CrossMark

Geometry of the Wiman–Edge pencil, I: algebro-geometric aspects

Igor Dolgachev[1] · Benson Farb[2] · Eduard Looijenga[3]

In memory of William L. Edge

Abstract In 1981 William L. Edge discovered and studied a pencil $\mathcal{C}$ of highly symmetric genus 6 projective curves with remarkable properties. Edge's work was based on an 1895 paper of Anders Wiman. Both papers were written in the satisfying style of 19th century algebraic geometry. In this paper and its sequel *Geometry of the Wiman–Edge pencil, II: hyperbolic, conformal and modular aspects* (in preparation), we consider $\mathcal{C}$ from a more modern, conceptual perspective, whereby explicit equations are reincarnated as geometric objects.

✉ Eduard Looijenga
   E.J.N.Looijenga@uu.nl; Eduard@math.tsinghua.edu.cn

   Igor Dolgachev
   idolga@umich.edu

   Benson Farb
   bensonfarb@gmail.com

[1] Department of Mathematics, University of Michigan, 525 East University Av., Ann Arbor,
    MI 48109-1109, USA

[2] Department of Mathematics, University of Chicago, 5734 University Av., Chicago,
    IL 60637-1514, USA

[3] Yau Mathematical Sciences Center, Tsinghua University, Hai Dian District, Beijing 100084,
    People's Republic of China

Contents

1 Introduction

The purpose of this paper is to explore what we call the *Wiman–Edge pencil* $\mathcal{C}$, a pencil of highly symmetric genus 6 projective curves with remarkable properties. The smooth members of $\mathcal{C}$ can be characterized (see Theorem 3.3) as those smooth genus 6 curves admitting a nontrivial action of alternating group $\mathfrak{A}_5$. This mathematical gem was discovered in 1981 by William L. Edge [10,11], who based his work on a discovery in 1895 by Anders Wiman of a non-hyperelliptic curve C_0 of genus 6 (now called the *Wiman curve*) with automorphism group isomorphic to the symmetric group $\mathfrak{S}_5$. The results of both Wiman and Edge are in the satisfying style of 19th century mathematics, with results in terms of explicit equations.

In this paper and its sequel [12], we consider $\mathcal{C}$ from a more modern, conceptual perspective, whereby explicit equations are reincarnated as geometric objects. In particular we will view $\mathcal{C}$ through a variety of lenses: from algebraic geometry to representation theory to hyperbolic and conformal geometry. We will address both modular and arithmetic aspects of $\mathcal{C}$, partly through Hodge theory. Given the richness and variety of structures supported by $\mathcal{C}$, we can say in hindsight that the Wiman–Edge

pencil deserved such a treatment, and indeed it seems odd that this had not happened previously.

In the introduction of his *Lectures on the Icosahedron* [18], Felix Klein writes: "A special difficulty, which presented itself in the execution of my plan, lay in the great variety of mathematical methods entering in the theory of the Icosahedron." We believe that this is still very much true today, in ways Klein probably never anticipated. This, by the way, is followed by the sentence: "On this account it seemed advisable to take granted no specific knowledge in any direction, but rather to introduce, where necessary, such explanations . . ." While we are writing only about one aspect of Klein's book, in this paper we have tried to take Klein's advice seriously.

The Wiman–Edge pencil

As mentioned above, the story starts with Wiman [24], who, while classifying algebraic curves of genus $g = 4, 5$ and 6 whose automorphisms group contains a simple group, discovered a curve C_0 of genus 6 with automorphism group isomorphic to the symmetric group $\mathfrak{S}_5$. On the last page of his paper, Wiman gives the equation of a 4-nodal plane sextic birationally isomorphic to W:

$$2 \sum x^4 yz + 2 \sum x^3 y^3 - 2 \sum x^4 y^2 + \sum x^3 y^2 z + \cdots - 6x^2 y^2 z^2 = 0,$$

He reproduces this equation on p. 208 of his later paper [25], related to the classification of finite subgroups of the plane Cremona group. Wiman states there that the group of birational automorphisms of C_0 is generated by a group of projective transformations isomorphic to the symmetric group $\mathfrak{S}_4$ together with the standard quadratic birational involution with fundamental points at its three singular points.

Wiman erroneously claims that his curve is the unique non-hyperelliptic curve of genus 6 whose automorphism group contains a non-cyclic simple group, the alternating group $\mathfrak{A}_5$ in his case. This mistake was corrected almost a hundred years later by Edge [10,11], who placed W inside a pencil

$$\lambda P(x, y, z) + \mu Q(x, y, z) = 0$$

of 4-nodal plane sextics each of whose members admits a group of automorphisms isomorphic to $\mathfrak{A}_5$. In projective coordinates different from the one chosen by Wiman, the pencil is generated by the curves defined by the homogeneous equations:

$$P(x, y, z) = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0$$

and

$$Q(x, y, z) = x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2 y^2 z^2 = 0,$$

where the Wiman curve W corresponds to the parameters $(\lambda : \mu) = (0 : 1)$.
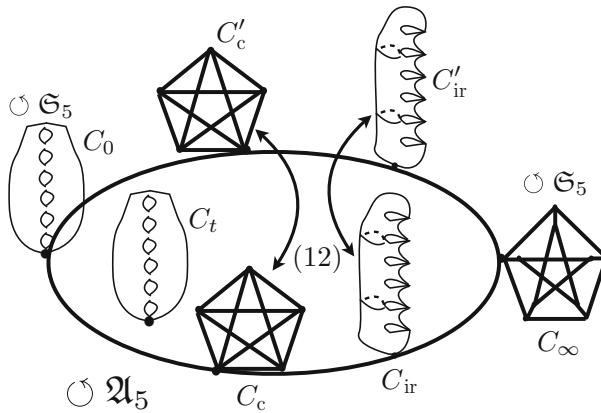
Fig. 1 A schematic of the Wiman–Edge pencil $\mathcal{C}$, consisting of genus 6 projective curves containing the alternating group $\mathfrak{A}_5$ in their automorphism group. The generic curve $C_t \in \mathcal{C}$ has $\mathrm{Aut}(C_t) \cong \mathfrak{A}_5$, while there is a unique smooth member $C_0 \in \mathcal{C}$, the *Wiman sextic*, with more symmetry: $\mathrm{Aut}(C_0) \cong \mathfrak{S}_5$. There are five singular members of $\mathcal{C}$: two irreducible curves, 6-noded rational curves C_{ir} and C'_{ir}; two curves C_{c} and C'_{c}, each consisting of five conics whose intersection graph is the complete graph on five vertices; and a union C_∞ of 10 lines whose intersection graph is the Petersen graph. The group $\mathfrak{S}_5$ acts on $\mathcal{C}$ with $\mathfrak{A}_5$ leaving each member of $\mathcal{C}$ invariant. This action has two $\mathfrak{S}_5$-invariant members, C_0 and C_∞, while any odd permutation switches C_{ir} with C'_{ir} and C_{c} with C'_{c}

Edge showed that it is more natural to view the pencil as a pencil $\mathcal{C}$ of curves on a quintic del Pezzo surface S obtained by blowing up the four base points of the pencil. In this picture, the lift C_0 to S of Wiman's curve W is, as Edge discovered, "a uniquely special canonical curve of genus 6" on S. That is, the standard action of $\mathfrak{S}_5$ on the quintic del Pezzo surface S permutes the 1-parameter family of smooth genus 6 curves on S but leaves invariant a unique such curve, namely the Wiman sextic C_0.

The action of $\mathfrak{S}_5$ on S induces an action of $\mathfrak{S}_5$ on $\mathcal{C}$, whereby the subgroup $\mathfrak{A}_5$ leaves each member of the pencil $\mathcal{C}$ invariant, and acts faithfully on each curve by automorphisms. In addition to C_0, the pencil $\mathcal{C}$ has precisely one other $\mathfrak{S}_5$-fixed member, a reducible curve that is a union of 10 lines intersecting in the pattern of the Petersen graph (see Fig. 2), with the $\mathfrak{S}_5$-action on this union inducing the standard $\mathfrak{S}_5$-action on the Petersen graph. The other four singular members of $\mathcal{C}$ come in pairs, the curves in each pair being switched by any odd permutation. A schematic of $\mathcal{C}$, and the $\mathfrak{S}_5$-action on it, is given (with explanation) in Fig. 1.

Since Edge's discovery, the pencil $\mathcal{C}$, which we propose to call the *Wiman–Edge pencil*, has appeared in several modern research papers. For example, its nonsingular members have been identified with the quotient of one of the two 1-parameter families of lines on a nonsingular member of the Dwork pencil of Calabi–Yau quintic threefolds:

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5\phi x_1 x_2 x_3 x_4 x_5 = 0$$

by the group of order 125 of obvious symmetries of the Dwork pencil (see [3,27]; we elaborate a bit on the connection in the present paper).The Wiman–Edge occupies a prominent place in the monograph [4] on birational geometry of algebraic varieties

with $\mathfrak{A}_5$-symmetry. It also appears in a recent posting by Cheltsov–Kuznetsov–Shramov [5].

This paper

The purpose of the present paper is two-fold. First, we reprove all previously known facts about the Wiman–Edge pencil that can be found in Edge's papers. Instead of computations based on the pencil's equation (used also in a later paper [14] whose authors it seems were not aware of Edges's work), our proofs rely on the representation theory of the symmetry group of the pencil, and also on the moduli-theoretical interpretation of the quintic del Pezzo surface. Although our approach is not new, and has been used also in [4] or [16], respectively, we would like to believe that our methods are more conceptual and geometric.

The other goal of the paper is to answer some natural questions that arise while one gets familiar with the pencil. Thus we give a purely geometric proof of the uniqueness of the Wiman–Edge pencil as an $\mathfrak{A}_5$-invariant family of stable curves of genus 6, and in particular, the uniqueness of the Wiman curve W as a non-hyperelliptic curve of genus 6 with the group of automorphisms isomorphic to $\mathfrak{S}_5$. In fact, we give two different proofs of this result: an algebraic geometrical one given in Theorem 3.3 below, and a second one in the sequel [12] that is essentially group-theoretical and topological.

Secondly, we give a lot of attention to the problem only barely mentioned by Edge and later addressed in the paper [20]: describe an $\mathfrak{A}_5$-equivariant projection of the Wiman–Edge pencil to a Klein plane realizing an irreducible 2-dimensional projective representation of $\mathfrak{A}_5$. It reveals a natural relation between the Wiman–Edge pencil and the symmetry of the icosahedron along the lines of Klein's book [18]. In particular, we relate the singular members of the pencil with some attributes of the geometry of the Clebsch diagonal cubic surface as well with some rational plane curves of degree 6 and 10 invariant with respect to a projective group of automorphisms isomorphic to $\mathfrak{A}_5$ which were discovered by Roy Winger [26].

In the sequel [12] we will discuss some other aspects of the Wiman paper related to hyperbolic geometry, the moduli space of curves and Shimura curves.

Section-by-section outline of this paper

Section 2 collects a number of mostly known facts regarding quintic del Pezzo surfaces, but with emphasis on naturality, so that it is straightforward to keep track of how the automorphism group of such a surface acts on the vector spaces associated to it. We also recall the incarnation of such a surface as the Deligne–Knudsen–Mumford compactification $\overline{\mathcal{M}}_{0,5}$ and mention how some its features can be recognized in either description. Perhaps new is Lemma 2.1 and its use to obtain Proposition 2.3.

Section 3 introduces the principal object of this paper, the Wiman–Edge pencil $\mathcal{C}$. Our main result here is that its smooth fibres define the universal family of genus 6 curves endowed with a faithful $\mathfrak{A}_5$-action. We also determine the singular fibers of $\mathcal{C}$. Each of these turns out to be a stable curve. Indeed, the whole pencil consists of all the stable genus 6 curves endowed with a faithful $\mathfrak{A}_5$-action that can be smoothed

as an $\mathfrak{A}_5$-curve. We further prepare for the next two sections by describing $\mathfrak{S}_5$-orbits in $\overline{\mathcal{M}}_{0,5}$, thus recovering a list due to Coble. This simplifies considerably when we restrict to the $\mathfrak{A}_5$-action and this what we will only need here.

The following two sections concern the projection to a Klein plane. These are $\mathfrak{A}_5$-equivariant projections as mentioned above. Section 4 concentrates on the global properties of this projection, and proves among other things that the ramification curve of such a projection is in fact a singular member of the Wiman–Edge pencil. It is perhaps worthwhile to point out that we prove this using the Thom–Boardman formula for cusp singularities. This is the only instance we are aware of where such a formula for a second order Thom–Boardman symbol is used to prove an algebro-geometric property. We also show that the image of the Wiman–Edge pencil in a Klein plane under the projection is no longer a pencil, but a curve of degree 5 with two singular points.

As its title makes clear, Sect. 5 focusses on the images in the Klein plane of special members of the Wiman–Edge pencil. We thus find ourselves suddenly staring at a gallery of planar representations of degree 10 of genus 6 curves with $\mathfrak{A}_5$-action, probably all known to our predecessors in the early 20th century if not earlier. Among them stand out (what we have called) the *Klein decimic* and the *Winger decimic*. Less exciting perhaps at first is the case of a conic with multiplicity 5; but this turns out to be the image of a member of the Wiman–Edge pencil which together with its $\mathfrak{S}_5$-conjugate is characterized by possessing a pencil of even theta characteristics.

In the final section, Sect. 6, we look at the $\mathfrak{S}_5$-orbit space of $\overline{\mathcal{M}}_{0,5}$ (which is just the Hilbert–Mumford compactification of the space of binary quintics given up to projective equivalence) and make the connection with the associated invariant theory.

Conventions

Throughout this paper the base field is $\mathbb{C}$ and S stands for a quintic del Pezzo surface (the definition is recalled in Sect. 2). The canonical line bundle Ω^n_M of a complex n-manifold M will be often denoted by ω_M. As this is also the dualizing sheaf for M, we use the same notation if M is possibly singular, but has such a sheaf (we only use this for curves with nodes).

For a vector space V, let $\mathbb{P}(V)$ denote the projective space of 1-dimensional subspaces of V and $\check{\mathbb{P}}(V) = \mathbb{P}(V^\vee)$ denotes the projective space of hyperplanes of V. We write $\mathrm{Sym}^d V$ for dth symmetric power of the vector space V. For a space or variety X we denote by $\mathrm{Sym}^d X$ the quotient of X^d by the permutation group $\mathfrak{S}_d$.

2 Del Pezzo surfaces of degree 5

2.1 A brief review

Here we recall some known facts about quintic del Pezzo surfaces, i.e., del Pezzo surfaces of degree 5, that one can find in many sources (such as [8]). By definition a *del Pezzo surface* is a smooth projective algebraic surface S with ample anticanonical bundle ω_S^{-1}. For such a surface S, the first Chern class map $\mathrm{Pic}(S) \rightarrow H^2(S; \mathbb{Z})$ is

an isomorphism. Denoting by $K_S \in H^2(S; \mathbb{Z})$ the canonical class of S, i.e., the first Chern class of ω_S (and hence minus the first Chern class of the tangent bundle of S), the self-intersection number $d = (-K_S)^2 = K_S^2$ is called the *degree* of S. With the exception of surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (which are del Pezzo surfaces of degree 8), a del Pezzo surface of degree d admits a birational morphism $\pi : S \to \mathbb{P}^2$ whose inverse is the blow-up of $9 - d$ distinct points (so we always have $d \leqslant 9$) satisfying some genericity conditions. But beware that when $d \leqslant 6$, there is more that one way to contract curves in S that produce a copy of $\mathbb{P}^2$.

When $d = 5$ (which we assume from now on), these genericity conditions amount to having no three (of the four) points lie on a line so that we can adapt our coordinates on $\mathbb{P}^2$ such that the points in question, after having them numbered (p_1, p_2, p_3, p_4), are the vertices of the coordinate system:

$$p_1 = (1{:}0{:}0), \quad p_2 = (0{:}1{:}0), \quad p_3 = (0{:}0{:}1), \quad p_4 = (1{:}1{:}1).$$

This shows that any two quintic del Pezzo surfaces are isomorphic and also proves that the automorphism group of S contains the permutation group $\mathfrak{S}_4$. The complete linear system $|-K_S| := \mathbb{P}(H^0(S, \omega_S^{-1}))$ defined by nonzero sections of the dual of the canonical line bundle of S is then the strict transform of the linear system in $\mathbb{P}^2$ of plane cubic curves passing through these points in the sense that a general member is the strict transform of such a cubic. This is a linear system of dimension $9 - 4 = 5$ and gives an embedding in a 5-dimensional projective space

$$S \hookrightarrow \mathbb{P}_S := \check{\mathbb{P}}(H^0(S, \omega_S^{-1}))$$

with image a surface of degree 5. We call S, thus embedded in a projective space, an *anticanonical model* of S.

The automorphism group of S is, however, bigger than $\mathfrak{S}_4$: the embedding $\mathfrak{S}_4 \hookrightarrow \mathrm{Aut}(S)$ extends to an isomorphism

$$\mathfrak{S}_5 \cong \mathrm{Aut}(S)$$

given by assigning to the transposition (45) the lift of the Cremona transformation

$$(t_0{:}t_1{:}t_2) \mapsto (t_0^{-1}{:}t_1^{-1}{:}t_2^{-1}),$$

which indeed lifts to an involution of S (see [8, Theorem 8.5.8]).

The image of a nonsingular rational curve on S with self-intersection number -1 (in other words, an *exceptional curve of the first kind*) is a line on the anticanonical model, and any such line is so obtained. Any line on the quintic del Pezzo surface S is either the strict transform of a line through p_i and p_j, $1 \leqslant i < j \leqslant 4$, or the preimage of p_i, $i = 1, \ldots, 4$, so that there are $\binom{4}{2} + 4 = 10$ lines on S.

The strict transform $\mathcal{P}_i$ of the pencil of lines through p_i makes a pencil $\mathcal{P}_i$ of rational curves on S whose members are pairwise disjoint: they are the fibers of a morphism from S to a projective line. The strict transform of the conics through $p_1, \ldots, p_4$ makes up a pencil $\mathcal{P}_5$ with the same property. The intersection number of a member

Fig. 2 The Petersen graph $\mathcal{G}$. This is the graph with vertices corresponding to 2-element subsets of $\{1, 2, 3, 4, 5\}$, with two vertices $\{i, j\}$ and $\{k, \ell\}$ connected by an edge precisely when $\{i, j\} \cap \{k, \ell\} = \varnothing$. The graph $\mathcal{G}$ is the intersection graph of the 10 lines on the quintic del Pezzo surface, with a vertex for each line and an edge connecting vertices corresponding to lines with nontrivial intersection
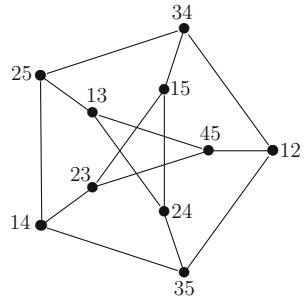
of $\mathcal{P}_i$ with $-K_S$ is equal to the intersection number of a cubic with a line in $\mathbb{P}^2$ minus 1 if $i \neq 5$, resp. cubic with a conic minus 4 if $i = 5$, hence is equal to 2. This is therefore also a conic in the anticanonical model, and that is why we refer to $\mathcal{P}_i$ as a *pencil of conics* and call the corresponding morphism from S to a projective line a *conic bundle*.

Every pencil of conics has exactly three singular fibers; these are unions of two different lines. For example, the pencil $\mathcal{P}_i$, $i \neq 5$, has singular members equal to the pre-images on S of the lines $\overline{p_i p_j}$ joining the point p_i with the point p_j, $j \neq i$. The singular members of pencil $\mathcal{P}_5$ are proper transforms of the pairs of lines $\overline{p_i p_j} + \overline{p_k p_l}$ with all indices distinct.

It follows that each of 10 lines on S is realized as an irreducible component of exactly three different pencils of conics. This allows one to label it with a 2-element subset of $\{1, \ldots, 5\}$, being the complement of the corresponding 3-element subset. We thus obtain a bijection between the set of 10 lines and the collection of 2-element subsets of $\{1, \ldots, 5\}$, with two lines intersecting if and only if the associated 2-element subsets are disjoint, or equivalently, when the two lines constitute a singular member of one of the five pencils of conics. Thus the *intersection graph* $\mathcal{G}$ of the set lines, i.e., the graph whose vertex set is the set of lines and in which two vertices are connected by an edge if and only if the associated lines intersect, is the Petersen graph (Fig. 2). We see here $\mathrm{Aut}(S)$ also represented as the automorphism group of $\mathcal{G}$.

The edges of the Petersen graph represent reducible conics on S, and there are indeed $3 \times 5 = 15$ of them. The three reducible conics of a conic bundle are then represented by three disjoint edges whose indices are three 2-element subsets of a subset of $\{1, 2, 3, 4, 5\}$ of cardinality 4. If we regard the missing item as a label for the conic pencil, we see that $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ is realized as the full permutation group of the set of conic pencils.

2.2 The modular incarnation

A quintic del Pezzo surface has the—for us very useful—incarnation as the Deligne–Mumford moduli space of stable 5-pointed rational curves, $\overline{\mathcal{M}}_{0,5}$, see [17]. In the present case, this is also the Hilbert–Mumford (or GIT) compactification of $\mathcal{M}_{0,5}$, which means that a point of $\overline{\mathcal{M}}_{0,5}$ is uniquely represented up to automorphism by a *smooth* rational curve C and a 5-tuple $(x_1, \ldots, x_5) \in C^5$ for which the divisor

$\sum_{i=1}^{5}(x_i)$ has all its multiplicities $\leqslant 2$. One of the advantages of either model is that the $\mathfrak{S}_5$-action is evident. The 10 lines appear as the irreducible components of the boundary $\partial\overline{\mathcal{M}}_{0,5} = \overline{\mathcal{M}}_{0,5} \smallsetminus \mathcal{M}_{0,5}$ of the compactification: these are the loci defined by $x_i = x_j$, where $\{i, j\}$ is a 2-element subset of $\{1, \ldots, 5\}$. We thus recover the bijection between the set of 10 lines and the collection of 2-element subsets of $\{1, \ldots, 5\}$.

The conic bundles also have a modular interpretation, namely as forgetful maps: if a point of $\overline{\mathcal{M}}_{0,5}$ is represented by a Deligne–Mumford stable curve $(C; x_1, \ldots, x_5)$, then forgetting x_i, $i = 1, \ldots, 5$, followed by contraction unstable components (and renumbering the points by $\{1, 2, 3, 4\}$ in an order preserving manner) yields an element of $\overline{\mathcal{M}}_{0,4}$. There is a similar definition in case a point of $\overline{\mathcal{M}}_{0,5}$ is represented in a Hilbert–Mumford stable manner, but beware that a Hilbert–Mumford stable representative of $\overline{\mathcal{M}}_{0,4}$ is given uniquely up to isomorphism by a 4-tuple $(x_1, \ldots, x_4) \in C^4$ for which the divisor $\sum_{i=1}^{4}(x_i)$ is either reduced or twice a reduced divisor; if for example $x_1 = x_2$, then we also require that $x_3 = x_4$ (but $x_2 \neq x_3$). The moduli space $\overline{\mathcal{M}}_{0,4}$ is a smooth rational curve and $\partial\overline{\mathcal{M}}_{0,4} = \overline{\mathcal{M}}_{0,4} \smallsetminus \mathcal{M}_{0,4}$ consists of three points indexed by the three ways we can partition $\{1, 2, 3, 4\}$ into two 2-element subsets. The resulting morphism $f_i \colon \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ represents $\mathcal{P}_i$ and over the three points of $\partial\overline{\mathcal{M}}_{0,4}$ lie the three singular fibres.

Let us focus for a moment on $\mathcal{P}_5$, in other words, on $f_5 \colon \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$. Each x_i, $i = 1, \ldots, 4$, then defines a section $\overline{\mathcal{M}}_{0,4} \to \overline{\mathcal{M}}_{0,5}$ of f_5, and f_5 endowed with these four sections can be understood as the "universal stable 4-pointed rational curve". Note that the images of these sections are irreducible components of $\partial\overline{\mathcal{M}}_{0,5}$ and hence lines in the anticanonical model. They are indexed by the unordered pairs $\{i, 5\}$, $i = 1, \ldots, 4$. The singular fibers of f_5 are those over the 3-element set $\partial\overline{\mathcal{M}}_{0,4}$; each such fiber is a union of two intersection lines, so that in this way all 10 lines are accounted for.

The $\mathfrak{S}_5$-stabilizer of the conic bundle defined by f_5 is clearly $\mathfrak{S}_4$. Let us take a closer look at how $\mathfrak{S}_4$ acts on f_5. First observe that every smooth fiber of f_5 (i.e., a strict transform of a smooth conic) meets every fiber of every f_i, $i \neq 5$, with multiplicity one. We also note that $\mathfrak{S}_4$ acts on the 3-element set $\partial\overline{\mathcal{M}}_{0,4}$ as its full permutation group, the kernel being the Klein Viergruppe. So the Viergruppe acts trivially on $\overline{\mathcal{M}}_{0,4}$, but its action on the universal pointed rational curve $\overline{\mathcal{M}}_{0,5}$ is of course faithful. The homomorphism $\mathfrak{S}_4 \to \mathfrak{S}_3$ can also be understood as follows. If we are given a smooth rational curve C endowed and a 4-element subset $X \subset C$, then the double cover $\widetilde{C}$ of C ramified at X is a smooth genus 1 curve. If we enumerate the points of X by $X = \{x_1, x_2, x_3, x_4\}$, then we can choose x_1 as origin, so that $(\widetilde{C}, x_1)$ becomes an elliptic curve. This then makes $\{x_1, x_2, x_3\}$ the set of elements of $(\widetilde{C}, x_1)$ of order 2. Thus $\mathcal{M}_{0,4}$ can be regarded as the moduli space of elliptic curves endowed with a principal level 2 structure. This also suggests that we should think of X as an affine plane over $\mathbb{F}_2$; for this interpretation the three boundary points of $\overline{\mathcal{M}}_{0,4}$ are cusps, and correspond to the three directions in this plane. We now can think of $\mathfrak{S}_4$ as the affine group of X, the Viergruppe as its translation subgroup, and the quotient $\mathfrak{S}_3$ as the projective linear group $\mathrm{PGL}_2(\mathbb{F}_2)$.

Choose an affine coordinate z for the base of $\overline{\mathcal{M}}_{0,4}$ such that $\partial\overline{\mathcal{M}}_{0,4}$ is the root set of $z^3 = 1$. Then the full permutation group of $\partial\overline{\mathcal{M}}_{0,4}$ is the group of Möbius

transformations consisting of the rotations $z \mapsto \zeta z$ and the involutions $z \mapsto \zeta/z$, with $\zeta^3 = 1$. From this we see that this action has three irregular orbits: two 3-element orbits: besides the root set of $z^3 = 1$ which is $\partial \overline{\mathcal{M}}_{0,4}$, the root set of $z^3 = -1$, and one 2-element orbit: $\{0, \infty\}$. The root set of $z^3 = -1$ is represented by $\mathbb{P}^1$ with an enumeration of the 4th roots of unity in $\mathbb{C} \subset \mathbb{P}^1$, and the third orbit is given by $\{0, \infty\}$.

We will be more concerned with the last orbit that gives rise to two *special conics* in the pencil. So every conic bundle has two special conics as fibers and $\mathfrak{S}_5$ permutes these 10 special conics transitively. Two distinct special conics intersect unless they are in the same pencil. The two points 0, ∞ have the same $\mathfrak{S}_3$-stabilizer in $\mathbb{P}^1$ of order 3. In the action of $\mathfrak{S}_4$ on the total space of the pencil, the two special conics are fixed by the subgroup $\mathfrak{A}_4$. It follows from the preceding that these two fibers define the locus represented by $(x_1, \ldots, x_5) \in C^5$ (with C a smooth rational curve) for which there exists an affine coordinate z on C such that $\{z(x_1), \ldots, z(x_4)\}$ is the union of $\{\infty\}$ and the root set of $z^3 = 1$. Our discussion also shows that the subgroup $\mathfrak{A}_5$ has two orbits in that set, with each orbit having exactly one special conic in each conic bundle.

2.3 Representation spaces of $\mathfrak{S}_5$

In what follows we make repeated use of the representation theory of $\mathfrak{S}_5$ and $\mathfrak{A}_5$, and so let us agree on its notation. In Tables 1 and 2, the columns are the conjugacy classes of the group, indicated by the choice of a representative. We partially followed Fulton–Harris [13] for the notation of the types of the irreducible representations of $\mathfrak{S}_5$. Here **1** denotes the trivial representation, sgn denotes the sign representation, V denotes the

Table 1 The character table of $\mathfrak{S}_5$

type	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
1	1	1	1	1	1	1	1
sgn	1	-1	1	1	-1	-1	1
V	4	2	0	1	-1	0	-1
$V \otimes \mathrm{sgn}$	4	-2	0	1	1	0	-1
W	5	1	1	-1	1	-1	0
$W \otimes \mathrm{sgn}$	5	-1	1	-1	-1	1	0
$E := \bigwedge^2 V$	6	0	-2	0	0	0	1

Table 2 The character table of $\mathfrak{A}_5$

type	(1)	(12)(34)	(123)	(12345)	(12354)
1	1	1	1	1	1
V	4	0	1	-1	-1
W	5	1	-1	0	0
I	3	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
I'	3	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$

Coxeter representation, that is, the standard 4-dimensional irreducible representation of $\mathfrak{S}_5$ and E stands for $\wedge^2 V$. Note that each of these representations is real and hence admits a nondegenerate $\mathfrak{S}_5$-invariant quadratic form, making it self-dual.

Our labeling of the irreducible representations of $\mathfrak{A}_5$ overlaps with that of $\mathfrak{S}_5$, and this is deliberately so: the $\mathfrak{S}_5$-representations V and $V \otimes \mathrm{sgn}$ become isomorphic when restricted to $\mathfrak{A}_5$, but remain irreducible and that is why we still denote them by V. The same applies to W and $W \otimes \mathrm{sgn}$ (and of course to $\mathbf{1}$ and sgn). On the other hand, the restriction to $\mathfrak{A}_5$ of $\mathfrak{S}_5$-representation $E := \wedge^2 V$ is no longer irreducible, but is isomorphic as an $\mathfrak{A}_5$-representation $I \oplus I'$ (cf. Table 2). The representations I and I' differ by the outer automorphism of $\mathfrak{A}_5$ induced by conjugation with an element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$. Both I and I' are realized as the group of isometries of Euclidean 3-space that preserve an icosahedron. In particular, they are real (and hence orthogonal).

We will often use the fact that the natural map $\mathfrak{S}_5 \to \mathrm{Aut}(\mathfrak{A}_5)$ (given by conjugation) is an isomorphism of groups. So the outer automorphism group of $\mathfrak{A}_5$ is of order 2 and representable by conjugation with an odd permutation.

The isomorphism $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ depends of course on the model of S as a blow-up $\mathbb{P}^2$, but since $\mathrm{Aut}(S)$ permutes these models transitively, this isomorphism is unique up to an inner automorphism. This implies that the characters of $\mathrm{Aut}(S)$, or equivalently, the isomorphism types of the finite dimensional irreducible representations of this group, are naturally identified with those of $\mathfrak{S}_5$.

Via this action of $\mathfrak{S}_5$ on S, all linear spaces naturally associated to S can be made into linear representations of $\mathfrak{S}_5$. For example, $H^2(S; \mathbb{C})$ contains the trivial representation, spanned by the anticanonical class $c_1(S) = -K_S$. The intersection pairing gives an integral, symmetric bilinear form on $H^2(S; \mathbb{Z})$. The orthogonal complement of K_S in $H^2(S; \mathbb{Z})$, denoted here by $H_0^2(S; \mathbb{Z})$, contains (and is spanned by) a root system of type A_4, the roots being the elements of self-intersection -2. A root basis is

$$\left(e_1 - e_2, \ e_2 - e_3, \ e_3 - e_4, \ e_0 - e_1 - e_2 - e_3 \right)$$

where e_0 is the class of a preimage of a line in $\mathbb{P}^2$ and e_i is the class of the line over p_i. This identifies $H_0^2(S; \mathbb{C})$ as an $\mathfrak{S}_5$-representation with the Coxeter representation V. The (classes of) lines themselves in $H^2(S; \mathbb{Z})$ make up an $\mathfrak{S}_5$-orbit. In the language of root systems, it is the orbit of a fundamental weight; for the given root basis this weight is represented by the orthogonal projection of e_4 in $H_0^2(S; \mathbb{Q})$. Thus $H_0^2(S; \mathbb{C})$ is isomorphic to V as an $\mathfrak{S}_5$-representation.

Another example is $H^0(S, \omega_S^{-1})$, a 6-dimensional representation of $\mathfrak{S}_5$. Using the explicit action of $\mathfrak{S}_5$ on S and hence on the space of cubic polynomials representing elements of $H^0(S, \omega_S^{-1})$, we find that it has the same character as $E = \wedge^2 V$. This is why we shall write E_S for $H^0(S, \omega_S^{-1})$, so that $|-K_S| = \mathbb{P}(E_S)$ and $\mathbb{P}_S = \check{\mathbb{P}}(E_S)$.

2.4 The Plücker embedding

We here show how S can be obtained in an intrinsic manner as a linear section of the Grassmannian of lines in projective 4-space. The naturality will make this automatically $\mathrm{Aut}(S)$-equivariant. Let C_∞ denote the union of the 10 lines on S. It is clear that

C_∞ is a normal crossing divisor in S (under the modular interpretation of $S \cong \overline{\mathcal{M}}_{0,5}$ it is the Deligne–Mumford boundary).

Lemma 2.1 *The sheaf of logarithmic differentials $\Omega^1_S(\log C_\infty)$ is globally generated. The action of $\mathfrak{S}_5$ on S makes the space of global sections*

$$W_S := H^0(S; \Omega^1_S(\log C_\infty))$$

an irreducible 5-dimensional $\mathfrak{S}_5$-representation isomorphic to W.

Proof Denote by $\mathcal{L}$ the 10-element set of lines on S and consider the *residue exact sequence*

$$0 \to \Omega^1_S \to \Omega^1_S(\log C_\infty) \xrightarrow{\text{res}} \bigoplus_{L \in \mathcal{L}} i_{L*}\mathcal{O}_L \to 0,$$

where $i_L : L \hookrightarrow S$ is the inclusion. All homomorphisms here are natural and the exact sequence of cohomology gives an exact sequence of $\mathfrak{S}_5$-representations:

$$0 \to H^0(S; \Omega^1_S(\log C_\infty)) \to \mathbb{C}^{\mathcal{L}} \xrightarrow{\phi} H^1(S; \Omega^1_S).$$

The $\mathfrak{S}_5$-representation $\mathbb{C}^{\mathcal{L}}$ is the permutation representation on the vertices of the Petersen graph. Its character is that of $\mathbf{1} \oplus V \oplus W$, as one can see by looking at pairs of complementary pentagons inside of the Petersen graph. We also note that $H^1(S; \Omega^1_S) \cong H^2(S; \mathbb{C})$ is as a representation isomorphic to $\mathbf{1} \oplus V$. The blow-up model makes it obvious that $H^2(S; \mathbb{C})$ is generated by the classes of lines, so that the homomorphism ϕ is surjective, and we obtain an isomorphism of representations

$$W_S = H^0(S; \Omega^1_S(\log C_\infty)) \cong W.$$

It remains to show that W_S generates $\Omega^1_S(\log C_\infty)$. For this, we use the description of S as the blow-up of $\mathbb{P}^2$ in the vertices of its coordinate simplex. In terms of the coordinates $(t_0:t_1:t_2)$, the logarithmic forms $dt_1/t_1 - dt_0/t_0$ and $dt_2/t_2 - dt_0/t_0$ generate the sheaf of differentials on $\mathbb{P}^2$ outside the coordinate simplex and so they define elements ζ_1 and ζ_2 of W_S which generate $\Omega_{S \setminus C_\infty}$. Since $\text{Aut}(S)$ is transitive on the vertices of the Petersen graph, it remains to prove generation along just one line, say the strict transform of $t_1 = 0$. A straightforward computation shows that ζ_1 and ζ_2 take care of this, except at the point at infinity (where $t_0 = 0$), but the origin is included. Since $\text{Aut}(S)$ is also transitive on the edges of the Petersen graph, this finishes the proof. $\square$

Remark 2.2 A local computation shows right away that $\det \Omega^1_S(\log C_\infty) = \omega_S(C_\infty)$. Since C_∞ is a divisor of ω_S^{-2}, the latter is isomorphic to ω_S^{-1}, so that $c_1(\Omega^1_S(\log C_\infty)) = -K_S$. The residue exact sequence allows one to compute $c_2(\Omega^1_S(\log C_\infty))$ also we have

$$c(\Omega_S^1(\log C_\infty)) = 1 + c_1 + c_2 = \frac{c(\Omega_S^1)}{\prod_{L \in \mathcal{L}}(1 - [L])}$$
$$= (1 + K_S + c_2(S)) \prod_{L \in \mathcal{L}}(1 + [L] + [L]^2).$$

As is well-known, $c_2(S)$ is the cohomology class in top degree whose value on the fundamental class is the Euler characteristic of S, which is 7 (since S is the blow-up of four points in $\mathbb{P}^2$). Using the identities $L^2 = -1$ and $\sum_{L \in \mathcal{L}}[L] = -2K_S$, together with the fact that the Petersen graph has 15 edges, we then find that $\langle c_2(\Omega_S^1(\log C_\infty)), [S] \rangle = 2$.

Since $\Omega_S^1(\log C_\infty)$ is globally generated, for every $x \in S$ the evaluation map $e_x \colon W_S \to \Omega_S^1(\log C_\infty)(x))$ is onto. So $\ker(e_s)$ is of dimension 3 and the annihilator of this kernel in $W_S^\vee$ is of dimension 2. We thus obtain a morphism

$$f \colon S \to G(2, W_S^\vee) = G_1(\mathbb{P}(W_S^\vee)) = G_1(\check{\mathbb{P}}(W_S)).$$

Under this map, the pull-back of the dual of the tautological subbundle on the Grassmannian becomes isomorphic to $\Omega_S^1(\log C_\infty)$.

Proposition 2.3 *The morphism f is a closed, $\mathfrak{S}_5$-equivariant embedding. Its composition with the Plücker embedding*

$$G_1(\mathbb{P}(W_S^\vee)) \hookrightarrow \check{\mathbb{P}}(\wedge^2 W_S) \cong \mathbb{P}^9$$

is the anticanonical embedding into the subspace $\check{\mathbb{P}}(E_S)$, where E_S is identified with a direct summand of the representation

$$\wedge^2 W_S \cong E_S \oplus (V \otimes \mathrm{sgn}). \tag{1}$$

In particular, the anticanonical model of S is $\mathfrak{S}_5$-equivariantly isomorphic to the intersection of $G_1(\mathbb{P}(W_S^\vee))$ with the subspace $\mathbb{P}(E)$ in its Plücker embedding.

Proof A locally free sheaf $\mathcal{E}$ of rank r on a compact variety X that is globally generated determines a morphism $X \to G(r, H^0(\mathcal{E})^\vee)$ whose composite with the Plücker embedding $G(r, H^0(\mathcal{E})^\vee) \hookrightarrow \mathbb{P}(H^0(\mathcal{E})^\vee)$ is given by the invertible sheaf $\det \mathcal{E}$. Applying this to our situation, we find that the composite of f with the Plücker embedding is given by the complete linear system $|-K_S|$. We know that it defines a closed embedding, from this the first claim follows.

To see the second claim, we compute the character of the representation of $\mathfrak{S}_5$ in $\wedge^2 W_S$ by means of the formula $\chi_{\wedge^2 W}(g) = (\chi_W(g)^2 - \chi_W(g^2))/2$. The standard character theory gives us the decomposition (1). □

Remark 2.4 The linear map $F \colon V \otimes \mathrm{sgn} \to \wedge^2 W^\vee$ of $\mathfrak{S}_5$-representations found above defines an equivariant linear embedding

$$\mathbb{P}^4 \cong \mathbb{P}(V) = \mathbb{P}(V \otimes \mathrm{sgn}) \hookrightarrow \check{\mathbb{P}}(\wedge^2 W) \cong \mathbb{P}^9$$

whose geometry is discussed in [8, Example 10.2.20]. We note that this gives rise to a rational map

$$\widetilde{F} \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$$

obtained as the composite of the Veronese map $\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^2 V)$ and the projectivization of the linear map

$$\mathrm{Sym}^2 V \xrightarrow{\wedge^2 F} \mathrm{Sym}^2(\wedge^2 W^\vee) \to \wedge^4 W^\vee \cong \det(W^\vee) \otimes W.$$

It is clear that $\widetilde{F}$ is given by a 4-dimensional linear system of quadrics. Its indeterminacy locus is where $F(v)$ has rank $\leqslant 2$ instead of 4 (as one might expect) and consists of five points in general position (so these are the vertices of a coordinate simplex of the 4-dimensional $\mathbb{P}(V)$) and $\widetilde{F}$ is then given by the linear system of quadrics through these. The $\mathfrak{S}_5$-equivariant map $\widetilde{F}$ appears in modular setting which we shall now recall (and which is discussed for example in [8, Section 9.4.4]).

The moduli space $\overline{\mathcal{M}}_{0,6}$ of stable 6-pointed genus zero curves is isomorphic to the blow-up of $\mathbb{P}^3$ at the five vertices of its coordinate simplex followed by the blow-up of the proper transforms of the lines of this simplex (see [17]). There is a natural map from $\overline{\mathcal{M}}_{0,6}$ to the Hilbert–Mumford compactification of $\mathcal{M}_{0,6}$. The latter appears as the image an $\overline{\mathcal{M}}_{0,6}$ in a 4-dimensional projective space via the linear system of quadrics in $\mathbb{P}^3$ through the five coordinate vertices so that this reproduces a copy of our map Φ above. The image of this map is a cubic hypersurface, called the *Segre cubic* $\mathcal{S}_3$.

The Segre cubic has 10 nodal points, each of which is the image of an exceptional divisor over a line of the coordinate simplex. Observe that the modular interpretation of the morphism $\overline{\mathcal{M}}_{0,6} \to \mathcal{S}_3$ makes evident an action of $\mathfrak{S}_6$, although in the above model, only its restriction to the subgroup $\mathfrak{S}_5$ is manifest. The ambient 4-dimensional projective space of the Segre cubic $\mathcal{S}_3$ is the projectivization of an irreducible 5-dimensional representation of $\mathfrak{S}_6$ corresponding to the partition $(3, 3)$ (see [8, p. 470]).   □

2.5 The anticanonical model

Some of what follows can be found in Shepherd-Barron [23]; see also Mukai [19].

We have just seen that an anticanonical model of S is obtained as linear section of a Grassmannian $G_1(W_S^\vee)$ for its Plücker embedding. It is well-known that the Plücker equations define the image of $G_1(W_S^\vee) \hookrightarrow \mathbb{P}(\wedge^2 W_S^\vee) = \check{\mathbb{P}}(\wedge^2 W_S)$ as the intersection of five quadrics: these are given by the map

$$\wedge^2 W_S^\vee \to \wedge^4 W_S^\vee = \det(W_S^\vee) \otimes W_S$$
$$\alpha \mapsto \alpha \wedge \alpha$$

or rather by its dual $\det(W_S) \otimes W_S^\vee \to \mathrm{Sym}^2(\wedge^2 W_S)$. The latter is a nonzero map of $\mathfrak{S}_5$-representations. As we mentioned, the irreducible representation W is self-dual,

and the character table of $\mathfrak{S}_5$ shows that the (-1)-eigenspace of the transposition $(12) \in \mathfrak{S}_5$ in W has dimension 2. Hence $\det(W_S)$ is the trivial representation and $\det(W_S) \otimes W_S^\vee$ can as an $\mathfrak{S}_5$-representation be identified with the irreducible representation W_S. Thus we have obtained an $\mathfrak{S}_5$-equivariant embedding

$$W_S \hookrightarrow \mathrm{Sym}^2(\wedge^2 W_S)$$

(unique up to scalars) and this realizes $\mathbb{P}(W_S)$ as the linear system of (Plücker) quadrics in $\check{\mathbb{P}}(\wedge^2 W_S)$ that define $G_1(W_S^\vee)$. By restriction to $E_S^\vee$, we can also understand W_S as defining the linear system of quadrics in $\check{\mathbb{P}}(E_S)$ that has $S = G_1(W_S^\vee) \cap \check{\mathbb{P}}(E_S)$ as its base locus. When thus interpreted, we will write $I_S(2)$ for W_S.

The character of $\mathrm{Sym}^2 E_S$ can be computed by means of the formula $\chi_{\mathrm{Sym}^2 E}(g) = (\chi_E(g^2) + \chi_E(g)^2)/2$. We then find that

$$\mathrm{Sym}^2 E_S \cong W^{\oplus 2} \oplus (W \otimes \mathrm{sgn}) \oplus V \oplus \mathbf{1} \oplus \mathrm{sgn}. \tag{2}$$

We have already singled out the subrepresentation $I_S(2)$ as a copy of W and so the remaining summands in (2) will add up to a representation that embeds in $H^0(S, \omega_S^{-2})$. An application of Riemann–Roch or the explicit discussion below shows that this is in fact an isomorphism, so that we have an exact sequence of $\mathfrak{S}_5$-representations

$$0 \to I_S(2) \to S^2 H^0(S, \omega_S^{-1}) \to H^0(S, \omega_S^{-2}) \to 0$$

and an $\mathfrak{S}_5$-equivariant isomorphism

$$|\omega_S^{-2}| \cong \check{\mathbb{P}}(W \oplus (W \otimes \mathrm{sgn}) \oplus V \oplus \mathbf{1} \oplus \mathrm{sgn}) \cong \mathbb{P}^{15}.$$

The summands $\mathbf{1}$ and sgn in $\mathrm{Sym}^2 E_S$ are explained by restricting the representation E_S to $\mathfrak{A}_5$: then E_S decomposes into two irreducible $\mathfrak{A}_5$-representations: $E_S = I \oplus I'$, each of which is orthogonal. This means that $(\mathrm{Sym}^2 I)^{\mathfrak{A}_5}$ and $(\mathrm{Sym}^2 I')^{\mathfrak{A}_5}$ are of dimension 1. If Q is a generator of $(\mathrm{Sym}^2 I)^{\mathfrak{A}_5}$, then its image Q' under an element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ is a generator of $(\mathrm{Sym}^2 I')^{\mathfrak{A}_5}$. So $Q + Q' \in \mathrm{Sym}^2 E_S$ is $\mathfrak{S}_5$-invariant (it spans the $\mathbf{1}$-summand) and $Q - Q' \in \mathrm{Sym}^2 E_S$ transforms according to the sign character (it spans the sgn-summand). The above discussion shows that their images in $H^0(S, \omega_S^{-2})$ remain independent.

Corollary 2.5 *The image of $Q - Q'$ in $H^0(S, \omega_S^{-2})$, which spans a copy of the sign representation, has divisor C_∞.*

Proof As is well-known, 'taking the residue at infinity' identifies the space of rational 3-forms on $\mathbb{C}^3$ that are invariant under scalar multiplication (i.e., are homogeneous of degree zero) with the space of rational 2-forms on $\mathbb{P}^2$. Thus

$$\alpha := \frac{(dt_0 \wedge dt_1 \wedge dt_2)^2}{t_0 t_1 t_2 (t_0 - t_1)(t_1 - t_2)(t_2 - t_0)}$$

can be understood as a rational section of $\omega_{\mathbb{P}^2}^2$ whose divisor is minus the sum of the six coordinate lines. It follows that $\pi^*\alpha$ can be regarded as a generating section of $\omega_S^2(C_\infty)$. Hence its inverse $\pi^*\alpha^{-1}$ becomes a generating section of $\omega_S^{-2}(-C_\infty)$, making $H^0(S, \omega_S^{-2}(-C_\infty))$ an $\mathfrak{S}_5$-invariant subspace of $H^0(S, \omega_S^{-2})$. Exchanging t_1 and t_2 clearly takes α to $-\alpha$ and so the span $H^0(S, \omega_S^{-2}(-C_\infty))$ of $\pi^*\alpha^{-1}$ is a copy of the sign representation in $H^0(S, \omega_S^{-2})$. It follows that $\pi^*\alpha^{-1}$ must be proportional to the image of $Q - Q'$ in $H^0(S, \omega_S^{-2})$, so that the divisor of $Q - Q'$ is C_∞.   □

Remark 2.6 We can make the canonical embedding and $I_2(S)$ explicit in terms of the blow-up model of S. Let us first observe that the Petersen graph has twelve 5-cycles (pentagons). The stabilizer subgroup of a 5-cycle is the dihedral subgroup D_{10} of $\mathfrak{S}_5$ of order 10. For example, the cycle formed by the vertices (12), (23), (34), (45), (15) is stabilized by the subgroup generated by permutations (12345) and (25)(34). A geometric interpretation of a 5-cycle is a hyperplane section of S which consists of a pentagon of lines. Note that they come in pairs with respect to taking the complementary subgraph. A pentagon of lines, viewed as reduced divisor on S, is a member of the anticanonical system of S. For our description of S as a blown-up $\mathbb{P}^2$ these must be transforms of triangles in $\mathbb{P}^2$. The list of these triangles is as follows:

$$
\begin{aligned}
f_0 &= t_1 t_2 (t_0 - t_2), & f_0' &= t_0(t_0 - t_1)(t_1 - t_2); \\
f_1 &= t_1(t_0 - t_1)(t_0 - t_2), & f_1' &= t_0 t_2(t_1 - t_2); \\
f_2 &= (t_0 - t_1)(t_0 - t_2)t_2, & f_2' &= t_0 t_1(t_1 - t_2); \\
f_3 &= t_1 t_2(t_0 - t_1), & f_3' &= t_0(t_0 - t_2)(t_1 - t_2); \\
f_4 &= t_1(t_0 - t_2)(t_1 - t_2), & f_4' &= t_0 t_2(t_0 - t_1); \\
f_5 &= t_2(t_1 - t_2)(t_0 - t_1), & f_5' &= t_0 t_1(t_0 - t_2).
\end{aligned}
$$

A direct check gives that the left-hand column can be linearly expressed in terms of the right-hand column (and vice versa), as follows:

$$
\begin{aligned}
f_0' &= f_1 - f_2 + f_5, & f_1' &= f_0 - f_3 + f_5, & f_2' &= f_0 - f_3 + f_4, \\
f_3' &= f_1 - f_2 + f_4, & f_4' &= f_2 + f_3 - f_5, & f_5' &= f_0 + f_1 + f_4.
\end{aligned}
$$

But beware that in order to get actual of sections of ω_S^{-1}, we need to multiply these elements with $(dt_0 \wedge dt_1 \wedge dt_2)^{-1}$: the resulting 3-vector fields are then invariant under scalar multiplication and have a residue at infinity that can be understood as an element of $H^0(S, \omega_S^{-1})$. Note that $(f_0, \ldots, f_5)$ is a basis of the linear space $H^0(S, \omega_S^{-1})$, and so these basic elements can serve as the coordinates of an anticanonical embedding $S \hookrightarrow \mathbb{P}^5$. Since $f_i f_i' = f_j f_j'$, we see that S is contained in the intersection of quadrics defined by equations $x_i x_i' - x_j x_j' = 0$, where x_i' is the linear form in $x_0, \ldots, x_5$ that expresses f_i' in terms of the $f_0, \ldots, f_5$. The five linear independent quadratic forms $x_i x_i' - x_0 x_0'$, $i = 1, \ldots, 5$, then give us the defining equations for S in $\mathbb{P}^5$. We easily check that the quadratic form $\sum_{i=1}^6 x_i x_i'$ corresponding to the sum of pentagons of lines spans the sgn summand. It cuts out on S the union C_∞ of lines on S. The linear

space spanned by quadratic forms $\{x_i x_i'\}_{i=1}^6$ decomposes as the direct sum $\mathrm{sgn} \oplus W$. As we have already observed, the space of quadrics $V(x_i x_i' - x_6 x_6')$ span the kernel $I_S(2)$ of $\mathrm{Sym}^2 H^0(S, \omega_S^{-1}) \to H^0(S, \omega_S^{-2})$.

It is a priori clear that $\mathfrak{S}_5$ permutes the pentagons of lines listed above, but we see that it in fact preserves the 12-element set and the preceding discussion makes explicit how.

3 The Wiman–Edge pencil and its modular interpretation

3.1 The Wiman–Edge pencil and its base locus

We found in Sect. 2.5 that there are exactly two $\mathfrak{S}_5$-invariant quadrics on $\check{\mathbb{P}}(E_S)$, one defined by $Q + Q'$ and spanning the $\mathbf{1}$-summand, the other by $Q - Q'$ and spanning the sgn-summand and (by Corollary 2.5) cutting out on S the 10-line union C_∞. The trivial summand spanned by $Q + Q'$ cuts out a curve that we shall call the *Wiman curve* and denote by C_0. We shall find it to be smooth of genus 6. We observe that the plane spanned by Q and Q' is the fixed point set of $\mathfrak{A}_5$ in $\mathrm{Sym}^2 E_S$ and hence defines a pencil $\mathcal{C}$ of curves on S whose members come with a (faithful) $\mathfrak{A}_5$-action. This pencil is of course spanned by C_0 and C_∞ and these are the only members that are in fact $\mathfrak{S}_5$-invariant. We refer to $\mathcal{C}$ as the *Wiman–Edge pencil*; we sometimes also use this term for its image in $\mathbb{P}^2$ under the natural map $\pi : S \to \mathbb{P}^2$.

Lemma 3.1 (Base locus) *The base locus $\Delta := C_0 \cap C_\infty$ of $\mathcal{C}$ is the unique 20-element $\mathfrak{S}_5$-orbit in C_∞. The curves C_0 and C_∞ intersect transversally so that each member of the Wiman–Edge pencil is smooth at Δ.*

Proof Since this $C_0 \cap C_\infty$ is $\mathfrak{S}_5$-invariant, it suffices to determine how a line L on S meets C_0. We first note that the intersection number $C_0 \cdot L$ (taken on S) is $-2K_S \cdot [L] = 2$. When we regard L as an irreducible component of C_∞, or rather, as defining a vertex of the Petersen graph, then we see that the other lines meet L in three distinct points and that the $\mathfrak{S}_5$-stabilizer of L acts on L as the full permutation group of these three points. So if we choose an affine coordinate z on L such that the three points in question are the third roots of 1, then we find that the $\mathfrak{S}_5$-stabilizer of L acts on L with three irregular orbits: two of size 3 (the roots of $z^3 - 1$ and the roots of $z^3 + 1$) and one of size 2 ($\{z = 0, z = \infty\}$). It follows that the C_0 meets L in the size 2 orbit. In particular, the intersection of C_0 with C_∞ is transversal and contained in the smooth part of C_∞.   □

3.2 Genus 6 curves with $\mathfrak{A}_5$-symmetry

Any reduced $C \in \mathcal{C}$, being a member of $|-2K_S|$, has its normal sheaf in S isomorphic to $\mathcal{O}_C \otimes \omega_S^{-2}$. By the adjunction formula, its dualizing sheaf ω_C is therefore isomorphic to

$$\mathcal{O}_C \otimes \omega_S^{-2} \otimes_{\mathcal{O}_S} \omega_S = \mathcal{O}_C \otimes \omega_S^{-1} = \mathcal{O}_C(1).$$

In particular

$$\deg(\omega_C) = \deg(\mathcal{O}_C \otimes \omega_S^{-1}) = (-2K_S) \cdot (-K_S) = 10.$$

It follows that C has arithmetic genus 6, that the natural map $E_S \to H^0(C, \omega_C)$ is an $\mathfrak{A}_5$-equivariant isomorphism and that C is canonically embedded in $\mathbb{P}_S$. Our goal is to give the Wiman–Edge pencil a modular interpretation.

Proposition 3.2 ($\mathfrak{A}_5$ and $\mathfrak{S}_5$ orbit spaces) *Let C be a smooth projective curve genus 6 endowed with a faithful action of $\mathfrak{A}_5$. Then $\mathfrak{A}_5 \backslash C$ is of genus zero and $\mathfrak{A}_5$ has four irregular orbits with isotropy orders* 3, 2, 2 *and* 2.

If the $\mathfrak{A}_5$-action extends to a faithful $\mathfrak{S}_5$-action, then $\mathfrak{S}_5 \backslash C$ is of genus zero and $\mathfrak{S}_5$ has three irregular orbits with isotropy orders 6, 4 *and* 2. *The union of these irregular $\mathfrak{S}_5$-orbits is also the union of the irregular $\mathfrak{A}_5$-orbits: the $\mathfrak{S}_5$-orbit with isotropy order 6, resp. 4, is an $\mathfrak{A}_5$-orbit with isotropy order 3, resp. 2, and the $\mathfrak{S}_5$-orbit with isotropy order 2 decomposes into two $\mathfrak{A}_5$-orbits with the same isotropy groups. (In other words, the double cover $\mathfrak{A}_5 \backslash C \to \mathfrak{S}_5 \backslash C$ only ramifies over the points of ramification of order 6 and 4.) Such a curve exists.*

See also [4, Theorem 5.1.5].

Proof The stabilizer of a point of a smooth curve under a faithful finite group action is cyclic. So when $G = \mathfrak{S}_5$, the stabilizer of a point of C is a cyclic group of order $i \leqslant 6$, but when $G = \mathfrak{A}_5$, it cannot be of order 4 or 6. Let for $i > 1$, k_i be the number of G-orbits of size $|G|/i$. By the Hurwitz formula we have

$$\frac{10}{|G|} = 2(g'-1) + \sum_{i=2}^{6} \frac{i-1}{i} k_i,$$

where g' is the genus of the quotient.

For $G = \mathfrak{S}_5$ we have $|G| = 120$ and so this shows right away that $g' = 0$. It follows that

$$25 = 6k_2 + 8k_3 + 9k_4 + \frac{48}{5} k_5 + 10k_6.$$

It is then immediate that $k_5 = 0$ and $k_6 \leqslant 1$. We find that the only solutions for (k_2, k_3, k_4, k_6) are $(1, 0, 1, 1)$ and $(0, 2, 1, 0)$. The latter possibility clearly does not occur because it would imply that an element of order 6 acts without fixed points, contradicting the Hurwitz equality $10 = 6(2g'-2)$.

For $G = \mathfrak{A}_5$, we have $|G| = 60$ and hence we then also have $g' = 0$. The formula now becomes $13 = 3k_2 + 4k_3 + 24k_5/5$, which has as only solution $(k_2, k_3, k_5) = (3, 1, 0)$.

The assertion concerning the map $\mathfrak{A}_5 \backslash C \to \mathfrak{S}_5 \backslash C$ formally follows from the above computations.

The last assertion will follow from the Riemann existence theorem, once we find a regular $\mathfrak{S}_5$-covering of $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with the simple loops yielding monodromies

α, β, $\gamma \in \mathfrak{S}_5$ of order 6, 4, 2 respectively, which generate $\mathfrak{S}_5$ and for which $\alpha\beta\gamma = 1$. This can be arranged: take $\alpha = (123)(45)$ and $\beta = (1245)$ and $\gamma = (14)(23)$. $\qquad\square$

We next show that any smooth projective curve genus 6 endowed with a faithful action of $\mathfrak{A}_5$ appears in the Wiman–Edge pencil. For this we shall invoke a theorem of Mukai [19], which states that a canonical smooth projective curve C of genus 6 lies on a quintic del Pezzo surface if and only if it is neither bielliptic (i.e., it does not double cover a genus 1 curve), nor trigonal (it does not triple cover a genus zero curve), nor isomorphic to a plane quintic.

Theorem 3.3 *Every smooth projective curve of genus* 6 *endowed with a faithful $\mathfrak{A}_5$-action is $\mathfrak{A}_5$-equivariantly isomorphic to a member of the Wiman–Edge pencil. This member is unique up to the natural action of the involution $\mathfrak{S}_5/\mathfrak{A}_5$.*

Proof Let C be such a curve. We first show that C is not hyperelliptic, so that we have a canonical model. If it were, then it is so in a unique manner so that the set of its 14 Weierstraß points is in invariant with respect to $\mathfrak{A}_5$. But we found in Proposition 3.2 that $\mathfrak{A}_5$ has in C one irregular orbit of size 20, three of size 30, and no others, and thus such an invariant subset cannot exist.

From now on we assume that C is canonical. We first show that it is neither trigonal, nor bielliptic, nor isomorphic to a plane quintic.

C is trigonal: This means that C admits a base point free pencil of degree 3. This pencil is then unique [2, p. 209] so that the $\mathfrak{A}_5$-action on C permutes the fibers. Consider the associated morphism $C \to \mathbb{P}^1$. The Riemann–Hurwitz formula then tells us that the ramification divisor of this morphism on C has degree 16. It must be $\mathfrak{A}_5$-invariant. But our list of orbit sizes precludes this possibility and so such a divisor cannot exist.

C is isomorphic to a plane quintic: It is then so in a unique manner [2, p. 209] and hence the $\mathfrak{A}_5$-action on C will extend as a projective representation to the ambient $\mathbb{P}^2$.

The resulting projective representation cannot be reducible, for then $\mathfrak{A}_5$ has a fixed point, $p \in \mathbb{P}^2$ say, and the action of $\mathfrak{A}_5$ on the tangent space $T_p\mathbb{P}^2$ will be faithful. But as Table 2 shows, $\mathfrak{A}_5$ has no faithful representation. So the projective representation is irreducible and hence the projectivization of copy of I or I'. Either representation is orthogonal and so the ambient projective plane contains an $\mathfrak{A}_5$-invariant conic. The quintic defines on this conic an effective divisor of degree 10. Since $\mathfrak{A}_5$ acts on the conic (a Riemann sphere) as the group of motions of a regular icosahedron, no $\mathfrak{A}_5$-orbit on this conic has fewer than 12 points and so such a divisor cannot exist.

C is bielliptic: This means that C comes with an involution ι whose orbit space is of genus 1. Let $G \subset \mathrm{Aut}(C)$ be the subgroup generated by $\mathfrak{A}_5$ and ι. By a theorem of Hurwitz, $|\mathrm{Aut}(C)| \leqslant 84(6 - 1) = 420$ and so $[G{:}\mathfrak{A}_5]$ can be 1, 2, 4 or 6.

Let us first deal with the index 6 case. For this we note that the G-action (by left translations) on $G/\mathfrak{A}_5$ has a kernel contained in $\mathfrak{A}_5$. This kernel is a normal subgroup and contained in $\mathfrak{A}_5$. Since $\mathfrak{A}_5$ is simple, this kernel is either trivial or all of $\mathfrak{A}_5$. It cannot be all of $\mathfrak{A}_5$, because $G/\mathfrak{A}_5$ is then cyclic of order 6 and hence cannot be generated by the image of ι. It follows that G acts faithfully on $G/\mathfrak{A}_5$ so that we get an embedding of G in $\mathfrak{S}_6$. Its image is then a subgroup of index 2 and so this image must be $\mathfrak{A}_6$: $G \cong \mathfrak{A}_6$. We now invoke the Hurwitz formula to this group action: the

stabilizer of a point is cyclic of order $\leqslant 6$ and so if for a divisor i of $|G|$, k_i is the number of G-orbits in C of size $|G|/i$, then

$$\frac{10}{360} = 2(g' - 1) + \sum_{i=2}^{6} \frac{i-1}{i} k_i,$$

where g' is the genus of the quotient. This implies that $g' = 0$ and then this comes down to

$$73 = 18k_2 + 24k_3 + 27k_4 + \frac{144}{5} k_5 + 30k_6.$$

It is clear that k_5 must be zero. Since the left-hand side is odd, we must have $k_4 = 1$, and we then find that no solution exists.

If the index $\leqslant 4$, then the argument above gives a map $G \to \mathfrak{S}_4$. Its kernel is contained in $\mathfrak{A}_5$, but cannot be trivial for reasons of cardinality. It follows that the kernel equals $\mathfrak{A}_5$, so that $\mathfrak{A}_5$ is normal in G. Since the image of ι generates $G/\mathfrak{A}_5$, it then follows that the index is 1 or 2.

In the last case, G is either isomorphic to $\mathfrak{A}_5 \times \mathfrak{C}_2$ or to $\mathfrak{S}_5$, depending on whether or not conjugation by ι induces in $\mathfrak{A}_5$ an inner automorphism. In the first case, the action of $\mathfrak{A}_5$ descends to a faithful action on the elliptic curve. This would make $\mathfrak{A}_5$ an extension of finite cyclic group by an abelian group, which is evidently not the case. It follows that $G \subset \mathfrak{S}_5$ and that ι is conjugate to (12) or to (12)(34).

Denote by χ the character of the $\mathfrak{S}_5$-representation $H^0(C, \Omega_C)$. Since the quotient of C by ι has genus 1, we have $\chi(\iota) = -5 + 1 = -4$. If ι is conjugate to (12), then we then read off from Table 1 that χ is the character of $\mathbf{1} \oplus \mathrm{sgn}^{\oplus 5}$ or $\mathrm{sgn}^{\oplus 2} \oplus (V \otimes \mathrm{sgn})$. Its restriction to $\mathfrak{A}_5$ is then the trivial character resp. $\mathbf{1}^{\oplus 2} \oplus V$. But this contradicts the fact that the $\mathfrak{A}_5$-orbit space of C has genus zero. If ι is conjugate to (12)(34), then we then read off from Table 2 that the $\mathfrak{A}_5$-representation $H^0(C, \Omega_C)$ takes on ι a value $\geqslant -2$, which contradicts the fact that this value equals -4.

According to Mukai [19], it now follows that C lies on a weak quintic del Pezzo surface S_C in $\check{\mathbb{P}}(H^0(C, \omega_C))$. It may have singular points, and in fact quadric sections of a weak del Pezzo quintic form a divisor D in the moduli space $\mathcal{M}_6$. However, we claim that C must lie on a smooth del Pezzo surface.

To prove this claim, first note that if C lies on a singular surface then it has fewer than five g_4^1's, where by a g_4^1 we mean a linear series of degree 4 and dimension 1. In the plane model this is because three points are collinear or two points coincide. The five g_4^1's are defined by four pencils through nodes of the sextic and the pencil of conics through four nodes. In the singular case, when, we choose three collinear points, there is no pencil of conics. The divisor D in $\mathcal{M}_6$ mentioned above is characterized by the fact that it has at most four g_4^1's. Now, $\mathfrak{A}_5$ acts on these g_4^1's, and hence leaves them all invariant. Thus it preserves a map $C \to \mathbb{P}^1$ of degree 4. This is impossible since there are no invariant subset of ramification points. This proves the claim.

It is also known [23] that S_C is unique. This uniqueness property implies that the faithful $\mathfrak{A}_5$-action on C, which extends naturally to $\check{\mathbb{P}}(H^0(C, \omega_C))$, will leave S_C

invariant. A choice of an $\mathfrak{A}_5$-equivariant isomorphism $h\colon S_C \xrightarrow{\cong} S$ will then identify C in an $\mathfrak{A}_5$-equivariant manner with a member of the Wiman–Edge pencil. Any two $\mathfrak{A}_5$-equivariant isomorphisms $h, h'\colon S_C \xrightarrow{\cong} S$ differ by an automorphism of S, so by an element $g \in \mathfrak{S}_5$. But the $\mathfrak{A}_5$-equivariance then amounts to g centralizing $\mathfrak{A}_5$. This can happen only when g is the identity. So h is unique. □

Let $\mathcal{B}$ denote the base of the Wiman–Edge pencil (a copy of $\mathbb{P}^1$) so that we have projective flat morphism $\mathcal{C} \to \mathcal{B}$. Recall that $\mathfrak{S}_5$ acts on the family in such a manner that the action on $\mathcal{C} \to \mathcal{B}$ is through an involution ι which has two fixed points. We denote by $\mathcal{B}^\circ \subset \mathcal{B}$ the locus over which this morphism is smooth. So the restriction over $\mathcal{B}^\circ$ is a family of smooth projective genus 6 curves endowed with a faithful $\mathfrak{A}_5$-action. It has the following modular interpretation.

Theorem 3.4 (Universal property) *The smooth part of the Wiman–Edge pencil, $\mathcal{C}_{\mathcal{B}^\circ} \to \mathcal{B}^\circ$, is universal in the sense that every family $\mathcal{C}' \to \mathcal{B}'$ of smooth projective genus 6 curves endowed with a fiberwise faithful $\mathfrak{A}_5$-action fits in a unique $\mathfrak{A}_5$-equivariant fiber square*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C}_{\mathcal{B}^\circ} \\ \downarrow & & \downarrow \\ \mathcal{B}' & \longrightarrow & \mathcal{B}^\circ. \end{array}$$

Moreover, the natural morphism $\mathcal{B}^\circ \to \mathcal{M}_6$ factors through an injection $\langle \iota \rangle \backslash \mathcal{B}^\circ \hookrightarrow \mathcal{M}_6$.

Proof Theorem 3.3 (and its proof) has an obvious extension to families of genus 6-curves with $\mathfrak{A}_5$-action. This yields the first assertion. If $t, t' \in \mathcal{B}^\circ$ are such that C_t and $C_{t'}$ are isomorphic as projective curves, then as we have seen, an isomorphism $C_t \cong C_{t'}$ is induced by an element of $\mathfrak{S}_5$ and so $t' \in \{t, \iota(t)\}$. □

We will find in Sect. 3.3 that the singular members of the Wiman–Edge pencil are all stable. We found already one such curve, namely the union of the 10 lines, and so this element will map in $\mathcal{M}_6$ to the boundary.

From now on we identify the base of Wiman–Edge pencil with $\mathcal{B}$.

Corollary 3.5 *The Wiman curve C_0 is smooth and is $\mathfrak{S}_5$-isomorphic to the curve found in Proposition 3.2. It defines the unique ι-fixed point of $\mathcal{B}^\circ$.*

Proof By Theorem 3.4, there is a unique member of the Wiman–Edge pencil whose base point maps the unique point of $\mathcal{B}^\circ$ which supports a smooth genus 6 curve with $\mathfrak{S}_5$-action. The Wiman–Edge pencil has two members with $\mathfrak{S}_5$-action, one is the union of the 10 lines and the other is the Wiman curve. So it must be the Wiman curve. □

Corollary 3.6 *Let C be a smooth projective curve genus 6 endowed with a faithful action of $\mathfrak{A}_5$. If the resulting map $\phi\colon \mathfrak{A}_5 \hookrightarrow \mathrm{Aut}(C)$ is not surjective, then it extends to an isomorphism $\mathfrak{S}_5 \cong \mathrm{Aut}(C)$, and $C = C_0$ is the Wiman curve. The $\mathfrak{A}_5$-representation, resp. $\mathfrak{S}_5$-representation, $H^0(C, \omega_C)$ is equivalent to $I \oplus I'$, resp. E, and $H^1(C; \mathbb{C})$ is equivalent to $I^{\oplus 2} \oplus I'^{\oplus 2}$, resp. $E^{\oplus 2}$.*

Proof Since the $\mathfrak{A}_5$-curve C is represented by a member of the Wiman–Edge pencil, we can assume it is a member of that pencil. Since C is canonically embedded, an automorphism of C extends naturally of the ambient projective space and hence also to S (as it is the unique quintic del Pezzo surface containing C). This implies that $\mathrm{Aut}(C) \subset \mathfrak{S}_5$. This inclusion is an equality precisely when C is the Wiman curve.

As for the last assertion, we know that the representation space $H^0(C, \omega_C)$ is as asserted, since it is a member of the Wiman curve. The representations $I \oplus I'$, resp. E, are self-dual, and since $H^1(C, \mathbb{C})$ contains $H^0(C, \omega_C)$ as an invariant subspace with quotient the (Serre-)dual of $H^0(C, \omega_C)$, the last assertion follows also. □

Remark 3.7 If an $\mathfrak{A}_5$-curve C represents a point $z \in \mathcal{B}^\circ$, then $\mathfrak{A}_5$ acts nontrivially (and hence faithfully) on $H^0(C, \omega_C^2)$ and hence also on its Serre dual $H^1(C, \omega_C^{-1})$. Since $\mathfrak{A}_5$ is not a complex reflection group, a theorem of Chevalley implies that the orbit space $\mathfrak{A}_5 \backslash H^1(C, \omega_C^{-1})$ must be singular at the image of the origin. The local deformation theory of curves tells us that when $\mathfrak{A}_5$ is the full automorphism group of C, the germ of this orbit space at the origin is isomorphic to the germ of $\mathcal{M}_6$ at the image of z. Hence $\mathcal{B}^\circ$ maps to the singular locus of $\mathcal{M}_6$. (This is a special case of a theorem of Rauch–Popp–Oort which states that any curve of genus $g \geqslant 4$ with nontrivial group of automorphisms defines a singular point of $\mathcal{M}_g$.)

3.3 Singular members of the Wiman–Edge pencil

The singular members of the Wiman–Edge pencil were found by Edge [11]. A modern proof of his result can be found in [4, Theorem 6.2.9]. Here we obtain them in a different manner as part of a slightly stronger result that we obtain with a minimum of computation.

Let us begin with an *a priori* characterization of the reducible genus 5 curves with $\mathfrak{A}_5$-action that occur in the Wiman–Edge pencil.

Lemma 3.8 *There are precisely two reducible members C_c and C'_c of $\mathcal{C}$ distinct from C_∞; each is is a stable union of five special conics whose intersection graph is $\mathfrak{A}_5$-equivariantly isomorphic to the full graph on a 5-element set. Any element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ exchanges C_c and C'_c.*

Proof Let C be such a member of $\mathcal{C}$ and let Y be an irreducible component of C. Then Y cannot be a line and so its degree $d := -Y \cdot K_S$ in the anticanonial model must be $\geqslant 2$. Since $\mathfrak{A}_5$ has no subgroup of index < 5, the number r of irreducible components in the $\mathfrak{A}_5$-orbit of Y, must be $\geqslant 5$. But we also must have $rd \leqslant \deg(C) = 10$, and so the only possibility is that $(d, r) = (2, 5)$.

An irreducible, degree 2 curve in a projective space is necessarily a smooth conic. Its $\mathfrak{A}_5$-stabilizer has index 5 in $\mathfrak{A}_5$, and so must be $\mathfrak{S}_5$-conjugate to $\mathfrak{A}_4$. This implies that Y is a special conic. The five irreducible components of its $\mathfrak{A}_5$-orbit lie in distinct conic bundles $\mathcal{P}_i$, and we number them accordingly $Y_1, \ldots, Y_5$. For $1 \leqslant i < j \leqslant 5$, any smooth member of $\mathcal{P}_i$ meets any smooth member of $\mathcal{P}_j$ with multiplicity one, and so this is in particular true for Y_i and Y_j. Hence the set of singular points of C is covered in an $\mathfrak{A}_5$-equivariant manner by the set of 2-element subsets of $\{1, \ldots, 5\}$.

The action of $\mathfrak{A}_5$ on this last set is transitive, and so either the singular set of C is a singleton $\{p\}$ (a point common to all the irreducible components) or the intersections $Y_i \cap Y_j$, $1 \leqslant i < j \leqslant 5$, are pairwise distinct. The first case is easily excluded, for p must then be a fixed point of the $\mathfrak{A}_5$-action and there is no such point. So C is as described by the lemma.

The sum of the classes of the five conic bundles is $\mathfrak{S}_5$-invariant and hence proportional to $-K_S$. The intersection product with $-K_S$ is $5.2 = 10$ and hence this class is equal to $-2K_S$, in other words, the class of the Wiman–Edge pencil. Since an $\mathfrak{A}_5$-orbit of a special conic takes precisely one member from every conic bundle, its follows that the sum of such an orbit indeed gives a member of $\mathcal{C}$. There two such orbits and so we get two such members.   □

There are precisely two faithful projective representations of $\mathfrak{A}_5$ on $\mathbb{P}^1$ up to equivalence, and they only differ by precomposition with an automorphism of $\mathfrak{A}_5$ that is not inner. We will refer to theses two representations as the *Schwarzian representations* of $\mathfrak{A}_5$. Both appear as the symmetry groups of a regular icosahedron drawn on the Riemann sphere. This action has three irregular orbits, corresponding to the vertices, the barycenters of the faces and the midpoints of the edges, and so are resp. 12, 20 and 30 in size. Their points come in (antipodal) pairs and the $\mathfrak{A}_5$-action preserves these pairs. We give a fuller discussion in Sect. 4.1.

Lemma 3.9 *There exists an irreducible stable curve C of genus 6 with six nodes endowed with a faithful $\mathfrak{A}_5$-action. Such a C is unique up to an automorphism of $\mathfrak{A}_5$.*

Proof Let C be such a curve with six nodes and denote by $D \subset C$ its singular set. Then the normalization of $\widetilde{C} \to C$ is of genus zero: $\widetilde{C} \cong \mathbb{P}^1$. If $\widetilde{D} \subset \widetilde{C}$ denotes the preimage of D, then $\widetilde{D}$ consists of 12 points that come in six pairs. The $\mathfrak{A}_5$-action on C lifts to $\widetilde{C}$ and will preserve $\widetilde{D}$ and its decomposition into six pairs. From the above remarks it follows that $\mathfrak{A}_5$ acts on $\widetilde{C}$ as the symmetry group of an icosahedron drawn on $\widetilde{C}$ which has $\widetilde{D}$ as vertex set and such that antipodal pairs are the fibers of $\widetilde{C} \to C$. This shows both existence and uniqueness up to an automorphism of $\mathfrak{A}_5$.   □

Proposition 3.10 *An irreducible singular member of the Wiman–Edge pencil is necessarily as in Lemma 3.9: a stable curve with six nodes and of geometric genus zero. It appears in the Wiman–Edge pencil together with its outer transform.*

Proof We have already encountered the singular members C_∞ and C_c, C_c'. A well-known formula (see, for example, [15, p. 509–510]) applied to the Wiman pencil gives

$$e(S) - e(C)e(\mathbb{P}^1) + C \cdot C = \sum_{t \in \mathcal{B}} (e(C_t) - e(C))$$

where $e(\cdot)$ stands for the Euler–Poincaré characteristic, C denotes a general fiber and C_t denotes a fiber over a point $t \in \mathcal{B}$. The left-hand side equals $27 - (-20) = 47$. The reducible fibers C_∞, C_c, C_c' contribute to the right hand side $15 + 2 \cdot 10 = 35$, so that there is 12 left as the contribution coming the irreducible fibers.

It is known that $e(C_t) - e(C)$ is equal to the sum of the Milnor numbers of singular points of C_t. We know that $\mathfrak{A}_5$ leaves invariant each fiber, but that no fiber other than

C_∞ (which is reducible) or the Wiman curve C_0 (which is smooth) is $\mathfrak{S}_5$-invariant. In other words, the irreducible fibers come in pairs. Since $\mathfrak{A}_5$ cannot fix a point on S (because it has no a nontrivial linear representations of dimension 2 and hence cannot act nontrivially on the tangent space at this point), and a proper subgroup of $\mathfrak{A}_5$ has index $\geqslant 5$, the irreducible fibers come as a single pair, with each member having exactly six singular points, all of Milnor number 1, that is, having six ordinary double points. Hence the normalization of such a fiber is a rational curve as in Lemma 3.9   □

We sum up with the following.

Corollary 3.11 (Classification of singular members of $\mathcal{C}$) *Each singular member of the Wiman–Edge pencil $\mathcal{C}$ is a stable curve of genus 6 with $\mathfrak{A}_5$-action. The set of these curves is a union of three sets:*

(lines) C_∞, *a union of 10 lines with intersection graph the Petersen graph.*
(conics) *a pair C_c, C_c', each of which is a union of five conics whose intersection graph is the complete graph on five vertices.*
(irred) *a pair C_{ir}, C_{ir}' of irreducible rational curves, each with six nodes.*

The action of $\mathfrak{S}_5$ on $\mathcal{C}$ leaves C_∞ invariant; the induced action on the set of 10 lines of C_∞ induces an action on the corresponding intersection graph that is isomorphic to the $\mathfrak{S}_5$-action on the Petersen graph. The action of any odd permutation of $\mathfrak{S}_5$ on $\mathcal{C}$ interchanges C_c and C_c', and also interchanges C_{ir} and C_{ir}'.

Remark 3.12 Our discussion in Sect. 2.1 shows that $C_c \cup C_c'$, when regarded as a curve on $\overline{\mathcal{M}}_{0,5}$, meets $\mathcal{M}_{0,5}$ in the locus parameterizing 5-pointed rational curves $(C; x_1, \ldots, x_5)$ with the property that there exists an affine coordinate z for C such that $\{x, \ldots, x_5\}$ contains the union of $\{0\}$ and the roots of $z^3 = 1$. So we can characterize the Wiman–Edge pencil on $\overline{\mathcal{M}}_{0,5}$ as the pencil which contains in $|\partial \overline{\mathcal{M}}_{0,5}|$ and these two loci. It is desirable to have a modular interpretation of this pencil.

3.4 Connection with the Dwork pencil

Since the singular members of the Wiman–Edge pencil also play a special role in the work of Candelas et al. [3], this is perhaps a good place to make the connection with that paper. Let $p = (p_1, \ldots, p_5)$ be distinct points of $\mathbb{P}^1$. As Zagier [27] points out, there exist linear forms ℓ_i on $\mathbb{C}^2$ defining p_i, such that

$$\ell_1^5 + \cdots + \ell_5^5 = 5\psi \ell_1 \ell_2 \ell_3 \ell_4 \ell_5$$

for some $\psi \in \mathbb{C}$ with the 5-tuple $(\ell_1^5, \ell_2^5, \ell_3^5, \ell_4^5, \ell_5^5)$ being unique up to a common scalar (so that ψ^5 only depends on $p_1, \ldots, p_5$). In other words,

$$\ell := [\ell_1 : \cdots : \ell_5] \colon \mathbb{P}^1 \to \mathbb{P}^4$$

maps to a line on a member of the *Dwork pencil*, that is, the pencil of quintic 3-folds $X_\psi \subset \mathbb{P}^4$ defined by $z_1^5 + \cdots + z_5^5 = 5\psi z_1 z_2 z_3 z_4 z_5$, such that the coordinate hyperplane $z_i = 0$ defines p_i.

This construction is essentially $PGL_2(\mathbb{C})$-equivariant and has some symmetries, perhaps the most obvious ones being its $\mathfrak{S}_5$-symmetry. But we also have acting the group $T[5] \cong \mu_5{}^4$ of order 5 elements in the diagonal torus T of $PGL_5(\mathbb{C})$. Together with $\mathfrak{S}_5$ this gives a faithful action of the subgroup $\mathfrak{S}_5 \ltimes T[5] \subset PGL_5(\mathbb{C})$ on the total space of the pencil. It acts on the parameter ψ through a character $\mathfrak{S}_5 \ltimes T[5] \to \mu_5$, the latter being given as a nontrivial $\mathfrak{S}_5$-invariant character $\chi : T[5] \to \mu_5$. So $\ker(\chi)$ is as a group isomorphic to $\mu_5{}^3$ and every X_ψ is stabilized by $\mathfrak{S}_5 \ltimes \ker(\chi)$. A good way to express this is that we have thus obtained the following:

(i) an unramified $T[5]$-covering $\mathcal{T}_{0,5} \to \mathcal{M}_{0,5}$ endowed with an action $\mathfrak{S}_5 \ltimes T[5]$ which extends the $T[5]$-action and is compatible with the $\mathfrak{S}_5$-action on $\mathcal{M}_{0,5}$,

(ii) a regular function $\psi : \mathcal{T}_{0,5} \to \mathbb{C}$, equivariant with respect to the above character (so that ψ^5 is defined as an $\mathfrak{S}_5$-invariant function on $\mathcal{M}_{0,5}$),

(iii) an $\mathfrak{S}_5 \ltimes T[5]$-equivariant lift of ψ from $\mathcal{T}_{0,5}$ to the Fano variety $\mathrm{Fano}(\mathcal{X}/\mathbb{P}^1)$ of lines on the Dwork pencil.

The main results of [3] can then be summed up as follows:

A fiber of $\mathcal{M}_{0,5} \xrightarrow{\psi^5} \mathbb{C}$ is the sum of a member of the Wiman pencil plus its transform under an odd permutation and for a suitable parametrization of the pencil base $\mathcal{B}$ by a parameter ϕ such that C_0, resp. C_∞, is defined by $\phi = 0$, resp. $\phi = \infty$, and we have $32\psi^{-5} = \phi^2 + 3/4$. Moreover, the morphisms above with their symmetry extend (uniquely) over the blow-up $\mathcal{C} \to \overline{\mathcal{M}}_{0,5}$ of $\overline{\mathcal{M}}_{0,5}$ in Δ (recall that this is simply the total space of the Wiman–Edge pencil $\phi : \mathcal{C} \to \mathbb{P}^1$): we have a $T[5]$-cover $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ with a compatible $\mathfrak{S}_5 \ltimes T[5]$-action and an equivariant extension of the Fano map to $\widetilde{\mathcal{C}}$ which makes the Dwork pencil and the symmetrized Wiman–Edge pencil fit in the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Fano}(\mathcal{X}/\mathbb{P}^1) & \xleftarrow{\;F\;} & \widetilde{\mathcal{C}} & \xrightarrow{\;\pi\;} & \mathcal{C} \\
& \searrow & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi^2} \\
& & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1.
\end{array}
$$

Here the slant map is the structural map and the bottom map is given by $\psi \mapsto 32\psi^{-5} - 3/4$. For $\psi \neq 0, \infty$, $\widetilde{C}_\psi$ parametrizes all the nonisolated points of $\mathrm{Fano}(X_\psi)$; for $\psi^5 \neq 2^7/3$ it consists of two irreducible curves, each of which is an unramified $\ker(\chi)$-cover of $C_{\pm\phi}$ (of arithmetic genus 626), but for $\psi^5 = 2^7/3$, there is only one irreducible curve (which covers the Wiman curve).

The singular members of the Wiman–Edge pencil are accounted for as follows: the Fermat quintic ($\psi = 0$) yields the sum of 10 lines with multiplicity 2 ($2C_\infty$), the sum of the five coordinate hyperplanes ($\psi = \infty$) yields $C_c + C'_c$, and if ψ is a 5th root of unity, we get $C_{ir} + C'_{ir}$.

3.5 Plane model of the Wiman–Edge pencil

Let $\pi : S \to \mathbb{P}^2$ be the blowing-down morphism. The Wiman–Edge pencil is the proper inverse transform of a pencil of curves of degree 6 with points of multiplicity $\geqslant 2$ at the

fundamental points invariant with respect to the Cremona group G of transformations isomorphic to $\mathfrak{A}_4$. Following Edge, we choose the fundamental points of π^{-1} to be the points

$$(-1:1:1), \quad (1:-1:1), \quad (1:1:-1), \quad (1:1:1).$$

The group generated by projective transformations that permute the coordinates and their sign changes is isomorphic to $\mathfrak{S}_4$, where the sign changes are even permutations of order 2. Together with the additional symmetry defined by the Cremona involution T with the first reference points, we obtain a subgroup of Cremona transformations isomorphic to $\mathfrak{S}_5$. The subgroup $\mathfrak{A}_5$ is generated by cyclic permutations, sign changes and the transformation T.

Let $F = 0$ be the equation of a curve from the Wiman–Edge pencil. The condition that F is invariant with respect the sign changes shows that F must be a combination of monomials $x^a y^b z^c$ with a, b, c even. This allows us to write

$$F = a(x^6 + y^6 + z^6) + b(x^4 y^2 + y^4 z^2 + z^4 x^2)$$
$$+ c(x^4 z^2 + y^4 x^2 + z^4 y^2) + d x^2 y^2 z^2.$$

The additional conditions that $3a + 3b + 3c + d = 0$ will guarantee that the fundamental points are singular points. The Cremona involution T is given by

$$\sigma : (x:y:z) \mapsto \big(-x^2 + y^2 + z^2 + xy + xz + yz : x^2 - y^2 + z^2 + xy + xz + yz$$
$$: x^2 + y^2 - z^2 + xy + xz + yz\big)$$

The invariance of F with respect to T gives (a, b, c) is a linear combination of $(2, 1, 1)$ and $(0, 1, -1, 0)$. This gives the equation of the Wiman–Edge pencil

$$F = \lambda\big(x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12 x^2 y^2 z^2\big)$$
$$+ \mu(x^2 - y^2)(y^2 - z^2)(x^2 - z^2) = 0$$

We check that the Wiman curve B is the member of the pencil with $(\lambda : \mu) = (1:0)$. Computing the partial derivatives, we find that the curve is indeed smooth confirming Corollary 5.10 (Fig. 3).

The base locus of the pencil should be a subscheme of length $6^2 = 36$. The four reference points contribute 16 to this number, the rest is the image of 20 base points of the Wiman–Edge pencil on S in the plane. It easy to find them. Each line through two fundamental points intersects C_0 at two nonsingular points, this gives us 12 points:

$$(\sqrt{-3} : \pm 1 : \pm 1) \quad \text{and} \quad (\pm 1 : \sqrt{-3} : \pm 1) \quad \text{and} \quad (\pm 1 : \pm 1 : \sqrt{-3}).$$

The remaining eight points are on the exceptional curves where they represent the directions of the tangent lines to the branches of singular points of B. The equations of the four pairs of these tangent lines are
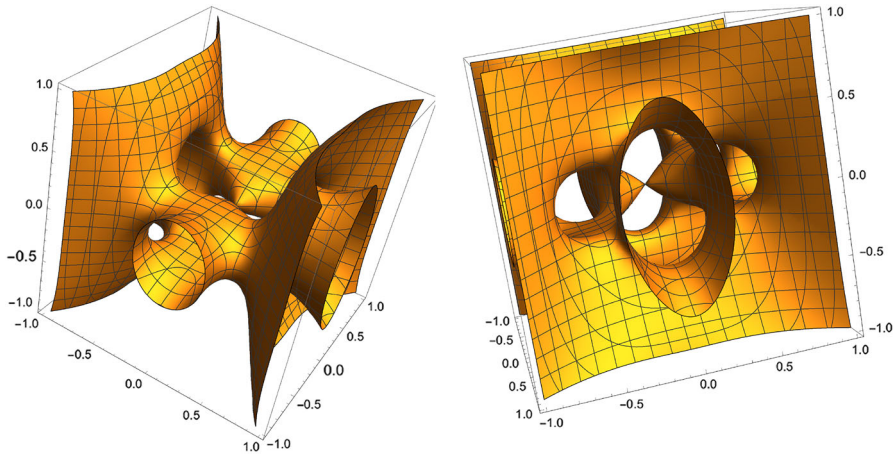
Fig. 3 Two views of a piece of the Wiman sextic (the image of the Wiman curve in $\mathbb{P}^2$)

$$(x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z) = 0,$$
$$(x + \epsilon y - \epsilon^2 z)(x + \epsilon^2 y - \epsilon z) = 0,$$
$$(x - \epsilon y + \epsilon^2 z)(x - \epsilon^2 y + \epsilon z) = 0,$$
$$(x - \epsilon y - \epsilon^2 z)(x - \epsilon^2 y - \epsilon z) = 0,$$

where $\epsilon = (-1 + \sqrt{-3})/2$ is a primitive root of 1. One checks that each tangent line passing through p_i together with three of the lines $\langle p_i, p_j \rangle$, $j \neq i$, form a harmonic 4-tuple of lines in the pencil of lines through p_i ([1]).

Choose one of the branch tangents, say $x + \epsilon y + \epsilon^2 z = 0$. It intersects the Wiman sextic B at two nonsingular points $(1:\sqrt{-3}:1)$ and $(1:-1:\sqrt{-3})$. We see that each double point of W is a *biflecnode*, i.e., the branch tangents at singular point intersect the curve at this point with multiplicity 4 instead of expected 3.

The singular irreducible members C_{ir}, resp. C'_{ir}, are proper transforms of the members of the pencil corresponding to the parameters $(\lambda:\mu) = (1:5\sqrt{5})$, resp. $(1:-5\sqrt{5})$. The singular points of C_{ir} besides the fundamental points are

$$(0:0:\lambda), \quad (0:1:-\lambda), \quad (1:0:\lambda'), \quad (1:0:-\lambda'), \quad (1:\lambda:0), \quad (1:-\lambda:0),$$

where $\lambda = (1 + \sqrt{5})/2$, $\lambda' = (1 - \sqrt{5})/2$. The six singular points of C'_{ir} are obtained from these points by replacing λ with λ'.

---

[1] Edge calls them *Hesse duads* for the following reason. It follows from observing the Petersen graph the stabilizer subgroup of each line acts on the line by permuting three intersection points with other lines. If one considers these points as the zero set of a binary form ϕ of degree 3, then its Hessian binary form of degree 2 given as the determinant of the matrix of second partial derivatives of ϕ has zeros at two points, the Hessian duad.

The reducible members C_c and C'_c of the Wiman–Edge pencil are proper transforms of the members of the pencil of plane curves

$$(x + \epsilon y + \epsilon^2 z)(x + \epsilon y - \epsilon^2 z)(x - \epsilon y + \epsilon^2 z)$$
$$\cdot (x - \epsilon y - \epsilon^2 z)(x^2 + \epsilon^2 y^2 + \epsilon z^2) = 0,$$

where $\epsilon \neq 1$, $\epsilon^3 = 1$. As shown already by Edge, they correspond to parameters $(\lambda : \mu) = (1 : \pm \sqrt{-3})$. We leave these computations to the reader, they are straightforward.

3.6 Irregular orbits in S

Recall that if a group acts on a set, an orbit is called *regular* if the stabilizer of one (and hence any) of its points is trivial; otherwise it is called an *irregular* orbit. For what follows it will helpful to have a catalogue of irregular an $\mathfrak{S}_5$-orbits and $\mathfrak{A}_5$-orbits in S. Here we can observe that an $\mathfrak{S}_5$-orbit is an $\mathfrak{A}_5$-orbit if and only if its $\mathfrak{S}_5$-stabilizer is not contained in $\mathfrak{A}_5$ (otherwise it splits into two $\mathfrak{A}_5$-orbits). So a determination of the irregular $\mathfrak{S}_5$-orbits determines one of the irregular $\mathfrak{A}_5$-orbits. The $\mathfrak{S}_5$-equivariant incarnation of S as the moduli space of stable, 5-pointed genus zero curves makes this determination (which is in fact due to Coble) rather straightforward, as we will now explain.

A point of $\overline{\mathcal{M}}_{0,5}$ is the same thing as a *stable map* $x \colon \{1, 2, \ldots, 5\} \to \mathbb{P}^1$ (where 'stable' means here that every fiber has at most two elements), given up to a composition with a Möbius transformation. The $\mathfrak{S}_5$-stabilizer of a stable map x consists of the set of $\sigma \in \mathfrak{S}_5$ for which there exists a $\rho(\sigma) \in \mathrm{PGL}_2(\mathbb{C})$ with the property that $x\sigma = \rho(\sigma)x$. Since x is stable, its image has at least three distinct points, and so $\rho(\sigma)$ will be unique. It follows that ρ will be a group homomorphism. Its image will be a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ with the property that it has in $\mathbb{P}^1$ an orbit of size $\leqslant 5$.

Klein determined the finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ up to conjugacy: they are the cyclic groups, represented by the group μ_n of nth roots of unity acting in $\mathbb{C} \subset \mathbb{P}^1$ as scalar multiplication; the dihedral groups, represented by the semidirect product of μ_n and the order 2 group generated by the inversion $z \mapsto z^{-1}$; and the tetrahedral, octahedral and icosahedral groups, which are isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$, $\mathfrak{A}_5$ respectively. The octahedral and icosahedral groups have no orbit of size $\leqslant 5$ in $\mathbb{P}^1$, and hence cannot occur here. The tetrahedral group has one such orbit: it is of size 4 (the vertices of a tetrahedron), but since we want a degree 5 divisor, it then must have a fixed point, and this is clearly not the case.

It remains to go through the dihedral and cyclic cases. We denote by $\mathfrak{C}_n$ a cyclic group of order n and by $\mathfrak{D}_{2n}$ the dihedral group of order $2n$ isomorphic to $\mathfrak{C}_n \rtimes \mathfrak{C}_2$ ([2]). The conjugacy classes of the nontrival cyclic subgroups of $\mathfrak{A}_5$ are represented by $\langle (12)(34) \rangle \cong \mathfrak{C}_2$ and $\langle (123) \rangle \cong \mathfrak{C}_3$, and for $\mathfrak{S}_5$ we have the two additional classes represented by $\langle (12) \rangle \cong \mathfrak{C}_2$ and $\langle (12)(345) \rangle \cong \mathfrak{C}_6$. Likewise, there are three conjugacy classes of dihedral subgroups in $\mathfrak{A}_5$ represented by

---

[2] We follow the now standard ATLAS notation for finite groups.

$$\langle (12)(34), (13)(23)\rangle \cong \mathfrak{D}_4, \quad \langle (12)(45), (123)\rangle \cong \mathfrak{D}_6 \cong \mathfrak{S}_3,$$
$$\langle (12)(34), (12345)\rangle \cong \mathfrak{D}_{10}$$

and additional three conjugacy classes in $\mathfrak{S}_5$:

$$\langle (12), (34)\rangle \cong \mathfrak{D}_4, \quad \langle (12), (123)\rangle \cong \mathfrak{D}_6, \quad \langle (12), (1324)\rangle \cong \mathfrak{D}_8.$$

To distinguish the conjugacy classes of dihedral subgroups of order 4 or 6 we denote them by $\mathfrak{D}_4^{\mathrm{ev}}$, $\mathfrak{S}_3^{\mathrm{ev}}$, resp. $\mathfrak{D}_4^{\mathrm{odd}}$, $\mathfrak{S}_3^{\mathrm{odd}}$ if they are contained in $\mathfrak{A}_5$, resp. in $\mathfrak{S}_5$, but not in $\mathfrak{A}_5$. A similar convention applies to $\mathfrak{C}_2$: we have $\mathfrak{C}_2^{\mathrm{odd}}$ and $\mathfrak{C}_2^{\mathrm{ev}}$.

We then end up with the following list, due to Arthur Coble [7, pp. 400–401].

Theorem 3.13 (The irregular orbits of $\mathfrak{S}_5$ acting on S) *The set of irregular orbits of $\mathfrak{S}_5$ acting on S is one of the following sets, named by the conjugacy class of a stabilizer subgroup:*

$\mathfrak{C}_2^{\mathrm{odd}}$ *For example, $\langle (12)\rangle$ is the stabilizer of $(0, 0, \infty, 1, z)$ when z is generic. An orbit of this type has size 60. It is a regular $\mathfrak{A}_5$-orbit.*

$\mathfrak{C}_2^{\mathrm{ev}}$ *For example, $\langle (12)(34)\rangle$ is the stabilizer of $(z, -z, 1, -1, \infty)$ when z is generic. An orbit of this type has size 60 and decomposes into two $\mathfrak{A}_5$-orbits of size 30.*

$\mathfrak{C}_4$ *For example, $\langle (1234)\rangle$ is the stabilizer of $(1, \sqrt{-1}, -1, -\sqrt{-1}, \infty)$. This is an $\mathfrak{S}_5$-orbit of size 30. It is also an $\mathfrak{A}_5$-orbit of type $\mathfrak{C}_2^{\mathrm{ev}}$ (take $z = \sqrt{-1}$).*

$\mathfrak{D}_4^{\mathrm{odd}}$ *For example $\langle (12), (34)\rangle$ is the stabilizer of $(0, 0, 1, -1, \infty)$. This is an $\mathfrak{S}_5$-orbit of size 30, which is also an $\mathfrak{A}_5$-orbit of type $\mathfrak{C}_2^{\mathrm{ev}}$ (let $z \to 0$).*

$\mathfrak{S}_3^{\mathrm{ev}}$ *For example $\langle (23)(45), (123)\rangle$ is the stabilizer of $(1, \zeta_3, \zeta_3^2, 0, \infty)$. This is an $\mathfrak{S}_5$-orbit of size 20 which splits into two $\mathfrak{A}_5$-orbits of size 10.*

$\mathfrak{D}_8$ *For example $\langle (12), (1324)\rangle$ is the stabilizer of $(0, 0, \infty, \infty, 1)$. This is the unique $\mathfrak{S}_5$-orbit of size 15. It is also an $\mathfrak{A}_5$-orbit of type $\mathfrak{D}_4^{\mathrm{ev}}$.*

$\mathfrak{C}_6$ *For example, $\langle (12)(345)\rangle$ is the stabilizer of $(\infty, \infty, 1, \zeta_3, \zeta_3^2)$. This is a single orbit of size 20 which is also an $\mathfrak{A}_5$-orbit of type $\mathfrak{C}_3$.*

$\mathfrak{D}_{10}$ *For example, $\langle (25)(34), (12345)\rangle$ is the stabilizer of $(1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$. The associated orbit is of size 12 and splits into two $\mathfrak{A}_5$-orbits of size 6.*

In particular, the irregular $\mathfrak{A}_5$-orbits are of type $\mathfrak{C}_2^{\mathrm{ev}}$ are parametrized by a punctured rational curve.

Remark 3.14 It is clear from Theorem 3.13 that we have a curve of irregular $\mathfrak{A}_5$-orbits of size 30. However the locus of such points in S has 15 irreducible components. This is because the preimage of an $\mathfrak{A}_5$-orbit under the map $w \in \mathbb{C} \mapsto (w, -w, 1, -1, \infty)$ is generically of the form $\{z, 1/z\}$. An example of the closure such an irreducible component is the preimage of the line defined in $\mathbb{P}^2$ by $t_2 = t_0 + t_1$ under the blow-up of the vertices of the coordinate vertex (it is pointwise fixed under an even linear permutation of these vertices). Since this line does not pass through any of the four vertices, this also shows that its preimage in S is a rational normal curve of degree 3. We thus obtain an $\mathfrak{S}_5$-invariant curve on S of degree 45 (defined by a section of ω_S^{-9}) with 15 irreducible components. *A priori* this section is $\mathfrak{A}_5$-invariant, but as Clebsch [6] showed, it is in fact $\mathfrak{S}_5$-invariant (see also Remark 6.6).

Since we have already determined the size of some orbits in the anticanonical model, we can now interpret the orbits thus found. We do this only insofar it concerns $\mathfrak{A}_5$-orbits, because that is all we need. Note that this is not a closed subset. Here is what remains, but stated in terms of the Wiman–Edge pencil:

Corollary 3.15 *The irregular $\mathfrak{A}_5$-orbits in $S \cong \overline{\mathcal{M}}_{0,5}$ of size 30 are parametrized by punctured rational curve and the others (named by the conjugacy class of a stabilizer subgroup) are as follows:*

$\mathfrak{C}_3$ *This 20-element orbit is the base point locus Δ of $\mathcal{C}$.*

$\mathfrak{D}_4^{ev}$ *This 15-element orbit is the singular locus of C_∞.*

$\mathfrak{S}_3^{ev}$ *This consists of two 10-element orbits in $\mathcal{M}_{0,5} \cong S \setminus C_\infty$ equal to $\mathrm{Sing}(C_c)$ and $\mathrm{Sing}(C_c')$.*

$\mathfrak{D}_{10}$ *This consists of two 6-element orbits, namely the singular loci of the two irreducible members of the Wiman–Edge pencil, $\mathrm{Sing}(C_{ir})$ and $\mathrm{Sing}(C_{ir}')$.*

The orbit pairs of type $\mathfrak{S}_3^{ev}$ and $\mathfrak{D}_{10}$ are swapped by an a conjugacy with an element of $\mathfrak{S}_5 \setminus \mathfrak{A}_5$ (which induces a nontrivial outer automorphism of $\mathfrak{A}_5$).

Proof Theorem 3.13 yields a complete list of the irregular $\mathfrak{A}_5$-orbits in terms of $\overline{\mathcal{M}}_{0,5}$ of size smaller than 30. We have already encountered some of these orbits as they appear in this corollary. All that is then left to do is to compare cardinalities.   □

4 Projection to a Klein plane

4.1 The Klein plane

The two representations I and I' of $\mathfrak{A}_5$ are the complexification of two real representations that realize $\mathfrak{A}_5$ as the group of motions of a regular icosahedron. They differ only in the way we have identified this group of motions with $\mathfrak{A}_5$. The full group of isometries of the regular icosahedron (including reflections) is a direct product $\{\pm 1\} \times \mathfrak{A}_5$, and is in fact a Coxeter group of type H_3, a property that will be quite helpful to us when we need to deal with the $\mathfrak{A}_5$-invariants in the symmetric algebra of I. Both $\mathfrak{A}_5$-actions give rise to $\mathfrak{A}_5$-actions on the unit sphere in Euclidean 3-space. Via the isomorphism $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{PU}_2$, they can also be considered as actions of $\mathfrak{A}_5$ on the Riemann sphere $\mathbb{P}^1$. These projective representations are what we have called the *Schwarzian representations* and are the only two nontrivial projective representations of $\mathfrak{A}_5$ on $\mathbb{P}^1$ up to isomorphism.

We observed above that a Schwarzian representation has three irregular orbits of sizes 12, 20 and 30, corresponding respectively to the vertices, the barycenters and the midpoints of the edges of a spherical icosahedron. The antipodal map, when considered as an involution $\mathbb{P}^1$, is antiholomorphic: it comes from assigning to a line in $\mathbb{C}^2$ its orthogonal complement. We can think of this as defining an $\mathfrak{A}_5$-invariant real structure on $\mathbb{P}^1$ without real points. In particular, the involution is not in the image of $\mathfrak{A}_5$. Yet it preserves the $\mathfrak{A}_5$-orbits, so that each orbit decomposes into pairs. The preimages of $\mathfrak{A}_5 \hookrightarrow \mathrm{Aut}(\mathbb{P}^1)$ under the degree 2 isogeny $\mathrm{SU}_2 \to \mathrm{PU}_2$ define two representations of degree 2 of an extension $\widetilde{\mathfrak{A}}_5$ of $\mathfrak{A}_5$ by a central subgroup of order

2, called *binary icosahedral group*. As above, these two representations of $\widetilde{\mathfrak{A}}_5$ differ by an outer automorphism. If we take the symmetric square of such a representation, then the kernel $\{\pm 1\}$ of this isogeny acts trivially and hence factors through a linear representation of $\mathfrak{A}_5$ of degree 3. This is an icosahedral representation of type I or I'.

We can phrase this solely in terms of a given Schwarzian representation of $\mathfrak{A}_5$ on a projective line K. For then the projective plane P underlying the associated icosahedral representation is the one of effective degree 2 divisors on K (the symmetric square of K) and K embeds $\mathfrak{A}_5$-equivariantly in P as the locus defined by points with multiplicity 2. This is of course the image of K under the Veronese embedding and makes K appear as a conic.

We will identify K with its image in P, and following Klein [18] we refer to this image as the *fundamental conic*. It is also defined by a non-degenerate $\mathfrak{A}_5$-invariant quadratic form on the icosahedral representation (which we know is self-dual). We call P a *Klein plane*. So an $x \in P \smallsetminus K$ can be understood as a 2-element subset of K. The latter spans a line in P and this is simply the polar that the conic K associates to x (when $x \in K$, this will be the tangent line of X at x).

Following Winger, we can now identify all the irregular $\mathfrak{A}_5$-orbits in P. As we have seen, K has exactly three irregular $\mathfrak{A}_5$-orbits having sizes 30, 20 and 12, each of which being invariant under an antipodal map. This antipodal invariance implies that the antipodal pairs in the above orbits span a collection of resp. 15, 10 and 6 lines in P, each of which makes up an $\mathfrak{A}_5$-orbit. When we regard these pairs as effective divisors of degree 2, they also yield $\mathfrak{A}_5$-orbits in $P \smallsetminus K$ of the same size (K parameterizes the nonreduced divisors). The bijection between lines and points is induced by polarity with respect to K. This yields all the irregular orbits in the Klein plane:

Lemma 4.1 (Winger [26, Section 1]) *There are unique irregular $\mathfrak{A}_5$-orbits in P having size* 12, 20 *(both in K),* 6, 10 *or* 15 *(all three in $P \smallsetminus K$). The remaining irregular orbits in P have size* 30 *and are parametrized by a punctured rational curve. Further, the points with stabilizer a fixed $\tau \in \mathfrak{A}_5$ of order 2 is open and dense in the image of the map $K \to P$ given by $z \in K \mapsto (z) + (\tau(z))$.*

Proof We can think of a point of K as a point on the icosahedron in Euclidean 3-space. An element of $P \smallsetminus K$ is represented by an effective degree 2 divisor on K which has the same $\mathfrak{A}_5$-stabilizer. If we identify K with its $\mathfrak{A}_5$-action as a spherical icosahedron in Euclidean 3-space with its group of motions, then such a divisor spans an affine line in Euclidean 3-space with the same stabilizer. When the line passes through the origin, we get the three orbits of sizes 30, 20 and 12, otherwise the stabilizer is of order 2. □

We call a member of the 6-element $\mathfrak{A}_5$-orbit in $P \smallsetminus K$, a *fundamental point* and denote this orbit by F. We call the polar line of such a point a *fundamental line*.

4.2 Two projections

The irreducible $\mathfrak{S}_5$-representation E_S splits into two 3-dimensional irreducible $\mathfrak{A}_5$-representations I_S and I'_S. The two summands give a pair of disjoint planes $\check{\mathbb{P}}(I_S)$ and $\check{\mathbb{P}}(I'_S)$ in $\check{\mathbb{P}}(E)$, to which we shall refer as *Klein planes*. We abbreviate them by P resp.

P', and denote by $K \subset P$ and $K' \subset P'$ the fundamental ($\mathfrak{A}_5$-invariant) conics. We have $\mathfrak{A}_5$-equivariant (Klein) projections $p\colon S \dashrightarrow P$ and $p'\colon S \dashrightarrow P'$ of the anticanonical model $S \subset \check{\mathbb{P}}(E)$ with center P', resp. P. Precomposition with an element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ exchanges these projections.

Proposition 4.2 *The Klein planes are disjoint with S and the Klein projections are finite morphisms of degree* 5. *Together they define a finite morphism*

$$f := (p, p')\colon S \to P \times P'$$

that is birational onto its image.

The first assertion of Proposition 4.2 is [20, Theorem 5].

Proof We focus on $p\colon S \to P$. Let us first note that $P' \cap S$ is $\mathfrak{A}_5$-invariant and equals the base locus of the linear system of anticanonical curves parametrized by $\mathbb{P}(I) \subset \mathbb{P}(E)$. The curve part Y of $P' \cap S$ has degree < 5. The class $[Y] \in H^2(S; \mathbb{Z})$ of Y is $\mathfrak{A}_5$-invariant. Since the span of K_S is supplemented in $H^2(S; \mathbb{C})$ by an irreducible $\mathfrak{A}_5$-representation isomorphic to V, we must have $[Y] = -dK_S$ for some integer $d \geqslant 0$. But then

$$5 > \deg(Y) = [Y] \cdot (-K_S) = 5d$$

and so $Y = \varnothing$. It follows that the base locus is finite. Since it is a linear section of S it has at most five points. But Lemma 4.1 shows that P' has no orbit of size $\leqslant 5$ and so the base locus is empty.

To see that $p\colon S \to P$ is surjective, suppose its image is a curve, say of degree m. The preimage of a general line in P in S is an anticanonical curve and hence connected. This implies that $m = 1$. But then S lies in hyperplane and this is a contradiction.

So p is a surjection of nonsingular surfaces. If some irreducible curve is contracted by p, then this curve will have negative self-intersection. The only curves on S with that property are the lines, and then a line is being contracted. Since $\mathfrak{A}_5$ acts transitively on the lines, all of them are then contracted. In other words, the exceptional set is the union C_∞ of the 10 lines on S. But C_∞ has self-intersection $(-2K_S)^2 = 20 > 0$ and hence cannot be contracted.

So the preimage of a point in P is finite. This is also the intersection of the quintic surface S with a codimension 2 linear subspace and so this fiber consists of five points, when counted with multiplicity.

For the last assertion, we notice that since one of the components of f has degree 5, the degree of $S \xrightarrow{f} f(S)$ must divide 5. So it is either 1 or 5. If it is 5 then p and p' will have the same generic fiber so that p' factors through p via an isomorphism $h\colon P \xrightarrow{\cong} P'$. But this would make the $\mathfrak{A}_5$-representations I and I' projectively equivalent, which is not the case. (We could alternatively observe that then the elements of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ preserve the fibers of p so that we get in fact an action of $\mathfrak{S}_5$ on P which makes p $\mathfrak{S}_5$-equivariant. But there is no projective representation of $\mathfrak{S}_5$ on $\mathbb{P}^2$.) $\qquad\square$

The proof of the next proposition makes use of the Thom–Boardman polynomial for the A_2-singularity locus. Let us first state the general result that we need. Let f be a

morphism between two compact, nonsingular complex surfaces. Assume first that f has a smooth locus of critical points (= ramification divisor) $\Sigma^1(f)$. As Whitney and Thom observed, $f|\Sigma^1(f)$ need not be a local immersion: generically it will have a finite set $\Sigma^{1,1}(f)$ of (Whitney) cusp singularities; at such a point f exhibits a stable map-germ which in local-analytic coordinates can be given by $(x, y) \mapsto (x^3 + xy, y)$.

In the more general case when $\Sigma^1(f)$ is a reduced divisor (when defined by the Jacobian determinant), $\Sigma^{1,1}(f)$ is defined as a 0-cycle on the source manifold. While is not so hard to prove that the degree of $\Sigma^{1,1}(f)$ is a characteristic number of the virtual normal bundle ν_f of f, it is another matter to obtain a closed formula for it. In the present case it is equal to $\langle c_1^2(\nu_f) + c_2(\nu_f), [S] \rangle$ (see for instance [21, Theorem 5.1], where this is listed as the case A_2).

Proposition 4.3 (The ramification curve) *The ramification curve R of the finite morphism $p\colon S \to P$ is a singular irreducible member of the Wiman–Edge pencil, and hence obtained as an $\mathfrak{A}_5$-curve by means of the procedure of Lemma* 3.9.

Proof The divisor class $[R]$ is given by the well-known formula $K_S - p^*(K_P)$. Since K_P is -3 times the class of a line, and p^* takes the class of a line to the class of hyperplane section (i.e., $-K_S$), we have $p^*(K_P) = 3K_S$. It follows that $[R] = -2K_S$. Since R is $\mathfrak{A}_5$-invariant, it must be a member of the Wiman–Edge pencil. In particular, R is reduced.

We now apply the above formula to $p\colon S \to P$. The Chern classes of the virtual normal bundle of p are easily computed: $c_1(\nu_p) = 2K_S$ and $c_2(\nu_p)$ takes the value -2 on the fundamental class $[S]$ of S. We find that

$$\langle c_1(\nu_p)^2 + c_2(\nu_p), [S] \rangle = 4 \cdot 5 - 2 = 18.$$

So $\Sigma^{1,1}(p)$ is a 0-cycle whose support is an $\mathfrak{A}_5$-invariant subset of S contained in R. In Sect. 3.6, we found that $\mathfrak{A}_5$ has two 6-element orbits in S (each being the singular locus of an irreducible member of the Wiman–Edge pencil) and that any other $\mathfrak{A}_5$-orbit has at least 10 elements. Since R is a member of the Wiman–Edge pencil, we conclude that that it must be one with six nodes (and that each node has multiplicity 3 in $\Sigma^{1,1}(p)$). This also proves that $p|R$ is a local isomorphism at the 20-element orbit Δ. $\square$

Remark 4.4 The assertion of Proposition 4.3 can be confirmed by computation. If $\pi\colon S \to \mathbb{P}^2$ is the blowing down morphism and $\phi\colon S \to \mathbb{P}^5 = \mathbb{P}(E)$ is the $\mathfrak{S}_5$-equivariant anti-canonical embedding, then the composition $\phi \circ \pi^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}(E)$ and the $\mathfrak{A}_5$-equivariant projection $\mathbb{P}(E) \dashrightarrow \mathbb{P}(I)$ is given by

$$(t_0 : t_1 : t_2) \mapsto \big(f_0(t_0, t_1, t_2) : f_1(t_0, t_1, t_2) : f_2(t_0, t_1, t_2)\big)$$

where

$$f_0 = -x^3 + y^3 + z^3 - \lambda x^2 y - \lambda' x^2 z + \lambda' x y^2 - \lambda y^2 z + \lambda x z^2 - \lambda' y z^2,$$
$$f_1 = \lambda' x^3 + \lambda y^3 - (\lambda + 1) x^2 y - (\lambda' + 1) y^2 x + x z^2 + y z^2,$$
$$f_2 = \lambda y^3 + z^3 - (\lambda + 1) x^2 y - \lambda' x^2 z - \lambda y^2 z + y z^2.$$

Here $\lambda = (1 + \sqrt{5})/2$ and $\lambda' = (1 - \sqrt{5})/2$ are two roots of the quadratic equation $t^2 - t - 1 = 0$. Here we use the coordinates in the plane in which the equation of the Wiman–Edge pencil is written. Computing the Jacobian $J(f_0, f_1, f_2)$ we find the equation of a irreducible singular member of the pencil with parameters $(1:5\sqrt{5})$.

Proposition 4.5 *The preimage $C_K = p^*K$ of the fundamental conic in P is a non-singular member of the Wiman–Edge pencil. The ordered 4-tuple $(C_0, C_\infty, C_K, C_{K'})$ consists of four distinct members of the Wiman–Edge pencil which lie in harmonic position: there is a unique affine coordinate for $\mathcal{C}$ that identifies this 4-tuple with $(0, \infty, 1, -1)$.*

Proof It is clear that C_K is an $\mathfrak{A}_5$-invariant member of $|-2K_S|$ and therefore a member of the Wiman–Edge pencil. It is defined by the quadric Q and similarly $C_{K'}$ is defined by the quadric Q'. The last clause of the proposition then follows as C_0, resp. C_∞, is defined by $Q + Q'$, resp. $Q - Q'$.

If C_K is singular, then it must be one of C_c, C_c', R, R'. We shall exclude each of these possibilities.

Suppose that $C_K = C_c$ or C_c'. Then let Y be an irreducible component of C_K. Now both Y and K are conics and since p is a linear projection, $Y \xrightarrow{p} K$ must be of degree 1. On the other hand, R meets C_K transversally in Δ, and so Y meets R transversally in four distinct points. This implies that $Y \xrightarrow{p} K$ must have degree 3 and we arrive at a contradiction.

We cannot have $C_K = R$, for then p would ramify along p^*K and so p^*K would be 2-divisible: this would make it twice an anticanonical divisor (a hyperplane section) and R is clearly not of that type. If $C_K = R'$, then $R' = C_K \xrightarrow{p} K$ must ramify in the singular part $\mathrm{Sing}(R')$ of R' and so R contains $\mathrm{Sing}(R')$. This contradicts the fact that the two members R and R' of the Wiman–Edge pencil intersect transversally.   □

We can also improve the statement about the birationality of the map (p, p').

Theorem 4.6 *The map*

$$(p, p'): S \to P \times P'$$

is a local isomorphism onto its image. Further,

$$f_!(1) = 5(h^2 + hh' + h'^2)$$

where (h, h') is the standard basis in $H^2(P \times P'; \mathbb{Z})$, and hence the composite of f with the Segre embedding

$$P \times P = \check{\mathbb{P}}(I) \times \check{\mathbb{P}}(I') \hookrightarrow \check{\mathbb{P}}(I \otimes I') \cong \mathbb{P}^8$$

has degree 20.

Proof We have already proved that the map is of degree 1 onto its image and hence coincides with the normalization of the image.

It remains to show that (p, p') is a local isomorphism. Suppose it is not. Since the projection p, resp. p', is a local isomorphism on $S \setminus R$, resp. $S \setminus R'$, the map could only

fail to be local isomorphism at some $x \in R \cap R'$. Let L be the kernel of the derivative at x. Then L is mapped to 0 under the composition with the both projections. This implies that L coincides with the tangent line of R and R' at x. But we know that two curves in the Wiman–Edge pencil intersect transversally at a base point. This contradiction proves the assertion.

Next represent h, resp. h', by general lines $\ell \subset P$, resp. $\ell' \subset P'$. Then $f^*(h.h')$ is represented by $p^*\ell . p'^*\ell'$. This is evidently a plane section of S and hence is a class of degree 5. Since p and p' are also of degree 5, it follows that $f_*(1)$ is as asserted. □

Remark 4.7 The map $(p, p') \colon S \to P \times P'$ is not injective. We will make this explicit in Remark 5.7, but we here give a general argument that implies that there must exist a whole curve of fibers consisting of more than a single point. Let us write f for (p, p'). The double point formula asserts that its virtual number of double points is equal to the degree of $f^*f_!(1) - c_2(\nu_f)$ [21]. The degree of

$$f^*f_!(1) = f^*(5(h^2 + hh' + h'^2))$$

is equal to $5\cdot5 + 5\cdot5 + 5\cdot5 = 75$. On the other hand, if $\mu_S \in H^4(S)$ is the fundamental class (so that $K_S^2 = 5\mu_S$) then $c(\theta_S) = 1 + K_S + 7\mu_S$ (the coefficient 7 is the topological Euler characteristic of S) and

$$p^*c(\theta_P) = p^*(1 + 3h + 3h^2) = 1 - 3K_S + 15\mu_S.$$

Thus

$$c(\nu_f) = c(\theta_S)^{-1}p^*c(\theta_P)p'^*c(\theta_{P'}) = (1 + K_S + 7\mu_S)^{-1}(1 - 3K_S + 15\mu_S)^2$$
$$= (1 - K_S - 2\mu_S)(1 - 6K_S + 75\mu_S) = 1 - 7K_S + 103\mu_S.$$

This tells us that the degree of ν_f is 103. It follows that the number of virtual double points is -28. This can only be interpreted as saying that the double point locus of f must contain a curve on S with negative self-intersection.

4.3 The projection of irregular orbits

Let us describe the images of the irregular $\mathfrak{A}_5$-orbits in S under the projection map $p \colon S \to P$. Since the projection is $\mathfrak{A}_5$-equivariant, the image of an irregular $\mathfrak{A}_5$-orbit in S is an irregular orbit in the Klein plane. According to Lemma 4.1, there are $\mathfrak{A}_5$-orbits in P of size 6, 10, 15 (all outside the fundamental conic) and of 12 and 20 (all on the fundamental conic), of which those of size 6 and 10 come in pairs, the others being unique. The other irregular orbits in P are of size 30 and are parametrized by a rational curve. On the other hand, by Corollary 3.15, in S there are two $\mathfrak{A}_5$-orbits in S of size 6 and 10, one of size 15 and 20, and an irreducible curve of orbits of size 30.

This immediately implies that the orbits on S of cardinalities 6, 10, 15 are mapped one-to-one to the orbits of the same cardinality in the Klein plane. This is also true for the 20-element orbit, since it consists of the base points of the Wiman–Edge pencil $\mathcal{C}$ and hence is mapped to the 20-element orbit on the conic. The size 12 orbit in K

Table 3 Irregular $\mathfrak{A}_5$-orbits in P of cardinality $\neq 30$ explained by $\mathfrak{A}_5$-orbits in S

special $\mathfrak{A}_5$-orbit $\mathcal{O}$ in S	#($\mathcal{O}$)	$p(\mathcal{O}) \subset K$?	#($p(\mathcal{O})$)
Base locus Δ of $\mathcal{C}$	20	Yes	20
A regular orbit in C_K	60	Yes	12
Singular part of C_∞	15	No	15
Singular part of C_c or C_c'	10 (2)	No	10
Singular part of R or R'	6 (2)	No	6

must be the image of an orbit in S whose size is divisible by 12 and so this can only be a regular orbit. Since the $\mathfrak{A}_5$-orbits of size 30 in S and in P are parametrized by an irreducible curve; p will map the generic point of the former to the generic point of the latter. This information, insofar relevant here is displayed in Table 3; it will help us to determine the p-images of the special members of the pencil.

4.4 The projection of the Wiman–Edge pencil

We will later investigate the p-images of the special members of the Wiman–Edge pencil, but at this point it is convenient to already make the following observation.

Proposition 4.8 *The divisor p_*C_∞ is the sum of the 10 lines that are spanned by the antipodal pairs in the 20-element orbit $p(\Delta)$ on K. Each singular point of p_*C_∞ lies on exactly two lines and the resulting $\binom{10}{2} = 45$ double points make up two irregular $\mathfrak{A}_5$-orbits, one of which is the unique 15-element orbit defined by pairs of lines which meet in S (and so the other has size 30).*

Proof The image of a line on S is a line in P and so p_*C_∞ is a sum of 10 lines. The polars of these lines make up an $\mathfrak{A}_5$-orbit in P of size $\leqslant 10$. There is only one such orbit and it has exactly 10 elements.

The singular locus of C_∞ is a 15-element orbit and we observed that this orbit maps bijectively onto the unique 15-element orbit in P. It follows from the discussion in Sect. 3.6, that the stabilizer of each singular point of C_∞ is the group $\mathfrak{D}_4^{\mathrm{ev}}$. Its projection has the same stabilizer group. Thus the image of $\mathrm{Sing}(C_\infty)$ consists of 15 points. Since there is only one orbit in P of cardinality 15, the remaining $45 - 15 = 30$ points form an orbit of $\mathfrak{A}_5$ in P. $\square$

This has implications for a *generic* member C of $\mathcal{C}$, as follows. The curve p_*C_∞ being reduced and of geometric genus 6, it follows that p_*C has the same property. As p is linear, p_*C is a plane curve of the same degree as C, namely 10. So the arithmetic genus of p_*C is $(10-1)(10-2)/2 = 36$, and hence its genus defect is 30. Since the singular set of C specializes to a subset of the singular set of C_∞, it follows that this singular set consists of 30 nodes and makes up an $\mathfrak{A}_5$-orbit (but remember that such orbits move in a curve and can degenerate into an orbit of smaller size).

So $C \in \mathcal{C} \mapsto p_*C$ defines a morphism from the base $\mathcal{B}$ of the Wiman–Edge pencil (a copy of $\mathbb{P}^1$) to $|\mathcal{O}_P(10)|$. We denote its image by $p_*\mathcal{B}$. It is clear that every point of $p_*\mathcal{B}$ will be an $\mathfrak{A}_5$-invariant curve. An $\mathfrak{A}_5$-invariant curve in P admits an $\mathfrak{A}_5$-invariant

equation (because every homomorphism $\mathfrak{A}_5 \to \mathbb{C}^\times$ is trivial), and so every member of $p_*\mathcal{C}$ lands in the projectivization of $(\mathrm{Sym}^{10} I)^{\mathfrak{A}_5}$ (recall that P is the projectivization of the dual of I).

Since $\mathfrak{A}_5$ acts on I as the group of orientation-preserving elements of a Coxeter group of type H_3, this space is easy to determine using the invariant theory of Coxeter groups: if Φ_2 is an equation for K and Φ_6, Φ_{10}, Φ_{15} an equation for the $\mathfrak{A}_5$-invariant union of resp. 6, 10 and 15 lines, then these generate the $\mathfrak{A}_5$-invariants in the symmetric algebra of I. The first three generate the invariants of the Coxeter group and are algebraically independent ($\Phi_{15}{}^2$ is a polynomial in these). So $(\mathrm{Sym}^{10} I)^{\mathfrak{A}_5}$ is of dimension 3 and has the basis $\{\Phi_2{}^5, \Phi_6\Phi_2{}^2, \Phi_{10}\}$. Thus every $\mathfrak{A}_5$-invariant decimic in P can be written by the equation

$$a\Phi_2{}^5 + b\Phi_6\Phi_2{}^2 + c\Phi_{10} = 0 \tag{3}$$

In particular, $p_*\mathcal{B}$ is a plane curve. Since $p_*\Delta$ is the transversal intersection of the 10 lines union and K (the common zero set of Φ_2 and Φ_{10}), we also see from equation (3) that every $\mathfrak{A}_5$-invariant decimic in P passes through $p_*\Delta$ and is there tangent to the 10 lines union unless it contains K as an irreducible component of multiplicity 2 (i.e., $c = 0$).

The formula (3) also proves the following

Proposition 4.9 (The net of $\mathfrak{A}_5$-decimics) *All members of the net of $\mathfrak{A}_5$-decimics intersect the fundamental conic transversally at* 20 *points and they are all tangent at these points to one of the* 10 *lines that joins two antipodal points.*

Of course, the 20 points in the statement of Proposition 4.9 are the projection of the set Δ of base points of the Wiman–Edge pencil.

Proposition 4.10 *The members of $p_*\mathcal{C}$ distinct from $5K$ are reduced and intersect K transversally in $p_*\Delta$. The map $\mathcal{B} \to p_*\mathcal{B}$ defined by $C \mapsto p_*C$ is injective, and $p_*\mathcal{B}$ is a curve of degree 5 in $|\mathcal{O}_P(10)|$.*

Proof If $C \in \mathcal{C}$ is not equal to C_K then

$$p_*C \cdot K = C \cdot C_K = (-2K_S)^2 = 20.$$

Since this is also the size of $p(C) \cap K = p(\Delta)$, it follows that p_*C is reduced and meets K transversally in $p(\Delta)$.

Let C_1, C_2 be members of the Wiman–Edge pencil distinct from C_K. Since p_*C_i is reduced, the map $p \colon C_i \to p_*C_i$ is a normalization and so the equality $p_*C_1 = p_*C_2$ lifts to an $\mathfrak{A}_5$-equivariant isomorphism $C_1 \cong C_2$. As $\mathcal{C}$ is the universal family, it follows that $C_1 = C_2$. So the map $\mathcal{B} \to p_*\mathcal{B}$ is injective.

It also follows that for $z \in P$ generic, then through each of the five points of $p^{-1}(z)$ passes exactly one member of $\mathcal{C}$ and these members are distinct and smooth. This means that the hyperplane in $|\mathcal{O}_P(10)|$ of decimics passing through z meets $p_*\mathcal{B}$ transversally in five points and so the curve in question has degree 5. $\qquad\square$

In particular, $p_*\mathcal{C}$ is not a pencil. We can be a bit more precise. If $a\Phi_2{}^5 + b\Phi_6\Phi_2{}^2 + c\Phi_{10} = 0$ represents $p_*C \in p_*\mathcal{C}$, with $p_*C \neq 5K$, then the fact that p_*C is transversal to K implies that $c \neq 0$. So such a curve has unique equation for which $c = 1$. In particular, p_*R has an equation of the form

$$\Phi_R := a_R\Phi_2{}^5 + b_R\Phi_6\Phi_2{}^2 + \Phi_{10}.$$

(We will see later that this curve is in fact the Klein decimic.) It follows that Φ_2, Φ_6, Φ_R, Φ_{15} still generate the algebraic $\mathfrak{A}_5$-invariants.

Proposition 4.11 *The plane curve $p_*\mathcal{B}$ has two singular points, namely the points represented by p_*R and by $5K$, where it has a singularity of type A_4, resp. E_8 (having local-analytic parameterizations $t \mapsto (t^2, t^5)$, resp. $t \mapsto (t^3, t^5)$).*

Proof Let $x \in \Delta$. Since p has simple ramification at x, we can find local-analytic coordinates (z_1, z_2) at x and (w_1, w_2) at $p(x)$ such that $p^*w_1 = z_1^2$ and $p^*w_2 = z_2$, and such that K is at $p(x)$ given by $w_2 = 0$. So the ramification locus R is given at x by $z_1 = 0$ and C_K by $z_2 = 0$.

A tangent direction at x not tangent to C_K has in the (z_1, z_2)-coordinates a unique generator of the form $(\lambda, 1)$. We therefore can regard λ as a coordinate for the complement in $\mathbb{P}(T_xS)$ of the point defined by T_xC_K and hence as a coordinate for the complement $\mathcal{B}^+ \subset \mathcal{B}$ of the point representing C_K. This means that the member of $\mathcal{C} \smallsetminus \{C_K\}$ corresponding to λ has a local parametrization at x given by $z_1 = \lambda z_2(1 + c_1(\lambda)z_2 + \cdots)$. Its image under p has then the local parametrization $w_1 = \lambda^2 w_2^2(1 + 2c_1(\lambda)w_2 + \cdots)$, which shows that λ^2, when regarded as a regular function on $\mathcal{B}^+$, is in fact a regular function on its image $p_*\mathcal{B}^+$.

Now let us make these coordinate choices compatible with the chosen basis of invariants. For this we choose a third root $\Phi_6^{1/3}$ of Φ_6 and take $w_1 = \Phi_R\Phi_6^{-5/3}$ (this means that z_1 must be a square root of this) and $w_2 = \Phi_2\Phi_6^{-1/3}$. We write a member of $p_*\mathcal{B}^+$ uniquely as

$$a(\lambda)\Phi_2{}^5 + b(\lambda)\Phi_6\Phi_2{}^2 = \Phi_R$$

with a and b polynomials of degree $\leqslant 5$ and 5 being attained. If we multiply this equation by $\Phi_6^{-5/3}$, then it becomes

$$a(\lambda)w_2{}^5 + b(\lambda)w_2{}^2 = w_1.$$

It follows from the preceding that $b(\lambda) = \lambda^2$. Hence a has degree 5. By Proposition 4.10, the curve $\lambda \mapsto (a(\lambda), \lambda^2)$ must be injective. This means that $a(\lambda) - a(-\lambda)$ is nonzero when $\lambda \neq 0$. This can only happen when there is at most one odd power of λ appearing in a. This power must then be 5, of course. It follows that $\mathbb{P}^1 \cong \mathcal{B} \to p_*\mathcal{B}$ is given by

$$[\lambda : \mu] \mapsto (\lambda^5 a_5 + \lambda^4\mu a_4 + \lambda^2\mu^3 a_2)\Phi_2{}^5 + (\lambda^2\mu^3)\Phi_6\Phi_2{}^2 - \mu^5\Phi_R,$$

where a_5, a_4, a_2 are constants with $a_5 \neq 0$. The proposition follows from this. $\qquad\square$

5 Images of some members of the Wiman–Edge pencil in the Klein plane

5.1 The preimage of a fundamental conic

We will characterize C_K and $C_{K'}$ as members of $\mathcal{C}$ by the fact that they support an exceptional even theta characteristic. Recall that a *theta characteristic* of a projective smooth curve C is a line bundle κ over C_K endowed with an isomorphism $\phi \colon \kappa^{\otimes 2} \cong \omega_C$. It is called even or odd according to the parity of the dimension of $H^0(C, \kappa)$.

The strong form of Clifford's theorem as stated in [2] implies that

$$\dim |\kappa| \leqslant \frac{1}{2}\,(g(C) - 1)$$

provided that C is not hyperelliptic. If $|\kappa|$ is 1-dimensional and without fixed points, then the associated morphism $C \to \check{\mathbb{P}}(H^0(C, \kappa)) \cong \mathbb{P}^1$ is of degree $\leqslant g(C) - 1$ and has as fibers the moving part of $|\kappa|$. We can obtain this morphism by means of a linear projection of the canonical image: the natural map $\mathrm{Sym}^2 H^0(C, \kappa) \to H^0(C, \omega_C)$ is injective (with image a 3-dimensional subspace), so that dually, we have a projection onto a projective plane

$$\check{\mathbb{P}}(H^0(C, \omega_C)) \dashrightarrow \check{\mathbb{P}}(\mathrm{Sym}^2 H^0(C, \kappa)).$$

The latter contains a conic that can be identified with the image of $\check{\mathbb{P}}(H^0(C, \kappa))$ under the Veronese map. The composition with the canonical map $C \to \check{\mathbb{P}}(H^0(C, \omega_C))$ has image in this conic and thus realizes $C \to \check{\mathbb{P}}(H^0(C, \kappa))$. Note that this conic can be understood as a quadric in $\check{\mathbb{P}}(H^0(C, \omega_C))$ of rank 3 that contains the canonical image of C.

Proposition 5.1 *The morphism $C_K \to K$ is obtained as the complete linear system of an even theta characteristic κ on C_K followed by the (Veronese) embedding $K \subset P$. The $\mathfrak{A}_5$-action on C_K lifts to an action of the binary icosahedral group $\widetilde{\mathfrak{A}}_5$ on κ in such a way that $H^0(C_K, \kappa)$ is an irreducible $\widetilde{\mathfrak{A}}_5$-representation of degree 2. Similarly for $C_{K'} \to K'$, albeit that $H^0(C_{K'}, \kappa')$ will be the other irreducible $\widetilde{\mathfrak{A}}_5$-representation of degree 2.*

There are no other pairs (C, θ), where C is a member of $\mathcal{C}$ and θ is an $\mathfrak{A}_5$-invariant theta characteristic with $\dim |\theta| = 1$.

Proof Observe that the preimage of a line in P meets C_K in a canonical divisor and that any effective degree 2 divisor on K spans a line in P. This implies that the fibers of $C_K \to K$ belong to the divisor class of a theta characteristic $(\kappa, \phi \colon \kappa^{\otimes 2} \cong \omega_{C_K})$. The $\mathfrak{A}_5$-action on C_K need not lift to such an action on κ, but its central extension, the binary icosahedral group $\widetilde{\mathfrak{A}}_5$, will (in a way that makes ϕ equivariant). Thus $H^0(C_K, \kappa)$ becomes an $\widetilde{\mathfrak{A}}_5$-representation. It contains a 2-dimensional (base point free) subrepresentation which accounts for the morphism $C_K \to K$. To see that this inclusion is an equality, we note that by Clifford's theorem as cited above, $\dim H^0(C_K, \kappa) \leqslant 3$. If it were equal to 3, then $\widetilde{\mathfrak{A}}_5$ would have a trivial summand in

$H^0(C_K, \kappa)$ and hence so would $H^0(C_K, \kappa^{\otimes 2}) = H^0(C_K, \omega_{C_K})$. This contradicts the fact that the latter is of type $I \oplus I'$ as an $\mathfrak{A}_5$-representation.

It is clear that we obtain $(C_{K'}, \kappa')$ as the transform/pull-back of (C_K, κ) under an element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$.

If (C, θ) is as in the proposition, then C lies on an $\mathfrak{A}_5$-invariant quadric of rank 3. This quadric will be defined by $\lambda Q + \lambda' Q'$ for some $(\lambda : \lambda')$ (with Q and Q' as in Sect. 2.5), and so the rank condition implies $\lambda = 0$ or $\lambda' = 0$. In other words, (C, θ) equals (C_K, κ) or $(C_{K'}, \kappa')$. $\qquad\square$

5.2 The image of reducible singular members

Recall that C_c and C_c' are the $\mathfrak{A}_5$-orbit of a special conic. So the following proposition tells us what their p-images are like.

Proposition 5.2 *The projection p maps each special conic isomorphically onto a conic in P that is tangent to four lines that are the projections of lines on S, and passes through four points of the 10-element $\mathfrak{A}_5$-orbit.*

Proof Let Y be a special conic. Since Y is part of C_c or C_c', Y meets Δ in four distinct points and $p_* Y$ is a degree 2 curve which meets $p(\Delta)$ in four points. In particular p maps Y isomorphically onto its image (not on a double line). For each $a \in Y \cap \Delta$, $p(a)$ and its antipode span a line in P and according to Proposition 4.10, $p(Y)$ is tangent to the line at $p(a)$. $\qquad\square$

5.3 The decimics of Klein and Winger

Let us call a projective plane endowed with a group G of automorphisms isomorphic to $\mathfrak{A}_5$ a *Klein plane* (but without specifying an isomorphism $G \cong \mathfrak{A}_5$). Such a plane is unique up to isomorphism (it is isomorphic to both P and P'). For what follows it is convenient to make the following definition.

Definition 5.3 (*Special decimics*) We call a reduced, G-invariant curve in a Klein plane of degree 10 a *special decimic* if it is singular at each fundamental point and its normalization is rational. If the singularity at a fundamental point is an ordinary node, we call it a *Winger decimic*; otherwise (so when it is worse than that) we call it a *Klein decimic*.

We will see that each of these decimics is unique up to isomorphism, and that $p(R)$ is a Klein decimic and $p(R')$ a Winger decimic. We will also show that the singularity of a Klein decimic at a fundamental point must be a double cusp (i.e., with local analytic equation $(x^3 - y^2)(y^3 - x^3)$). As to our naming: a Klein curve appears in [18, Chapter 4, Section 3] (p. 218 in the cited edition), and a Winger curve appears in [26, Section 9].

Let us first establish the relation between special decimics and the Wiman–Edge pencil. Let K be a copy of a $\mathbb{P}^1$ and let $G \subset \mathrm{Aut}(K)$ be a subgroup isomorphic to $\mathfrak{A}_5$. Recall that K has a unique G-orbit $F^\#$ of size 12 (think of this as the vertex set of a regular icosahedron) which comes in six antipodal pairs. Let us denote by

$z \in F^{\#} \mapsto z' \in F^{\#}$ the antipodal involution, and let $\overline{K}$ be obtained from $\widetilde{Y}$ by identifying every $z \in F^{\#}$ with $z' \in F^{\#}$ as to produce an ordinary node. This is just the curve of arithmetic genus 6 that we constructed in Lemma 3.9.

Notice that the normalization of a special decimic factors through this quotient of K: it is the image of $\overline{K}$ under a 2-dimensional linear system of degree 10 divisors on $\overline{K}$ that comes from a 3-dimensional irreducible subrepresentation of $H^0(\overline{K}, \mathcal{O}_{\overline{K}}(10))$. In order to identify these subrepresentations, we focus our attention on the dualizing sheaf $\omega_{\overline{K}}$ of $\overline{K}$. This is the subsheaf of the direct image of $\omega_K(F^{\#})$ characterized by the property that the sum of the residues in a fiber add up to zero. So $H^0(\overline{K}, \omega_{\overline{K}})$ is the subspace of $H^0(K, \omega_K(F^{\#}))$ consisting of differentials whose residues at the two points of any antipodal pair add up to zero.

Lemma 5.4 *There is an exact sequence of G-representations*

$$0 \to H^0(\overline{K}, \omega_{\overline{K}}) \to H^0(K, \omega_K(F^{\#})) \to \widetilde{H}^0(F; \mathbb{C}) \to 0$$

where the last term denotes the reduced cohomology of F. Further, $\widetilde{H}^0(F; \mathbb{C})$ is an irreducible G-representation of dimension 5 (hence of type W).

Proof For every $x \in F$ there is a linear form on $H^0(K, \omega_K(F^{\#}))$ that assigns to $\alpha \in H^0(K, \omega_K(F^{\#}))$ the sum of the residues at the associated antipodal pair on K. By the residue formula, these six linear forms add up to zero. Apart from that, residues can be arbitrarily described and we thus obtain the exact sequence. The character of the permutation representation on F (which can be thought of as the set of six lines through opposite vertices of the icosahedron) is computed to be that of trivial representation plus that of W. This implies that $\widetilde{H}^0(F; \mathbb{C}) \cong W$ as G-representations. □

We know by Propositions 3.10 and 4.3 that the dualizing sheaf of $\overline{K}$ defines the canonical embedding for $\overline{K}$ and that the 6-dimensional G-representation $H^0(\overline{K}, \omega_{\overline{K}})$ decomposes into two irreducible subrepresentations of dimension 3 that are not of the same type. This enables us to prove that there are only two isomorphism types of special decimics.

Corollary 5.5 (Classification of special decimics) *Let $Y \subset P$ be a special decimic and let $\overline{K} \to Y$ define a partial G-equivariant normalization (so that the six nodes of $\overline{K}$ lie over the six fundamental points of P). Then Y can be identified in a G-equivariant manner with the image of $\overline{K}$ under the linear system that comes from one of the two irreducible 3-dimensional G-subrepresentations of $H^0(\overline{K}, \omega_{\overline{K}})$. In fact, p_*R and p'_*R are special decimics and every special decimic is isomorphic to one of them.*

Proof The line bundle $\omega_K(F^{\#})$ is of degree 10. The fact that $\omega_{\overline{K}}$ is of degree $-2+12 = 10$ implies that $\mathbb{P}(H^0(K, \omega_K(F^{\#})))$ is the complete linear system of degree 10. As the G-embedding $Y \subset P$ is of degree 10, the G-embedding $Y \subset P$ is definable by a 3-dimensional G-invariant subspace of $H^0(K, \omega_K(F^{\#}))$. It follows from Lemma 5.4 that this subrepresentation must be contained in $H^0(\overline{K}, \omega_{\overline{K}})$, and hence is given by one of its 3-dimensional summands.

Let us write E_K for $H^0(\overline{K}, \omega_{\overline{K}})$ and $\mathbb{P}_K$ for $\check{\mathbb{P}}(E_K)$. The canonical map $\overline{K} \to \mathbb{P}_K$ is an embedding and realizes $\overline{K}$ as a member of the Wiman–Edge pencil: if we regard this embedding as an inclusion, then the $\mathfrak{A}_5$-embedding $R \subset \mathbb{P}_S$ is obtained from the G-embedding $\overline{K} \subset \mathbb{P}_K$ via a compatible pair of isomorphisms $(G, E_K) \cong (\mathfrak{A}_5, E_S)$. The preimage $S_K \subset \mathbb{P}_K$ of $S \subset \mathbb{P}_S$ is a G-invariant quintic del Pezzo surface which contains $\overline{K}$. We have a decomposition $E_K = I_K \oplus I'_K$ into two irreducible subrepresentations such that the two associated linear systems of dimension 2 reproduce the projections $p|R$ and $p'|R$. So p_*R and p'_*R represent the two types of special decimics. $\qquad\square$

Theorem 5.6 *The preimage $p^{-1}F$ of the set of fundamental points is the disjoint union of $\mathrm{Sing}(R)$ and $\mathrm{Sing}(R')$. Moreover, p_*R' is a Winger decimic and p_*R is a Klein decimic.*

Proof We know that the singular set of R makes up an $\mathfrak{A}_5$-orbit and that such an $\mathfrak{A}_5$-orbit is mapped to F. The same is true for R' and so each fundamental point is a singular point of both p_*R and p_*R'. For every singular point x of R, $p^{-1}p(x) \smallsetminus \{x\}$ consists of $k \leqslant 5 - 2 = 3$ points. So $p^{-1}p(\mathrm{Sing}(R)) \smallsetminus \mathrm{Sing}(R)$ is an $\mathfrak{A}_5$-invariant set of size $6k \in \{6, 12, 18\}$.

Our irregular orbit catalogue for the $\mathfrak{A}_5$-action on S (see Sect. 3.6) shows that only $k = 1$ is possible, so that this must be the 6-element orbit in S different from $\mathrm{Sing}(R)$, i.e., $\mathrm{Sing}(R')$. This proves that the (disjoint) union of $\mathrm{Sing}(R)$ and $\mathrm{Sing}(R')$ make up a fiber of p. In particular, p_*R' has an ordinary double point at $p(x)$ and hence is a Winger decimic. On the other hand, p_*R has multiplicity at least 4 at $p(x)$: this is because the restriction of p to a local branch of R has a singularity at x (because of the presence of the other branch). So p_*R must be a Klein decimic. $\qquad\square$

Remark 5.7 Theorem 5.6 shows that there is a unique $\mathfrak{A}_5$-equivariant bijection

$$\mathrm{Sing}(C_{\mathrm{ir}}) \xrightarrow{\cong} \mathrm{Sing}(C'_{\mathrm{ir}})$$

that commutes with p. This gives rise to a bijection $\sigma \colon F \cong F'$ between the set of fundamental points in P and those in P'. Similarly, p' will determine an $\mathfrak{A}_5$-equivariant bijection $\sigma' \colon F' \to F$. The composition $\sigma' \circ \sigma$ is a permutation of F that commutes with the $\mathfrak{A}_5$-action. Since that $\mathfrak{A}_5$-action contains 5-cycles (which have just one fixed point in F), it follows that this permutation must be the identity: the two bijections are each others inverse. It follows that for every $x \in \mathrm{Sing}(C_{\mathrm{ir}})$, the set $\{x, \sigma(x)\}$ is a fiber of $p \times p' \colon S \to P \times P'$.

We can now also say a bit more about the special decimics.

Proposition 5.8 *The singular set of a Winger decimic consists of 36 ordinary double points, and a local branch at each fundamental point has that node as a hyperflex. The singular points that are not fundamental make up a 30-element orbit.*

Proof For the first assertion, we essentially follow Winger's argument. The normalization $q \colon \widetilde{Y} \to Y \subset P$ is a rational curve that comes with an automorphism group

$G \cong \mathfrak{A}_5$. The points of $\widetilde{Y}$ mapping to a flex point or worse make up a divisor D of degree $3(10 - 2) = 24$ (if t is an affine coordinate of $\widetilde{Y}$, and $(z_0:z_1:z_2)$ is a coordinate system for P such that q^*z_0, resp. q^*z_i, is a polynomial of degree $10 - i$, resp. $\leqslant 10 - i$, then D is the defined by the Wronskian determinant (z_0, z_1, z_2) and is viewed as having degree $(10 + 9 + 8) - 3 = 24$). If a point of $\widetilde{Y}$ maps to cusp or a hyperflex (or worse) then it appears with multiplicity $\geqslant 2$. We prove that each of the two points of $\widetilde{Y}$ lying over a fundamental point of P has multiplicity $\geqslant 2$ in D. This implies that D is $\geqslant$ twice the 12-element G-orbit in $\widetilde{Y}$ and hence, in view of its degree, must be equal to this. It will then follow that Y has only nodal singularities and that these will be 36 in number by the genus formula.

Let us first note that since G has no nontrivial 1-dimensional character, Y admits a G-invariant equation. Let $x \in P$ be a fundamental point. Let $U \subset P$ be the affine plane complementary to the polar of x and make it a vector space by choosing x as its origin. Then G_x acts linearly on U. The stabilizer G_x is a dihedral group of order 10, and we can choose coordinates (w_1, w_2) such that $(w_1, w_2) \mapsto (w_2, w_1)$ and $(w_1, w_2) \mapsto (\zeta_5 w_1, \zeta_5^{-1} w_2)$ define generators of G_x. The algebra of G_x-invariant polynomials is then generated by $w_1 w_2$ and $w_1^5 + w_2^5$.

The germ of Y at x admits G_x-invariant equation and this is given by $w_1 w_2$+order $\geqslant 2$. So the tangent lines of the branches of Y at x are the coordinate lines. The subgroup of G_x generated by $(w_1, w_2) \mapsto (\zeta_5 w_1, \zeta_5^{-1} w_2)$ fixes each branch. So the branch whose tangent line is $w_1 = 0$ has a local equation of the type

$$w_1 + a w_1^2 w_2 + b w_1^3 w_2^2 + c w_2^4 + \text{(terms of order } \geqslant 5) = 0.$$

This branch meets $w_1 = 0$ at the origin with multiplicity $\geqslant 4$ and hence has there a hyperflex.

So $\mathrm{Sing}(p_* R') \smallsetminus F$ is an $\mathfrak{A}_5$-invariant 30-element subset of P and it suffices to prove that this is an $\mathfrak{A}_5$-orbit. This set does not meet the fundamental conic K (for $p_* R'$ is transversal to that conic). There are unique $\mathfrak{A}_5$-orbits in $P \smallsetminus K$ of size 6 and 10; the others have size 30 and 60. Hence $\mathrm{Sing}(p_* R') \smallsetminus F$ is a 30-element orbit in $P \smallsetminus K$. $\square$

5.4 Construction of the Klein decimic

We now give a construction of the Klein decimic, which will at the same time show that it will have double cusp singularities at the fundamental points.

Let P be a Klein plane. We first observe that its fundamental set F does not lie on a conic. For if it did, then this conic would be unique, hence $\mathfrak{A}_5$-invariant and so its intersection with K would then produce an orbit with $\leqslant 4$ elements, which we know does not exist. This implies that for each $x \in F$ there is a conic K_x which meets F in $F \smallsetminus \{x\}$.

Now blow up F. The result is a cubic surface X for which the strict transform $\widetilde{K}_x$ of K_x is a line (an exceptional curve) and these lines are pairwise disjoint([3]). The set

---

[3] This surface is isomorphic to the *Clebsch diagonal cubic surface* in $\mathbb{P}^4$ defined by $\sum_{i=0}^{4} Z_i^3 = 0$, $\sum_{i=0}^{4} Z_i = 0$; the evident $\mathfrak{S}_5$-symmetry accounts for its full automorphism group and its isomorphism

of six exceptional curves arising from the blow-up and the $\widetilde{K}_x$ form what is called a *double six* in X. So they can be simultaneously blown down to form a copy $P^\dagger$ of a projective plane. The naturality of this construction implies that the $\mathfrak{A}_5$-symmetry is preserved. In particular, $P^\dagger$ comes with a nontrivial $\mathfrak{A}_5$-action, and so we have defined a fundamental conic $K^\dagger \subset P^\dagger$. The image of $\bigcup_x \widetilde{K}_x$ in $P^\dagger$ is an $\mathfrak{A}_5$-invariant subset of size 6 and so this must be the fundamental set $F^\dagger \subset P^\dagger$. So the diagram $P \xleftarrow{q} X \xrightarrow{q^\dagger} P^\dagger$ is involutive. The birational map

$$(q^\dagger)^{-1}q : P \dashrightarrow P^\dagger$$

is defined by the linear system I_K of quintics in P which have a node (or worse) at each fundamental point, so that $P^\dagger$ gets identified with the projective space dual to $|I_K|$.

Lemma 5.9 *If ℓ_x is the polar line of x, then $K_x \cap K = \ell_x \cap K$ and hence K_x is tangent to K at these two points.*

Proof The divisor $\sum_{x \in F} K_x \cdot K$ in K is $\mathfrak{A}_5$-invariant and of degree $6 \cdot 4 = 24$. Given the $\mathfrak{A}_5$-orbit sizes in K (12, 20, 30, 60), this implies that that this divisor is twice the 12-element orbit. Hence K_x meets K in two distinct points of this orbit and is there tangent to K. Assigning to x the polar of $K \cap K_x$ then gives us a map from F onto a 6-element orbit of P. This orbit can only be F itself. We thus obtain a permutation of F which commutes with the $\mathfrak{A}_5$-action. Since every point of F is characterized by its $\mathfrak{A}_5$-stabilizer, this permutation must be the identity.   □

Corollary 5.10 *The Cremona transformation $q^\dagger q^{-1} : P \dashrightarrow P^\dagger$ takes $K \subset P$ to a Klein decimic in $P^\dagger$. This Klein decimic has a double cusp (with local-analytic equation $(x^2 - y^3)(x^3 - y^2) = 0$) at each fundamental point, and is smooth elsewhere.*

Proof Since q is an isomorphism over K, we can identify K with $q^{-1}K$. Any singular point of $q^\dagger q^{-1}K$ will of course be the image of some K_x. Since we can regard K_x as the projectivized tangent space of the fundamental point in $P^\dagger$ to which it maps, Lemma 5.9 shows that the image of K at this fundamental point is as asserted: we have two cusps meeting there with different tangent lines.   □

Note that by the involutive nature of this construction, $q(q^\dagger)^{-1}$ will take $K^\dagger$ to the Klein decimic in P.

Remark 5.11 Theorem 5.6 and the preceding corollary imply that the double cusp singularity appears as a discriminant curve of a finite morphism between surface germs of degree 4. Here is a local description for it that also takes into account the $\mathfrak{D}_5$-symmetry: there exist a local-analytic coordinate system (z_1, z_2) for S at a singular point of R and a local-analytic coordinate system (w_1, w_2) at its image in P such that

$$p^*w_1 = z_1^2 + z_2^3, \quad p^*w_2 = z_1^3 + z_2^2.$$

Footnote 3 continued

type is characterized by that property. The intersection of this cubic surface with the quadric defined by $\sum_{i=0}^4 Z_i^2 = 0$ is the *Bring curve* mentioned in Remark 5.12; its automorphism group is also $\mathfrak{S}_5$.

Note that it is indeed of degree 4 at the origin. The ramification locus is defined by $z_1 z_2 (9z_1 z_2 - 4) = 0$, hence is near the origin given by $z_1 z_2 = 0$. The image of the ramification locus at the origin is the double cusp, for the z_1-axis, parametrized by $(z_1, z_2) = (t, 0)$ is the the parametrized cusp $(w_1, w_2) = (t^2, t^3)$ and likewise z_2-axis maps to the parametrized cusp $(w_1, w_2) = (t^3, t^2)$.

The $\mathfrak{A}_5$-isotropy groups of the two points is a dihedral isotropy group of order 10 and so the map-germ must have this symmetry as well. We can see this being realized here as in fact coming from two linear representations of this group. Let $\mathfrak{D}_5$ be the dihedral group of order 10, thought of as the semi-direct product of the group μ_5 of the 5th roots of unity and an order 2 group whose nontrivial element s acts by inversion: $s\zeta = \zeta^{-1}s$. Then letting $\mathfrak{D}_5$ act on the source, resp. target, by

$$s(z_1, z_2) = (z_2, z_1); \qquad \zeta(z_1, z_2) = (\zeta z_1, \zeta^{-1} z_2);$$
$$s(w_1, w_2) = (w_2, w_1); \qquad \zeta(w_1, w_2) = (\zeta^2 w_1, \zeta^{-2} w_2)$$

makes the map-germ $\mathfrak{D}_5$-equivariant. It can also be verified that the $\Sigma^{1,1}$-multiplicity of this germ is 3, as it must be, in view of the proof of Proposition 4.3.

Remark 5.12 We have seen that the curve of Wiman decimics in P contains both $5K$ and the $\mathfrak{A}_5$-invariant sum of 10 lines. It has as a remarkable counterpart: the pencil of sextics spanned by $3K$ and the $\mathfrak{A}_5$-invariant sum of six lines. This pencil was studied in detail by Winger [26] and is discussed in [8, 9.5.11]. The six lines meet K in the 12-element orbit and this intersection is evidently transversal. It follows that this orbit is the base locus of the *Winger pencil* (as we will call it) and that all members of the pencil except $3K$ have a base point as a flex point (with tangent line the fundamental line passing through it). Note that a general member W of the pencil is smooth of genus 10.

Winger shows that this pencil has, besides its two generators, which are evidently singular, two other singular members. Both are irreducible; one of them has as its singular locus a node at each fundamental point (the 6-element orbit of $\mathfrak{A}_5$) and the other, which we shall call the *Winger sextic*, has as its singular locus a node at each point of the 10-element orbit of $\mathfrak{A}_5$, and each local branch at such a point has this point as a flex point. So their normalizations have genus 4 and 0 respectively. The former turns out be isomorphic to the Bring curve (whose automorphism group is known to be isomorphic to $\mathfrak{S}_5$) and the latter will be isomorphic as an $\mathfrak{A}_5$-curve to K.

In fact, as an abstract curve with $\mathfrak{A}_5$-action, the Winger sextic can be obtained in much the same way as the curve constructed in Lemma 3.9, simply by identifying the antipodal pairs of its 20-element orbit so as to produce a curve with 10 nodes. This plane sextic is also discussed in [9]. With the help of the Plücker formula (e.g., [8, formula 1.50]), it then follows that the dual curve of the Winger sextic is a Klein decimic. (A priori this curve lies in the dual of P, but in the presence of the fundamental conic $K \subset P$ we can regard it as a curve in P: just assign to each point of the Winger curve the K-polar of its tangent line.)

Other remarkable members of this pencil include the two nonsingular *Valentiner sextics* with automorphism group isomorphic to $\mathfrak{A}_6$ ([4]).

6 Passage to $\mathfrak{S}_5$-orbit spaces

The goal of this section to characterize the Wiman curve as a curve on $\overline{\mathcal{M}}_{0,5}$, or rather its $\mathfrak{S}_5$-orbit space in $\mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5}$. The latter is simply the moduli space of stable effective divisors of degree 5 on $\mathbb{P}^1$.

6.1 Conical structure of $\mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5}$

We first want to understand how the $\mathfrak{S}_5$-stabilizer $\mathfrak{S}_{5,z}$ of a point $z \in \overline{\mathcal{M}}_{0,5}$ acts near z. The finite group action on the complex-analytic germ of $\overline{\mathcal{M}}_{0,5}$ at z can be linearized in the sense that it is complex-analytically equivalent to the action on the tangent space. It then follows from a theorem of Chevalley that the local orbit space at $\mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5}$ is nonsingular at the image of z if and only if $\mathfrak{S}_{5,z}$ acts on $T_z\overline{\mathcal{M}}_{0,5}$ as a complex reflection group.

Lemma 6.1 *The $\mathfrak{S}_5$-stabilizer of any point not in Δ is a Coxeter group (and hence a complex reflection group) so that $\mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5}$ is nonsingular away from the point δ that represents the orbit Δ.*

Proof If z is represented in the Hilbert–Mumford model by $(x_1, \ldots, x_5) \in (\mathbb{P}^1)^5$, then the tangent space $T_z\overline{\mathcal{M}}_{0,5}$ fits naturally in a short exact sequence:

$$0 \rightarrow H^0(\mathbb{P}^1, \theta_{\mathbb{P}^1}) \rightarrow \bigoplus_{i=1}^{5} T_{x_i}\mathbb{P}^1 \rightarrow T_z\overline{\mathcal{M}}_{0,5} \rightarrow 0 \tag{4}$$

where $H^0(\mathbb{P}^1, \theta_{\mathbb{P}^1})$ is the space of vector fields on $\mathbb{P}^1$ (which in terms of an affine coordinate t has basis $\partial/\partial t, t\partial/\partial t, t^2\partial/\partial t$) and the map is given by evaluation at $x_1, \ldots, x_5$ respectively. With the help of this formula it is fairly straightforward to determine the action of $\mathfrak{S}_{5,z}$ on $T_z\overline{\mathcal{M}}_{0,5}$. When $\mathfrak{S}_{5,z}$ is of order 2, we need to verify that $\mathfrak{S}_{5,z}$ leaves a line in $T_z\overline{\mathcal{M}}_{0,5}$ pointwise fixed. This is left to the reader. There are three other cases to consider:

Case 1: $z = (1, \zeta_3, \zeta_3^2, 0, \infty)$, and $\mathfrak{S}_{5,z} \cong \mathfrak{S}_3^{\mathrm{ev}}$ is the group generated by $t \mapsto \zeta_3 t$ and the involution $t \mapsto t^{-1}$ (which gives the exchanges $\zeta_3 \leftrightarrow \zeta_3^2$ and $0 \leftrightarrow \infty$). The fact that $H^0(\mathbb{P}^1, \theta_{\mathbb{P}^1})$ maps onto the direct sum of tangent spaces at the third root of unity implies that

$$T_0\mathbb{P}^1 \oplus T_\infty\mathbb{P}^1 \rightarrow T_z\overline{\mathcal{M}}_{0,5}$$

---

is an isomorphism. Multiplication by ζ_3 acts here as $(v_0, v_\infty) \mapsto (\zeta_3 v_0, \zeta_3^{-1} v_\infty)$ and inversion exchanges v_0 and v_∞. We thus find that $\mathfrak{S}_{5,z}$ acts as the Coxeter group A_2 on $T_z \overline{\mathcal{M}}_{0,5}$.

Case 2: $z = (0, 0, 1, \infty, \infty)$, and $\mathfrak{S}_{5,z} \cong \mathfrak{D}_8$ acts in the obvious manner. We again find that $T_0 \mathbb{P}^1 \oplus T_\infty \mathbb{P}^1 \to T_z \overline{\mathcal{M}}_{0,5}$ is an isomorphism. The group $\mathfrak{S}_{5,z}$ now acts as the Coxeter group B_2: It consists of the signed permutations of (v_0, v_∞).

Case 3: $z = (1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$, and $\mathfrak{S}_{5,z} \cong \mathfrak{D}_{10}$ is generated by $t \mapsto \zeta_5 t$ and the involution $t \mapsto t^{-1}$ (which gives the exchanges $\zeta_5^i \leftrightarrow \zeta_5^{-i}$). For $i \geqslant -1$ let $v^{(i)} \in \bigoplus_{\zeta^5=1} T_\zeta \mathbb{P}^1$ be the restriction of the vector field $t^{i+1} d/dt$. Then scalar multiplication by ζ_5 multiplies $v^{(i)}$ by ζ_5^{-i}, so that this is a basis of eigenvectors for this action. The vectors $v^{(-1)}, v^{(0)}, v^{(1)}$ span the image of $H^0(\mathbb{P}^1, \theta_{\mathbb{P}^1})$, and hence $v^{(2)}$ and $v^{(3)}$ span $T_z \overline{\mathcal{M}}_{0,5}$. Multiplication by ζ_5 sends $(v^{(2)}, v^{(3)})$ to $(\zeta_5^{-2} v^{(2)}, \zeta_5^{-3} v^{(3)})$ and the involution sends it to $(-v^{(3)}, -v^{(2)})$. Thus it acts through the Coxeter group $I_2(5)$. □

It is clear from Proposition 3.2 that the $\mathfrak{A}_5$-orbit space of every smooth member of $\mathcal{C}$ is a smooth rational curve. This is also true for a singular member, for we have seen that $\mathfrak{A}_5$ is transitive on its irreducible components and that an irreducible component is rational (a quotient curve of a rational curve is rational). It is clear that $\mathfrak{S}_5 \backslash \overline{\mathcal{M}}_{0,5}$ will inherit this structure. Since the base locus Δ is a single $\mathfrak{A}_5$-orbit and all members of the Wiman–Edge pencil are nonsingular at Δ, it then follows that the image of the Wiman–Edge pencil gives $\mathfrak{A}_5 \backslash \overline{\mathcal{M}}_{0,5}$ the structure of a quasi-cone with vertex δ.

Proposition 6.2 *Regard $\mathbb{P}^2$ as a projective cone with vertex the origin o of the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$. Let $\mathfrak{C}_6$ be the automorphism group of this cone generated by*

$$(u_0 : u_1 : u_2) \mapsto (u_0 : \zeta_3 u_1, -\zeta_3 u_2).$$

Then $\mathfrak{S}_5 \backslash \overline{\mathcal{M}}_{0,5}$ is as a quasi-cone isomorphic to $\mathfrak{C}_6 \backslash \mathbb{P}^2$ (which has an A_2-singularity at its vertex) and embeds in a projective space by means of the 5-dimensional linear system defined by the degree 6 subspace of $\mathbb{C}[u_0, u_1, u_2]^{\mathfrak{C}_6}$. This embedding is of degree 6 and is such that C_0 maps onto a line; any other member of $\mathcal{C}$ maps onto a conic.

Proof The type of this cone can be determined by a local study at a point of Δ in the way we did this in the proof of Lemma 6.1. One such point is represented in the Hilbert–Mumford model by $z := (0, 0, 1, \zeta_3, \zeta_3^2) \in (\mathbb{P}^1)^5$. The $\mathfrak{S}_5$-stabilizer of z is cyclic of order 6 with a generator σ with

$$\sigma(x_1, x_2, x_3, x_4, x_5) = (x_2, x_1, \zeta_3 x_3, \zeta_3 x_4, \zeta_3 x_5)$$

and the tangent space $T_z \overline{\mathcal{M}}_{0,5}$ is identified with $T_0 \mathbb{P}^1 \oplus T_0 \mathbb{P}^1$. If we identify $T_0 \mathbb{P}^1$ with $\mathbb{C}$ by taking the coefficient of $\partial/\partial t|_{t=0}$, then we see that σ acts as $(t_1, t_2) \in \mathbb{C}^2 \mapsto (\zeta_3 t_2, \zeta_3 t_1) \in \mathbb{C}^2$. So if we pass to $(u_1, u_2) = (t_1 + t_2, t_1 - t_2)$, then $\sigma(u_1, u_2) = (\zeta_3 u_1, -\zeta_3 u_2)$. An affine neighborhood of the vertex of the cone $\mathfrak{S}_5 \backslash \overline{\mathcal{M}}_{0,5}$ is the quotient of $\mathbb{C}^2$ by the automorphism group generated by σ.

The diagonal form of σ shows that it leaves invariant only two lines through the origin: the u_2-axis defined by $u_1 = 0$, on which it acts faithfully with order 6, and the u_1-axis defined by $u_2 = 0$ on which it acts with order 3. Since C_0 and C_∞ are the only members of $\mathcal{C}$ that are $\mathfrak{S}_5$-invariant, these axes must define their germs at z. The $\mathfrak{S}_5$-stabilizer of z (which we have identified with $\mathfrak{C}_6$) acts on the germ of C_∞ at 0 through a subgroup of $\mathfrak{S}_3$ (for it preserves a line on the quintic del Pezzo surface) and so $\mathfrak{C}_6$ cannot act faithfully on it. It follows that C_0 is defined by $u_1 = 0$.

Since σ^3 sends (u_1, u_2) to $(u_1, -u_2)$, its algebra of invariants is $\mathbb{C}[u_1, u_2^2]$. Now σ sends (u_1, u_2^2) to $(\zeta_3 u_1, \zeta_3^{-1} u_2^2)$, so that its orbit space produces a Kleinian singularity of type A_2. Following Klein, its algebra of invariants is generated by $u_1^3, u_1 u_2^2, u_2^6$. So the degree 6 part of $\mathbb{C}[u_0, u_1, u_2]^{\mathfrak{C}_6}$ has basis $u_0^6, u_0^3 u_1^3, u_0^3 u_1 u_2^2, u_2^6, u_1^6, u_1^2 u_2^4, u_1^4 u_2^2$, and this $\mathfrak{C}_6 \backslash \mathbb{P}^2$ in $\mathbb{P}^5$. Its image is clearly of degree 6. We have seen that the image of C_0 is given by $u_1 = 0$ and the above basis restricted to nonzero monomials are u_0^6 and u_2^6 and so its image is a line. If we substitute $u_2 = \lambda u_1$ (with λ fixed), then the monomials in question span $u_0^6, u_0^3 u_1^3, u_1^6$ and so the image of this line is indeed a conic.   □

Remark 6.3 The conclusion of the previous lemma and proposition agrees with the explicit isomorphism $\mathfrak{S}_5 \backslash \overline{\mathcal{M}}_{0,5} \cong \mathbb{P}(1, 2, 3)$ from Remark 6.6 below. The only singular point of the quotient is the point $(0:0:1)$, which is a rational double point of type A_2. The points projecting to the singular point satisfy $I_4 = I_8 = 0$. We know that $I_4 = 0$ defines the Wiman curve on S and $I_8 = 0$ defines the union $C_K + C'_K$. They intersect at 20 base points with the subgroup $\mathfrak{S}_3^{\mathrm{odd}}$ as the stabilizer subgroup.

The weighted projective space $\mathbb{P}(1, 2, 3)$ is obtained as the quotient of $\mathbb{P}^2$ by a cyclic group of order 6 acting on coordinates as

$$(x_0:x_1:x_2) \mapsto (x_0:\zeta_6^3 x_1:\zeta_6^2 x_2).$$

The linear space of $\mathfrak{C}_3$-invariant homogeneous polynomials of degree 6 is spanned by six monomials $x_0^6, x_1^6, x_0^4 x_1^2, x_0^2 x_1^4, x_2^6, x_0^3 x_2^3$. They define a $\mathfrak{C}_6$-invariant map $\mathbb{P}^2 \to \mathbb{P}^5$ which factors through the quotient $\mathbb{P}(1, 2, 3) = \mathbb{P}^2/\mathfrak{C}_6$. In terms of weighted homogeneous coordinates t_0, t_1, t_2 in $\mathbb{P}(1, 2, 3)$ it is given by monomials of the map $t_0^6, t_1^3, t_0^4 t_1, t_0^2 t_1^2, t_2^2, t_0^3 t_2$. The restriction of the map to the coordinate line $t_0 = 0$ is given by the monomials t_1^3, t_2^2. This shows that the image of the Wiman curve under the composition

$$S = \mathcal{M}_{0,5} \to \mathbb{P}(1, 2, 3) \to \mathbb{P}^5$$

is a line. One can interpret this map as the map

$$f : \mathcal{M}_{0,5} \to \mathbb{P}(V(5)) = \mathfrak{S}_5 \backslash (\mathbb{P}^1)^5 \cong \mathbb{P}^5,$$

where $V(5)$ is the space of binary quintic. According to [1], this map can be given explicitly by

$$(a_1, \ldots, a_5) \mapsto R(a_1, \ldots, a_5) = \prod_{i=1}^{5} (T_i z - S_i^3 w) = \sum_{i=0}^{5} B_i z^{5-i} w^i,$$

where S, T are fundamental invariants of degrees 2 and 3 of binary forms of degree 4 satisfying $S^3 - 27T^2$ equal to the discriminant. The reference gives explicit expressions for the coefficients B_i as polynomials of degree 24 in basic invariants I_4, I_8, I_{12} of binary quintic forms.

6.2 The cross ratio map

The embedding of $\mathfrak{S}_5 \backslash \overline{\mathcal{M}}_{0,5}$ in projective 5-space that we found in Proposition 6.2 has a modular interpretation, which we now give.

As was mentioned in Sect. 2.1, the conic bundles of S have such an interpretation on $\overline{\mathcal{M}}_{0,5}$ as forgetful morphisms

$$f_i : \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}{}^{(i)} \cong \mathbb{P}^1$$

(it forgets the ith point). The notation $\overline{\mathcal{M}}_{0,4}{}^{(i)}$ used here means that we do not renumber the four points and let them be indexed by $\{1, \ldots, 5\} \smallsetminus \{i\}$; this helps to make manifest the $\mathfrak{S}_5$-action on the product $\prod_{i=1}^5 \overline{\mathcal{M}}_{0,4}{}^{(i)}$ for which the morphism

$$f = (f_1, \ldots, f_5) : \overline{\mathcal{M}}_{0,5} \to \prod_{i=1}^5 \overline{\mathcal{M}}_{0,4}{}^{(i)}$$

is $\mathfrak{S}_5$-equivariant. The following is well known.

Lemma 6.4 *The map $f = (f_1, \ldots, f_5) : \overline{\mathcal{M}}_{0,5} \to \prod_{i=1}^5 \overline{\mathcal{M}}_{0,4}{}^{(i)}$ is injective.*

Proof On $\mathcal{M}_{0,5}$ this is clear, since given an ordered 5-element subset $(x_1, \ldots, x_5)$ of a smooth rational curve C, there is a unique affine coordinate z on C such that $(z(x_1), z(x_2), z(x_3)) = (0, \infty, 1)$, and then $z(x_i) = f_i(C; x_1, x_2, x_3, x_4)$ for $i = 4, 5$. Since this even allows $x_4 = x_5$, we have in fact injectivity away from the zero dimensional strata. A typical zero dimensional stratum is represented in a Hilbert–Mumford stable manner by a 5-term sequence in $\{1, 2, 3\}$ with at exactly two repetitions, e.g., $(1, 2, 3, 1, 2)$ and then the value of f_i on it is computed by removing the ith term and stipulating that of the four remaining two items two that are not repeated are made equal (and are then renumbered in an order preserving manner such that all terms lie in $\{1, 2\}$). (In our example the value of f_1 is $(2, 1, 1, 2)$, the value of f_2, f_3, f_4 is $(1, 2, 1, 2)$, and the value of f_5 is $(1, 2, 2, 1)$.) Elementary combinatorics shows that f is here injective, too. □

Recall that $\overline{\mathcal{M}}_{0,4}{}^{(i)}$ is naturally a 3-pointed smooth rational curve and that the $\mathfrak{S}_5$-stabilizer of the factor $\overline{\mathcal{M}}_{0,4}{}^{(i)}$, $\mathfrak{S}_{5,i}$ (the permutation group of $\{1, \ldots, 5\} \smallsetminus \{i\}$), acts on $\overline{\mathcal{M}}_{0,4}{}^{(i)}$ through the full permutation group of these three points. Since $\mathfrak{S}_{5,i} \backslash \overline{\mathcal{M}}_{0,4}{}^{(i)}$ is independent of any numbering, it is canonically isomorphic to $\mathfrak{S}_4 \backslash \overline{\mathcal{M}}_{0,4}$, and hence with $\mathbb{P}^1$ (such an isomorphism can be given by the j-invariant of the double cover of $\mathbb{P}^1$ ramifying in a prescribed 4-element subset). So the $\mathfrak{S}_5$-orbit space of $\prod_{i=1}^5 \overline{\mathcal{M}}_{0,4}{}^{(i)}$ is then identified with $\mathrm{Sym}^5(\mathbb{P}^1)$. Since we can regard $\mathrm{Sym}^5(\mathbb{P}^1)$ as the linear system of degree 5 on $\mathbb{P}^1$, it is a projective space of dimension 5. Let

$$\Phi\colon \mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5} \to \mathfrak{S}_5\backslash\prod_{i=1}^{5}\overline{\mathcal{M}}_{0,4}{}^{(i)} \cong \mathrm{Sym}^5(\mathbb{P}^1)$$

be obtained from f by passing to the $\mathfrak{S}_5$-orbit spaces and denote by

$$\widetilde{\Phi}\colon \overline{\mathcal{M}}_{0,5} \to \mathrm{Sym}^5(\mathbb{P}^1)$$

its precomposite with the quotient morphism. It follows from Lemma 6.4 that Φ is injective and Proposition 6.2 tells us that $\mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5}$ naturally embeds in a 5-dimensional projective space as a quasi-cone.

Theorem 6.5 *The morphism $\widetilde{\Phi}^*$ takes the hyperplane class of $\mathrm{Sym}^5(\mathbb{P}^1)$ to $-12K_S$, and Φ is projectively equivalent to the projective embedding found in Proposition 6.2. In particular, $\widetilde{\Phi}(C_\infty)$ is a line in $\mathrm{Sym}^5(\mathbb{P}^1)$.*

Proof The hyperplane class of $\mathrm{Sym}^5(\mathbb{P}^1)$ pulled back to $(\mathbb{P}^1)^5$ is the sum of the degree 1 classes coming from the factors. The map

$$\overline{\mathcal{M}}_{0,4} \to \mathfrak{S}_4\backslash\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$$

is of degree 6. The sum of the two special conics that appear as fibers of $f_5\colon \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ represent the pull-back of a degree 2 class on $\overline{\mathcal{M}}_{0,4}$ and so the pull-back of the degree 1 class on $\mathbb{P}^1$ is 3 times the sum of these two special conics. Hence the hyperplane class pulled back to $\overline{\mathcal{M}}_{0,5}$ is equal to $3(C_c + C_c') = -12K_S$. The degree of the image of Φ is then computed as

$$\frac{(-12K_S)^2}{\#(\mathfrak{S}_5)} = \frac{12^2 \cdot 5}{120} = 6.$$

This is also the degree of the image of $\mathfrak{S}_5\backslash\overline{\mathcal{M}}_{0,5}$ in $\mathbb{P}^5$ embedded as a quasi-cone. One verifies in straightforward manner that $\mathfrak{C}_6\backslash\mathbb{P}^2$ is simply-connected and so $\mathrm{Pic}(\mathfrak{C}_6\backslash\mathbb{P}^2) \cong H^2(\mathfrak{C}_6\backslash\mathbb{P}^2; \mathbb{Z})$ is infinite cyclic. It follows that the two morphisms must be projectively equivalent. □

Remark 6.6 Let $V(5)$ be the space of binary quintic forms. It is classically known (and a fact that can be found in any text book on invariant theory) that the subalgebra of SL_2-invariants in the polynomial algebra $\mathbb{C}[V(5)]$ is generated by the Clebsch invariants I_4, I_8, I_{12}, I_{18} of degrees indicated in the subscripts (the last invariant is skew, i.e., it is not a GL_2-invariant). Their explicit form can be found in Salmon's book [22]. There is one basic relation among these invariants, which is of the form

$$I_{18}^2 = P(I_4, I_8, I_{12}).$$

Since the subspace spanned by elements of degrees $4k$ is the subalgebra freely generated by the first three invariants, we see that the GIT-quotient $SL_2\backslash\backslash\mathbb{P}(V(5))$ is

isomorphic to the weighted projective plane $\mathbb{P}(1, 2, 3)$. So in view of the discussion above, we obtain

$$\mathfrak{S}_5 \backslash S = \mathfrak{S}_5 \backslash \overline{\mathcal{M}}_{0,5} \cong \mathbb{P}(1, 2, 3).$$

With the help of exact sequence (4) we can identify $H^0(S, \omega_S^{-k})^{\mathfrak{S}_5}$ with $\mathbb{C}[V(5)]_{2k}$. In other words, the zero set (or, more precisely, closed subscheme) of any invariant of degree $2k$ pulls back to a curve on S defined by an $\mathfrak{S}_5$-invariant section of ω_S^{-k}. It then makes sense to ask what the curves on S are corresponding to the basic invariants I_4, I_8, I_{12}, I_{18}, as these correspond to $\mathfrak{S}_5$-invariant sections of $\omega_S^{-2}, \omega_S^{-4}, \omega_S^{-6}, \omega_S^{-9}$ respectively. We see for example right away that I_4 corresponds to our Wiman curve.

The discriminant Δ of a binary quintic is a polynomial in coefficients of degree 8, and as such it must be a linear combination of I_4^2 and I_8. In fact, it is known that $\Delta = I_4^2 - 128 I_8$. It is also a square of an element $D \in \mathbb{C}[V(5)]_4$ and so D must correspond to the curve C_∞. It is clear that any SL_2-invariant in $\mathbb{C}[V(5)]_8$ is a linear combination of $\Delta = D^2$ and I_4^2, in other words, has the form $(aD + bI_4)(aD - bI_4)$ for some $(a, b) \neq (0, 0)$. So this defines on S a reducible $\mathfrak{S}_5$-invariant divisor which is the sum of two $\mathfrak{A}_5$-invariant divisors. In other words, it represents the union of two $\mathfrak{S}_5$-conjugate members of the Wiman–Edge pencil.

The invariant I_{18}, being the, up to proportionality unique, invariant of degree 18 must represent a unique curve on S defined by an $\mathfrak{S}_5$-invariant section of ω_S^{-9}. According to Clebsch [6, p. 298], the invariant I_{18} vanishes on a binary form when its four zeros are invariant with respect to an involution of $\mathbb{P}^1$ and the remaining zero is a fixed point of this involution. This is precisely the locus parametrized by $\mathfrak{C}_2^{\mathrm{ev}}$ and according to Remark 3.14 indeed defined by a section of ω_S^{-9}.

References

1. Abdesselam, A.: A computational solution to a question by Beauville on the invariants of the binary quintic. J. Algebra **303**(2), 771–788 (2006)
2. Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J.: Geometry of Algebraic Curves. Vol. I. Grundlehren der Mathematischen Wissenschaften, vol. 267. Springer, New York (1985)
3. Candelas, Ph., de la Ossa, X., van Geemen, B., van Straten, D.: Lines on the Dwork pencil of quintic threefolds. Adv. Theor. Math. Phys. **16**(6), 1779–1836 (2012)
4. Cheltsov, I., Shramov, C.: Cremona Groups and the Icosahedron. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton (2016)
5. Cheltsov, I., Kuznetsov, A., Shramov, C.: Coble fourfold, S_6-invariant quartic threefolds, and Wiman–Edge Sextics (2017). arXiv:1712.08906
6. Clebsch, A.: Ueber die Anwendung der quadratischen Substitution auf die Gleichungen fünften Grades und die geometrische Theorie des ebenen Fünfecks. Math. Ann. **4**(2), 284–345 (1871)
7. Coble, A.B.: An application of the form-problems associated with certain Cremona groups to the solution of equations of higher degree. Trans. Amer. Math. Soc. **9**(4), 396–424 (1908)

8. Dolgachev, I.V.: Classical Algebraic Geometry. Cambridge University Press, Cambridge (2012)
9. Dolgachev, I.V.: Quartic surfaces with icosahedral symmetry. Adv. Geom. **18**(1), 119–132 (2018). arXiv:1604.02567
10. Edge, W.L.: A pencil of four-nodal plane sextics. Proc. Cambridge Philos. Soc. **89**(3), 413–421 (1981)
11. Edge, W.L.: A pencil of specialized canonical curves. Proc. Cambridge Philos. Soc. **90**(2), 239–249 (1981)
12. Farb, B., Looijenga, E.: Geometry of the Wiman–Edge pencil, II: hyperbolic, conformal and modular aspects (in preparation)
13. Fulton, W., Harris, J.: Representation Theory. Graduate Texts in Mathematics, vol. 129. Readings in Mathematics. Springer, New York (1991)
14. González-Aguilera, V., Rodríguez, R.E.: A pencil in $\widetilde{\mathfrak{M}}_6$ with three points at the boundary. Geom. Dedic. **42**(3), 255–265 (1992)
15. Griffiths, Ph., Harris, J.: Principles of Algebraic Geometry. Wiley, New York (1994)
16. Inoue, N., Kato, F.: On the geometry of Wiman's sextic. J. Math. Kyoto Univ. **45**(4), 743–757 (2005)
17. Kapranov, M.M.: Chow Quotients of Grassmannians. I. In: Gel'fand, S., Gindikin, S. (eds.) I.M. Gel'fand Seminar. Advances in Soviet Mathematics, vol. 16.2, pp. 29–110. American Mathematical Society, Providence (1993)
18. Klein, F.: Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade. Reprint of the 1884 original. Edited, with an introduction and commentary by Peter Slodowy. Birkhäuser, Basel (1993)
19. Mukai, S.: Curves and Grassmannians. In: Yang, J.-H., Namikawa, Y., Uneo, K. (eds.) Algebraic Geometry and Related Topics. Conference Proceedings and Lecture Notes in Algebraic Geometry, vol. 1, pp. 19–40. International Press, Cambridge (1993)
20. Rauschning, J., Slodowy, P.: An aspect of icosahedral symmetry. Canadian Math. Bull. **45**(4), 686–696 (2002)
21. Rimányi, R.: Thom polynomials, symmetries and incidences of singularities. Invent. Math. **143**(3), 499–521 (2001)
22. Salmon, G.: Lessons Introductory to the Modern Higher Algebra. Hodges, Foster and Co., Dublin (1876) (5th edn. reprinted by Chelsea, New York (1986))
23. Shepherd-Barron, N.I.: Invariant theory for S_5 and the rationality of M_6. Compositio Math. **70**(1), 13–25 (1989)
24. Wiman, A.: Ueber die algebraischen Curven von den Geschlechtern $p = 4, 5$ and 6 welche eindeutige Transformationen in sich besitzen. Bihang till Kongl. Svenska Vetenskapsakad. Handlingar **21**(3), 2–41 (1895)
25. Wiman, A.: Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene. Math. Ann. **48**(1–2), 195–240 (1896)
26. Winger, R.M.: On the invariants of the ternary icosahedral group. Math. Ann. **93**(1), 210–216 (1925)
27. Zagier, D.: Lines on the Dwork quintic pencil and its higher degree analogues. J. Differential Geom. **97**(1), 177–189 (2014)