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Certain graphs under Pythagorean fuzzy environment

Muhammad Akram[1] · Jawaria Mohsan Dar[1] · Sumera Naz[1]

Abstract

Graph theory plays crucial role in structuring many real-world problems including, medical sciences, control theory, expert systems and network security. Product in graphs, an operation that consider two graphs and produce a new graph by simple or complex changes, has wide range of applications in games theory, automata theory, structural mechanics and networking system. An intuitionistic fuzzy model is used to handle the vagueness and uncertainty in network problems. A Pythagorean fuzzy model is a powerful tool for describing vagueness and uncertainty more accurately as compared to intuitionistic fuzzy model. The objective of this paper is to apply the concept of Pythagorean fuzzy sets to graphs and then combine two Pythagorean fuzzy graphs (PFGs) using two new graph products namely, maximal product and the residue product. This research paper investigates the regularity for these products. Moreover, it discusses some eminent properties such as strongness, connectedness and completeness. Further, it proposes some necessary and sufficient conditions for $\mathscr{G}_1 * \mathscr{G}_2$ and $\mathscr{G}_1 \cdot \mathscr{G}_2$ to be regular. Finally, decision-making problems concerning evaluation of best company for investment and alliance partner selection of a software company are solved to better understand PFGs.

Keywords Pythagorean fuzzy graphs · Regular Pythagorean fuzzy graphs · Maximal product · Residue product

Introduction

Fuzzy set theory [1] is the most efficient tool having the capability to deal with imprecise and incomplete information in different disciplines, including engineering, mathematics, statistics, artificial intelligence, medical and social sciences. To cope with imprecise and incomplete information, consisting of doubts in human judgement, the fuzzy set shows some restrictions. So, for characterizing the hesitancy more explicitly, fuzzy sets were extended to intuitionistic fuzzy sets (IFSs) by Atanassov [2], which assign a membership grade (μ) and a non-membership grade (ν) to the objects, satisfying the condition $\mu + \nu \leq 1$ and the hesitancy part $\pi = 1 - \mu - \nu$. The IFSs have gained extensive attentions and have been broadly applied in different areas of real life. The limitation $\mu + \nu \leq 1$ confines the choice of the mem-

bership and non-membership grades in IFS. To evade this situation, Yager [3–5] initiated the idea of Pythagorean fuzzy set (PFS), depicted by a membership grade (μ) and a non-membership grade (ν) with the condition $\mu^2 + \nu^2 \leq 1$. Zhang and Xu [6] introduced the concept of Pythagorean fuzzy number (PFN) for interpreting the dual aspects of an element. In a decision-making environment, a specialist gives the preference information about an alternative with the membership grade 0.9 and the non-membership grade 0.3; it is noted that the IFN fails to address this situation, as $0.9 + 0.3 > 1$, but $(0.9)^2 + (0.3)^2 \leq 1$. Thus, the PFS has much stronger ability than IFS to model fuzziness in the practical MCDM problems. Under Pythagorean fuzzy environment, many researchers have initiated work in different directions and acquired various eminent results [7]. Some operations on PFSs [8] and Pythagorean fuzzy TODIM approach to multi-criteria decision-making [9] have been discussed. Furthermore, the PFS has been investigated from different perspectives, including aggregation operators [10,11]. Garg [12–18] explored applications of Pythagorean fuzzy sets in decision-making. Lately, the concept of Pythagorean fuzzy set has been extended to interval-valued Pythagorean fuzzy set and hesitant Pythagorean fuzzy set. Garg [19,20] elaborated exponential operational laws and their aggregation operators under interval-valued Pythagorean fuzzy informa-

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tion. Yu et al. [21] discussed hesitant Pythagorean fuzzy Maclaurin symmetric mean operators and its applications to multi-attribute decision-making.

Graphs are the pictorial representation that bond the objects and highlight their information. Graph theory is rapidly moving into the core of mathematics due to its applications in various fields, including physics, biochemistry, biology, electrical engineering, astronomy, operations research and computer science. To emphasize on a real-world problem, the bondedness between the objects occur due to some relations. But when there exists uncertainty and haziness in the bonding, then the corresponding graph model can be taken as fuzzy graph model. For example, a social network may be represented as a graph, where vertices represent members and edges represent relation between members. If the relations among the members are to be measured as good or bad, then fuzziness should be added to the representation. This and many other problems motivated to define fuzzy graph. The concept of fuzzy graphs was presented by Kaufmann [22], based on Zadeh's fuzzy relation in 1971. Later, Rosenfeld [23] discussed several basic graph theoretical concepts, including paths, cycles, bridges and connectedness in fuzzy environment. Nagoor Gani and Radha [24] initiated the concept of regular fuzzy graph in 2008. Mordeson and Peng [25] introduced some operations on fuzzy graphs and studied their properties. Further, Nirmala and Vijaya [26] explored new operations on fuzzy graphs. Parvathi and Karunambigai [27] considered intuitionistic fuzzy graphs (IFGs). Later, Akram and Davvaz [28] discussed IFGs. An algorithm for computing sum distance matrix, eccentricity of vertices, diameter and radius in IFGs was presented by sarwar and Akram [29]. Akram and Dudek [30] described intuitionistic fuzzy hypergraphs with applications. Recently, Naz et al. [31] originally proposed the concept of PFGs, a generalization of the notion of Akram and Davvaz's IFGs [28], along with its applications in decision-making. Akram and Naz [32] studied energy of PFGs with applications. Akram et al. [33] introduced the concept of Pythagorean fuzzy Planar graphs. Dhavudh and Srinivasan [35] dealt with IFGs2T. Ghorai and Pal [37,38] studied some properties of m-polar fuzzy graphs. Recently, Akram et al. [34] introduced some new operations including rejection, symmetric difference, maximal product and residue product of PFGs. This paper describes two new operations namely, maximal product and residue product of PFGs that can allow the mathematical design of network. These products can be applied to construct and analyze several real-world networks such as road networks and communication networks. The work explores some significant properties such as regularity, strongness, completeness and connectedness. As regularity plays a substantial role in designing reliable communication networks, so the main focus is to familiarize the regularity of these product with illustrative examples. The work also proposes

some necessary and sufficient conditions for these two products to be regular. Finally, it discusses some applications of PFGs in decision-making.

For better understanding, we present prerequisite terminologies and notations:

Definition 1.1 [1] A *fuzzy set* (FS) on a universe $\mathscr{X}$ is an object of the form

$$\mathscr{A} = \{\langle u, \mu_{\mathscr{A}}(u)\rangle | u \in \mathscr{X}\},$$

where $\mu_{\mathscr{A}} : \mathscr{X} \longrightarrow [0, 1]$ represents the membership functions of $\mathscr{A}$.

Definition 1.2 [22] A *fuzzy graph* (FG) on a non-empty set $\mathscr{X}$ is a pair $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ with $\mathscr{A}$ a FS on $\mathscr{X}$ and $\mathscr{B}$ a fuzzy relation on $\mathscr{X}$ such that

$$\mu_{\mathscr{B}}(uv) \leq \mu_{\mathscr{A}}(u) \wedge \mu_{\mathscr{A}}(v)$$

for all $u, v \in \mathscr{X}$, where $\mathscr{A} : \mathscr{X} \longrightarrow [0, 1]$ and $\mathscr{B} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$.

Definition 1.3 [2] An *intuitionistic fuzzy set* (IFS) on a universe $\mathscr{X}$ is an object of the form

$$\mathscr{A} = \{\langle u, \mu_{\mathscr{A}}(u), \nu_{\mathscr{A}}(u)\rangle | u \in \mathscr{X}\},$$

where $\mu_{\mathscr{A}} : \mathscr{X} \longrightarrow [0, 1]$ and $\nu_{\mathscr{A}} : \mathscr{X} \longrightarrow [0, 1]$ represents the membership and non-membership functions of $\mathscr{A}$, and $\mu_{\mathscr{A}}, \nu_{\mathscr{A}}$ satisfies the condition $0 \leq \mu_{\mathscr{A}}(u) + \nu_{\mathscr{A}}(u) \leq 1$ for all $u \in \mathscr{X}$.

Definition 1.4 [27] An *intuitionistic fuzzy graph* (IFG) on a non-empty set $\mathscr{X}$ is a pair $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ with $\mathscr{A}$ an IFS on $\mathscr{X}$ and $\mathscr{B}$ an IFR on $\mathscr{X}$ such that

$$\mu_{\mathscr{B}}(uv) \leq \mu_{\mathscr{A}}(u) \wedge \mu_{\mathscr{A}}(v), \ \ \nu_{\mathscr{B}}(uv) \geq \nu_{\mathscr{A}}(u) \vee \nu_{\mathscr{A}}(v)$$

and $0 \leq \mu_{\mathscr{B}}(uv) + \nu_{\mathscr{B}}(uv) \leq 1$ for all $u, v \in \mathscr{X}$, where, $\mu_{\mathscr{B}} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$ and $\nu_{\mathscr{B}} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$ represents the membership and non-membership functions of $\mathscr{B}$, respectively.

Definition 1.5 [3] A *Pythagorean fuzzy set* (PFS) on a universe $\mathscr{X}$ is an object of the form

$$\mathscr{A} = \{\langle u, \mu_{\mathscr{A}}(u), \nu_{\mathscr{A}}(u)\rangle | u \in \mathscr{X}\},$$

where $\mu_{\mathscr{A}} : \mathscr{X} \longrightarrow [0, 1]$ and $\nu_{\mathscr{A}} : \mathscr{X} \longrightarrow [0, 1]$ represents the membership and non-membership functions of $\mathscr{A}$, and $\mu_{\mathscr{A}}, \nu_{\mathscr{A}}$ satisfies the condition $0 \leq \mu_{\mathscr{A}}^2(u) + \nu_{\mathscr{A}}^2(u) \leq 1$ for all $u \in \mathscr{X}$.

Fig. 1 Pythagorean fuzzy graph
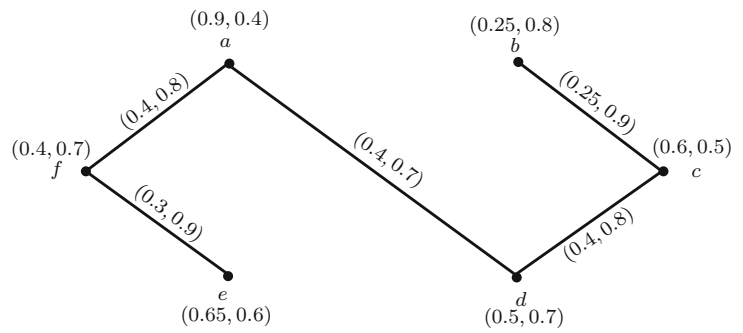
Definition 1.6 Let $\mathscr{A}_1 = \{\langle u, \mu_{\mathscr{A}_1}(u), \nu_{\mathscr{A}_1}(u)\rangle | u \in \mathscr{X}\}$ and $\mathscr{A}_2 = \{\langle u, \mu_{\mathscr{A}_2}(u), \nu_{\mathscr{A}_2}(u)\rangle | u \in \mathscr{X}\}$ be two Pythagorean fuzzy sets, then some set operations can be defined as

1. $\mathscr{A}_1 \subseteq \mathscr{A}_2$ if and only if $\mu_{\mathscr{A}_1}(u) \leq \mu_{\mathscr{A}_2}(u)$ and $\nu_{\mathscr{A}_1}(u) \geq \nu_{\mathscr{A}_2}(u)$,
2. $\mathscr{A}_1 = \mathscr{A}_2$ if and only if $\mathscr{A}_1(u) \subseteq \mathscr{A}_2(u)$ and $\mathscr{A}_2(u) \subseteq \mathscr{A}_1(u)$,
3. $\mathscr{A}_1^c = \{\langle u, \nu_{\mathscr{A}_1}(u), \mu_{\mathscr{A}_1}(u)\rangle | u \in \mathscr{X}\}$,
4. $\mathscr{A}_1 \cap \mathscr{A}_2 = \{\langle u, \min\{\mu_{\mathscr{A}_1}(u), \mu_{\mathscr{A}_2}(u)\}, \max\{\nu_{\mathscr{A}_1}(u), \nu_{\mathscr{A}_2}(u)\}\rangle | u \in \mathscr{X}\}$,
5. $\mathscr{A}_1 \cup \mathscr{A}_2 = \{\langle u, \max\{\mu_{\mathscr{A}_1}(u), \mu_{\mathscr{A}_2}(u)\}, \min\{\nu_{\mathscr{A}_1}(u), \nu_{\mathscr{A}_2}(u)\}\rangle | u \in \mathscr{X}\}$.

Definition 1.7 [31] A Pythagorean fuzzy set $\mathscr{B}$ on $\mathscr{X} \times \mathscr{X}$ is said to be a *Pythagorean fuzzy relation* (PFR) on $\mathscr{X}$, denoted by

$$\mathscr{B} = \{\langle uv, \mu_{\mathscr{B}}(uv), \nu_{\mathscr{B}}(uv)\rangle | uv \in \mathscr{X} \times \mathscr{X}\},$$

where $\mu_{\mathscr{B}} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$ and $\nu_{\mathscr{B}} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$ represents the membership and non-membership functions of $\mathscr{B}$, and $\mu_{\mathscr{B}}, \nu_{\mathscr{B}}$ satisfies the condition $0 \leq \mu_{\mathscr{B}}^2(uv) + \nu_{\mathscr{B}}^2(uv) \leq 1$ for all $uv \in \mathscr{X} \times \mathscr{X}$.

Definition 1.8 [31] A *Pythagorean fuzzy graph* (PFG) on a non-empty set $\mathscr{X}$ is a pair $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ with $\mathscr{A}$ a PFS on $\mathscr{X}$ and $\mathscr{B}$ a PFR on $\mathscr{X}$ such that

$$\mu_{\mathscr{B}}(uv) \leq \mu_{\mathscr{A}}(u) \wedge \mu_{\mathscr{A}}(v), \quad \nu_{\mathscr{B}}(uv) \geq \nu_{\mathscr{A}}(u) \vee \nu_{\mathscr{A}}(v)$$

and $0 \leq \mu_{\mathscr{B}}^2(uv) + \nu_{\mathscr{B}}^2(uv) \leq 1$ for all $u, v \in \mathscr{X}$, where, $\mu_{\mathscr{B}} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$ and $\nu_{\mathscr{B}} : \mathscr{X} \times \mathscr{X} \longrightarrow [0, 1]$ represents the membership and non-membership functions of $\mathscr{B}$, respectively.

Example 1.1 Consider a graph $G = (V, E)$ where $V = \{a, b, c, d, e, f\}$ and $E = \{ad, bc, cd, af, ef\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$, respectively.

$$\mathscr{A} = \left\langle \left(\frac{a}{0.9}, \frac{b}{0.25}, \frac{c}{0.6}, \frac{d}{0.5}, \frac{e}{0.65}, \frac{f}{0.4}\right), \right.$$
$$\left. \times \left(\frac{a}{0.4}, \frac{b}{0.8}, \frac{c}{0.5}, , \frac{d}{0.7}, \frac{e}{0.6} \frac{f}{0.7}\right) \right\rangle \text{ and}$$
$$\mathscr{B} = \left\langle \left(\frac{ad}{0.4}, \frac{bc}{0.25}, \frac{cd}{0.4}, \frac{af}{0.4}, \frac{fe}{0.3}\right), \right.$$
$$\left. \times \left(\frac{ad}{0.7}, \frac{bc}{0.9}, \frac{cd}{0.8}, \frac{af}{0.8}, \frac{fe}{0.9}\right) \right\rangle.$$

By routine calculation, one can see from Fig. 1 that it is a Pythagorean fuzzy graph.

Definition 1.9 A Pythagorean fuzzy preference relation (PFPR) on the set of alternatives $\mathscr{X} = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $R = (r_{ij})_{n \times n}$, where $r_{ij} = (x_i x_j, \mu(x_i x_j), \nu(x_i x_j))$ for all $i, j = 1, 2, \ldots, n$. For convenience, let $r_{ij} = (\mu_{ij}, \nu_{ij})$ where μ_{ij} indicates the degree to which the object x_i is preferred to the object x_j, ν_{ij} denotes the degree to which the object x_i is not preferred to the object x_j, and $\pi_{ij} = \sqrt{1 - \mu_{ij}^2 - \nu_{ij}^2}$ is interpreted as a hesitancy degree, with the conditions:

$$\mu_{ij}, \nu_{ij} \in [0, 1], \ \mu_{ij}^2 + \nu_{ij}^2 \leq 1, \ \mu_{ij} = \nu_{ji},$$
$$\mu_{ii} = \nu_{ii} = 0.5, \text{ for all } i, j = 1, 2, \ldots, n.$$

Regular maximal product in Pythagorean fuzzy graphs

Definition 2.1 Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be two PFGs of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. The *maximal product* of $\mathscr{G}_1$ and $\mathscr{G}_2$ is denoted by $\mathscr{G}_1 * \mathscr{G}_2 = (\mathscr{A}_1 * \mathscr{A}_2, \mathscr{B}_1 * \mathscr{B}_2)$ and defined as:

(i) $\begin{cases} (\mu_{\mathscr{A}_1} * \mu_{\mathscr{A}_2})(u_1, u_2) = \mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{A}_2}(u_2) \\ (\nu_{\mathscr{A}_1} * \nu_{\mathscr{A}_2})(u_1, u_2) = \nu_{\mathscr{A}_1}(u_1) \wedge \nu_{\mathscr{A}_2}(u_2) \end{cases}$
for all $(u_1, u_2) \in V_1 \times V_2$,

(ii) $\begin{cases} (\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u, u_2)(u, v_2)) = \mu_{\mathscr{A}_1}(u) \vee \mu_{\mathscr{B}_2}(u_2 v_2) \\ (\nu_{\mathscr{B}_1} * \nu_{\mathscr{B}_2})((u, u_2)(u, v_2)) = \nu_{\mathscr{A}_1}(u) \wedge \nu_{\mathscr{B}_2}(u_2 v_2) \end{cases}$
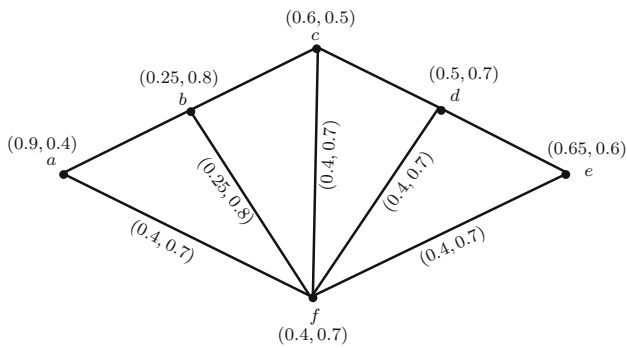for all $u \in V_1$ and $u_2 v_2 \in E_2$,

(0.6, 0.5)
c

(0.25, 0.8)
b

(0.5, 0.7)
d

(0.9, 0.4)
a

(0.65, 0.6)
e

(0.25, 0.8)

(0.4, 0.7)

(0.4, 0.7)

(0.4, 0.7)

(0.4, 0.7)

(0.4, 0.7)

f
(0.4, 0.7)

Fig. 2 Strong Pythagorean fuzzy graph

(iii) $\begin{cases} (\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u_1, z)(v_1, z)) = \mu_{\mathscr{B}_1}(u_1 v_1) \vee \mu_{\mathscr{A}_2}(z) \\ (\nu_{\mathscr{B}_1} * \nu_{\mathscr{B}_2})((u_1, z)(v_1, z)) = \nu_{\mathscr{B}_1}(u_1 v_1) \wedge \nu_{\mathscr{A}_2}(z) \end{cases}$
for all $z \in V_2$ and $u_1 v_1 \in E_1$.

Definition 2.2 A Pythagorean fuzzy graph $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is said to be a *strong* Pythagorean fuzzy graph of underlying crisp graph $G = (V, E)$ if

$$\mu_{\mathscr{B}}(uv) = \mu_{\mathscr{A}}(u) \wedge \mu_{\mathscr{A}}(v),$$
$$\nu_{\mathscr{B}}(uv) = \nu_{\mathscr{A}}(u) \vee \nu_{\mathscr{A}}(v) \quad \text{for all } uv \in E.$$

Example 2.1 Consider a graph $G = (V, E)$ where $V = \{a, b, c, d, e, f\}$ and $E = \{af, bf, cf, df, ef\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$, respectively.

$$\mathscr{A} = \left\langle \left(\frac{a}{0.9}, \frac{b}{0.25}, \frac{c}{0.6}, \frac{d}{0.5}, \frac{e}{0.65}, \frac{f}{0.4} \right), \right.$$
$$\left. \times \left(\frac{a}{0.4}, \frac{b}{0.8}, \frac{c}{0.5}, , \frac{d}{0.7}, \frac{e}{0.6}, \frac{f}{0.7} \right) \right\rangle \text{ and}$$
$$\mathscr{B} = \left\langle \left(\frac{af}{0.4}, \frac{bf}{0.25}, \frac{cf}{0.4}, \frac{df}{0.4}, \frac{ef}{0.4} \right), \right.$$
$$\left. \times \left(\frac{af}{0.7}, \frac{bf}{0.8}, \frac{cf}{0.7}, \frac{df}{0.7}, \frac{ef}{0.7} \right) \right\rangle.$$

By routine calculation, one can see from Fig. 2 that it is a strong Pythagorean fuzzy graph.

Theorem 2.1 *If $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are two strong PFGs, then their maximal product is also a strong PFG.*

Proof Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be two strong PFGs of the graph G_1 and G_2, respectively. Then,

$$\mu_{\mathscr{B}_1}(u_1 u_2) = \mu_{\mathscr{A}_1}(u_1) \wedge \mu_{\mathscr{A}_1}(u_2),$$
$$\nu_{\mathscr{B}_1}(u_1 u_2) = \nu_{\mathscr{A}_1}(u_1) \vee \nu_{\mathscr{A}_1}(u_2) \quad for \ all \ u_1 u_2 \in E_1,$$
$$\mu_{\mathscr{B}_2}(v_1 v_2) = \mu_{\mathscr{A}_2}(v_1) \wedge \mu_{\mathscr{A}_2}(v_2),$$
$$\nu_{\mathscr{B}_2}(v_1 v_2) = \nu_{\mathscr{A}_2}(v_1) \vee \nu_{\mathscr{A}_2}(v_2) \quad for \ all \ v_1 v_2 \in E_2.$$

By definition of maximal product, we have
If $u_1 = u_2$ and $v_1 v_2 \in E_2$,

$$(\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u_1, v_1)(u_2, v_2))$$
$$= \mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{B}_2}(v_1 v_2)$$
$$= \mu_{\mathscr{A}_1}(u_1) \vee \{\mu_{\mathscr{A}_2}(v_1) \wedge \mu_{\mathscr{A}_2}(v_2)\}$$
$$= \{\mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{A}_2}(v_1)\} \wedge \{\mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{A}_2}(v_2)\}$$
$$= (\mu_{\mathscr{A}_1} * \mu_{\mathscr{A}_2})(u_1, v_1) \wedge (\mu_{\mathscr{A}_1} * \mu_{\mathscr{A}_2})(u_2, v_2),$$
$$(\nu_{\mathscr{B}_1} * \nu_{\mathscr{B}_2})((u_1, v_1)(u_2, v_2))$$
$$= \nu_{\mathscr{A}_1}(u_1) \wedge \nu_{\mathscr{B}_2}(v_1 v_2)$$
$$= \nu_{\mathscr{A}_1}(u_1) \wedge \{\nu_{\mathscr{A}_2}(v_1) \vee \nu_{\mathscr{A}_2}(v_2)\}$$
$$= \{\nu_{\mathscr{A}_1}(u_1) \wedge \nu_{\mathscr{A}_2}(v_1)\} \vee \{\nu_{\mathscr{A}_1}(u_1) \wedge \nu_{\mathscr{A}_2}(v_2)\}$$
$$= (\nu_{\mathscr{A}_1} * \nu_{\mathscr{A}_2})(u_1, v_1) \vee (\nu_{\mathscr{A}_1} * \nu_{\mathscr{A}_2})(u_2, v_2).$$

If $v_1 = v_2$ and $u_1 u_2 \in E_1$,

$$(\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u_1, v_1)(u_2, v_2))$$
$$= \mu_{\mathscr{B}_1}(u_1 u_2) \vee \mu_{\mathscr{A}_2}(v_1)$$
$$= \{\mu_{\mathscr{A}_1}(u_1) \wedge \mu_{\mathscr{A}_1}(u_2)\} \vee \mu_{\mathscr{A}_2}(v_1)$$
$$= \{\mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{A}_2}(v_1)\} \wedge \{\mu_{\mathscr{A}_1}(u_2) \vee \mu_{\mathscr{A}_2}(v_1)\}$$
$$= (\mu_{\mathscr{A}_1} * \mu_{\mathscr{A}_2})(u_1, v_1) \wedge (\mu_{\mathscr{A}_1} * \mu_{\mathscr{A}_2})(u_2, v_2),$$
$$(\nu_{\mathscr{B}_1} * \nu_{\mathscr{B}_2})((u_1, v_1)(u_2, v_2))$$
$$= \nu_{\mathscr{B}_1}(u_1 u_2) \wedge \nu_{\mathscr{A}_2}(v_1)$$
$$= \{\nu_{\mathscr{A}_1}(u_1) \vee \nu_{\mathscr{A}_1}(u_2)\} \wedge \nu_{\mathscr{A}_2}(v_1)$$
$$= \{\nu_{\mathscr{A}_1}(u_1) \wedge \nu_{\mathscr{A}_2}(v_1)\} \vee \{\nu_{\mathscr{A}_1}(u_2) \wedge \nu_{\mathscr{A}_2}(v_1)\}$$
$$= (\nu_{\mathscr{A}_1} * \nu_{\mathscr{A}_2})(u_1, v_1) \vee (\nu_{\mathscr{A}_1} * \nu_{\mathscr{A}_2})(u_2, v_2).$$

Hence, the maximal product $\mathscr{G}_1 * \mathscr{G}_2$ of two strong PFGs is a strong PFG. □

Remark 2.1 Converse of the Theorem 2.1 may not be true as it can be seen in the following example.
Consider two Pythagorean fuzzy graphs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ as shown in Fig. 3, their maximal product is given in Fig. 4.
Here $\mu_{\mathscr{B}_1}(u_1 u_2) \neq \mu_{\mathscr{A}_1}(u_1) \wedge \mu_{\mathscr{A}_1}(u_2)$, $\nu_{\mathscr{B}_1}(u_1 u_2) \neq \nu_{\mathscr{A}_1}(u_1) \vee \nu_{\mathscr{A}_1}(u_2)$ and $\mu_{\mathscr{B}_2}(v_1 v_2) \neq \mu_{\mathscr{A}_2}(v_1) \wedge \mu_{\mathscr{A}_2}(v_2)$, $\nu_{\mathscr{B}_2}(v_1 v_2) \neq \nu_{\mathscr{A}_2}(v_1) \vee \nu_{\mathscr{A}_2}(v_2)$. Hence, $\mathscr{G}_1$ and $\mathscr{G}_2$ are not strong PFGs.
But $(\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u, v)(x, y)) = \mu_{\mathscr{A}_1}(u, v) \wedge \mu_{\mathscr{A}_2}(x, y)$, $(\nu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u, v)(x, y)) = \nu_{\mathscr{A}_1}(u, v) \vee \nu_{\mathscr{A}_2}(x, y)$ for all edge $(u, v)(x, y)$ in E. Thus, their maximal product $\mathscr{G}_1 * \mathscr{G}_2$ is a strong PFG.

Definition 2.3 Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. $\mathcal{G}$ is said to be *connected*, if for every pair of vertices, there exist at least one non-zero path, that is, for all $u, v \in V$, the μ-strength of connectedness
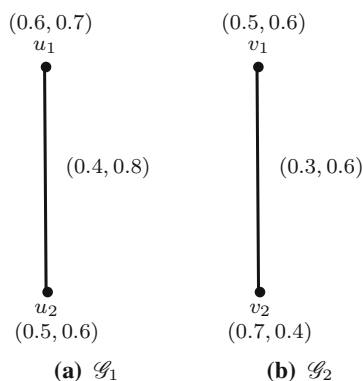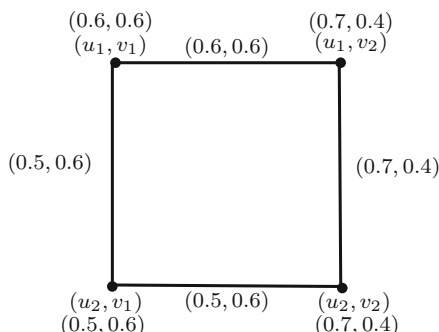
Fig. 3 Pythagorean fuzzy graphs

Fig. 4 Maximal product of PFGs

$$\mu^\infty(u, v) = \sup \{ \mu^l(u, v) : \text{for some l}\}$$

and ν-strength of connectedness

$$\nu^\infty(u, v) = \inf \{ \nu^l(u, v) : \text{for some l}\}$$

satisfy one of the following conditions:

$$\mu^\infty(u, v) > 0 \text{ and } \nu^\infty(u, v) > 0$$
$$\text{or } \mu^\infty(u, v) = 0 \text{ and } \nu^\infty(u, v) > 0$$
$$\text{or } \mu^\infty(u, v) > 0 \text{ and } \nu^\infty(u, v) = 0.$$

where,

$$\mu^l(u, v) = \mu(uu_1) \wedge \mu(u_1u_2) \wedge \ldots \wedge \mu(u_{l-1}v),$$
$$\nu^l(u, v) = \nu(u, u_1) \vee \nu(u_1u_2) \vee \ldots \vee \nu(u_{l-1}v) \text{ for some l}$$

represents μ-strength of path and ν-strength of path of length l, respectively.

Example 2.2 Consider a Pythagorean fuzzy graph $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ as displayed in the following Fig. 5.

For $b, f \in V$, using Definition 2.3 we have

$$\mu^5(b, f) = \mu_{\mathscr{B}}(b, c) \wedge \mu_{\mathscr{B}}(c, i) \wedge \mu_{\mathscr{B}}(i, h) \wedge \mu_{\mathscr{B}}(h, g)$$
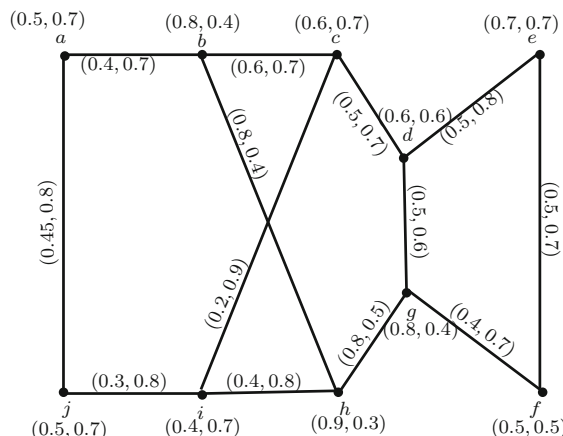$$\wedge \mu_{\mathscr{B}}(g, f) = 0.6 \wedge 0.2 \wedge 0.4$$

Fig. 5 $\mathscr{G} = (\mathscr{A}, \mathscr{B})$

$$\wedge 0.8 \wedge 0.4 = 0.2,$$
$$\mu^4(b, f) = \mu_{\mathscr{B}}(b, c) \wedge \mu_{\mathscr{B}}(c, d) \wedge \mu_{\mathscr{B}}(d, e) \wedge \mu_{\mathscr{B}}(e, f)$$
$$= 0.6 \wedge 0.5 \wedge 0.5 \wedge 0.5 = 0.5,$$
$$\mu^4(b, f) = \mu_{\mathscr{B}}(b, c) \wedge \mu_{\mathscr{B}}(c, d) \wedge \mu_{\mathscr{B}}(d, g) \wedge \mu_{\mathscr{B}}(g, f)$$
$$= 0.6 \wedge 0.5 \wedge 0.5 \wedge 0.4 = 0.4,$$
$$\mu^3(b, f) = \mu_{\mathscr{B}}(b, h) \wedge \mu_{\mathscr{B}}(h, g) \wedge \mu_{\mathscr{B}}(g, f)$$
$$= 0.8 \wedge 0.8 \wedge 0.4 = 0.4.$$

This implies

$$\mu^\infty(b, f) = \sup\{\mu^5(b, f), \mu^4(b, f), \mu^4(b, f), \mu^3(b, f)\}$$
$$= \sup\{0.2, 0.5, 0.4, 0.4\} = 0.5 > 0,$$

and

$$\nu^5(b, f) = \nu_{\mathscr{B}}(b, c) \vee \nu_{\mathscr{B}}(c, i) \vee \nu_{\mathscr{B}}(i, h) \vee \nu_{\mathscr{B}}(h, g)$$
$$\vee \nu_{\mathscr{B}}(g, f) = 0.7 \vee 0.9 \vee 0.8 \vee 0.5 \vee 0.7 = 0.9,$$
$$\nu^4(b, f) = \nu_{\mathscr{B}}(b, c) \vee \nu_{\mathscr{B}}(c, d) \vee \nu_{\mathscr{B}}(d, e) \vee \nu_{\mathscr{B}}(e, f)$$
$$= 0.4 \vee 0.7 \vee 0.8 \vee 0.7 = 0.8,$$
$$\nu^4(b, f) = \mu_{\mathscr{B}}(b, c) \vee \nu_{\mathscr{B}}(c, d) \vee \nu_{\mathscr{B}}(d, g) \vee \nu_{\mathscr{B}}(g, f)$$
$$= 0.4 \vee 0.7 \vee 0.6 \vee 0.7 = 0.7,$$
$$\nu^3(b, f) = \nu_{\mathscr{B}}(b, h) \vee \nu_{\mathscr{B}}(h, g) \vee \nu_{\mathscr{B}}(g, f)$$
$$= 0.4 \vee 0.5 \vee 0.7 = 0.7.$$

This implies

$$\nu^\infty(b, f) = \inf\{\nu^5(b, f), \nu^4(b, f), \nu^4(b, f), \nu^3(b, f)\}$$
$$= \inf\{0.9, 0.8, 0.7, 0.7\} = 0.7 > 0.$$

That is, there exist a non-zero path between b and f. Therefore, $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ in Fig. 5, is connected Pythagorean fuzzy graph.

Theorem 2.2 *The maximal product* $\mathscr{G}_1 * \mathscr{G}_2$ *of two connected PFGs is always a connected PFG.*

Proof Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be two connected PFGs with underlying crisp graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then $\mu_1^\infty(u_i u_j) > 0$, $\nu_1^\infty(u_i u_j) > 0$ for all $u_i, u_j \in V_1$ and $\mu_2^\infty(v_i v_j) > 0$, $\nu_2^\infty(v_i v_j) > 0$ for all $v_i, v_j \in V_2$.

The maximal product of $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ can be taken as $\mathscr{G} = (\mathscr{A}, \mathscr{B})$. Consider the k subgraphs of $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ with the vertex sets $V_2 = \{u_i v_1, u_i v_2, \ldots, u_i v_n\}$ for $i = 1, 2, \ldots, m$. Each of these subgraphs of $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is connected as $u_i's$ are same.

Since $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are connected, each u_i and v_i are adjacent to at least one of the vertices in V_1 and V_2. Therefore, there exists at least one edge between any pair of the above k subgraphs.

Thus, we have

$$\mu^\infty((u_i v_j)(u_k, v_l)) > 0,$$
$$\nu^\infty((u_i v_j)(u_k, v_l)) > 0 \quad for \ all \ (u_i, v_j)(u_k, v_l) \in E.$$

Hence, $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is a connected PFG. □

Definition 2.4 A Pythagorean fuzzy graph $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is said to be a *complete* Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$ if

$$\mu_{\mathscr{B}}(uv) = \mu_{\mathscr{A}}(u) \wedge \mu_{\mathscr{A}}(v),$$
$$\nu_{\mathscr{B}}(uv) = \nu_{\mathscr{A}}(u) \vee \nu_{\mathscr{A}}(v) \quad \text{for all } u, v \in V.$$

Example 2.3 Consider a graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{ab, ac, ad, ae, bc, bd, be, cd, ce, de\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$, respectively.

$$\mathscr{A} = \left\langle \left(\frac{a}{0.25}, \frac{b}{0.8}, \frac{c}{0.65}, \frac{d}{0.4}, \frac{e}{0.9} \right), \right.$$
$$\left. \times \left(\frac{a}{0.8}, \frac{b}{0.7}, \frac{c}{0.6}, , \frac{d}{0.7}, \frac{e}{0.4} \right) \right\rangle \text{ and}$$

$$\mathscr{B} = \left\langle \left(\frac{ab}{0.4}, \frac{ac}{0.25}, \frac{ad}{0.4}, \frac{ae}{0.4}, \frac{bc}{0.4}, \frac{bd}{0.4}, \frac{be}{0.4}, \frac{cd}{0.4}, \frac{ce}{0.4}, \frac{de}{0.4} \right), \right.$$
$$\left. \times \left(\frac{ab}{0.7}, \frac{ac}{0.8}, \frac{ad}{0.7}, \frac{ae}{0.7}, \frac{bc}{0.7}, \frac{bd}{0.4}, \frac{be}{0.4}, \frac{cd}{0.4}, \frac{ce}{0.4}, \frac{de}{0.4} \right) \right\rangle.$$

By routine calculation, one can see from Fig. 6 that it is a complete Pythagorean fuzzy graph.

Remark 2.2 If two Pythagorean fuzzy graphs are complete, their maximal product may not be a complete PFG, as it can be seen in the following example. Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{u_1, u_2\}$ and
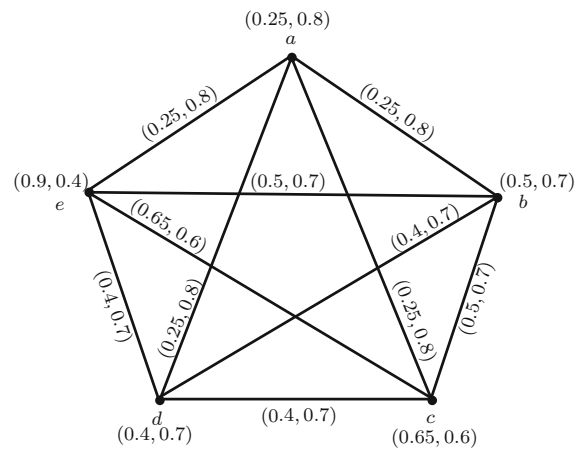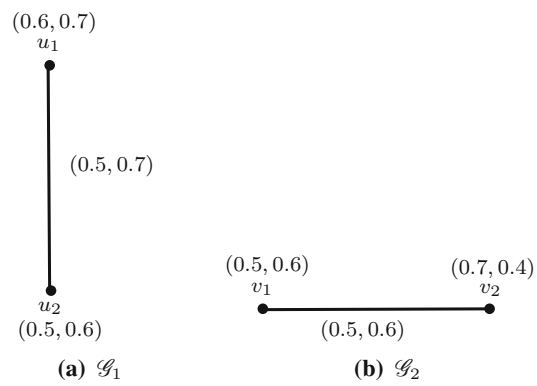
Fig. 6 Complete Pythagorean fuzzy graph
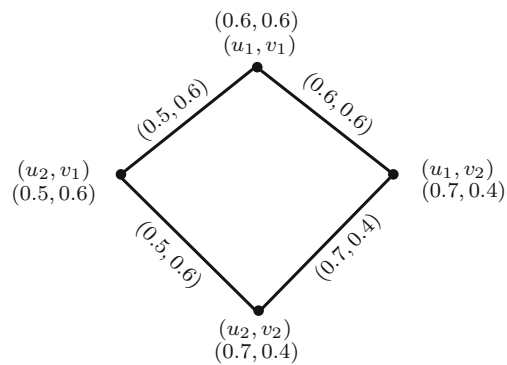

Fig. 7 Pythagorean fuzzy graphs

Fig. 8 Maximal product of two PFGs

$V_2 = \{v_1, v_2\}$, respectively, as shown in Fig. 7. Their maximal product $\mathscr{G}_1 * \mathscr{G}_2$ is shown in Fig. 8.

By routine calculation, one can see from Fig. 7 that $\mathscr{G}_1$ and $\mathscr{G}_2$ are complete PFGs. While notice that $\mathscr{G}_1 * \mathscr{G}_2$ is not a complete PFG, as the case $u_1 u_2 \in E_1$ and $v_1 v_2 \in E_2$ is not included in the definition of the maximal product. Further, one can notice that the maximal product of two complete PFGs is a strong PFG (Fig. 9).
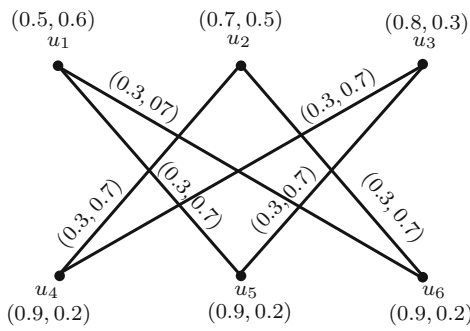
Fig. 9 $\mathscr{G} = (\mathscr{A}, \mathscr{B})$

Definition 2.5 Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. If

$$d_\mu(u) = \sum_{u,v \neq u \in V} \mu_{\mathscr{B}}(uv) = k,$$

$$d_\nu(u) = \sum_{u,v \neq u \in V} \nu_{\mathscr{B}}(uv) = l \quad \text{for all } u \in V,$$

then $\mathscr{G}$ is said to be *regular* Pythagorean fuzzy graph of degree (k, l) or (k, l)-regular PFG.

Example 2.4 Consider a graph $G = (V, E)$ where $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $E = \{u_1u_5, u_1u_6, , u_2u_4, u_2u_6, u_3u_4, u_3u_5\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$, respectively.

$$\mathscr{A} = \left\langle \left(\frac{u_1}{0.5}, \frac{u_2}{0.7}, \frac{u_3}{0.8}, \frac{u_4}{0.9}, \frac{u_5}{0.9}, \frac{u_6}{0.9} \right), \right.$$
$$\left. \times \left(\frac{u_1}{0.6}, \frac{u_2}{0.5}, \frac{u_3}{0.3}, \frac{u_4}{0.2}, \frac{u_5}{0.2}, \frac{u_6}{0.2} \right) \right\rangle \text{ and}$$

$$\mathscr{B} = \left\langle \left(\frac{u_1u_5}{0.3}, \frac{u_1u_6}{0.3}, \frac{u_2u_4}{0.3}, \frac{u_2u_6}{0.3}, \frac{u_3u_4}{0.3}, \frac{u_3u_5}{0.3} \right), \right.$$
$$\left. \times \left(\frac{u_1u_5}{0.7}, \frac{u_1u_6}{0.7}, \frac{u_2u_4}{0.7}, \frac{u_2u_6}{0.7}, \frac{u_3u_4}{0.7}, \frac{u_3u_5}{0.7} \right) \right\rangle.$$

Since $d_\mu(u_i) = 0.6$ and $d_\nu(u_i) = 1.4$ for all $u_i \in V$ and $i = 1, 2, \ldots, 6$, $\mathscr{G}$ is a regular Pythagorean fuzzy graph of degree $(0.3, 0.7)$ or $(0.3, 0.7)$-regular PFG.

Remark 2.3 If $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are two regular PFGs, then their maximal product $\mathscr{G}_1 * \mathscr{G}_2$ may not be regular PFG as it can be seen in this example. Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{a, b\}$ and $V_2 = \{c, d, e, f\}$, respectively, as shown in Fig. 10. Their maximal product $\mathscr{G}_1 * \mathscr{G}_2$ is shown in Fig. 11.

By routine calculation, one can see from Fig. 10 that $\mathscr{G}_1$ and $\mathscr{G}_2$ are regular PFGs. While notice that

$$(d_\mu)_{\mathscr{G}_1 * \mathscr{G}_2}(a, c) = \{\mu_{\mathscr{A}_1}(a) \vee \mu_{\mathscr{B}_2}(cd)\}$$

$$+ \{\mu_{\mathscr{A}_1}(a) \vee \mu_{\mathscr{B}_2}(cf)\} + \{\mu_{\mathscr{B}_1}(ab) \vee \mu_{\mathscr{A}_2}(c)\} = 1.4,$$

$$(d_\nu)_{\mathscr{G}_1 * \mathscr{G}_2}(a, c) = \{\nu_{\mathscr{A}_1}(a) \wedge \nu_{\mathscr{B}_2}(cd)\}$$

$$+ \{\nu_{\mathscr{A}_1}(a) \wedge \nu_{\mathscr{B}_2}(cf)\} + \{\nu_{\mathscr{B}_1}(ab) \wedge \nu_{\mathscr{A}_2}(c)\} = 1.8,$$

$$(d_\mu)_{\mathscr{G}_1 * \mathscr{G}_2}(b, f) = \{\mu_{\mathscr{A}_1}(b) \vee \mu_{\mathscr{B}_2}(fc)\}$$

$$+ \{\mu_{\mathscr{A}_1}(b) \vee \mu_{\mathscr{B}_2}(fe)\} + \{\mu_{\mathscr{B}_1}(ab) \vee \mu_{\mathscr{A}_2}(f)\} = 1.2,$$

$$(d_\nu)_{\mathscr{G}_1 * \mathscr{G}_2}(b, f) = \{\nu_{\mathscr{A}_1}(b) \wedge \nu_{\mathscr{B}_2}(fc)\}$$

$$+ \{\nu_{\mathscr{A}_1}(b) \wedge \nu_{\mathscr{B}_2}(fe)\} + \{\nu_{\mathscr{B}_1}(ab) \wedge \nu_{\mathscr{A}_2}(f)\} = 1.7.$$

That is, $(d)_{\mathscr{G}_1 * \mathscr{G}_2}(a, c) \neq (d)_{\mathscr{G}_1 * \mathscr{G}_2}(b, f)$. Therefore, $\mathscr{G}_1 * \mathscr{G}_2$ is not regular PFG.

Definition 2.6 Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. Then, $\mathscr{G}$ is said to be a *partially* regular Pythagorean fuzzy graph if $G = (V, E)$ is a regular graph.

Example 2.5 Consider a Pythagorean fuzzy graph $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ as displayed in the Fig. 12.

Since $d_\mu(a) = 1.65 \neq 2 = d_\mu(a)$ and $d_\nu(a) = 2.35 \neq 1.8 = d_\nu(b)$. Hence, $\mathscr{G}$ is not a regular Pythagorean fuzzy graph but G is a regular graph as the degree of each vertex is equal. Thus, $\mathscr{G}$ is a partially regular Pythagorean fuzzy graph.

The following theorems explain the conditions for the maximal product of two regular PFGs to be regular.

Theorem 2.3 *If $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ is a partially regular PFG and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ is a PFG such that $\mu_{\mathscr{A}_1} \leq \mu_{\mathscr{B}_2}$, $\nu_{\mathscr{A}_1} \geq \nu_{\mathscr{B}_2}$ and $\mu_{\mathscr{A}_2}, \nu_{\mathscr{A}_2}$ are constant functions of values c_1 and c_2, respectively, then their maximal product is regular if and only if $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ is regular PFG.*

Proof Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ be a partially regular PFG such that $G_1 = (V_1, E_1)$ is r-regular and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be any PFG with $\mu_{\mathscr{A}_1} \leq \mu_{\mathscr{B}_2}$, $\nu_{\mathscr{A}_1} \geq \nu_{\mathscr{B}_2}$ then $\mu_{\mathscr{A}_2} \geq \mu_{\mathscr{B}_1}$, $\nu_{\mathscr{A}_2} \leq \nu_{\mathscr{B}_1}$ and $\mu_{\mathscr{A}_2}, \nu_{\mathscr{A}_2}$ are constant functions of values c_1 and c_2, respectively.

Now assume that $\mathscr{G}_2 = (\mathscr{A}_1, \mathscr{B}_1)$ is a (k, l)-regular PFG. Then, the degree of any vertex in maximal product is given by,

$$(d_\mu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_1, u_2) = \sum_{(u_1, u_2)(v_1, v_2) \in E_1 \times E_2} (\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u_1, u_2)(v_1, v_2))$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_2} \mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{B}_2}(u_2 v_2)$$

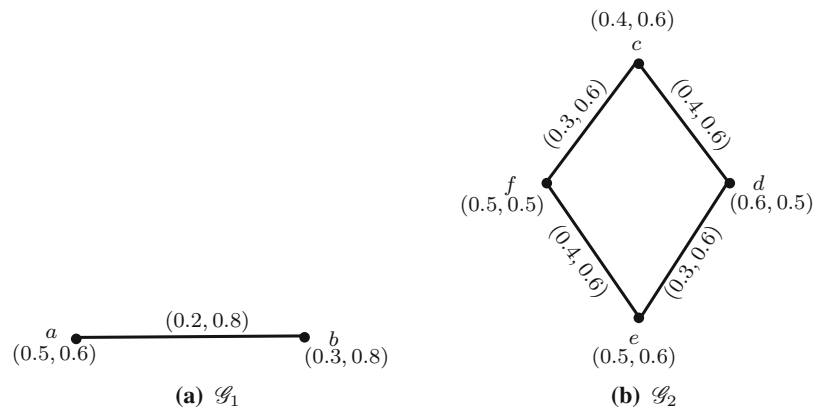$$+ \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu_{\mathscr{B}_1}(u_1 v_1) \vee \mu_{\mathscr{A}_2}(u_2)$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_2} \mu_{\mathscr{B}_2}(u_2 v_2)$$

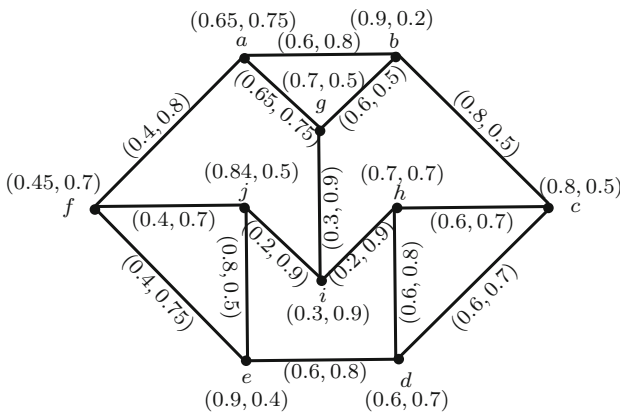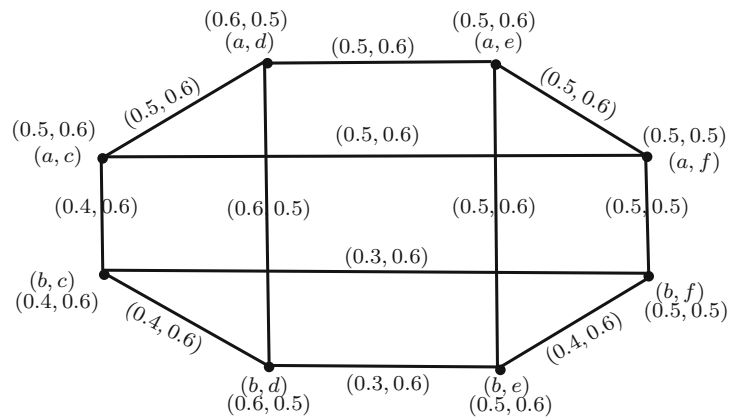$$+ \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu_{\mathscr{A}_2}(u_2)$$

$$= (d_\mu)_{\mathscr{G}_2}(u_2) + d_{G_1}(u_1)\mu_{\mathscr{A}_2}(u_2)$$

$$= rc_1 + k,$$

Fig. 10 Pythagorean fuzzy graphs

(a) $\mathscr{G}_1$

(b) $\mathscr{G}_2$

Fig. 11 Maximal product of two PFGs

Fig. 12 Pythagorean fuzzy graph

$$(d_\nu)_{\mathscr{G}_1*\mathscr{G}_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E_1\times E_2} (\nu_{\mathscr{B}_1}*\nu_{\mathscr{B}_2})((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1, u_2v_2\in E_2} \nu_{\mathscr{A}_1}(u_1)\wedge \nu_{\mathscr{B}_2}(u_2v_2)$$

$$+ \sum_{u_2=v_2, u_1v_1\in E_1} \nu_{\mathscr{B}_1}(u_1v_1)\wedge \nu_{\mathscr{A}_2}(u_2)$$

$$= \sum_{u_1=v_1, u_2v_2\in E_2} \nu_{\mathscr{B}_2}(u_2v_2)$$

$$+ \sum_{u_2=v_2, u_1v_1\in E_1} \nu_{\mathscr{A}_2}(u_2)$$

$$= (d_\mu)_{\mathscr{G}_2}(u_2) + d_{G_1}(u_1)\nu_{\mathscr{A}_2}(u_2)$$

$$= rc_2 + l.$$

This is constant for all vertices in $V_1 \times V_2$. Hence, $\mathscr{G}_1 * \mathscr{G}_2$ is a regular PFG.

Conversely, assume that $\mathscr{G}_1 * \mathscr{G}_2$ is a regular PFG. Then, for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$$(d_\mu)_{\mathscr{G}_1*\mathscr{G}_2}(u_1, v_1) = (d_\mu)_{\mathscr{G}_1*\mathscr{G}_2}(u_2, v_2)$$
$$(d_\mu)_{\mathscr{G}_2}(v_1) + d_{G_1}(u_1)\mu_{\mathscr{A}_2}(v_1) = (d_\mu)_{\mathscr{G}_2}(v_2)$$
$$+ d_{G_1}(u_2)\mu_{\mathscr{A}_2}(v_2)$$
$$rc_1 + (d_\mu)_{\mathscr{G}_2}(v_1) = rc_1 + (d_\mu)_{\mathscr{G}_2}(v_2)$$
$$(d_\mu)_{\mathscr{G}_2}(v_1) = (d_\mu)_{\mathscr{G}_2}(v_2),$$
$$(d_\nu)_{\mathscr{G}_1*\mathscr{G}_2}(u_1, v_1) = (d_\nu)_{\mathscr{G}_1*\mathscr{G}_2}(u_2, v_2)$$
$$(d_\nu)_{\mathscr{G}_2}(v_1) + d_{G_1}(u_1)\nu_{\mathscr{A}_2}(v_1) = (d_\nu)_{\mathscr{G}_2}(v_2)$$
$$+ d_{G_1}(u_2)\nu_{\mathscr{A}_2}(v_2)$$
$$rc_2 + (d_\nu)_{\mathscr{G}_2}(v_1) = rc_2 + (d_\nu)_{\mathscr{G}_2}(v_2)$$
$$(d_\nu)_{\mathscr{G}_2}(v_1) = (d_\nu)_{\mathscr{G}_2}(v_2).$$

This is true for all vertices in $\mathscr{G}_2$. Hence, $\mathscr{G}_2$ is regular PFG. □

Theorem 2.4 *If* $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ *and* $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ *are two partially regular PFG such that* $\mu_{\mathscr{A}_1} \geq \mu_{\mathscr{B}_2}$, $\nu_{\mathscr{A}_1} \leq \nu_{\mathscr{B}_2}$,

$\mu_{\mathscr{A}_2} \geq \mu_{\mathscr{B}_1}$, $\nu_{\mathscr{A}_2} \leq \nu_{\mathscr{B}_1}$ *and* $\mu_{\mathscr{A}_2}$, $\nu_{\mathscr{A}_2}$ *are constant functions of values* c_1 *and* c_2, *respectively, then their maximal product is regular if and only if* $\mu_{\mathscr{A}_1}$ *and* $\nu_{\mathscr{A}_1}$ *are constant functions.*

Proof Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are two partially regular PFG such that $\mu_{\mathscr{A}_1} \geq \mu_{\mathscr{B}_2}$, $\nu_{\mathscr{A}_1} \leq \nu_{\mathscr{B}_2}$, $\mu_{\mathscr{A}_2} \geq \mu_{\mathscr{B}_1}$, $\nu_{\mathscr{A}_2} \leq \nu_{\mathscr{B}_1}$ and $\mu_{\mathscr{A}_2}$, $\nu_{\mathscr{A}_2}$ are constant functions of values c_1 and c_2, respectively, with G_i is r_i-regular, $i = 1, 2$. Now assume that $\mu_{\mathscr{A}_1}$ and $\nu_{\mathscr{A}_1}$ are constant functions of values c_3 and c_4, respectively. Then, the degree of any vertex in maximal product is given by,

$$(d_\mu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E_1\times E_2} (\mu_{\mathscr{B}_1} * \mu_{\mathscr{B}_2})((u_1,u_2)(v_1,v_2))$$
$$= \sum_{u_1=v_1, u_2v_2\in E_2} \mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{B}_2}(u_2v_2)$$
$$+ \sum_{u_2=v_2, u_1v_1\in E_1} \mu_{\mathscr{B}_1}(u_1v_1) \vee \mu_{\mathscr{A}_2}(u_2)$$
$$= \sum_{u_1=v_1, u_2v_2\in E_2} \mu_{\mathscr{A}_1}(u_1)$$
$$+ \sum_{u_2=v_2, u_1v_1\in E_1} \mu_{\mathscr{A}_2}(u_2)$$
$$= d_{G_2}(u_2)\mu_{\mathscr{A}_1}(u_1) + d_{G_1}(u_1)\mu_{\mathscr{A}_2}(u_2)$$
$$= r_1 c_1 + r_2 c_3,$$

$$(d_\nu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E_1\times E_2} (\nu_{\mathscr{B}_1} * \nu_{\mathscr{B}_2})((u_1,u_2)(v_1,v_2))$$
$$= \sum_{u_1=v_1, u_2v_2\in E_2} \nu_{\mathscr{A}_1}(u_1) \wedge \nu_{\mathscr{B}_2}(u_2v_2)$$
$$+ \sum_{u_2=v_2, u_1v_1\in E_1} \nu_{\mathscr{B}_1}(u_1v_1) \wedge \nu_{\mathscr{A}_2}(u_2)$$
$$= \sum_{u_1=v_1, u_2v_2\in E_2} \nu_{\mathscr{A}_1}(u_1)$$
$$+ \sum_{u_2=v_2, u_1v_1\in E_1} \nu_{\mathscr{A}_2}(u_2)$$
$$= d_{G_2}(u_2)\nu_{\mathscr{A}_1}(u_1) + d_{G_1}(u_1)\nu_{\mathscr{A}_2}(u_2)$$
$$= r_1 c_2 + r_2 c_4.$$

This is constant for all vertices in $V_1 \times V_2$. Hence, $\mathscr{G}_1 * \mathscr{G}_2$ is a regular PFG.

Conversely, assume that $\mathscr{G}_1 * \mathscr{G}_2$ is a regular PFG. Then, for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$$(d_\mu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_1, v_1) = (d_\mu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_2, v_2)$$
$$d_{G_1}(u_1)\mu_{\mathscr{A}_2}(v_1) + d_{G_2}(v_1)\mu_{\mathscr{A}_1}(u_1) = d_{G_1}(u_2)\mu_{\mathscr{A}_2}(v_2)$$
$$+ d_{G_2}(v_2)\mu_{\mathscr{A}_1}(u_2)$$
$$rc_1 + r_2\mu_{\mathscr{A}_1}(u_1) = rc_1 + r_2\mu_{\mathscr{A}_1}(u_2)$$
$$\mu_{\mathscr{A}_1}(u_1) = \mu_{\mathscr{A}_1}(u_2),$$
$$(d_\nu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_1, v_1) = (d_\nu)_{\mathscr{G}_1 * \mathscr{G}_2}(u_2, v_2)$$
$$d_{G_1}(u_1)\nu_{\mathscr{A}_2}(v_1) + d_{G_2}(v_1)\nu_{\mathscr{A}_1}(u_1) = d_{G_1}(u_2)\nu_{\mathscr{A}_2}(v_2)$$
$$+ d_{G_2}(v_2)\nu_{\mathscr{A}_1}(u_2)$$
$$rc_2 + r_2\nu_{\mathscr{A}_1}(u_1) = rc_2 + r_2\nu_{\mathscr{A}_1}(u_2)$$
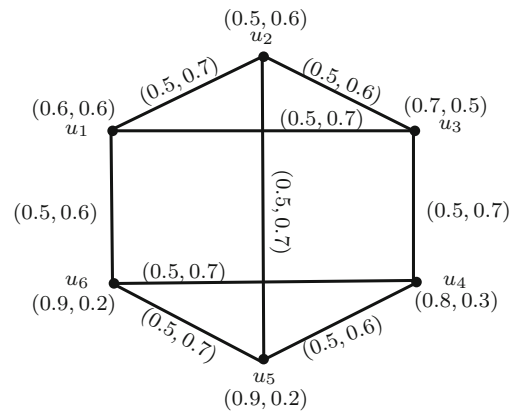$$\nu_{\mathscr{A}_1}(u_1) = \nu_{\mathscr{A}_1}(u_2).$$

Fig. 13 Pythagorean fuzzy graph

This is true for all vertices in $\mathscr{G}_1$. Hence, $\mu_{\mathscr{A}_1}$ and $\nu_{\mathscr{A}_1}$ are constant functions. $\qquad \square$

Definition 2.7 Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. Then $\mathscr{G}$ is said to be a *full* regular Pythagorean fuzzy graph if $\mathscr{G}$ is both regular and partially regular graph.

Example 2.6 Consider a Pythagorean fuzzy graph $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ as displayed in the Fig. 13.

Since $d_\mu(u_i) = 1.5$ and $d_\nu(u_i) = 2$ for all $u_i \in V$, where $i = 1, \ldots, 6$. Hence, $\mathscr{G}$ is a regular Pythagorean fuzzy graph of degree $(1.5, 2)$. Also, G is a regular graph as the degree of each vertex is equal. Thus, $\mathscr{G}$ is a full regular Pythagorean fuzzy graph.

Remark 2.4 If two Pythagorean fuzzy graphs are full regular, their maximal product may not be full regular PFG as it can be seen in this example. Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{a, b\}$ and $V_2 = \{c, d, e\}$, respectively, as shown in Fig. 14. Their maximal product $\mathscr{G}_1 * \mathscr{G}_2$ is shown in Fig. 15.

By routine calculation, one can see that $\mathscr{G}_1$ and $\mathscr{G}_2$ are full regular PFGs. But $(d)_{\mathscr{G}_1 * \mathscr{G}_2}(a, d) \neq (d)_{\mathscr{G}_1 * \mathscr{G}_2}(b, d)$. Hence, $\mathscr{G}_1 * \mathscr{G}_2$ is not full regular PFG.

Remark 2.5 The maximal product of two regular PFGs on complete graphs is partially regular. Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{u_1, v_1\}$ and $V_2 = \{u_2, v_2\}$, respectively, as shown in Fig. 16. Their maximal product $\mathscr{G}_1 * \mathscr{G}_2$ is shown in Fig. 17.

By routine calculation, one can see that $\mathscr{G}_1$ and $\mathscr{G}_2$ are regular PFGs. But $(d)_{\mathscr{G}_1 * \mathscr{G}_2}(u_1, u_2) \neq (d)_{\mathscr{G}_1 * \mathscr{G}_2}(v_1, v_2)$. Hence, $\mathscr{G}_1 * \mathscr{G}_2$ is a partially regular PFG as crisp graph is regular.
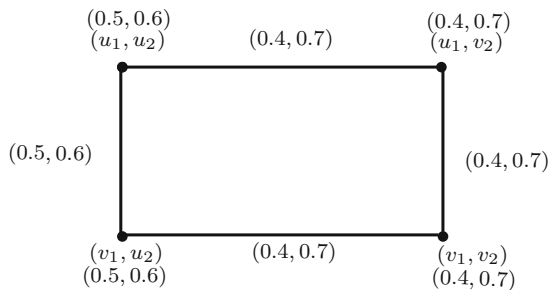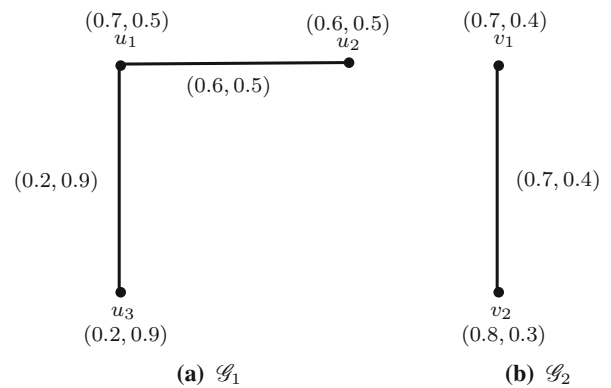
Fig. 14 Pythagorean fuzzy graphs

(a) $\mathcal{G}_1$ **(b)** $\mathcal{G}_2$
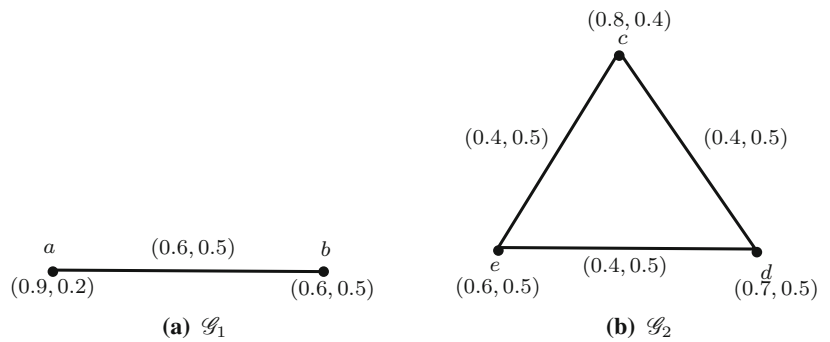
Fig. 15 Maximal product of two PFGs
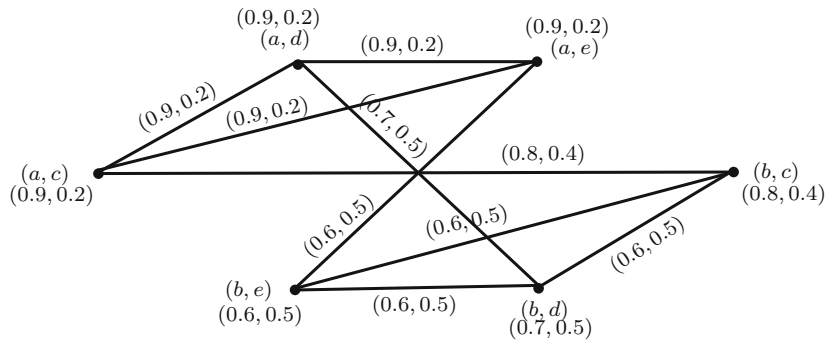
Fig. 16 Pythagorean fuzzy graph

(a) $\mathcal{G}_1$ **(b)** $\mathcal{G}_2$

Fig. 17 Maximal product of two PFGs

Regular residue product in Pythagorean fuzzy graphs

Definition 3.1 Let $\mathcal{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathcal{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be two PFGs of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. The *residue product* of $\mathcal{G}_1$ and $\mathcal{G}_2$ is denoted by $\mathcal{G}_1 \cdot \mathcal{G}_2 = (\mathcal{A}_1 \cdot \mathcal{A}_2, \mathcal{B}_1 \cdot \mathcal{B}_2)$ and defined as:

(i) $\begin{cases} (\mu_{\mathcal{A}_1} \cdot \mu_{\mathcal{A}_2})(u_1, u_2) = \mu_{\mathcal{A}_1}(u_1) \vee \mu_{\mathcal{A}_2}(u_2) \\ (\nu_{\mathcal{A}_1} \cdot \nu_{\mathcal{A}_2})(u_1, u_2) = \nu_{\mathcal{A}_1}(u_1) \wedge \nu_{\mathcal{A}_2}(u_2) \\ \text{for all } (u_1, u_2) \in V_1 \times V_2, \end{cases}$

(ii) $\begin{cases} (\mu_{\mathcal{B}_1} \cdot \mu_{\mathcal{B}_2})(u_1, u_2)(v_1, v_2) = \mu_{\mathcal{B}_1}(u_1 v_1) \\ (\nu_{\mathcal{B}_1} \cdot \nu_{\mathcal{B}_2})(u_1, u_2)(v_1, v_2) = \nu_{\mathcal{B}_1}(u_1 v_1) \\ \text{for all } u_1 v_1 \in E_1, u_2 \neq v_2. \end{cases}$

Fig. 18 Pythagorean fuzzy graphs

Remark 3.1 In general, the residue product of two strong PFGs may not be strong PFG as it is explained in this example. Consider two Pythagorean fuzzy graphs $\mathcal{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathcal{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ as shown in Fig. 18.

Here $\mu_{\mathcal{B}_1}(u_1 u_2) = \mu_{\mathcal{A}_1}(u_1) \wedge \mu_{\mathcal{A}_1}(u_2)$, $\nu_{\mathcal{B}_1}(u_1 u_2) = \nu_{\mathcal{A}_1}(u_1) \vee \nu_{\mathcal{A}_1}(u_2)$ for all $u_1 u_2 \in E_1$ and $\mu_{\mathcal{B}_2}(v_1 v_2) = \mu_{\mathcal{A}_2}(v_1) \wedge \mu_{\mathcal{A}_2}(v_2)$, $\nu_{\mathcal{B}_2}(v_1 v_2) = \nu_{\mathcal{A}_2}(v_1) \vee \nu_{\mathcal{A}_2}(v_2)$ for all $v_1 v_2 \in E_2$. Hence, $\mathcal{G}_1$ and $\mathcal{G}_2$ are strong PFGs.

But $(\mu_{\mathcal{B}_1} \cdot \mu_{\mathcal{B}_2})((u_1, v_2)(u_2, v_1)) = 0.6 \neq \mu_{\mathcal{A}_1}(u_1, v_2) \wedge \mu_{\mathcal{A}_2}(u_2, v_1) = 0.7$ and $(\nu_{\mathcal{B}_1} \cdot \nu_{\mathcal{B}_2})((u_1, v_2)(u_2, v_1))$

$= 0.5 \neq v_{\mathscr{A}_1}(u_1, v_2) \wedge v_{\mathscr{A}_2}(u_2, v_1) = 0.4$. Thus, their residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ is not a strong PFG.

The following theorem gives the condition under which the residue product is strong.

Theorem 3.1 *The residue product of a strong PFG $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ with any PFG $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ is a strong PFG if $\mu_{\mathscr{A}_1} \geq \mu_{\mathscr{A}_2}, v_{\mathscr{A}_1} \leq v_{\mathscr{A}_2}$.*

Proof Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ be a strong PFG and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be any PFG with $\mu_{\mathscr{A}_1} \geq \mu_{\mathscr{A}_2}, v_{\mathscr{A}_1} \leq v_{\mathscr{A}_2}$. Then

$$\mu_{\mathscr{B}_1}(u_1 u_2) = \mu_{\mathscr{A}_1}(u_1) \wedge \mu_{\mathscr{A}_1}(u_2),$$
$$v_{\mathscr{B}_1}(u_1 u_2) = \mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{A}_1}(u_2) \ for \ all \ u_1 u_2 \in E_1.$$

By definition of residue product, we have
If $u_1 u_2 \in E_1$ and $v_1 \neq v_2$, then

$$\begin{aligned}
&(\mu_{\mathscr{B}_1} \cdot \mu_{\mathscr{B}_2})((u_1, v_1)(u_2, v_2)) \\
&= \mu_{\mathscr{B}_1}(u_1 u_2) \\
&= \mu_{\mathscr{A}_1}(u_1) \wedge \mu_{\mathscr{A}_1}(u_2) \\
&= \{\mu_{\mathscr{A}_1}(u_1) \vee \mu_{\mathscr{A}_2}(v_1)\} \wedge \{\mu_{\mathscr{A}_1}(u_2) \vee \mu_{\mathscr{A}_2}(v_2)\} \\
&= (\mu_{\mathscr{A}_1} \cdot \mu_{\mathscr{A}_2})(u_1, v_1) \wedge (\mu_{\mathscr{A}_1} \cdot \mu_{\mathscr{A}_2})(u_2, v_2), \\
&(v_{\mathscr{B}_1} \cdot v_{\mathscr{B}_2})((u_1, v_1)(u_2, v_2)) \\
&= v_{\mathscr{B}_1}(u_1 u_2) \\
&= v_{\mathscr{A}_1}(u_1) \vee v_{\mathscr{A}_1}(u_2) \\
&= \{v_{\mathscr{A}_1}(u_1) \wedge v_{\mathscr{A}_2}(v_1)\} \vee \{v_{\mathscr{A}_1}(u_2) \wedge v_{\mathscr{A}_2}(v_2)\} \\
&= (v_{\mathscr{A}_1} \cdot v_{\mathscr{A}_2})(u_1, v_1) \vee (v_{\mathscr{A}_1} \cdot v_{\mathscr{A}_2})(u_2, v_2).
\end{aligned}$$

Hence, $\mathscr{G}_1 \cdot \mathscr{G}_2$ is a strong PFG. □

Theorem 3.2 *The residue product $\mathscr{G}_1 * \mathscr{G}_2$ of two connected PFGs is always a connected PFG.*

Proof Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be two connected PFGs with underlying crisp graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$. Then $\mu_1^\infty(u_i u_j) > 0$, $v_1^\infty(u_i u_j) > 0$ for all $u_i, u_j \in V_1$ and $\mu_2^\infty(v_i v_j) > 0$, $v_2^\infty(v_i v_j) > 0$ for all $v_i, v_j \in V_2$.

The residue product of $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ can be taken as $\mathscr{G} = (\mathscr{A}, \mathscr{B})$. Consider the k subgraphs of $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ with the vertex sets $V_2 = \{u_i v_1, u_i v_2, \dots, u_i v_n\}$ for $i = 1, 2, \dots, m$. Each of these subgraphs of $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is connected as $u_i's$ are same.

Since $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are connected, each u_i and v_i are adjacent to at least one of the vertices in V_1 and V_2. Therefore, there exists at least one edge between any pair of the above k subgraphs.
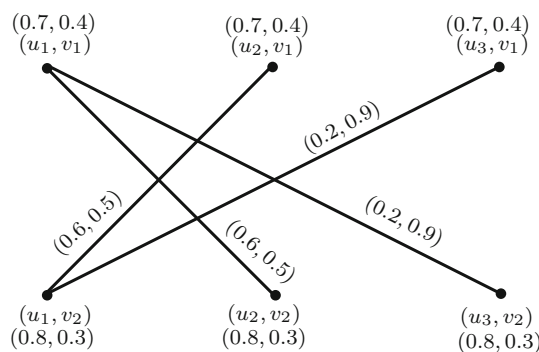Thus, we have

$$\mu^\infty((u_i v_j)(u_k, v_l)) > 0,$$

Fig. 19 Residue product of two PFGs

$$v^\infty((u_i v_j)(u_k, v_l)) > 0 \quad for \ all \ (u_i, v_j)(u_k, v_l) \in E.$$

Hence, $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is a connected PFG (Fig. 19). □

Remark 3.2 The residue product of two complete PFGs is not a complete PFG. We explain with an example. Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{a, b, c\}$ and $V_2 = \{d, e\}$, respectively, as shown in Fig. 20. Their residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ is shown in Fig. 21.

By routine calculation, one can see from Fig. 20 that $\mathscr{G}_1$ and $\mathscr{G}_2$ are complete PFGs. While notice that $\mathscr{G}_1 \cdot \mathscr{G}_2$ is not a complete PFG as the only case $u_1 u_2 \in E_1$, is included in the definition of the residue product.

Remark 3.3 If $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are two regular PFGs, then their residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ may not to be regular PFG as it is explained in this example. Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{a, b\}$ and $V_2 = \{c, d\}$, respectively, as shown in Fig. 22. Their residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ is shown in Fig. 23.

By routine calculation, one can see from Fig. 22 that $\mathscr{G}_1$ and $\mathscr{G}_2$ are regular PFGs. While notice that $(d)_{\mathscr{G}_1 . \mathscr{G}_2}(a, d) \neq (d)_{\mathscr{G}_1 * \mathscr{G}_2}(a, c)$. Therefore, $\mathscr{G}_1 \cdot \mathscr{G}_2$ is not regular PFG.

The following theorems explain the conditions for the residue product of two regular PFGs to be regular.

Theorem 3.3 *The residue product $\mathscr{G}_1 \cdot \mathscr{G}_1$ of any PFG $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ with a PFG $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ such that $|V_2| = 1$, is always a PFG with no edge.*
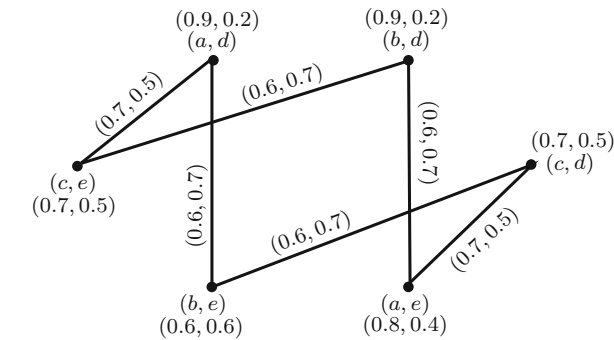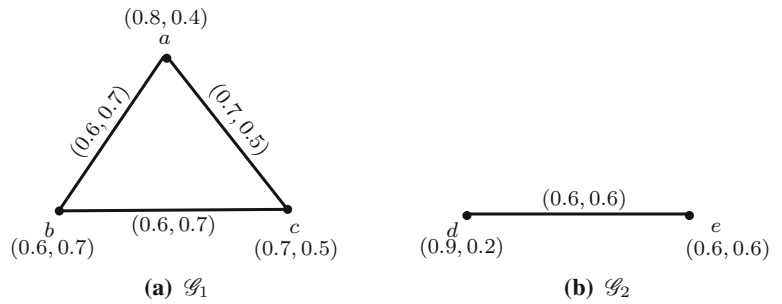
Proof The proof follows from the Definition 3.1. □

Theorem 3.4 *If $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are two PFGs such that $|V_2| > 1$, then their residue product is regular if and only if $\mathscr{G}_1$ is regular.*

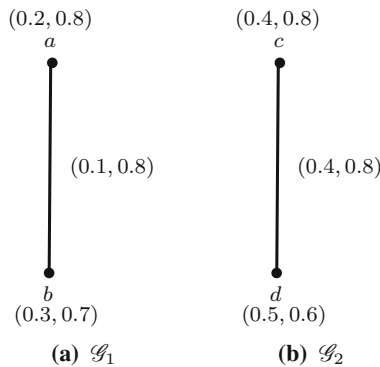**Proof** Let $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ be a (k, l)-regular PFG and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ be any PFG with $|V_2| > 1$.
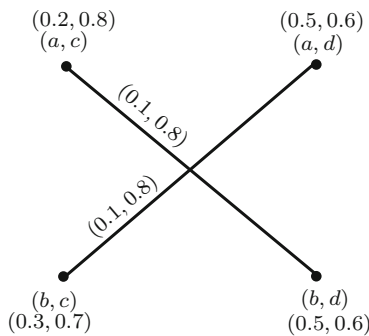If $|V_2| > 1$, then the degree of any vertex in Residue product is given by,

Fig. 20 Pythagorean fuzzy graphs

(a) $\mathscr{G}_1$

(b) $\mathscr{G}_2$

Fig. 21 Residue product of two PFGs

$$(d_\mu)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_1, u_2) = \sum_{u_1 v_1 \in E_1, u_2 \neq y_2} \mu_{\mathscr{B}_1}(u_1 v_1)$$
$$= (d_\mu)_{\mathscr{G}_1}(u_1) = k,$$
$$(d_v)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_1, u_2) = \sum_{u_1 v_1 \in E_1, u_2 \neq y_2} v_{\mathscr{B}_1}(u_1 v_1)$$
$$= (d_v)_{\mathscr{G}_1}(u_1) = l.$$

This is a constant for all vertices in $V_1 \times V_2$. Hence, $\mathscr{G}_1 \cdot \mathscr{G}_2$ is a regular PFG.

Conversely, assume that $\mathscr{G}_1 \cdot \mathscr{G}_2$ is a regular PFG. Then, for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$$(d_\mu)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_1, v_1) = (d_\mu)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_2, v_2)$$
$$(d_\mu)_{\mathscr{G}_1}(u_1) = (d_\mu)_{\mathscr{G}_1}(u_2),$$
$$(d_v)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_1, v_1) = (d_v)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_2, v_2)$$
$$(d_v)_{\mathscr{G}_1}(u_1) = (d_v)_{\mathscr{G}_1}(u_2).$$

This is true for all vertices in V_1. Hence, $\mathscr{G}_1$ is a regular PFG. $\square$

Fig. 22 Pythagorean fuzzy graphs

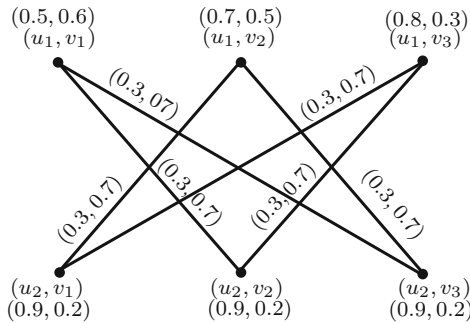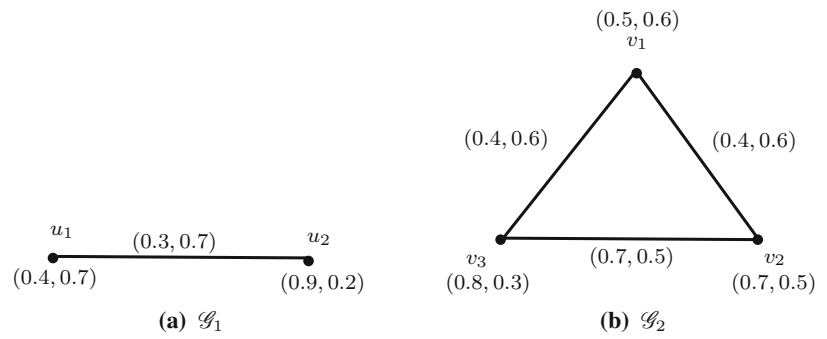**(a)** $\mathscr{G}_1$

(b) $\mathscr{G}_2$

Example 3.1 Consider two PFGs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ on $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$, respectively, as shown in Fig. 24. Their residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ is shown in Fig. 25.

Since $(d)_{\mathscr{G}_1}(u_i) = (0.3, 0.7)$ for i=1,2 and $(d)_{\mathscr{G}_2}(v_1) \neq (d)_{\mathscr{G}_2}(v_2)$, therefore $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ is a regular PFG and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ is not regular PFG with $|V_2| > 1$. Now, $(d)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_i, v_j) = (0.6, 1.4)$ for $i = 1, 2$ and $j = 1, 2, 3$. Thus, the residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ is a regular PFG.

Theorem 3.5 *If $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ are two PFGs such that $\mu_{\mathscr{A}_1} \geq \mu_{\mathscr{A}_2}$, $v_{\mathscr{A}_1} \leq v_{\mathscr{A}_2}$, then the total degree of any vertex in the residue product $\mathscr{G}_1 \cdot \mathscr{G}_2$ is given as,*

$$(td_\mu)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_i, v_j) = \begin{cases} (td_\mu)_{\mathscr{G}_1}(u_i) & \text{if } |V_1| > 1, \\ \mu_{\mathscr{A}_1}(u_i) & \text{if } |V_1| = 1, \end{cases}$$
$$(td_v)_{\mathscr{G}_1 \cdot \mathscr{G}_2}(u_i, v_j) = \begin{cases} (td_v)_{\mathscr{G}_1}(u_i) & \text{if } |V_1| > 1, \\ v_{\mathscr{A}_1}(u_i) & \text{if } |V_1| = 1. \end{cases}$$

Fig. 23 Residue product of two PFGs

Fig. 24 Pythagorean fuzzy graphs

(a) $\mathcal{G}_1$

(b) $\mathcal{G}_2$

Fig. 25 Residue product of two PFGs
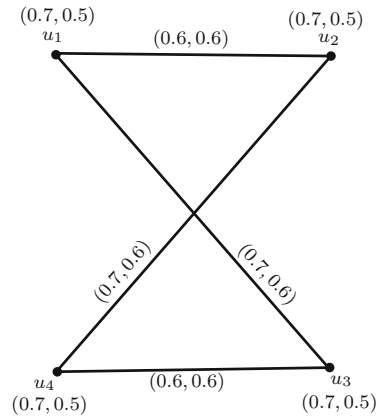
Proof If $\mu_{\mathcal{A}_1} \geq \mu_{\mathcal{A}_2}$, $\nu_{\mathcal{A}_1} \leq \nu_{\mathcal{A}_2}$ and $|V_2| > 1$, then

$$
\begin{aligned}
&(td_\mu)_{\mathcal{G}_1.\mathcal{G}_2}(u_i, v_j) \\
&= \sum_{(u_i,v_j)(u_k,v_l) \in E_1 \times E_2} (\mu_{\mathcal{B}_1} \cdot \mu_{\mathcal{B}_2})((u_i,v_j)(u_k,v_l)) \\
&\quad + (\mu_{\mathcal{A}_1} \cdot \mu_{\mathcal{A}_2})(u_i, v_j) \\
&= \sum_{u_i u_k \in E_1, v_j \neq v_l} \mu_{\mathcal{B}_1}(u_i u_k) + \mu_{\mathcal{A}_1}(u_i) \vee \mu_{\mathcal{A}_2}(v_j) \\
&= \begin{cases} (d_\mu)_{\mathcal{G}_1}(u_i) + \mu_{\mathcal{A}_1}(u_i) & \text{if } |V_2| > 1, \\ \mu_{\mathcal{A}_1}(u_i) & \text{if } |V_2| = 1, \end{cases} \\
&= \begin{cases} (td_\mu)_{\mathcal{G}_1}(u_i), & \text{if } |V_2| > 1, \\ \mu_{\mathcal{A}_1}(u_i), & \text{if } |V_2| = 1. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&(td_\nu)_{\mathcal{G}_1.\mathcal{G}_2}(u_i, v_j) \\
&= \sum_{(u_i,v_j)(u_k,v_l) \in E_1 \times E_2} (\nu_{\mathcal{B}_1} \cdot \nu_{\mathcal{B}_1})((u_i,v_j)(u_k,v_l)) \\
&\quad + (\nu_{\mathcal{A}_1} \cdot \nu_{\mathcal{A}_1})(u_i, v_j) \\
&= \sum_{u_i u_k \in E_1, v_j \neq v_l} \nu_{\mathcal{B}_1}(u_i u_k) \\
&\quad + \nu_{\mathcal{A}_1}(u_i) \wedge \nu_{\mathcal{A}_2}(v_j) \\
&= \begin{cases} (d_\nu)_{\mathcal{G}_1}(u_i) + \nu_{\mathcal{A}_1}(u_i) & \text{if } |V_2| > 1, \\ \nu_{\mathcal{A}_1}(u_i) & \text{if } |V_2| = 1, \end{cases} \\
&= \begin{cases} (td_\nu)_{\mathcal{G}_1}(u_i) & \text{if } |V_2| > 1, \\ \nu_{\mathcal{A}_1}(u_i) & \text{if } |V_2| = 1. \end{cases}
\end{aligned}
$$

$\square$

Fig. 26 $\mathcal{G} = (\mathcal{A}, \mathcal{B})$

Definition 3.2 Let $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. If

$$
\begin{aligned}
td_\mu(u) &= \sum_{u,v \neq u \in V} \mu_{\mathcal{B}}(uv) + \mu(u) = k, \\
td_\nu(u) &= \sum_{u,v \neq u \in V} \nu_{\mathcal{B}}(uv) + \nu(u) = l \quad \text{for all } u \in V,
\end{aligned}
$$

then $\mathcal{G}$ is said to be *totally regular* Pythagorean fuzzy graph of total degree (k, l) or (k, l)-totally regular PFG.

Example 3.2 Consider a graph $G = (V, E)$ where $V = \{u_1, u_2, u_3, u_4\}$ and $E = \{u_1 u_2, u_1 u_3, u_2 u_4, u_3 u_4\}$. Let $\mathcal{A}$ and $\mathcal{B}$ be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$, respectively (Fig. 26).

$$
\begin{aligned}
\mathcal{A} &= \left\langle \left(\frac{u_1}{0.7}, \frac{u_2}{0.7}, \frac{u_3}{0.7}, \frac{u_4}{0.7} \right), \right. \\
&\quad \left. \times \left(\frac{u_1}{0.5}, \frac{u_2}{0.5}, \frac{u_3}{0.5}, \frac{u_4}{0.5} \right) \right\rangle \text{ and} \\
\mathcal{B} &= \left\langle \left(\frac{u_1 u_2}{0.6}, \frac{u_1 u_3}{0.7}, \frac{u_2 u_4}{0.7}, \frac{u_3 u_4}{0.6} \right), \right. \\
&\quad \left. \times \left(\frac{u_1 u_2}{0.6}, \frac{u_1 u_3}{0.6}, \frac{u_2 u_4}{0.6}, \frac{u_3 u_4}{0.6} \right) \right\rangle.
\end{aligned}
$$

Since $td_\mu(u_i) = 1.8$ and $td_\nu(u_i) = 1.7$ for all $u_i \in V$ and $i = 1, 2, 3, 4$. Hence, $\mathcal{G}$ is a totally regular Pythagorean fuzzy graph of total degree $(1.8, 1.7)$ or $(1.8, 1.7)$-totally regular PFG.

Theorem 3.6 *If* $\mathcal{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ *is a totally regular PFG and* $\mathcal{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ *is a PFG such that* $\mu_{\mathcal{A}_1} \geq \mu_{\mathcal{A}_2}$, $\nu_{\mathcal{A}_1} \leq \nu_{\mathcal{A}_2}$ *and* $|V_2| > 1$, *then the residue product is totally regular PFG.*

Proof Let $\mathcal{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ be a (k, l)-totally regular PFG and $\mathcal{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be a PFG such that $\mu_{\mathcal{A}_1} \geq \mu_{\mathcal{A}_2}$, $\nu_{\mathcal{A}_1} \leq \nu_{\mathcal{A}_2}$ and $|V_2| > 1$. Then $(td_\mu)_{\mathcal{G}_1}(u_i) = k$, $(td_\nu)_{\mathcal{G}_1}(u_i) = l$, for all u_i in V_1 and $(\mu_{\mathcal{A}_1} \cdot \mu_{\mathcal{A}_2})(u_1, v_1) = \mu_{\mathcal{A}_1}(u_1)$, $(\nu_{\mathcal{A}_1} \cdot \nu_{\mathcal{A}_2})(u_1, v_1) = \nu_{\mathcal{A}_1}(u_1)$, for all (u_1, v_1) in $V_1 \times V_2$. Now,

$$(td_\mu)_{\mathcal{G}_1 \cdot \mathcal{G}_2}(u_1, v_1)$$
$$= (d_\mu)_{\mathcal{G}_1 \cdot \mathcal{G}_2}(u_1, v_1) + (\mu_{\mathcal{A}_1} \cdot \mu_{\mathcal{A}_2})(u_1, v_1)$$
$$= (d_\mu)_{\mathcal{G}_1}(u_1) + \mu_{\mathcal{A}_1}(u_1)$$
$$= (td_\mu)_{\mathcal{G}_1}(u_1)$$
$$= k,$$
$$(td_\nu)_{\mathcal{G}_1 \cdot \mathcal{G}_2}(u_1, v_1)$$
$$= (d_\nu)_{\mathcal{G}_1 \cdot \mathcal{G}_2}(u_1, v_1) + (\nu_{\mathcal{A}_1} \cdot \nu_{\mathcal{A}_2})(u_1, v_1)$$
$$= (d_\nu)_{\mathcal{G}_1}(u_1) + \nu_{\mathcal{A}_1}(u_1)$$
$$= (td_\nu)_{\mathcal{G}_1}(u_1)$$
$$= l.$$

This is constant for all vertices in $V_1 \times V_2$. Hence, $\mathcal{G}_1 \cdot \mathcal{G}_2$ is a (k, l)-totally regular PFG. □

Applications to decision-making

In this section, we apply the concept of PFGs to decision-making problems. Two decision-making problems concerning the 'evaluation of best company for investment' and 'alliance partner selection of a software company' are solved to illustrate the applicability of the proposed concept of PFGs in realistic scenario based on Pythagorean fuzzy preference relations (PFPRs) [31]. The algorithm for the alliance partner selection of a software company within the framework of PFPR is outlined in Algorithm 1.

Evaluation of best company for investment

To get the highest possible profit, assume that an investor wants to invest some money in an enterprise. Initially, he considers six possible alternatives: x_1 is a manufacturing company, x_2 is a fishing company, x_3 is a clothing company, x_4 is a feed company, x_5 is a seed company and x_6 is a building materials company. To prioritize these six possible
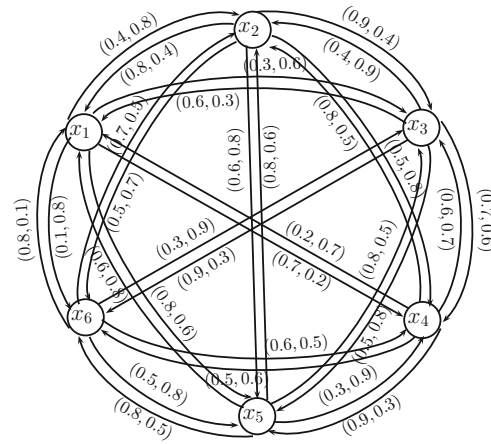
Fig. 27 Directed network of the Pythagorean fuzzy preference relation

Table 1 PFPR of the decision-maker

R	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$(0.5, 0.5)$	$(0.4, 0.8)$	$(0.3, 0.6)$	$(0.7, 0.2)$	$(0.8, 0.6)$	$(0.1, 0.8)$
x_2	$(0.8, 0.4)$	$(0.5, 0.5)$	$(0.9, 0.4)$	$(0.5, 0.8)$	$(0.6, 0.8)$	$(0.5, 0.7)$
x_3	$(0.6, 0.3)$	$(0.4, 0.9)$	$(0.5, 0.5)$	$(0.7, 0.6)$	$(0.5, 0.8)$	$(0.9, 0.3)$
x_4	$(0.2, 0.7)$	$(0.8, 0.5)$	$(0.6, 0.7)$	$(0.5, 0.5)$	$(0.9, 0.3)$	$(0.5, 0.6)$
x_5	$(0.6, 0.8)$	$(0.8, 0.6)$	$(0.8, 0.5)$	$(0.3, 0.9)$	$(0.5, 0.5)$	$(0.8, 0.5)$
x_6	$(0.8, 0.1)$	$(0.7, 0.5)$	$(0.3, 0.9)$	$(0.6, 0.5)$	$(0.5, 0.8)$	$(0.5, 0.5)$

alternatives $\mathcal{X} = \{x_1, x_1, x_3, x_4, x_5, x_6\}$, a decision-maker is invited to assess them. The decision-maker compares each pair of these alternatives and provides the judgements by a directed network of the PFPR whose vertices represent these alternatives as shown in Fig. 27 and corresponding PFPRs [31] are shown in Table 1:

Using Pythagorean fuzzy averaging (PFA) operator [5],

$$p_i = \text{PFA}(p_{i1}, p_{i2}, \ldots, p_{in})$$
$$= \left(\sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \mu_{ij}^2\right) \right)^{1/n}}, \left(\prod_{j=1}^{n} \nu_{ij} \right)^{1/n} \right), \quad i = 1, 2, 3, \ldots, n,$$

we aggregate all p_{ij}, $j = 1, 2, \ldots, 6$ corresponding to the company x_i, and then get the complex Pythagorean fuzzy number p_i of the company x_i, over all the other companies:

$p_1 = (0.5604, 0.5334)$, $p_2 = (0.6970, 0.5742)$,
$p_3 = (0.6669, 0.5185)$,
$p_4 = (0.6814, 0.5295)$, $p_5 = (0.6909, 0.6148)$,
$p_6 = (0.6116, 0.4561)$.

Table 2 PFPR of the expert from the business process outsourcing department

R_1	x_1	x_2	x_3	x_4	x_5
x_1	(0.5, 0.5)	(0.5, 0.7)	(0.8, 0.4)	(0.7, 0.6)	(0.3, 0.6)
x_2	(0.7, 0.5)	(0.5, 0.5)	(0.2, 0.9)	(0.7, 0.5)	(0.8, 0.3)
x_3	(0.4, 0.8)	(0.9, 0.2)	(0.5, 0.5)	(0.2, 0.6)	(0.9, 0.2)
x_4	(0.6, 0.7)	(0.5, 0.7)	(0.6, 0.2)	(0.5, 0.5)	(0.3, 0.7)
x_5	(0.6, 0.3)	(0.3, 0.8)	(0.2, 0.9)	(0.7, 0.3)	(0.5, 0.5)

Utilize the score function $s(p_i) = \mu_i^2 - \nu_i^2$ [6] in order to calculate the scores of p_i, $i = 1, 2, \ldots, 6$:

$$s(p_1) = 0.0295, \ s(p_2) = 0.1562, \ s(p_3) = 0.1759,$$
$$s(p_4) = 0.1839, \ s(p_5) = 0.0994, \ s(p_6) = 0.1660.$$

According to $s(p_i)$, $i = 1, 2, \ldots, 6$, we get the ranking of the partners x_i, $i = 1, 2, \ldots, 6$ as:

$$x_4 \succ x_3 \succ x_6 \succ x_2 \succ x_5 \succ x_1.$$

Therefore, the best company is x_4 to get the highest possible profit.

Alliance partner selection of a software company

Eastsoft is one of the top five software companies in China [36]. It offers a rich portfolio of businesses, including product engineering solutions, industry solutions, and related software products and platform and services. It is dedicated to becoming a globally leading IT solutions and services provider through continuous improvement of organization and process, competence development of leadership and employees, and alliance and open innovation. To improve the operation and competitiveness capability in the global market, Eastsoft plans to establish a strategic alliance with a transnational corporation. After numerous consultations, five transnational corporations (alternatives) would like to establish a strategic alliance with Eastsoft; they are HP x_1, PHILIPS x_2, EMC x_3, SAP x_4 and LK x_5. To select the desirable strategic alliance partner, a committee comprising three decision-makers $m_k (k = 1, 2, 3)$ (whose weight vector is $w = (0.3, 0.5, 0.7)$) is founded, come from the business process outsourcing department, the operation management department and the engineering management department of Eastsoft, respectively. Based on their experiences, the decision-makers compare each pair of alternatives and give individual judgments using the following PFPRs $R_k = (p_{ij}^{(k)})_{5 \times 5}$ $(k = 1, 2, 3)$:

The Pythagorean fuzzy digraphs $\mathcal{D}_k$ corresponding to PFPRs R_k $(k = 1, 2, 3)$ given in Tables 2, 3 and 4, are shown in Fig. 28.

Table 3 PFPR of the expert from the operation management department

R_2	x_1	x_2	x_3	x_4	x_5
x_1	(0.5, 0.5)	(0.9, 0.3)	(0.7, 0.2)	(0.3, 0.8)	(0.5, 0.8)
x_2	(0.3, 0.9)	(0.5, 0.5)	(0.6, 0.7)	(0.1, 0.5)	(0.8, 0.6)
x_3	(0.2, 0.7)	(0.7, 0.6)	(0.5, 0.5)	(0.7, 0.5)	(0.3, 0.7)
x_4	(0.8, 0.3)	(0.5, 0.1)	(0.5, 0.7)	(0.5, 0.5)	(0.8, 0.3)
x_5	(0.8, 0.5)	(0.6, 0.8)	(0.7, 0.3)	(0.3, 0.8)	(0.5, 0.5)

Table 4 PFPR of the expert from the engineering management department

R_3	x_1	x_2	x_3	x_4	x_5
x_1	(0.5, 0.5)	(0.7, 0.6)	(0.5, 0.8)	(0.3, 0.9)	(0.7, 0.6)
x_2	(0.6, 0.7)	(0.5, 0.5)	(0.8, 0.6)	(0.1, 0.7)	(0.3, 0.8)
x_3	(0.8, 0.5)	(0.6, 0.8)	(0.5, 0.5)	(0.4, 0.8)	(0.5, 0.7)
x_4	(0.9, 0.3)	(0.7, 0.1)	(0.8, 0.4)	(0.5, 0.5)	(0.8, 0.4)
x_5	(0.6, 0.7)	(0.8, 0.3)	(0.7, 0.5)	(0.4, 0.8)	(0.5, 0.5)

Compute the averaged pythagorean fuzzy element (PFE) $p_i^{(k)}$ of the alternative x_i over all the other alternatives for the experts $m_k(k = 1, 2, 3)$ by the Pythagorean fuzzy averaging (PFA) operator:

$$p_i^{(k)} = PFA(p_{i1}^{(k)}, p_{i2}^{(k)}, \ldots, p_{in}^{(k)})$$
$$= \left(\sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \mu_{ij}^2 \right) \right)^{1/n}}, \left(\prod_{j=1}^{n} \nu_{ij} \right)^{1/n} \right), \quad i = 1, 2, 3, \ldots, n.$$

The aggregation results of the experts m_k $(k = 1, 2, 3)$ are as follows:

$m_1:$ $p_1^{(1)} = (0.6139, 0.5502)$, $p_2^{(1)} = (0.6457, 0.5078)$, $p_3^{(1)} = (0.7312, 0.3949)$, $p_4^{(1)} = (0.5180, 0.5094)$, $p_5^{(1)} = (0.5152, 0.5036)$;

$m_2:$ $p_1^{(2)} = (0.6720, 0.4536)$, $p_2^{(2)} = (0.5574, 0.6239)$, $p_3^{(2)} = (0.5459, 0.5933)$, $p_4^{(2)} = (0.6639, 0.3160)$, $p_5^{(2)} = (0.6295, 0.5448)$;

$m_3:$ $p_1^{(3)} = (0.5761, 0.6645)$, $p_2^{(3)} = (0.5574, 0.6518)$, $p_3^{(3)} = (0.5985, 0.6454)$, $p_4^{(3)} = (0.7789, 0.2993)$, $p_5^{(3)} = (0.6371, 0.5305)$.

Compute a collective PFE p_i $(i = 1, 2, \ldots, n)$ of the alternative x_i over all the other alternatives using the Pythagorean fuzzy weighted averaging (PFWA) operator [5]:

$$p_i = \text{PFWA}(p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(s)})$$
$$= \left(\sqrt{1 - \prod_{k=1}^{s} \left(1 - (\mu_k^2) \right)^{w_k}}, \prod_{k=1}^{s} (\nu_k)^{w_k} \right).$$

Fig. 28 Pythagorean fuzzy digraphs
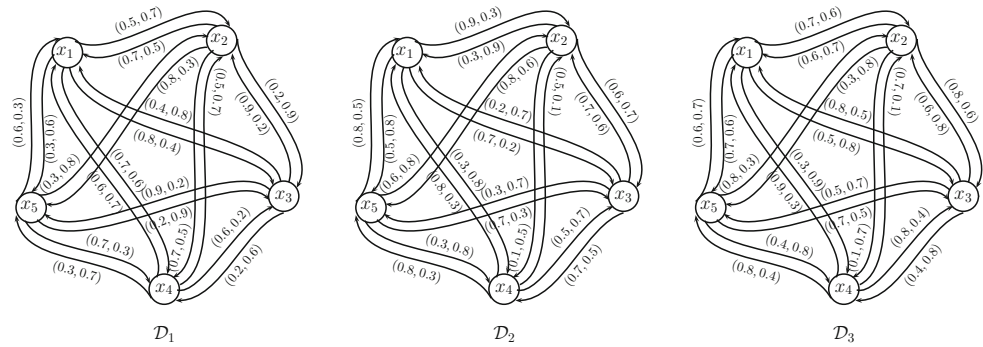
Table 5 The decision results of the alternatives using the different methods

Methods	Score of alternatives	Ranking of alternatives
Zhao et al. [40]	$-0.6800 \quad -2.1800 \quad -0.9400 \quad 3.8200 \quad -0.0200$	$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2$
Our proposed method	$0.3367 \quad 0.2276 \quad 0.3276 \quad 0.6067 \quad 0.3595$	$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2$

That is,

$$p_1 = (0.7180, 0.4229), \ p_2 = (0.6751, 0.4777),$$
$$p_3 = (0.7153, 0.4290),$$
$$p_4 = (0.8035, 0.1973), \ p_5 = (0.7128, 0.3855).$$

Compute the score functions $s(p_i) = \mu_i^2 - \nu_i^2$ [6] of $p_i (i = 1, 2, 3, 4, 5)$, and rank all the alternatives $x_i (i = 1, 2, 3, 4, 5)$ according to the values of $s(p_i)(i = 1, 2, 3, 4, 5)$.

$$s(p_1) = 0.3367, \ s(p_2) = 0.2276, \ s(p_3) = 0.3276,$$
$$s(p_4) = 0.6067, \ s(p_5) = 0.3595.$$

Then $x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_2$. Thus, the optimal choice is SAP x_4.

Comparative Analysis: We individually compute and compare the decision results using the method of Ref. [40] and our proposed method. They are shown in Table 5. From Table 5, the decision results of the method of Ref. [40] are consistent with our proposed method. It implies that our proposed method considers all the Pythagorean fuzzy evaluation information.

Conclusions and further work

Graph theory has vast range of applications in solving various networking problems encountered in different fields such as signal processing, transportation and error-correcting codes. For modeling the obscurity and uncertainties in practical decision-making and graphical networking problems, Pythagorean fuzzy graphs (PFGs) have better ability due to the increment of spaces in membership and non-membership

Algorithm 1 The algorithm for selection of alliance partner of a software company.

INPUT: A discrete set of alternatives $\mathscr{X} = \{x_1, x_2, \ldots, x_n\}$, a set of experts $e = \{m_1, m_2, \ldots, m_s\}$ with a weight vector $w = \{w_1, w_1, \ldots, w_s\}$ and construction of PFPR $R_k = (p_{ij}^{(k)})_{n \times n}$ for each expert.

OUTPUT: The selection of optimal alternative.

1. **begin**
2. Aggregate all $p_{ij}^{(k)} (j = 1, 2, \ldots, n)$ corresponding to the alternative x_i and get the PFE $p_i^{(k)}$ of the alternative x_i over all the other alternatives for the expert m_k by using the PFA operator.

$$p_i^{(k)} = PFA(p_{i1}^{(k)}, p_{i2}^{(k)}, \ldots, p_{in}^{(k)})$$
$$= \left(\sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \mu_{ij}^2 \right) \right)^{1/n}}, \left(\prod_{j=1}^{n} \nu_{ij} \right)^{1/n} \right),$$
$$i = 1, 2, 3, \ldots, n.$$

3. Aggregate all $p_i^{(k)} (k = 1, 2, \ldots, s)$ into a collective PFE p_i for the alternative x_i using the PFWA operator.

$$p_i = PFWA(p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(s)})$$
$$= \left(\sqrt{1 - \prod_{k=1}^{s} \left(1 - (\mu_k^2) \right)^{w_k}}, \prod_{k=1}^{s} (\nu_k)^{w_k} \right).$$

4. Compute the score functions $s(p_i) = \mu_i^2 - \nu_i^2$ of $p_i (i = 1, 2, \ldots, n)$.
5. Rank all the alternatives $x_i (i = 1, 2, \ldots, n)$ according to $s(p_i)(i = 1, 2, \ldots, n)$.
6. Output the optimal alternative.
7. **end**

grades. This paper has introduced two new graph products specifically, maximal product and residue product, for uniting two Pythagorean fuzzy graphs. Using these products, different types of structural models can be combined to

produce a better one. They may be useful for the configuration processing of space structures. Further, the paper has explored some crucial properties like strongness, connectedness, completeness and regularity. A special focus on the regularity has been given as it can be applied widely in designing reliable communication and computer networks. Applications of PFGs in decision-making concerning evaluation of best company for investment and alliance partner selection of a software company have been presented.

Further work We are working to extend our study to : (1) Pythagorean fuzzy soft graphs; (2) Rough Pythagorean fuzzy graphs; (3) Simplified interval-valued Pythagorean fuzzy graphs and; (4) Hesitant Pythagorean fuzzy graphs

Compliance with ethical standards

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