)$,
$$F = t_3^2 t_1 + t_0^3 + t_1^3 + t_2^3 + \lambda t_0 t_1 t_2;$$

(iv) $E_6(a_1)$, *order* 9, $g(t_0:t_1:t_2:t_3) = (t_0:\epsilon_9^4 t_1:\epsilon_9 t_2:\epsilon_9^7 t_3)$,
$$F = t_3^2 t_1 + t_1^2 t_2 + t_2^2 t_3 + t_0^3;$$

(v) E_6, *order* 12, $g(t_0:t_1:t_2:t_3) = (t_0:\epsilon_3 t_1:\epsilon_{12} t_2:\epsilon_6^5 t_3)$,
$$F = t_3^2 t_1 + t_2^2 t_3 + t_0^3 + t_1^3.$$

We proceed with an analysis case by case.

Type $3A_2$. G fixes the nonsingular cubic curve
$$C = \left\{t_3 = t_0^3 + t_1^3 + t_2^3 + \alpha t_0 t_1 t_2 = 0\right\}.$$

S is not G-birationally superrigid and the group $\mathrm{Bir}^G(S)$ is generated by biregular G-automorphisms of S and infinitely many Geiser involutions whose base locus points lie on the nonsingular cubic curve given by $t_3 = 0$.

The normaliser $N_{\mathrm{Aut}(S)}(G)$ of G in $\mathrm{Aut}(S)$ is the group $\mathrm{Aut}^G(S)$ of biregular G-automorphisms. If C is equianharmonic, *i.e.*, it has an automorphism of order 6, then S is the Fermat cubic surface and $\mathrm{Aut}(S) \simeq 3^3 \rtimes S_4$ (*cf.* Sect. 4.1): the normaliser $N_{\mathrm{Aut}(S)}(G)$ is isomorphic to $3^3 \rtimes S_3$. Otherwise, g is a central element of $\mathrm{Aut}(S)$, which is isomorphic to $H_3(3) \rtimes 4$ or $H_3(3) \rtimes 2$, where $H_3(3)$ is the Heisenberg group of unipotent 3×3-matrices over the finite field $\mathbb{F}_3$ (cubic surfaces of type III or IV; see [5, Table 4]). Then, the group $\mathrm{Aut}^G(S)$ coincides with $\mathrm{Aut}(S)$.

Type $E_6(a_2)$. The line $l_2 = \{t_2 = 0, t_3 = 0\} \subseteq \mathbb{P}^3$ is fixed. The intersection

$$l_2 \cap S = \left\{(1:-1:0:0), (1:-\epsilon_3:0:0), (1:-\epsilon_3^2:0:0)\right\}$$

consists of three fixed points. The intersections of their tangent spaces with the cubic surface are respectively

$$\begin{aligned}&\left\{t_0 + t_1 = t_3^3 + (\alpha - 1)\, t_2^2 t_0 = 0\right\},\\ &\left\{\epsilon_3 t_0 + t_1 = t_3^3 + (\alpha - \epsilon_3^2)\, t_2^2 t_0 = 0\right\},\\ &\left\{\epsilon_3^2 t_0 + t_1 = t_3^3 + (\alpha - \epsilon_3)\, t_2^2 t_0 = 0\right\},\end{aligned}$$

which are three cuspidal cubic curves (we can suppose without loss of generality that $\alpha^3 \neq 1$, otherwise S would be singular). There is only one further isolated fixed point on S, namely $(0:0:1:0)$, which is an Eckardt point and whose tangent space is given by the equation $\alpha t_0 + t_1 = 0$.

An invariant line, which is not $l_1 = \{t_0 = t_1 = 0\}$, belongs either to the pencil $\mathcal{P}_{(0:0:0:1)}$ of lines through $(0:0:0:1)$ intersecting the line l_2 or to the pencil $\mathcal{P}_{(0:0:1:0)}$ of lines through $(0:0:1:0)$ intersecting the line l_2. These pencils span respectively the planes $t_2 = 0$ and $t_3 = 0$. Orbits of length 2 lie on invariant lines, neither on l_1 (since it is tangent to the Eckardt point $(0:0:1:0)$, thus $l_1 \cap S = \{p\}$), nor on a line through $\mathcal{P}_{(0:0:0:1)}$ (since the group G modulo the stabiliser of the plane $t_2 = 0$ acts on it as a cyclic group of order 3). On the other hand, the nonsingular cubic curve

$$C = \left\{t_3 = t_0^3 + t_1^3 + t_2^2(\alpha t_0 + t_1) = 0\right\}$$

is covered by orbits of length 2, since the group G modulo the stabiliser of the plane $t_3 = 0$ acts on it as a cyclic group of order 2.

We conclude that S is not G-birationally superrigid and that the group $\mathrm{Bir}^G(S)$ is generated by G-biregular automorphisms of S, three Geiser involutions with base loci contained in $l_2 \cap S$, and infinitely many Bertini involutions, whose base locus points lie on the nonsingular cubic curve given by $t_3 = 0$. We complete the list of generators, computing the normaliser $N_{\mathrm{Aut}(S)}(G)$ of G in $\mathrm{Aut}(S)$.

Lemma 4.8

$$N_{\mathrm{Aut}(S)}(G) = \begin{cases} 3^2 \times 2 & \textit{if } S \textit{ is the Fermat cubic surface;}\\ \mathrm{Aut}(S) & \textit{otherwise.}\end{cases}$$

Proof Note that S is a cyclic cover of degree 3 of $\mathbb{P}^2$ branched along a nonsingular cubic curve C, and G is generated by $g_1 g_2$, where g_1 is the deck transformation of the cover and g_2 is the lift of the involution on C.

If S is the Fermat cubic surface, then G is generated by the element $(\sigma\rho\theta, (12)) \in 3^3 \rtimes S_4$ in the notation of Sect. 4.1 (surface of type I with $K = G \cap 3^3$ of dimension

1 and type II $3A_2$; see [5, Section 6.5.]). Given $(\sigma^{a_0}\rho^{a_1}\theta^{a_2}, \tau) \in N_{\mathrm{Aut}(S)}(G) \subseteq \mathrm{Aut}(S) = 3^3 \rtimes S_4$, we observe that $\tau \in N_{S_4}((12)) \simeq K_4$. Denoting the conjugation of g via $h \in 3^3 \rtimes S_4$ by $c_h(g)$, we write

$$c_{(\sigma^{a_0}\rho^{a_1}\theta^{a_2},(12))}(g)(t_0:t_1:t_2:t_3) = \left(\epsilon_3^{a_0-a_1}t_1:\epsilon_3^{a_1-a_0}t_0:t_2:\epsilon_3^2 t_3\right),$$
$$c_{(\sigma^{a_0}\rho^{a_1}\theta^{a_2},(34))}(g)(t_0:t_1:t_2:t_3) = \left(\epsilon_3^{a_1-a_0}t_1:\epsilon_3^{a_0-a_1}t_0:\epsilon_3^2 t_2:t_3\right).$$

Hence, $N_{\mathrm{Aut}(S)}(G)$ is generated by the permutation (12) and the subspace of $3^3_{(a_0,a_1,a_2)}$ satisfying the equation $a_0 \equiv a_1 \pmod 3$. In particular, $N_{\mathrm{Aut}(S)}(G) \simeq 3^2 \times 2$.

If S is not the Fermat cubic surface, then S is a surface of type III or IV [5, Table 4] and $\mathrm{Aut}(S)$ is a central extension of $\mathrm{Aut}(C)$ via g_1. Therefore, $N_{\mathrm{Aut}(S)}(G)$ is a central extension of $N_{\mathrm{Aut}(C)}(g_2)$ via g_1, but since g_2 is central in $\mathrm{Aut}(C)$, we conclude that $N_{\mathrm{Aut}(S)}(G) = \mathrm{Aut}(S)$, or equivalently that $G \lhd \mathrm{Aut}(S)$. □

Types $A_5 + A_1$, $E_6(a_1)$ **and** E_6. In the last few cases, *i.e.*, $A_5 + A_1$, $E_6(a_1)$ and E_6, the group G acts on $\mathbb{P}^3$ by means of four distinct characters. In particular, the points $p_0 := (1:0:0:0)$, $p_1 := (0:1:0:0)$, $p_2 := (0:0:1:0)$ and $p_3 := (0:0:0:1)$ are the only fixed points in $\mathbb{P}^3$. The only invariant lines are those interpolating pairs of points (p_i, p_j), where $i \neq j$, shortly written $l_{p_i p_j}$. Note that eventual orbits of length 2 lie on $l_{p_i p_j} \cap S$.

Type $A_5 + A_1$. The only fixed point in S is the Eckardt point p_3. In the following table, we list all the invariant lines and the orbits that they cut on S.

Invariant lines $l_{p_i p_j}$	$l_{p_i p_j} \cap S$	Orbits in $l_{p_i p_j} \cap S$
$l_{p_0 p_1} = \{t_2 = t_3 = 0\}$	$t_0^3 + t_1^3 = 0$	orbit of length 3
$l_{p_0 p_2} = \{t_1 = t_3 = 0\}$	$t_0^3 + t_2^3 = 0$	orbit of length 3
$l_{p_0 p_3} = \{t_1 = t_2 = 0\}$	$t_0^3 = 0$	fixed Eckardt point p_3
$l_{p_1 p_2} = \{t_0 = t_3 = 0\}$	$t_1^3 + t_2^3 = 0$	orbit of length 3
$l_{p_1 p_3} = \{t_0 = t_2 = 0\}$	$t_3^2 t_1 + t_1^3 = 0$	fixed Eckardt point p_3 and orbit of length 2 given by: $q_1 := (0:i:0:1)$, $q_2 := (0:-i:0:1)$.
$l_{p_2 p_3} = \{t_1 = t_0 = 0\}$	$t_2^3 = 0$	fixed Eckardt point p_3

Note that the conic

$$C = \left\{t_0 + t_2 = t_3^2 + t_1^2 - \lambda t_2^2 = 0\right\} \subseteq S$$

contains the only orbit of length 2 and the only fixed point in S is contained in a line. We conclude that S is G-birationally superrigid.

Type $E_6(a_1)$. All the fixed points in S are the points p_1, p_2 and p_3. They are not Eckardt points: by cyclic permutation of the variable (t_1, t_2, t_3) it is enough to check that $T_{p_1} S \cap S$ is an irreducible cubic curve. Indeed,

$$T_{p_1} S \cap S = \left\{t_2 = t_3^2 t_1 + t_0^3 = 0\right\}.$$

The invariant lines $l_{p_1 p_2}$, $l_{p_2 p_3}$ and $l_{p_1 p_3}$ intersect S in two fixed points, one of them necessarily with multiplicity 2. The invariant lines $l_{p_0 p_i}$, with $i = 1, 2, 3$, are principal tangent lines at the singular point of the cuspidal cubic curves $T_{p_i} S \cap S$. We conclude that S is not G-birationally superrigid and the group $\mathrm{Bir}^G(S)$ is finitely generated by G-biregular automorphisms of S and three Geiser involutions with base loci p_1, p_2 and p_3 respectively. More explicitly, the Geiser involutions are given by

$$\begin{aligned}
\varphi_{p_1}(t_0 : t_1 : t_2 : t_3) &= \left(t_0 : -t_1 - \frac{t_3^2}{t_2} : t_2 : t_3\right),\\
\varphi_{p_2}(t_0 : t_1 : t_2 : t_3) &= \left(t_0 : t_1 : -t_2 - \frac{t_1^2}{t_3} : t_3\right),\\
\varphi_{p_3}(t_0 : t_1 : t_2 : t_3) &= \left(t_0 : t_1 : t_2 : -t_3 - \frac{t_2^2}{t_1}\right).
\end{aligned}$$

Although finitely generated, $\mathrm{Bir}^G(S)$ is not a finite group, as we show in the following lemma.

Lemma 4.9 *The group* $\mathrm{Bir}^G(S)$ *is not a finite group.*

Proof It is enough to prove that the composition $\varphi_{p_2} \circ \varphi_{p_1}$ has infinite order. To this aim, recall the following facts:

- for any $p \in S$ the point $\varphi_{p_i}(p)$ is aligned with p_i and p;
- the involutions φ_{p_1} and φ_{p_2} fix the pencil of cubic curves

$$C_{(\lambda:\mu)} = \left\{\lambda t_0 - \mu t_3 = t_3^2 t_1 + t_1^2 t_2 + t_2^2 t_3 + t_3^3 = 0\right\}.$$

Fix $(\lambda : \mu) \in \mathbb{P}^1_{(\lambda:\mu)}$ such that $C_{(\lambda:\mu)}$ is nonsingular and choose O an inflection point on $C_{(\lambda:\mu)}$. Due to the previous facts, the following relations for the elliptic curve $(C_{(\lambda:\mu)}, O)$ hold:

$$\begin{aligned}
2p_2 + p_1 &= 0;\\
p_1 + p + \varphi_{p_1}(p) &= 0;\\
p_2 + \varphi_{p_1}(p) + \varphi_{p_2} \circ \varphi_{p_1}(p) &= 0.
\end{aligned}$$

In particular,

$$\varphi_{p_2} \circ \varphi_{p_1}(p) = p - 3p_2.$$

One can check (use MAGMA) that for a suitable choice of $(\lambda:\mu)$ (*e.g.* $(1:1)$), the point p_2 is not a torsion point. This implies that $\varphi_{p_2}\circ\varphi_{p_1}$ has infinite order. □

We complete the list of generators of the group $\mathrm{Bir}^G(S)$, describing the group of G-biregular automorphisms of S. Note first that via the following change of coordinates

$$\begin{aligned}(s_0:s_1:s_2:s_3)\\ &= \left(\sqrt[3]{9}\,t_0 : t_1+t_2+t_3 : \epsilon_9(t_1+\epsilon_9^6 t_2+\epsilon_9^3 t_3) : \epsilon_9^2(t_1+\epsilon_9^3 t_2+\epsilon_9^6 t_3)\right),\end{aligned}$$

we can suppose that S is given by the equation

$$s_0^3+s_1^3+s_2^3+s_3^3=0$$

and a generator g of G acts via

$$g(s_0:s_1:s_2:s_3)=(s_0:\epsilon_3 s_2:s_3:s_1).$$

Lemma 4.10 *The normaliser of G in* $\mathrm{Aut}(S)$*, denoted $N_{\mathrm{Aut}(S)}(G)$, is isomorphic to the dihedral group D_{18}.*

Proof Recall that the automorphism group of a Fermat cubic is the group $3^3\rtimes S_4$. Let G' be the image of G in S_4, generated by the permutation (234), and $K := G\cap 3^3$, generated by $h(s_0:s_1:s_2:s_3)=(s_0:\epsilon_3 s_1:\epsilon_3 s_2:\epsilon_3 s_3)$. The image of $N_{\mathrm{Aut}(S)}(G)$ is contained in $N_{S_4}((234))$, which is generated by (234) and (23) and isomorphic to S_3. Therefore, $N_{\mathrm{Aut}(S)}(G)$ is a subgroup of $3^3\rtimes S_3$ and admits a subgroup isomorphic to S_3.

The kernel of the projection $N_{\mathrm{Aut}(S)}(G)\to S_3$ is $3^3\cap N_{\mathrm{Aut}(S)}(G)=K$. Indeed, the conjugation of g via an element $\sigma^{a_0}\rho^{a_1}\theta^{a_2}\in 3^3$ is

$$c_{\sigma^{a_0}\rho^{a_1}\theta^{a_2}}(g)(s_0:s_1:s_2:s_3)=\left(s_0:\epsilon_3^{a_2-a_1+1}s_2:\epsilon_3^{-a_2}s_3:\epsilon_3^{a_1}s_1\right),$$

i.e., $3^3\cap N_{\mathrm{Aut}(S)}(G)=\{\sigma^{a_0}\rho^{a_1}\theta^{a_2}\in 3^3\,|\,a_1=a_2=0\}=K$. Since G is a subgroup of index 2 of $3\rtimes S_3$, we conclude that $N_{\mathrm{Aut}(S)}(G)=3\rtimes S_3\simeq D_{18}$. □

Type E_6. In the following tables, we list fixed points and invariant lines and the orbits that they cut on S.

Fixed points	$T_{p_i}S$	$T_{p_i}S\cap S$	Eckardt point
p_2	$t_3=0$	$t_0^3+t_1^3=0$	yes
p_3	$t_1=0$	$t_2^2t_3+t_0^3=0$	no

Invariant lines $l_{p_i p_j}$	$l_{p_i p_j} \cap S$	orbits in $l_{p_i p_j} \cap S$
$l_{p_0 p_1} = \{t_2 = t_3 = 0\}$	$t_0^3 + t_1^3 = 0$	orbit of length 3
$l_{p_0 p_2} = \{t_1 = t_3 = 0\}$	$t_0^3 = 0$	fixed Eckardt point p_2
$l_{p_0 p_3} = \{t_1 = t_2 = 0\}$	$t_0^3 = 0$	fixed point p_3
$l_{p_1 p_2} = \{t_0 = t_3 = 0\}$	$t_1^3 = 0$	fixed Eckardt point p_2
$l_{p_1 p_3} = \{t_0 = t_2 = 0\}$	$t_3^2 t_1 + t_1^3 = 0$	fixed point p_3 and orbit of length 2 given by: $q_1 := (0:i:0:1)$, $q_2 := (0:-i:0:1)$.
$l_{p_2 p_3} = \{t_1 = t_0 = 0\}$	$t_2^2 t_3 = 0$	fixed Eckardt point p_2 and fixed point p_3

Observe that the hypothesis of Lemma 4.2 (ii) holds for the orbit $\{q_1, q_2\}$. Indeed, the set $\{q_1, q_2\}$ is the only orbit of length 2 and q_i are not Eckardt points, since

$$T_{q_i} S \cap S = \left\{t_1 \pm i t_3 = t_2^2 t_3 + t_0^3 = 0\right\}$$

are cuspidal cubic curves. Moreover, the pencil of planes containing $\{q_1, q_2\}$ does not cut any conic on S and q_i is not contained in the tangent space of q_j, for $i \neq j$, by Remark 4.4. We conclude that S is not G-birationally superrigid and the group $\mathrm{Bir}^G(S)$ is generated by G-biregular automorphisms of S, a Bertini involution and a Geiser involution whose base loci are aligned: $\{q_1, q_2\}$ and p_3 respectively.

The Bertini involution with base points q_1 and q_2 is the deck transformation of the double cover

$$\begin{aligned} \psi : S &\to \mathbb{P}^3, \\ (t_0 : t_1 : t_2 : t_3) &\mapsto \left(t_1^2 + t_3^2 : t_0^2 : t_0 t_2 : t_2^2\right), \end{aligned}$$

and it is given explicitly by

$$\varphi_{q_1 q_2}(t_0 : t_1 : t_2 : t_3) = (t_0 : t_1' : t_2 : t_3'),$$

where

$$t_1' := -t_1 - \frac{2(t_1^2 + t_3^2) t_0^3}{t_2^4 + (t_1^2 + t_3^2)^2} \quad \text{and} \quad t_3' := -t_3 - \frac{2 t_2^2 t_0^3}{t_2^4 + (t_1^2 + t_3^2)^2}.$$

The Geiser involution with base point p_3 can be written as

$$\varphi_{p_3}(t_0 : t_1 : t_2 : t_3) = \left(t_0 : t_1 : t_2 : -t_3 - \frac{t_2^2}{t_1}\right).$$

Lemma 4.11 *The group* $\mathrm{Bir}^G(S)$ *is not a finite group.*

Proof The proof is analogous to the one of Lemma 4.9. It is enough to prove that the composition $\varphi_{p_3} \circ \varphi_{q_1q_2}$ has infinite order. Note that:

- for any $p \in S$ the point $\varphi_{p_3}(p)$ is aligned with p_3 and p;
- for any $p \neq p_3$, the points $\varphi_{q_1q_2}(p)$ and p belong to a conic contained in the plane $\Pi_{q_1q_2p}$, spanned by q_1, q_2 and p, and tangent to $S \cap \Pi_{q_1q_2p}$ at q_1 and q_2;
- the involutions φ_{p_3} and $\varphi_{q_1q_2}$ fix the pencil of cubic curves

$$C_{(\lambda:\mu)} = \left\{\lambda t_0 - \mu t_2 = t_3^2 t_1 + t_2^2 t_3 + t_1^3 + t_2^3 = 0\right\}.$$

Fix $(\lambda:\mu) \in \mathbb{P}^1_{(\lambda:\mu)}$ such that $C_{(\lambda:\mu)}$ is nonsingular and choose O an inflection point on $C_{(\lambda:\mu)}$. Due to the previous facts, the following relations for the elliptic curve $(C_{(\lambda:\mu)}, O)$ hold:

$$\begin{aligned} q_1 + q_2 + p_3 &= 0;\\ 2q_1 + 2q_2 + p + \varphi_{q_1q_2}(p) &= 0;\\ p_3 + \varphi_{q_1q_2}(p) + \varphi_{p_3} \circ \varphi_{q_1q_2}(p) &= 0. \end{aligned}$$

In particular,

$$\varphi_{p_3} \circ \varphi_{q_1q_2}(p) = p - 3p_3.$$

One can check (use MAGMA) that for a suitable choice of $(\lambda:\mu)$ (*e.g.* $(1:1)$), the point p_3 is not a torsion point. This implies that $\varphi_{p_3} \circ \varphi_{q_1q_2}$ has infinite order. □

We complete the list of generators of the group $\mathrm{Bir}^G(S)$, observing that the only G-biregular automorphisms of S are the elements of G itself. Note that up to a change of coordinates [5, 6.5, Case 3, Type III], we can suppose that S is given by the equation

$$s_0^3 + s_1^3 + s_2^3 + s_3^3 + 3(\sqrt{3} - 1)\, s_1 s_2 s_3 = 0$$

and a generator g of G acts via

$$\begin{aligned} &g(s_0:s_1:s_2:s_3)\\ &\quad = \left(\sqrt{3}\,\epsilon_3 s_0 : s_1 + s_2 + s_3 : s_1 + \epsilon_3 s_2 + \epsilon_3^2 s_3 : s_1 + \epsilon_3^2 s_2 + \epsilon_3 s_3\right). \end{aligned}$$

The automorphism group of S is $H_3(3) \rtimes 4$, where $H_3(3)$ is the Heisenberg group of unipotent 3×3-matrices over the finite field $\mathbb{F}_3$, generated by

$$\begin{aligned} \tilde{g}_1(s_0:s_1:s_2:s_3) &= (s_0:s_1:\epsilon_3 s_2:\epsilon_3^2 s_3),\\ \tilde{g}_2(s_0:s_1:s_2:s_3) &= (s_0:s_2:s_3:s_1) \end{aligned}$$

and 4 is the cyclic group generated by

$$\tilde{g}_4(s_0\!:\!s_1\!:\!s_2\!:\!s_3) = \left(\sqrt{3}\,s_0\!:\!s_1+s_2+s_3\!:\!s_1+\epsilon_3 s_2+\epsilon_3^2 s_3\!:\!s_1+\epsilon_3^2 s_2+\epsilon_3 s_3\right),$$

see [5, Theorem 6.14, Type III]. The group G is isomorphic to $3 \rtimes 4 \simeq 12$, where 3 is generated by $[\tilde{g}_1, \tilde{g}_2](s_0\!:\!s_1\!:\!s_2\!:\!s_3) = (\epsilon_3 s_0\!:\!s_1\!:\!s_2\!:\!s_3)$, *i.e.*, the centre of $H_3(3)$.

Lemma 4.12 *The group G is self-normalising in* $\mathrm{Aut}(S)$*, i.e., the normaliser of G in* $\mathrm{Aut}(S)$ *is G itself.*

Proof If $G \subsetneq N_{\mathrm{Aut}(S)}(G)$, then $[H_3(3), H_3(3)] = G \cap H_3(3) \subsetneq N_{\mathrm{Aut}(S)}(G) \cap H_3(3)$ as $\langle \tilde{g}_4 \rangle \subseteq G$. In particular, the image of $N_{\mathrm{Aut}(S)}(G) \cap H_3(3)$ via the quotient map $H_3(3) \to H_3(3)/[H_3(3), H_3(3)] \simeq 3^2$, generated by the image of $\tilde{g}_1$ and $\tilde{g}_2$, is non-trivial. Note that the element $\tilde{g}_4$ acts on $H_3(3)$ by conjugation via $(\tilde{g}_1, \tilde{g}_2) \to (\tilde{g}_2^2, \tilde{g}_1)$, see [5, Theorem 6.14, Type III]. As a result, we have

$$\begin{aligned} \tilde{g}_1^{-1}\tilde{g}_4\tilde{g}_1 &= \tilde{g}_1^{-1}\tilde{g}_2^{-1}\tilde{g}_4 \notin G, \\ \tilde{g}_2^{-1}\tilde{g}_4\tilde{g}_2 &= \tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_4 \notin G, \\ (\tilde{g}_1\tilde{g}_2)^{-1}\tilde{g}_4(\tilde{g}_1\tilde{g}_2) &= \tilde{g}_2^{-1}\tilde{g}_1^{-1}\tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_4 \notin G, \\ (\tilde{g}_1\tilde{g}_2^2)^{-1}\tilde{g}_4(\tilde{g}_1\tilde{g}_2^2) &= \tilde{g}_2\tilde{g}_1^{-1}\tilde{g}_2^{-1}\tilde{g}_1^{-1}\tilde{g}_4 \notin G. \end{aligned}$$

This implies that $N_{\mathrm{Aut}(S)}(G) \cap H_3(3)/[H_3(3), H_3(3)] = 1$, which yields a contradiction. We conclude that G is self-normalising in $\mathrm{Aut}(S)$. □

The results of this section are summarised in Theorem 1.4.

5 *G*-birational superrigidity of Del Pezzo surfaces of degree 2

In this section we prove Theorem 1.6 and we classify the Del Pezzo G-surfaces of degree 2 which are not G-birationally superrigid. Recall that a Del Pezzo surface S of degree 2 is a double cover of $\mathbb{P}^2$ branched over a nonsingular quartic curve. The surface S is a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ given by the equation

$$F = t_3^2 + F_4(t_0, t_1, t_2),$$

where F_4 is a polynomial of degree 4. The covering map $\nu\colon S \to \mathbb{P}^2$ is then given by the projection on the first three coordinates and the ramification curve R is the intersection of S with $\{t_3 = 0\}$.

As in the previous section, the proof of the Segre–Manin theorem (Theorem 3.1) implies that a minimal Del Pezzo G-surface of degree 2 is not G-birationally superrigid if and only if it admits a G-equivariant Bertini involution.

Lemma 5.1 *Let S be a Del Pezzo surface of degree 2. Then, a point p is the base locus of a Bertini involution if and only if p lies neither on a (-1)-curve nor on the ramification locus of the double cover $\nu\colon S \to \mathbb{P}^2$.*

Proof The proof is analogous to that of Lemma 4.2. Recall that a Del Pezzo surface of degree 2 is a blow-up of $\mathbb{P}^2$ at points $q_1, \ldots, q_7$ in general position, see [1, Exercise IV.8.(10).(a)]. We need to check that the blow-up $\widetilde{S}$ of S at p is a Del Pezzo surface, or equivalently that the seven points q_i and the image of p via the blow-down are in general position. We prove that if this is not the case, then p lies on a (-1)-curve or on the ramification locus. Indeed, note that the strict transform of a line passing through two of the points q_i or that of a conic through five of them or that of a singular cubic curve through seven of them, with one of the q_i at the singular point, is a (-1)-curve. Similarly, the strict transform of a singular cubic curve through all of the q_i, singular at p, is an anticanonical divisor, hence the pullback of a line via ν. Since this curve is singular at p, then p lies on the ramification locus.

Conversely, if p lies on a (-1)-curve, the canonical class of the blow-up S' of S at p has trivial intersection with the strict transform of the line, hence $-K_{S'}$ is not ample. On the other hand, if p lies on the ramification locus, then the preimage of the tangent line to the branch locus via ν is either an irreducible anticanonical divisor, singular only at p, *i.e.*, the strict transform of a singular cubic passing through q_i, or the union of two (-1)-curves, if the line is bitangent to the branch locus. □

Our strategy to identify birational superrigid G-surfaces will then consist in finding the fixed points of the given G-action and checking if these points lie on the ramification locus or on (-1)-curves. Recall that (-1)-curves on Del Pezzo surfaces of degree 2 are contained in the preimage of a bitangent line of the branched quartic in $\mathbb{P}^2$.

5.1 *G*-birational superrigidity for non-cyclic groups

The minimal non-cyclic groups G acting on S and fixing a point have been classified by Dolgachev and Duncan, the possible fixed points lie either on the ramification curve or they are the intersection of four (-1)-curves, see cases 2A and 2B of [4, Theorem 1.1]. Therefore, S is G-birationally superrigid by Theorem 3.1 and Lemma 5.1. This concludes the proof of Theorem 1.5. It remains to analyse cyclic groups.

5.2 *G*-birational superrigidity for cyclic groups

We describe the fixed locus of minimal cyclic groups G according to Dolgachev and Iskoviskikh classification. As before, we stick to their notation. Recall in particular that ϵ_n is a primitive n-th root of the unit and F_i is a polynomial of degree i.

Proposition 5.2 ([5, Section 6.6]) *Let $S = V(F)$ be a Del Pezzo surface of degree 2, endowed with a minimal action of a cyclic group G of automorphisms, generated by g. Then, one can choose coordinates in such a way that g and F are given in the following list:*

(i) A_1^7, *order* 2, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}t_2{:}-t_3)$,

$$F = t_3^2 + F_4(t_0, t_1, t_2);$$

(ii) $2A_3 + A_1$, *order* 4, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}it_2{:}t_3)$,

$$F = t_3^2 + t_2^4 + F_4(t_0, t_1);$$

(iii) $E_7(a_4)$, *order* 6, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}\epsilon_3 t_2{:}-t_3)$,

$$F = t_3^2 + t_2^3 F_1(t_0, t_1) + F_4(t_0, t_1);$$

(iv) $A_5 + A_1$, *order* 6, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}-t_1{:}\epsilon_3 t_2{:}-t_3)$,

$$F = t_3^2 + t_2^3 t_0 + t_0^4 + t_1^4 + a t_0^2 t_1^2;$$

(v) $D_6(a_2) + A_1$, *order* 6, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}\epsilon_3 t_1{:}\epsilon_3^2 t_2{:}-t_3)$,

$$F = t_3^2 + t_0(t_0^3 + t_1^3 + t_2^3) + t_1 t_2(\alpha t_0^2 + \beta t_1 t_2);$$

(vi) $E_7(a_2)$, *order* 12, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}it_1{:}\epsilon_3 t_2{:}t_3)$,

$$F = t_3^2 + t_0^4 + t_1^4 + t_0 t_2^3;$$

(vii) $E_7(a_1)$, *order* 14, $g(t_0{:}t_1{:}t_2{:}t_3) = (\epsilon_7 t_0{:}\epsilon_7^4 t_1{:}\epsilon_7^2 t_2{:}-t_3)$,

$$F = t_3^2 + t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0;$$

(viii) E_7, *order* 18, $g(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}\epsilon_3 t_1{:}\epsilon_9^2 t_2{:}-t_3)$,

$$F = t_3^2 + t_0^4 + t_0 t_1^3 + t_2^3 t_1.$$

We proceed with an analysis case by case.

Type A_1^7. The generator g is the standard Geiser involution of the surface S leaving the ramification curve $\{t_3 = F_4(t_0, t_1, t_2) = 0\}$ fixed. Hence, the surface is G-birationally superrigid.

Type $2A_3 + A_1$. The curve $S \cap \{t_2 = 0\}$ is fixed by the action of G. It is the preimage of the line $l = \{t_2 = 0\}$ under the double cover ν. The intersection of l with the branched quartic

$$C = \left\{t_2^4 + F_4(t_0, t_1) = 0\right\}$$

is simply given by $F_4(t_0, t_1) = 0$. Notice that the polynomial F_4 has four distinct roots as C is nonsingular, hence there are four distinct intersection points and l is not a bitangent line of C. This implies that every point in the preimage of l is the base locus of a Bertini involution with the exception of the preimages of the four points of intersection with C and of the points of intersection with the bitangent lines of C. In other words, $\mathrm{Bir}^G(S)$ is generated by G-automorphisms and infinitely many Bertini involutions, in particular S is not G-birationally superrigid.

To complete the list of generators of $\mathrm{Bir}^G(S)$ it suffices to compute the normaliser $N_{\mathrm{Aut}(S)}(G)$. Notice that up to a linear change of coordinates in the variables t_0, t_1, the equation F can be written as

$$F = t_3^2 + t_2^4 + t_0^4 + at_0^2t_1^2 + t_1^4.$$

The automorphism group $\mathrm{Aut}(S)$ depends on the parameter a and in each case we compute the normaliser $N_{\mathrm{Aut}(S)}(G)$ of G in $\mathrm{Aut}(S)$:

- if $a = 0$, then $\mathrm{Aut}(S) \simeq 2 \times (4^2 \rtimes S_3)$ (*cf.* [5, Theorem 6.17, Type II]), where 2 is generated by $\gamma(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}t_2{:}-t_3)$, the symmetric group S_3 is generated by the transpositions

$$\tau(t_0{:}t_1{:}t_2{:}t_3) = (t_1{:}t_0{:}t_2{:}t_3),$$
$$\sigma(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_2{:}t_1{:}t_3)$$

  and 4^2 is generated by

$$g_1(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}it_1{:}t_2{:}-t_3),$$
$$g_2(t_0{:}t_1{:}t_2{:}t_3) = \sigma g_1 \sigma(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}it_2{:}-t_3),$$

  subject to the following relations:

$$\tau g_2 \tau = g_2, \quad \tau g_1 \tau = g_1^{-1} g_2^{-1} = g_1^3 g_2^3.$$

  In particular, the group G is generated by $g = \gamma g_2$. Notice that $\langle g_2 \rangle$ is central in $4^2 \rtimes 2 = \langle \tau, g_1, g_2 \rangle$ and therefore it is central in $2 \times 4^2 \rtimes 2$. Since

$$(\sigma\tau) g (\sigma\tau)^{-1} = \gamma g_1 \notin G,$$
$$(\tau\sigma\tau) g (\tau\sigma\tau)^{-1} = (\tau\sigma) g (\tau\sigma)^{-1} = \gamma g_1^3 g_2^3 \notin G,$$

  we conclude that $N_{\mathrm{Aut}(S)}(G) = \langle \gamma, \tau, g_1, g_2 \rangle \simeq 2 \times 4^2 \rtimes 2$.
- if $a = \pm 2\sqrt{3}\, i$, then $\mathrm{Aut}(S) \simeq 2 \times 4A_4$ (*cf.* [5, Theorem 6.17, Type III]), where 2 is generated by $\gamma(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}t_2{:}-t_3)$ and $4A_4$ is a central extension of the alternating group A_4 generated by

$$g_1(t_0{:}t_1{:}t_2{:}t_3) = (t_1{:}t_0{:}t_2{:}-t_3),$$
$$g_2(t_0{:}t_1{:}t_2{:}t_3) = (it_1{:}-it_0{:}t_2{:}-t_3),$$
$$g_3(t_0{:}t_1{:}t_2{:}t_3) = \left(\epsilon_8^7 t_0 + \epsilon_8^7 t_1{:}\epsilon_8^5 t_0 + \epsilon_8 t_1{:}\sqrt{2}\,\epsilon_{12}{:}2\epsilon_6 t_3\right),$$
$$c(t_0{:}t_1{:}t_2{:}t_3) = (t_0{:}t_1{:}it_2{:}-t_3).$$

Since c is central in $\mathrm{Aut}(S)$ and $g = \gamma c$, we conclude that $N_{\mathrm{Aut}(S)}(G) = \mathrm{Aut}(S)$.

- if $a \neq 0, \pm 2\sqrt{3}\, i$, then $\mathrm{Aut}(S) \simeq 2 \times AS_{16}$, where AS_{16} is a non-abelian group of order 16 isomorphic to $2 \times 4 \rtimes 2$ (*cf.* [5, Tables 1 & 6]). The generators of $\mathrm{Aut}(S)$ coincide with that of the previous case with the exception of the generator g_3. Hence, as in the previous case, g is a central element and $N_{\mathrm{Aut}(S)}(G) = \mathrm{Aut}(S)$.

Type $E_7(a_4)$. The fixed locus is given by $S \cap \{t_2 = t_3 = 0\}$ and $(0{:}0{:}1{:}0)$. In particular all fixed points lie on the ramification curve and therefore they do not give rise to Bertini involutions, thus S is G-birationally superrigid.

Types $A_5 + A_1$, $D_6(a_2) + A_1$, E_7 **and** $E_7(a_1)$. The fixed locus of each of these groups is contained in the set

$$\big\{(1{:}0{:}0{:}0),\ (0{:}1{:}0{:}0),\ (0{:}0{:}1{:}0)\big\}$$

of points on the ramification curve, hence S does not admit any Bertini involution and it is G-birationally superrigid.

Type $E_7(a_2)$. The fixed locus consists of the point $(0{:}0{:}1{:}0)$ lying on the ramification curve and two points

$$p_1 = (1{:}0{:}0{:}i), \quad p_2 = (1{:}0{:}0{:}-i).$$

These points are mapped of the point $p = (1{:}0{:}0)$ via of the covering map ν. The branch locus is given by

$$C = \big\{t_0^4 + t_1^4 + t_0 t_2^3 = 0\big\}.$$

Suppose $q = (q_0{:}q_1{:}q_2)$ is a point in C whose tangent line

$$T_q C = \big\{(4q_0^3 + q_2^3)\, t_0 + 4q_1^3 t_1 + 3q_0 q_2^2 t_2 = 0\big\}$$

passes through p, then $q \in C \cap \{t_2^3 = -4t_0^3\}$. This intersection consists of 12 distinct points

$$\big(1{:}3^{1/4} i^j{:}4^{1/3}\epsilon_6^k\big), \quad j = 1, 2, 3, 4, \quad k = 1, 3, 5.$$

In other words, the lines tangent to C and passing through the possible q are given by

$$\big\{-2\cdot 3^{3/4} i^j t_1 + 3\cdot 2^{1/3}\epsilon_3^k t_2 = 0\big\}, \quad j = 1, 2, 3, 4, \quad k = 1, 2, 3,$$

which are pairwise distinct and intersect C in three distinct points each, and hence, are not bitangent lines. Therefore p_1 and p_2 are not in any (-1)-curve and it follows that S is not G-birationally superrigid.

The Bertini involution with base point p_1 is the deck transformation of the map

$$\begin{aligned} \psi_1 : S &\to \mathbb{P}^3, \\ (t_0{:}t_1{:}t_2{:}t_3) &\mapsto \big(t_1^2{:}t_1 t_2{:}t_2^2{:}t_3 - i t_0^2\big), \end{aligned}$$

and it is given explicitly by

$$\varphi_{p_1}(t_0 \colon t_1 \colon t_2 \colon t_3) = (t_0' \colon t_1 \colon t_2 \colon t_3'),$$

where

$$t_0' = -t_0 + \frac{i t_2^3}{2(t_3 - i t_0^2)} \quad \text{and} \quad t_3' = -i t_0^2 - \frac{t_1^4}{t_3 - i t_0^2} - \frac{i t_2^6}{4(t_3 - i t_0^2)^2}.$$

Similarly, the involution with base point p_2 is the deck transformation of the map

$$\begin{aligned} \psi_2 \colon S &\to \mathbb{P}^3, \\ (t_0 \colon t_1 \colon t_2 \colon t_3) &\mapsto \left(t_1^2 \colon t_1 t_2 \colon t_2^2 \colon t_3 + i t_0^2\right), \end{aligned}$$

therefore

$$\varphi_{p_2}(t_0 \colon t_1 \colon t_2 \colon t_3) = (t_0' \colon t_1 \colon t_2 \colon t_3'),$$

where

$$t_0' = -t_0 - \frac{i t_2^3}{2(t_3 + i t_0^2)} \quad \text{and} \quad t_3' = i t_0^2 - \frac{t_1^4}{t_3 + i t_0^2} + \frac{i t_2^6}{4(t_3 + i t_0^2)^2}.$$

Lemma 5.3 *The group* $\mathrm{Bir}^G(S)$ *is not a finite group.*

Proof The proof is analogous to the one of Lemmas 4.9 and 4.11. It is enough to prove that the composition $\varphi_{p_1} \circ \varphi_{p_2}$ has infinite order. To this aim, note that the involutions φ_{p_1} and φ_{p_2} fix the pencil of curves of genus one

$$C_{(\lambda:\mu)} = \left\{\lambda t_1 - \mu t_2 = t_3^2 + t_0^4 + t_1^4 + t_0 t_2^3 = 0\right\}.$$

In particular, for a general choice of $(\lambda \colon \mu)$, we have

$$C_{(\lambda:\mu)} = \left\{\lambda t_1 - \mu t_2 = \mu^3(t_3^2 + t_0^4 + t_1^4) + \lambda^3 t_0 t_1^3 = 0\right\} \subseteq \mathbb{P}(1,1,2)_{(t_0:t_1:t_3)}.$$

In the chart $\{s_1 := t_1 - r_0 t_0 \neq 0\}$, where r_0 is a root of the polynomial $F_{(\lambda:\mu)}(t_1) = \mu^3(1 + t_1^4) + \lambda^3 t_1^3$, the affine curve $C^\circ_{(\lambda:\mu)} := C_{(\lambda:\mu)} \cap \{s_1 \neq 0\}$ is the zero locus of a cubic equation in $\mathbb{C}^2$, and $C_{(\lambda:\mu)}$ is birational (thus isomorphic) to the (nonsingular) projective closure of $C^\circ_{(\lambda:\mu)}$ in $\mathbb{P}^2$ with coordinates $(t_0 \colon s_1 \colon t_3)$. Hence, we can identify the two curves. Let $p \in C^\circ_{(\lambda:\mu)}$. By restricting the linear system defining the double cover ψ_i to $C_{(\lambda:\mu)} \subseteq \mathbb{P}^2_{(t_0:s_1:t_3)}$, one can check that the points p_i and $\varphi_{p_i}(p)$ are contained in a conic, tangent to $C_{(\lambda:\mu)}$ at p_i and $O := (0:0:1)$. As in Lemmas 4.9 and 4.11, we deduce the following relations for the elliptic curve $(C_{(\lambda:\mu)}, O) \subseteq \mathbb{P}^2$:

$$\begin{aligned} p_1 + p_2 &= 0; \\ 2p_2 + p + \varphi_{p_2}(p) &= 0; \\ 2p_1 + \varphi_{p_2}(p) + \varphi_{p_1} \circ \varphi_{p_2}(p) &= 0. \end{aligned}$$

In particular,

$$\varphi_{p_1} \circ \varphi_{p_2}(p) = p + 4p_2.$$

One can check (use MAGMA) that for a suitable choice of $(\lambda : \mu)$ (*e.g.* $2\lambda^3 + 17\mu^3 = 0$ and $r_0 = 1/2$), the point p_2 is not a torsion point. This implies that $\varphi_{p_1} \circ \varphi_{p_2}$ has infinite order. □

The automorphism group of S is $\mathrm{Aut}(S) = 2 \times 4A_4$, see [5, Table 6, Theorem 6.17, Type III]. Here $4A_4$ is a non-split central extension of A_4 by a cyclic group of order 4, more explicitly there exists an exact sequence

$$0 \to 4 \to 4A_4 \to A_4 \to 0.$$

Let G' be the image of G in A_4 under the composition of quotient homomorphisms $2 \times 4A_4 \to 4A_4 \to A_4$. Notice $G' \simeq 3$, since G' is necessarily a cyclic group of A_4 whose order is a multiple of 3. It follows the image of $N_{2\times 4A_4}(G)$ is contained in $N_{A_4}(3)$. Moreover, notice that $N_{A_4}(3) = 3$, as there are no proper normal subgroups in A_4 containing 3 and 3 is not normal in A_4. Finally, since $2 \times 4A_4$ is a central extension of a central extension of A_4, one obtains $N_{2\times 4A_4}(12) = 2 \times 12$. The group $\mathrm{Bir}^G(S)$ is generated by G, the standard Geiser involution γ and two Bertini involutions with base locus p_1 and p_2 respectively.

The cases above yield the proof of Theorem 1.6.

Acknowledgements We would like to express our gratitude to Ivan Cheltsov for suggesting the problem at the summer school on *Rationality, Stable Rationality and Birationally Rigidity of Complex Algebraic Varieties* held in Udine in September 2017. Most of the lemmas in Sect. 3 were solved during the exercise classes in Udine. We would like to thank Fabio Bernasconi, Damián Gvirtz, Costya Shramov and Christian Urech for useful discussions. We would like to thank also our advisors Vladimir Guletskiĭ and Paolo Cascini. We are grateful to the anonymous referee for his/her helpful suggestions.

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