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ORIGINAL ARTICLE

On the x-coordinates of Pell equations that are sums of two Padovan numbers

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Abstract

Let $(P_n)_{n\geq 0}$ be the sequence of Padovan numbers defined by $P_0 = 0$, $P_1 = P_2 = 1$, and $P_{n+3} = P_{n+1} + P_n$ for all $n \geq 0$. In this paper, we find all positive square-free integers d such that the Pell equations $x^2 - dy^2 = N$ with $N \in \{\pm 1, \pm 4\}$, have at least two positive integer solutions (x, y) and (x', y') such that both x and x' are sums of two Padovan numbers.

Keywords Padovan number · Pell equation · Linear form in logarithms · Reduction method

Mathematics Subject Classification Primary 11B39 · 11D45 · Secondary 11D61 · 11J86

1 Introduction

Let $(P_n)_{n\geq 0}$ be the sequence of Padovan numbers defined by the linear recurrence

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1,\ \text{and } P_{n+3} = P_{n+1} + P_n \text{ for all } n \geq 0.$$

The Padovan sequence appears as sequence $A000931$ on the On-Line Encyclopedia of Integer Sequences (OEIS) [20]. The first few terms of this sequence are

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$$0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351\ldots.$$

Let $d \geq 2$ be a positive square-free integer. It is well known that the Pell equations

$$x^2 - dy^2 = \pm 1, \tag{1}$$

and

$$X^2 - dY^2 = \pm 4, \tag{2}$$

have infinitely many positive integer solutions (x, y) and (X, Y) , respectively. By putting (x_1, y_1) and (X_1, Y_1) for the smallest positive solutions to (1) and (2), respectively, all the solutions (x_k, y_k) and (X_k, Y_k) have the form

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k \qquad \text{for all} \quad k \in \mathbb{Z}^+,$$

and

$$\frac{X_k + Y_k\sqrt{d}}{2} = \left(\frac{X_1 + Y_1\sqrt{d}}{2}\right)^k \qquad \text{for all} \quad k \in \mathbb{Z}^+.$$

Furthermore, $(x_k)_{k \geq 1}$ and $(X_k)_{k \geq 1}$ are binary recurrent sequences. More exactly, the following formulae

$$x_k = \frac{(x_1 + y_1\sqrt{d})^k + (x_1 - y_1\sqrt{d})^k}{2}, \tag{3}$$

and

$$X_k = \left(\frac{X_1 + Y_1\sqrt{d}}{2}\right)^k + \left(\frac{X_1 - Y_1\sqrt{d}}{2}\right)^k \tag{4}$$

hold for all positive integers k.

In the recent years, Luca et al. [17] considered the Diophantine equation

$$x_k = T_n, \tag{5}$$

where x_k is given by (3) and $(T_n)_{n \geq 0}$ is the Tribonacci sequence defined by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$. The Tribonacci sequence appears as sequence $A000073$ on the OEIS [20]. The authors in [17] proved that Eq. (5) has at most one solution (k, n) in positive integers for all d except for $d = 2$ when Eq. (5) has the three solutions $(k, n) = \{(1, 1), (1, 2), (3, 5)\}$ and when $d = 3$ case in which Eq. (5) has the two solutions $(k, n) = \{(1, 3), (2, 5)\}$.

Inspired by the main result of Luca et al. [17], E. F. Bravo et al. [3, 4] studied the Diophantine equation

$$x_k = T_m + T_n. \tag{6}$$

They proved that for each square-free integer $d \geq 2$, there is at most one positive integer k such that x_k admits the representation (6) for some nonnegative integers

$0 \leq m \leq n$, except for $d \in \{2, 3, 5, 15, 26\}$. Furthermore, they explicitly stated all the solutions for these exceptional cases.

In the same spirit of the main result of Luca et al. [17], Rihane et al. [21] studied the Diophantine equations

$$x_k = P_n \qquad \text{and} \qquad X_k = P_n, \tag{7}$$

where x_k and X_k are given by (3) and (4), respectively. They proved that for each square-free integer $d \geq 2$, there is at most one positive integer x participating in the Pell equation (1) and at most one positive integer X participating in the Pell equation (2) that is a Padovan number with a few exceptions of d that they effectively computed. Furthermore, the exceptional cases were $d \in \{2, 3, 5, 6\}$ and $d = 5$ for the the first and second equations in (7), respectively. Several other related problems have been studied where x_k belongs to some interesting positive integer sequences. For example, see [8, 9, 11, 12, 14–16, 18].

2 Main results

In this paper, we study the same problem considered by E. F. Bravo et al.[3, 4] but with Padovan numbers instead of Tribonacci numbers. We also extend the results from the Pell equation (1) to the Pell equation (2). In both cases we find that there are only finitely many solutions that we effectively compute. Since $P_1 = P_2 = P_3 = 1$, we discard the situations when $n = 1$ and $n = 2$ and just count the solutions for $n = 3$. Similarly, $P_4 = P_5 = 2$, so we just count the solutions for $n = 5$.

The main aim of this paper is to prove the following results.

Theorem 1 *For each square-free integer $d \geq 2$, there is at most one positive integer k such that*

$$x_k = P_n + P_m \tag{8}$$

except when $d \in \{2, 3, 6, 15, 110, 483\}$ in the $+1$ case and $d \in \{2, 5, 10, 17\}$ in the -1 case.

Theorem 2 *For each square-free integer $d \geq 2$, there is at most one positive integer k such that*

$$X_k = P_n + P_m \tag{9}$$

except when $d \in \{3, 5, 21\}$ in the $+4$ case and $d \in \{2, 5\}$ in the -4 case.

For the exceptional values of d listed in Theorem 1 and Theorem 2, all solutions (k, n, m) are listed at the end of the proof of each result. The main tools used in this paper are lower bounds for nonzero linear forms in logarithms of algebraic numbers “á la Baker” and the Baker-Davenport reduction procedure, as well as the elementary properties of the Padovan sequence and solutions to Pell equations. Computations are done with the help of a computer program in *Mathematica*.

3 Preliminary results

3.1 The Padovan sequence

Here, we recall some important properties of the Padovan sequence $(P_n)_{n\geq 0}$. The characteristic equation

$$x^3 - x - 1 = 0,$$

has roots $\alpha, \beta, \gamma = \bar{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \qquad \beta = \frac{-(r_1 + r_2) + \sqrt{-3}(r_1 - r_2)}{12}, \tag{10}$$

with

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}. \tag{11}$$

Furthermore, a Binet-like formula for Padovan numbers is given by

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \qquad \text{for all} \quad n \geq 0, \tag{12}$$

where

$$a = \frac{\alpha + 1}{(\alpha - \beta)(\alpha - \gamma)}, \quad b = \frac{\beta + 1}{(\beta - \alpha)(\beta - \gamma)}, \quad c = \frac{\gamma + 1}{(\gamma - \alpha)(\gamma - \beta)} = \bar{b}. \tag{13}$$

Numerically, the following estimates hold:

$$\begin{aligned} &1.32 < \alpha < 1.33, \\ &0.86 < |\beta| = |\gamma| = \alpha^{-\frac{1}{2}} < 0.87, \\ &0.54 < a < 0.55, \\ &0.28 < |b| = |c| < 0.29. \end{aligned} \tag{14}$$

From (10), (11), and (14), it is easy to see that the contribution of the complex conjugate roots β and γ, to the right-hand side of (12), is very small. More exactly, setting $e(n) := P_n - a\alpha^n$ and taking into account the facts that $|\beta| = |\gamma| = \alpha^{-\frac{1}{2}}$ and $|b| = |c| < 0.29$ (by (14)), it follows that, for any $n \geq 1$,

$$|e(n)| = |b\beta^n + c\gamma^n| \leq |b||\beta|^n + |c||\gamma|^n = |b|\alpha^{-\frac{n}{2}} + |c|\alpha^{-\frac{n}{2}} < 2 \cdot 0.29 \cdot \alpha^{-\frac{n}{2}} < \frac{1}{\alpha^{n/2}}. \tag{15}$$

Finally, one can prove by induction that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 4. \tag{16}$$

3.2 Linear forms in logarithms

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then, the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$h(\eta_1 \pm \eta_2) \leq h(\eta_1) + h(\eta_2) + \log 2,$$
$$h(\eta_1\eta_2^{\pm 1}) \leq h(\eta_1) + h(\eta_2),$$
$$h(\eta^s) = |s|h(\eta) \quad (s \in \mathbb{Z}).$$

We recall the result of Bugeaud et al. (see [6], Theorem 9.4), which is a modified version of the result of Matveev [19], which is one of our main tools in this paper.

Theorem 3 (*Matveev according to Bugeaud et al.*, [6, 19]) *Let $\eta_1, \ldots, \eta_t$ be nonzero elements of an algebraic number field $\mathbb{K} \subset \mathbb{R}$ of degree $D_{\mathbb{K}}$ over $\mathbb{Q}$, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1,$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D_{\mathbb{K}}^2(1 + \log D_{\mathbb{K}})(1 + \log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{D_{\mathbb{K}}h(\eta_i), |\log\eta_i|, 0.16\}, \qquad \text{for all} \qquad i = 1, \ldots, t.$$

3.3 Reduction procedure

During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the following well-known classical result in the theory of Diophantine approximation. For further details, we refer the reader to the books of Baker and Wüstholz [2] and Cohen [7].

Lemma 1 (*Legendre*, [2, 7]). *Let τ be an irrational number, $p_0/q_0, p_1/q_1, p_2/q_2, \ldots$ be all the convergents of the continued fraction expansion of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [10], Lemma 5a) and itself is a generalization of a result of Baker and Davenport [1]. In this paper, we use an immediate variation of the result of Dujella and Pethő [10] due to J.J. Bravo et al. (see [5], Lemma 1). For a real number X, we write $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 2 (*Dujella, Pethő according to J. J. Bravo et al.* [5, *10*]) *Let M be a positive integer, p/q be a convergent of the continued fraction expansion of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case, we use the LLL algorithm that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i.$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \mathbf{e}_t \quad \text{for} \quad 1 \le j \le t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3 (*LLL-algorithm*, [7]) *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ be a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\mathbf{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\mathbf{b}_i^*\}$. We set*

$$c_1 := \max_{1 \le i \le t} \frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_i^*\|}, \quad \theta := \frac{\|\mathbf{b}_1\|}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2, \quad \text{and} \quad R := \frac{1}{2}\left(1 + \sum_{i=1}^{t} X_i\right).$$

If the integers x_i are such that $|x_i| \le X_i$, for $1 \le i \le t$ and $\theta^2 \ge Q + R^2$, then we have

$$\left| \sum_{i=1}^{t} x_i \tau_i \right| \ge \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen (see [7], Proposition 2.3.20).

Finally, the following Lemma is also useful. It is Lemma 7 in [13].

Lemma 4 (*Gúzman Sánchez, Luca*, [13]) *Let r, H, and L be positive real numbers. If $r \ge 1$, $H > (4r^2)^r$, and $H > L/(\log L)^r$, then*

$$L < 2^r H (\log H)^r.$$

4 Proof of Theorem 1

Let (x_1, y_1) be the smallest positive integer solution to the Pell equation (1). We put

$$\delta := x_1 + y_1\sqrt{d} \quad \text{and} \quad \sigma := x_1 - y_1\sqrt{d}. \tag{17}$$

From which we get that

$$\delta \cdot \sigma = x_1^2 - dy_1^2 =: N, \quad \text{where} \quad N \in \{\pm 1\}. \tag{18}$$

Then,

$$x_k = \frac{1}{2}(\delta^k + \sigma^k). \tag{19}$$

Since $\delta \ge 1 + \sqrt{2}$, it follows that the estimate

$$\frac{\delta^k}{\alpha^4} \le x_k \le \delta^k \quad \text{holds for all} \quad k \ge 1. \tag{20}$$

We assume that (k_1, n_1, m_1) and (k_2, n_2, m_2) are triples of integers such that

$$x_{k_1} = P_{n_1} + P_{m_1} \quad \text{and} \quad x_{k_2} = P_{n_2} + P_{m_2}. \tag{21}$$

We assume that $1 \le k_1 < k_2$. We also assume that $3 \le m_i < n_i$ for $i = 1, 2$. We set $(k, n, m) := (k_i, n_i, m_i)$, for $i = 1, 2$. Using the inequalities (16) and (20), we get from (21) that

$$\frac{\delta^k}{\alpha^4} \le x_k = P_n + P_m \le 2\alpha^{n-1} \quad \text{and} \quad \alpha^{n-2} \le P_n + P_m = x_k \le \delta^k.$$

The above inequalities give

$$(n-2)\log\alpha < k\log\delta < (n+3)\log\alpha + \log 2.$$

Dividing through by $\log\alpha$ and setting $c_2 := 1/\log\alpha$, we get that

$$-2 < c_2 k \log\delta - n < 3 + c_2\log 2,$$

and since $\alpha^3 > 2$, we get

$$|n - c_2 k \log\delta| < 6. \tag{22}$$

Furthermore, $k < n$, for if not, we would then get that

$$\delta^n \le \delta^k < 2\alpha^{n+3}, \quad \text{implying} \quad \left(\frac{\delta}{\alpha}\right)^n < 2\alpha^3,$$

which is false since $\delta \ge 1 + \sqrt{2}$, $\alpha \in (1.32, 1.33)$ (by (14)), and $n \ge 4$.

Besides, given that $k_1 < k_2$, we have by (16) and (21) that

$$\alpha^{n_1-2} \le P_{n_1} \le P_{n_1} + P_{m_1} = x_{k_1} < x_{k_2} = P_{n_2} + P_{m_2} \le 2P_{n_2} < 2\alpha^{n_2-1}.$$

Thus, we get that

$$n_1 < n_2 + 4. \tag{23}$$

4.1 An inequality for n and k

Using the Eqs. (8), (12), (19), and (21), we have

$$\frac{1}{2}(\delta^k + \sigma^k) = P_n + P_m = a\alpha^n + e(n) + a\alpha^m + e(m).$$

Therefore,

$$\frac{1}{2}\delta^k - a(\alpha^n + \alpha^m) = -\frac{1}{2}\sigma^k + e(n) + e(m),$$

and by (15), we have

$$|\delta^k(2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1| \le \frac{1}{2\delta^k a(\alpha^n+\alpha^m)}+\frac{2|b|}{\alpha^{n/2}a(\alpha^n+\alpha^m)} + \frac{2|b|}{\alpha^{m/2}a(\alpha^n+\alpha^m)} \le \frac{1}{a\alpha^n}\left(\frac{1}{2\delta^k}+\frac{2|b|}{\alpha^{n/2}}+\frac{2|b|}{\alpha^{m/2}}\right) < \frac{1.5}{\alpha^n}.$$

Thus, we have

$$|\delta^k(2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1| < \frac{1.5}{\alpha^n}. \tag{24}$$

Put

$$\Lambda_1 := \delta^k(2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1,$$

and

$$\Gamma_1 := k\log\delta - \log(2a) - n\log\alpha - \log(1+\alpha^{m-n}).$$

Since $|\Lambda_1| = |e^{\Gamma_1}-1| < 1/2$ for $n \ge 4$ (because $1.5/\alpha^4 < 1/2$), and the inequality $|y| < 2|e^y - 1|$ holds for all $y \in (-1/2, 1/2)$, it follows that $e^{|\Gamma_1|} < 2$ and so

$$|\Gamma_1| < e^{|\Gamma_1|}|e^{\Gamma_1}-1| < \frac{3}{\alpha^n}.$$

Thus, we get that

$$|k\log\delta - \log(2a) - n\log\alpha - \log(1+\alpha^{m-n})| < \frac{3}{\alpha^n}. \tag{25}$$

We apply Theorem 3 on the left-hand side of (24) with the data:

$$t := 4,\quad \eta_1 := \delta,\quad \eta_2 := 2a,\quad \eta_3 := \alpha,\quad \eta_4 := 1+\alpha^{m-n},$$
$$b_1 := k,\quad b_2 := -1,\quad b_3 := -n,\quad b_4 := -1.$$

Furthermore, we take the number field $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $D_{\mathbb{K}} := 6$. Since $\max\{1, k, n\} \le n$, we take $B := n$. First, we note that the left-hand side of (24) is nonzero, since otherwise,

$$\delta^k = 2a(\alpha^n + \alpha^m).$$

The left-hand side belongs to the quadratic field $\mathbb{Q}(\sqrt{d})$ while the right-hand side belongs to the cubic field $\mathbb{Q}(\alpha)$. These fields only intersect when both sides are rational numbers. Since δ^k is a positive algebraic integer and a unit, we get that to $\delta^k = 1$. Hence, $k = 0$, which is a contradiction. Thus, $\Lambda_1 \ne 0$ and we can apply Theorem 3.

We have $h(\eta_1) = h(\delta) = (\log\delta)/2$ and $h(\eta_3) = h(\alpha) = (\log\alpha)/3$. Furthermore,

$$2a = \frac{2\alpha(\alpha+1)}{2\alpha+3}.$$

The minimal polynomial of $2a$ is $23x^3 - 20x - 8$ and has roots $2a$, $2b$, $2c$. Since $2|b| = 2|c| < 1$ (by (14)), then

$$h(\eta_2) = h(2a) = \frac{1}{3}(\log 23 + \log(2a)).$$

On the other hand,

$$\begin{aligned} h(\eta_4) &= h(1+\alpha^{m-n}) \leq h(1) + h(\alpha^{m-n}) + \log 2 \\ &= (n-m)h(\alpha) + \log 2 = \frac{1}{3}(n-m)\log\alpha + \log 2. \end{aligned}$$

Thus, we can take $A_1 := 3\log\delta$,

$$A_2 := 2(\log 23 + \log(2a)), \quad A_3 := 2\log\alpha, \quad A_4 := 2(n-m)\log\alpha + 6\log 2.$$

Now, Theorem 3 tells us that

$$\begin{aligned} \log|\Lambda_1| > & -1.4 \times 30^7 \times 4^{4.5} \times 6^2 (1+\log 6)(1+\log n)(3\log\delta) \\ & \times (2(\log 23 + \log(2a))(2\log\alpha)(2(n-m)\log\alpha + 6\log 2) \\ > & -2.33 \times 10^{17}(n-m)(\log n)(\log\delta). \end{aligned}$$

Comparing the above inequality with (24), we get

$$n\log\alpha - \log 1.5 < 2.33 \times 10^{17}(n-m)(\log n)(\log\delta).$$

Hence, we get that

$$n < 8.30 \times 10^{17}(n-m)(\log n)(\log\delta). \tag{26}$$

We now return to the Diophantine equation (8) and rewrite it as

$$\frac{1}{2}\delta^k - a\alpha^n = -\frac{1}{2}\sigma^k + e(n) + P_m,$$

we obtain

$$|\delta^k(2a)^{-1}\alpha^{-n} - 1| \leq \frac{1}{a\alpha^{n-m}}\left(\frac{1}{\alpha} + \frac{1}{\alpha^{m+n/2}} + \frac{1}{2\delta^k\alpha^m}\right) < \frac{2.5}{\alpha^{n-m}}. \tag{27}$$

Put

$$\Lambda_2 := \delta^k(2a)^{-1}\alpha^{-n} - 1, \quad \Gamma_2 := k\log\delta - \log(2a) - n\log\alpha.$$

We assume for technical reasons that $n - m \geq 10$. Therefore, $|e^{\Lambda_2} - 1| < 1/2$. It follows that

$$|k\log\delta-\log(2a)-n\log\alpha|=|\Gamma_2|<e^{|\Lambda_2|}|e^{\Lambda_2}-1|<\frac{5}{\alpha^{n-m}}. \tag{28}$$

Furthermore, $\Lambda_2\neq 0$ (so $\Gamma_2\neq 0$), since $\delta^k\notin\mathbb{Q}(\alpha)$ by a previous argument.

We now apply Theorem 3 to the left-hand side of (27) with the data

$$t:=3,\quad \eta_1:=\delta,\quad \eta_2:=2a,\quad \eta_3:=\alpha,\quad b_1:=k,\quad b_2:=-1,\quad b_3:=-n.$$

Thus, we have the same A_1, A_2, A_3, as before. Then, by Theorem 3, we conclude that

$$\log|\Lambda|>-9.82\times 10^{14}(\log\delta)(\log n)(\log\alpha).$$

By comparing with (27), we get

$$n-m<9.84\times 10^{14}(\log\delta)(\log n). \tag{29}$$

This was obtained under the assumption that $n-m\geq 10$, but if $n-m<10$, then the above inequality also holds as well. We replace $n-m$ in (26) by its upper bound that we obtained in (29) and use the fact that $\delta^k\leq 2\alpha^{n+3}$, to obtain bounds on n and k in terms of $\log n$ and $\log\delta$. We now record what we have proved so far.

Lemma 5 *Let* (k,n,m) *be a solution to the Diophantine equation* (8) *with* $3\leq m<n$, *then*

$$k<2.5\times 10^{32}(\log n)^2(\log\delta)\quad\text{and}\quad n<8.2\times 10^{32}(\log n)^2(\log\delta)^2. \tag{30}$$

4.2 Absolute bounds

We recall that $(k,n,m)=(k_i,n_i,m_i)$, where $3\leq m_i<n_i$, for $i=1,2$ and $1\leq k_1<k_2$. Furthermore, $n_i\geq 4$ for $i=1,2$. We return to (28) and write

$$|\Gamma_2^{(i)}|:=|k_i\log\delta-\log(2a)-n_i\log\alpha|<\frac{5}{\alpha^{n_i-m_i}},\qquad\text{for}\qquad i=1,2.$$

We do a suitable cross product between $\Gamma_2^{(1)}$, $\Gamma_2^{(2)}$ and k_1,k_2 to eliminate the term involving $\log\delta$ in the above linear forms in logarithms:

$$\begin{aligned}|\Gamma_3|:&=|(k_1-k_2)\log(2a)+(k_1n_2-k_2n_1)\log\alpha|=|k_2\Gamma_2^{(1)}-k_1\Gamma_2^{(2)}|\\ &\leq k_2|\Gamma_2^{(1)}|+k_1|\Gamma_2^{(2)}|\leq\frac{5k_2}{\alpha^{n_1-m_1}}+\frac{5k_1}{\alpha^{n_2-m_2}}\leq\frac{10n_2}{\alpha^{\lambda}},\end{aligned} \tag{31}$$

where $\lambda:=\min_{1\leq i\leq 2}\{n_i-m_i\}$.

We need to find an upper bound for λ. If $10n_2/\alpha^{\lambda}>1/2$, we then get

$$\lambda<\frac{\log(20n_2)}{\log\alpha}<4\log(20n_2). \tag{32}$$

Otherwise, $|\Gamma_3|<1/2$, so

$$|e^{\Gamma_3} - 1| = |(2a)^{k_1-k_2}\alpha^{k_1n_2-k_2n_1} - 1| < 2|\Gamma_3| < \frac{20n_2}{\alpha^{\lambda}}. \tag{33}$$

We apply Theorem 3 with the data: $t := 2$, $\eta_1 := 2a$, $\eta_2 := \alpha$, $b_1 := k_1 - k_2$, $b_2 := k_1n_2 - k_2n_1$. We take the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $D_{\mathbb{K}} := 3$. We begin by checking that $e^{\Gamma_3} - 1 \neq 0$ (so $\Gamma_3 \neq 0$). This is true, because α and $2a$ are multiplicatively independent, since α is a unit in the ring of integers $\mathbb{Q}(\alpha)$ while the norm of $2a$ is 8/23.

We note that $|k_1 - k_2| < k_2 < n_2$. Furthermore, from (31), we have

$$|k_2n_1 - k_1n_2| < (k_2 - k_1)\frac{|\log(2a)|}{\log\alpha} + \frac{10k_2}{\alpha^{\lambda}\log\alpha} < 11k_2 < 11n_2$$

given that $\lambda \geq 1$. Therefore, we can take $B := 11n_2$. By Theorem 3, with the same $A_1 := \log 23$ and $A_2 := \log\alpha$, we have that

$$\log|e^{\Gamma_3} - 1| > -1.55 \times 10^{11}(\log n_2)(\log\alpha).$$

By comparing this with (33), we get

$$\lambda < 1.56 \times 10^{11}\log n_2. \tag{34}$$

Note that (34) is a better bound than (32), so (34) always holds. Without loss of generality, we can assume that $\lambda = n_i - m_i$, for $i = 1, 2$ fixed.

We set $\{i, j\} = \{1, 2\}$ and return to (25) to replace $(k, n, m) = (k_i, n_i, m_i)$:

$$|\Gamma_1^{(i)}| = |k_i\log\delta - \log(2a) - n_i\log\alpha - \log(1 + \alpha^{m_i-n_i})| < \frac{3}{\alpha^{n_i}}, \tag{35}$$

and also return to (28), with $(k, n, m) = (k_j, n_j, m_j)$:

$$|\Gamma_2^{(j)}| = |k_j\log\delta - \log(2a) - n_j\log\alpha| < \frac{5}{\alpha^{n_j-m_j}}. \tag{36}$$

We perform a cross product on (35) and (36) to eliminate the terms on $\log\delta$:

$$\begin{aligned} |\Gamma_4| :&= |(k_j - k_i)\log(2a) + (k_jn_i - k_in_j)\log\alpha + k_j\log(1 + \alpha^{m_i-n_i})| \\ &= |k_i\Gamma_2^{(j)} - k_j\Gamma_1^{(i)}| \leq k_i|\Gamma_2^{(j)}| + k_j|\Gamma_1^{(i)}| < \frac{5k_i}{\alpha^{n_j-m_j}} + \frac{3k_j}{\alpha^{n_i}} < \frac{8n_2}{\alpha^{\nu}} \end{aligned} \tag{37}$$

with $\nu := \min\{n_i, n_j - m_j\}$. As before, we need to find an upper bound on ν. If $8n_2/\alpha^{\nu} > 1/2$, then we get

$$\nu < \frac{\log(16n_2)}{\log\alpha} < 4\log(16n_2). \tag{38}$$

Otherwise, $|\Gamma_4| < 1/2$, so we have

$$|e^{\Gamma_4} - 1| \leq 2|\Gamma_4| < \frac{16n_2}{\alpha^{\nu}}. \tag{39}$$

To apply Theorem 3, first if $e^{\Gamma_4} = 1$, we obtain

$$(2a)^{k_i-k_j} = \alpha^{k_jn_i-k_in_j}(1+\alpha^{-\lambda})^{k_j}.$$

Since α is a unit, the right-hand side in above is an algebraic integer. This is a contradiction, because $k_1 < k_2$ so $k_i - k_j \neq 0$, and neither $(2a)$ nor $(2a)^{-1}$ are algebraic integers. Hence, $e^{\Gamma_4} \neq 1$. By assuming that $v \geq 100$, we apply Theorem 3 with the data:

$$t := 3, \quad \eta_1 := 2a, \quad \eta_2 := \alpha, \quad \eta_3 := 1+\alpha^{-\lambda},$$
$$b_1 := k_j - k_i, \quad b_2 := k_jn_i - k_in_j, \quad b_3 := k_j,$$

and the inequalities (34) and (39). We get

$$v = \min\{n_i, n_j - m_j\} < 1.14 \times 10^{14}\lambda \log n_2 < 1.78 \times 10^{25}(\log n_2)^2.$$

The above inequality also holds when $v < 100$. Furthermore, it also holds when the inequality (38) holds. Therefore, the above inequality holds in all cases. Note that the case $(i,j) = (2,1)$ leads to $n_1 - m_1 \leq n_1 \leq n_2 + 4$ whereas $(i,j) = (1,2)$ leads to $v = \min\{n_1, n_2 - m_2\}$. Hence, either the minimum is n_1, so

$$n_1 < 1.78 \times 10^{25}(\log n_2)^2, \tag{40}$$

or the minimum is $n_j - m_j$ and from the inequality (34), we get that

$$\max_{1\leq j\leq 2}\{n_j - m_j\} < 1.78 \times 10^{25}(\log n_2)^2. \tag{41}$$

Next, we assume that we are in the case (41). We evaluate (35), for $i = 1,2$ and make a suitable cross-product to eliminate the terms involving $\log\delta$:

$$\begin{aligned} |\Gamma_5| :&= |(k_2-k_1)\log(2a) + (k_2n_1 - k_1n_2)\log\alpha \\ &\quad + k_2\log(1+\alpha^{m_1-n_1}) - k_1\log(1+\alpha^{m_2-n_2})| \\ &= |k_1\Gamma_1^{(2)} - k_2\Gamma_1^{(1)}| \leq k_1|\Gamma_1^{(2)}| + k_2|\Gamma_1^{(1)}| < \frac{6n_2}{\alpha^{n_1}}. \end{aligned} \tag{42}$$

In the above inequality, we used the inequality (23) to conclude that $\min\{n_1, n_2\} \geq n_1 - 4$ as well as the fact that $n_i \geq 4$ for $i = 1.2$. Next, we apply a linear form in four logarithms to obtain an upper bound to n_1. As in the previous calculations, we pass from (42) to

$$|e^{\Gamma_5} - 1| < \frac{12n_2}{\alpha^{n_1}}, \tag{43}$$

which is implied by (42) except if n_1 is very small, say

$$n_1 \leq 4\log(12n_2). \tag{44}$$

Thus, we assume that (44) does not hold, therefore (43) holds. Then, to apply Theorem 3, we first justify that $e^{\Gamma_5} \neq 1$. Otherwise,

$$(2a)^{k_1-k_2} = \alpha^{k_2n_1-k_1n_2}(1+\alpha^{n_1-m_1})^{k_2}(1+\alpha^{n_2-m_2})^{-k_1}.$$

By the fact that $k_1 < k_2$, the norm $\mathbf{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(2a) = 8/23$ and that α is a unit, we have that 23 divides the norm $\mathbf{N}_{\mathbb{K}/\mathbb{Q}}(1+\alpha^{n_1-m_1})$. The factorization of the ideal generated by 23 in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ is $(23) = \mathfrak{p}_1^2\mathfrak{p}_2$, where $\mathfrak{p}_1 = (23, \alpha+13)$ and $\mathfrak{p}_2 = (23, \alpha+20)$. Hence, $\mathfrak{p}_2$ divides $\alpha^{n_1-m_1}+1$. Given that $\alpha \equiv -20 \pmod{\mathfrak{p}_2}$, then $(-20)^{n_1-m_1} \equiv -1 \pmod{\mathfrak{p}_2}$. Taking the norm $\mathbf{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}$, we obtain that $(-20)^{n_1-m_1} \equiv -1 \pmod{23}$. If $n_1 - m_1$ is even, -1 is a quadratic residue modulo 23 and if $n_1 - m_1$ is odd then 20 is a quadratic residue modulo 23. But, neither -1 nor 20 are quadratic residues modulo 23. Thus, $e^{\Gamma_5} \neq 1$.

Then, we apply Theorem 3 on the left-hand side of the inequality (43) with the data

$$t := 4, \quad \eta_1 := 2a, \quad \eta_2 := \alpha, \quad \eta_3 := 1+\alpha^{m_1-n_1}, \quad \eta_4 := 1+\alpha^{m_2-n_2},$$
$$b_1 := k_2 - k_1, \quad b_2 := k_2 n_1 - k_1 n_2, \quad b_3 := k_2, \quad b_4 := k_1.$$

Combining the right-hand side of (43) with the inequalities (34) and (41), Theorem 3 gives

$$n_1 < 3.02 \times 10^{16}(n_1 - m_1)(n_2 - m_2)(\log n_2) < 8.33 \times 10^{52}(\log n_2)^4. \tag{45}$$

In the above, we used the facts that

$$\min_{1\le i\le 2}\{n_i - m_i\} < 1.56 \times 10^{11}\log n_2 \quad \text{and} \quad \max_{1\le i\le 2}\{n_i - m_i\} < 1.78 \times 10^{25}(\log n_2)^2.$$

This was obtained under the assumption that the inequality (44) does not hold. If (44) holds, then so does (45). Thus, we have that inequality (45) holds provided that inequality (41) holds. Otherwise, inequality (40) holds which is a better bound than (45). Hence, we conclude that (45) holds in all possible cases.

By the inequality (22),

$$\log\delta \le k_1 \log\delta \le n_1 \log\alpha + \log 6 < 2.38 \times 10^{52}(\log n_2)^4.$$

By substituting this into (30) we get $n_2 < 4.64 \times 10^{137}(\log n_2)^{10}$, and then, by Lemma 4, with the data $r := 10$, $H := 4.64 \times 10^{137}$ and $L := n_2$, we get that $n_2 < 4.87 \times 10^{165}$. This immediately gives that $n_1 < 1.76 \times 10^{63}$.

We record what we have proved.

Lemma 6 *Let (k_i, n_i, m_i) be a solution to the Diophantine equation (8), with $3 \le m_i < n_i$ for $i \in \{1,2\}$ and $1 \le k_1 < k_2$, then*

$$\max\{k_1, m_1\} < n_1 < 1.76 \times 10^{63}, \quad \text{and} \quad \max\{k_2, m_2\} < n_2 < 4.87 \times 10^{165}.$$

5 Reducing the bounds for n_1 and n_2

In this section, we reduce the bounds for n_1 and n_2 given in Lemma 6 to cases that can be computationally treated. For this, we return to the inequalities for Γ_3, Γ_4, and Γ_5.

5.1 The first reduction

We divide both sides of the inequality (31) by $(k_2 - k_1)\log\alpha$. We get that

$$\left|\frac{\log(2a)}{\log\alpha} - \frac{k_2n_1 - k_1n_2}{k_2 - k_1}\right| < \frac{36n_2}{\alpha^{\lambda}(k_2 - k_1)} \quad \text{with} \quad \lambda := \min_{1\leq i\leq 2}\{n_i - m_i\}. \tag{46}$$

We assume that $\lambda \geq 10$. Below we apply Lemma 1. We put $\tau := \log(2a)/\log\alpha$, which is irrational and compute its continued fraction

$$[a_0, a_1, a_2, \ldots] = [1, 3, 3, 1, 11, 1, 2, 1, 1, 1, 3, 1, 1, 1, 2, 5, 1, 15, 2, 19, 1, 1, 2, 2, \ldots],$$

and its convergents

$$\left[\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots\right] = \left[1, \frac{4}{3}, \frac{13}{10}, \frac{17}{13}, \frac{200}{153}, \frac{217}{166}, \frac{634}{485}, \frac{851}{651}, \frac{1485}{1136}, \frac{2336}{1787}, \frac{8493}{6497}, \ldots\right].$$

Furthermore, we note that taking $M := 4.87 \times 10^{165}$ (by Lemma 6), it follows that

$$q_{315} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \leq i \leq 315\} = a_{282} = 2107.$$

Thus, by Lemma 1, we have that

$$\left|\tau - \frac{k_2n_1 - k_1n_2}{k_2 - k_1}\right| > \frac{1}{2109(k_2 - k_1)^2}. \tag{47}$$

Hence, combining the inequalities (46) and (47), we obtain

$$\alpha^{\lambda} < 75924n_2(k_2 - k_1) < 1.75 \times 10^{336},$$

so $\lambda \leq 2714$. This was obtained under the assumption that $\lambda \geq 10$. Otherwise, $\lambda < 10 < 2714$ holds as well.

Now, for each $n_i - m_i = \lambda \in [1, 2714]$, we estimate a lower bound for $|\Gamma_4|$, with

$$\Gamma_4 = (k_j - k_i)\log(2a) + (k_jn_i - k_in_j)\log\alpha + k_j\log(1 + \alpha^{m_i - n_i}) \tag{48}$$

given in the inequality (37), via the procedure described in Sect. 3.3 (LLL-algorithm). We recall that $\Gamma_4 \neq 0$. We apply Lemma 3 with the data:

$$t := 3, \quad \tau_1 := \log(2a), \quad \tau_2 := \log\alpha, \quad \tau_3 := \log(1 + \alpha^{-\lambda}),$$
$$x_1 := k_j - k_i, \quad x_2 := k_jn_i - k_in_j, \quad x_3 := k_j.$$

We set $X := 5.4 \times 10^{166}$ as an upper bound to $|x_i| < 11n_2$ for all $i = 1, 2, 3$, and

$C := (20X)^5$. A computer search using *Mathematica* allows us to conclude, together with the inequality (37), that

$$2 \times 10^{-671} < \min_{1 \le \lambda \le 2714} |\Gamma_4| < 8n_2\alpha^{-\nu}, \quad \text{with} \quad \nu := \min\{n_i, n_j - m_j\}$$

which leads to $\nu \le 6760$. As we have noted before, $\nu = n_1$ (so $n_1 \le 6760$) or $\nu = n_j - m_j$.

Next, we suppose that $n_j - m_j = \nu \le 6760$. Since $\lambda \le 2714$, we have

$$\lambda := \min_{1 \le i \le 2}\{n_i - m_i\} \le 2714 \quad \text{and} \quad \chi := \max_{1 \le i \le 2}\{n_i - m_i\} \le 6760.$$

Now, returning to the inequality (42) which involves

$$\begin{aligned}\Gamma_5 &:= (k_2 - k_1)\log(2a) + (k_2 n_1 - k_1 n_2)\log\alpha \\ &\quad + k_2\log(1 + \alpha^{m_1 - n_1}) - k_1\log(1 + \alpha^{m_2 - n_2}) \neq 0.\end{aligned} \tag{49}$$

We use again the LLL algorithm to estimate the lower bound for $|\Gamma_5|$ and thus, find a bound for n_1 that is better than the one given in Lemma 6.

We distinguish the cases $\lambda < \chi$ and $\lambda = \chi$.

5.2 The case $\lambda < \chi$

We take $\lambda \in [1, 2714]$ and $\chi \in [\lambda + 1, 6760]$ and apply Lemma 3 with the data: $t := 4$,

$$\begin{aligned}&\tau_1 := \log(2a), \quad \tau_2 := \log\alpha, \quad \tau_3 := \log(1 + \alpha^{m_1 - n_1}), \quad \tau_4 := \log(1 + \alpha^{m_2 - n_2}),\\ &x_1 := k_2 - k_1, \quad x_2 := k_2 n_1 - k_1 n_2, \quad x_3 := k_2, \quad x_4 := -k_1.\end{aligned}$$

We also put $X := 5.4 \times 10^{166}$ and $C := (20X)^9$. After a computer search in *Mathematica* together with the inequality (42), we can confirm that

$$8 \times 10^{-1342} < \min_{\substack{1 \le \lambda \le 2714 \\ \lambda + 1 \le \chi \le 6760}} |\Gamma_5| < 6n_2\alpha^{-n_1}.$$

This leads to the inequality

$$\alpha^{n_1} < 7.5 \times 10^{1341} n_2.$$

Substituting for the bound n_2 given in Lemma 6, we get that $n_1 \le 12172$.

5.3 The case $\lambda = \chi$

In this case, we have

$$\Lambda_5 := (k_2 - k_1)(\log(2a) + \log(1 + \alpha^{m_1 - n_1})) + (k_2 n_1 - k_1 n_2)\log\alpha \neq 0.$$

We divide through the inequality (42) by $(k_2 - k_1)\log\alpha$ to obtain

$$\left|\frac{\log(2a)+\log(1+\alpha^{m_1-n_1})}{\log\alpha}-\frac{k_2n_1-k_1n_2}{k_2-k_1}\right|<\frac{21n_2}{\alpha^{n_1}(k_2-k_1)}. \tag{50}$$

We now put

$$\tau_\lambda := \frac{\log(2a)+\log(1+\alpha^{-\lambda})}{\log\alpha},$$

and compute its continued fractions $[a_0^{(\lambda)}, a_1^{(\lambda)}, a_2^{(\lambda)}, \ldots]$, and its convergents $[p_0^{(\lambda)}/q_0^{(\lambda)}, p_1^{(\lambda)}/q_1^{(\lambda)}, p_2^{(\lambda)}/q_2^{(\lambda)}, \ldots]$, for each $\lambda \in [1, 2714]$. Furthermore, for each case we find an integer t_λ such that $q_{t_\lambda}^{(\lambda)} > M := 4.87 \times 10^{165} > n_2 > k_2 - k_1$ and calculate

$$a(M) := \max_{1\le\lambda\le 2714}\left\{a_i^{(\lambda)} : 0 \le i \le t_\lambda\right\}.$$

A computer search in *Mathematica* reveals that for $\lambda = 321$, $t_\lambda = 330$ and $i = 263$, we have that $a(M) = a_{321}^{(330)} = 306269$. Hence, combining the conclusion of Lemma 1 and the inequality (50), we get

$$\alpha^{n_1} < 21 \times 306271 n_2(k_2-k_1) < 1.525 \times 10^{338},$$

so $n_1 \le 2730$. Hence, we obtain that $n_1 \le 12172$ holds in all cases ($v = n_1$, $\lambda < \chi$ or $\lambda = \chi$). By the inequality (22), we have that

$$\log\delta \le k_1 \log\delta \le n_1 \log\alpha + \log 6 < 3475.$$

By considering the second inequality in (30), we can conclude that $n_2 \le 9.9 \times 10^{39}(\log n_2)^2$, which immediately yields $n_2 < 3.36 \times 10^{44}$, by a simple application of Lemma 4. We summarise the first cycle of our reduction process as follows:

$$n_1 \le 12172 \quad \text{and} \quad n_2 \le 3.36 \times 10^{44}.$$

From the above inequalities, we note that the upper bound on n_2 represents a very good reduction of the bound given in Lemma 6. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get a better bound on n_1. Thus, we return to the inequality (46) and take $M := 3.36 \times 10^{44}$. A computer search in *Mathematica* reveals that

$$q_{88} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \le i \le 88\} = a_{54} = 373,$$

from which it follows that $\lambda \le 752$. We now return to (48) and we put $X := 3.36 \times 10^{44}$ and $C := (10X)^5$ and then apply the LLL algorithm in Lemma 3 to $\lambda \in [1, 752]$. After a computer search, we get

$$5.33\times 10^{-184} < \min_{1\le\lambda\le 752}|\Gamma_4| < 8n_2\alpha^{-\nu},$$

then $\nu\le 1846$. By continuing under the assumption that $n_j - m_j = \nu \le 1846$, we return to (49) and put $X := 3.36\times 10^{44}$, $C := (10X)^9$ and $M := 3.36\times 10^{44}$ for the case $\lambda<\chi$ and $\lambda=\chi$. After a computer search, we confirm that

$$2\times 10^{-366} < \min_{\substack{1\le\lambda\le 752\\ \lambda+1\le\chi\le 1846}} |\Gamma_5| < 6n_2\alpha^{-n_1},$$

gives $n_1\le 3318$, and $a(M) = a_{175}^{(205)} = 206961$, which leads to $n_1\le 772$. Hence, in both cases $n_1\le 3318$ holds. This gives $n_2\le 5\times 10^{42}$ by a similar procedure as before, and $k_1\le 3125$.

We record what we have proved.

Lemma 7 *Let (k_i, n_i, m_i) be a solution to the Diophantine equation (8), with $3\le m_i<n_i$ for $i=1,2$ and $1\le k_1<k_2$, then*

$$m_1<n_1\le 3318,\quad k_1\le 3125,\quad \text{and}\quad n_2\le 5\times 10^{42}.$$

5.4 The final reduction

Returning to (17) and (19) and using the fact that (x_1, y_1) is the smallest positive solution to the Pell equation (1), we obtain

$$\begin{aligned} x_k = \frac{1}{2}(\delta^k+\sigma^k) &= \frac{1}{2}\left(\left(x_1+y_1\sqrt{d}\right)^k+\left(x_1-y_1\sqrt{d}\right)^k\right)\\ &= \frac{1}{2}\left(\left(x_1+\sqrt{x_1^2\mp 1}\right)^k+\left(x_1-\sqrt{x_1^2\mp 1}\right)^k\right) := Q_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = P_{n_1}+P_{m_1}$ and consider the equations

$$Q_{k_1}^{+}(x_1) = P_{n_1}+P_{m_1}\quad\text{and}\quad Q_{k_1}^{-}(x_1) = P_{n_1}+P_{m_1}, \tag{51}$$

with $k_1\in[1,3125]$, $m_1\in[3,3318]$ and $n_1\in[m_1+1,3318]$.

Besides the trivial case $k_1=1$, with the help of a computer search in *Mathematica* on the above equations in (51), we list the only nontrivial solutions in Table 1. We also note that $3+2\sqrt{2} = (1+\sqrt{2})^2$, so these solutions come from the same Pell equation when $d=2$.

From Table 1, we set each $\delta := \delta_t$ for $t = 1,2,\ldots,17$. We then work on the linear forms in logarithms Γ_1 and Γ_2, to reduce the bound on n_2 given in Lemma 7. From the inequality (28), for $(k,n,m) := (k_2,n_2,m_2)$, we write

Table 1 Solutions to $Q_{k_1}^{\pm}(x_1) = P_{n_1} + P_{m_1}$

$Q_{k_1}^{+}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	1	3	$2+\sqrt{3}$
2	3	2	2	$3+2\sqrt{2}$
2	4	1	15	$4+\sqrt{15}$
2	5	2	6	$5+2\sqrt{6}$
2	21	2	110	$21+2\sqrt{110}$
2	22	1	483	$22+\sqrt{483}$
2	47	4	138	$47+4\sqrt{138}$
$Q_{k_1}^{-}(x_1)$				
k_1	x_1	y_1	d	δ
2	1	1	2	$1+\sqrt{2}$
2	2	1	5	$2+\sqrt{5}$
2	3	1	10	$3+\sqrt{10}$
2	4	1	17	$4+\sqrt{17}$
2	5	1	26	$5+\sqrt{26}$
2	9	1	82	$9+\sqrt{82}$
2	10	1	101	$10+\sqrt{101}$
2	17	1	290	$17+\sqrt{290}$
2	42	1	1765	$42+\sqrt{1765}$
2	47	1	2210	$47+\sqrt{2210}$
2	63	1	3970	$63+\sqrt{3970}$

$$\left|k_2\frac{\log\delta_t}{\log\alpha} - n_2 + \frac{\log(2a)}{\log(\alpha^{-1})}\right| < \left(\frac{5}{\log\alpha}\right)\alpha^{-(n_2-m_2)}, \quad \text{for} \quad t = 1,2,\ldots,17. \tag{52}$$

We put

$$\tau_t := \frac{\log\delta_t}{\log\alpha}, \qquad \mu_t := \frac{\log(2a)}{\log(\alpha^{-1})} \qquad \text{and} \quad (A_t, B_t) := \left(\frac{5}{\log\alpha}, \alpha\right).$$

We note that τ_t is transcendental by Gelfond-Schneider Theorem (see [2], Theorem 2.1). Thus, τ_t is irrational. We can rewrite the inequality (52) as

$$0 < |k_2\tau_t - n_2 + \mu_t| < A_t B_t^{-(n_2-m_2)}, \quad \text{for} \quad t = 1,2,\ldots,17. \tag{53}$$

We take $M := 5\times 10^{42}$ which is the upper bound on n_2 according to Lemma 7 and apply Lemma 2 to the inequality (53). As before, for each τ_t with $t = 1,2,\ldots,17$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents

$p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in *Mathematica*, we find an integer s_t such that

$$q_{s_t}^{(t)} > 3 \times 10^{43} = 6M \qquad \text{and} \qquad \epsilon_t := \|\mu_t q^{(t)}\| - M\|\tau_t q^{(t)}\| > 0.$$

We finally compute all the values of $b_t := \lfloor \log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t \rfloor$. The values of b_t correspond to the upper bounds on $n_2 - m_2$, for each $t = 1, 2, \ldots, 17$, according to Lemma 2. The results of the computation for each t are recorded in Table 2.

By replacing $(k, n, m) := (k_2, n_2, m_2)$ in the inequality (25), we can write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - n_2 + \frac{\log(2a(1 + \alpha^{-(n_2 - m_2)}))}{\log(\alpha^{-1})} \right| < \left(\frac{3}{\log \alpha} \right) \alpha^{-n_2}, \quad \text{for} \quad t = 1, 2, \ldots, 17. \tag{54}$$

We now put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \quad \mu_{t, n_2 - m_2} := \frac{\log(2a(1 + \alpha^{-(n_2 - m_2)}))}{\log(\alpha^{-1})} \quad \text{and} \quad (A_t, B_t) := \left(\frac{3}{\log \alpha}, \alpha \right).$$

With the above notations, we can rewrite the inequality (54) as

$$0 < |k_2 \tau_t - n_2 + \mu_{t, n_2 - m_2}| < A_t B_t^{-n_2}, \quad \text{for} \quad t = 1, 2, \ldots, 17. \tag{55}$$

We again apply Lemma 2 to the inequality (55), for

Table 2 First reduction computation results

t	δ_t	s_t	q_{s_t}	$\epsilon_t >$	b_t
1	$2 + \sqrt{3}$	85	8.93366×10^{43}	0.3100	374
2	$4 + \sqrt{15}$	90	3.90052×10^{43}	0.3124	371
3	$5 + 2\sqrt{6}$	80	3.16032×10^{43}	0.0122	382
4	$21 + 2\sqrt{110}$	88	6.33080×10^{43}	0.2200	374
5	$22 + \sqrt{483}$	75	4.19689×10^{43}	0.2361	372
6	$47 + 4\sqrt{138}$	96	7.76442×10^{43}	0.3732	373
7	$1 + \sqrt{2}$	78	1.46195×10^{44}	0.3328	375
8	$2 + \sqrt{5}$	94	1.48837×10^{44}	0.2146	377
9	$3 + \sqrt{10}$	88	4.21425×10^{43}	0.1347	374
10	$4 + \sqrt{17}$	92	1.11753×10^{44}	0.2529	375
11	$5 + \sqrt{26}$	98	3.23107×10^{43}	0.1043	374
12	$9 + \sqrt{82}$	74	5.25207×10^{43}	0.2181	373
13	$10 + \sqrt{101}$	94	1.86122×10^{44}	0.2672	377
14	$17 + \sqrt{290}$	87	1.06422×10^{44}	0.0193	384
15	$42 + \sqrt{1765}$	78	3.81406×10^{43}	0.1768	373
16	$47 + \sqrt{2210}$	94	3.92482×10^{43}	0.4476	370
17	$63 + \sqrt{3970}$	85	6.00550×10^{43}	0.4056	371

$$t=1,2,\ldots,17,\quad n_2-m_2=1,2,\ldots,b_t,\quad \text{with}\quad M:=5\times 10^{43}.$$

We take

$$\epsilon_{t,n_2-m_2}:=\|\mu_t q^{(t,n_2-m_2)}\|-M\|\tau_t q^{(t,n_2-m_2)}\|>0,$$

and

$$b_t=b_{t,n_2-m_2}:=\lfloor\log(A_t q_{s_t}^{(t,n_2-m_2)}/\epsilon_{t,n_2-m_2})/\log B_t\rfloor.$$

With the help of *Mathematica*, we obtain the results in Table 3.

Thus, $\max\{b_{t,n_2-m_2}: t=1,2,\ldots,17 \text{ and } n_2-m_2=1,2,\ldots b_t\}\leq 408$. So, by Lemma 2, we have that $n_2\leq 408$, for all $t=1,2,\ldots,17$, and by the inequality (23) we have that $n_1\leq n_2+4$. From the fact that $\delta^k\leq 2\alpha^{n+3}$, we can conclude that $k_1<k_2\leq 133$. Collecting everything together, our problem is reduced to search for the solutions for (21) in the following range

$$1\leq k_1<k_2\leq 133,\quad 0\leq m_1<n_1\in[3,408],\quad \text{and}\quad 0\leq m_2<n_2\in[3,408].$$

After a computer search for the solutions to the Diophantine equations in (21) on the range above, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 1:

For the $+1$ case:

$$\begin{aligned}
(d=2)\,x_1&=3=P_6+P_0=P_5+P_3,\quad x_2=17=P_{12}+P_3;\\
(d=3)\,x_1&=2=P_3+P_0=P_3+P_3,\, x_2=7=P_9+P_0=P_7+P_6,\\
x_3&=26=P_{13}+P_8;\\
(d=6)\,x_1&=5=P_8+P_0=P_7+P_3=P_6+P_5,\\
x_2&=49=P_{16}+P_0=P_{15}+P_{12}=P_{14}+P_{13};\\
(d=15)\,x_1&=4=P_7+P_0=P_6+P_3=P_5+P_5,\quad x_2=31=P_{14}+P_6;\\
(d=110)\,x_1&=21=P_{13}+P_0=P_{12}+P_8=P_{11}+P_{10},\\
x_2&=881=P_{26}+P_{17}=P_{25}+P_{22};\\
(d=483)\,x_1&=22=P_{13}+P_3,\quad x_2=967=P_{26}+P_{20}=P_{25}+P_{23}.
\end{aligned}$$

For the -1 case:

Table 3 Final reduction computation results

t	1	2	3	4	5	6	7	8	9
b_{t,n_2-m_2}	388	389	394	394	393	394	396	392	392

t	10	11	12	13	14	15	16	17
b_{t,n_2-m_2}	396	392	408	390	396	396	388	389

$$
\begin{aligned}
(d=2)\, x_1 &= 1 = P_3 + P_0, \quad x_2 = 7 = P_9 + P_0 = P_8 + P_5 = P_7 + P_6,\\
x_3 &= 41 = P_{15} + P_7 = P_{14} + P_{10} = P_{13} + P_{12};\\
(d=5)\, x_1 &= 2 = P_5 + P_0 = P_3 + P_3, \quad x_2 = 38 = P_{15} + P_3;\\
(d=10)\, x_1 &= 3 = P_6 + P_0 = P_5 + P_3, \quad x_2 = 117 = P_{19} + P_6;\\
(d=17)\, x_1 &= 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5, \quad x_2 = P_{22} + P_6.
\end{aligned}
$$

This completes the proof of Theorem 1.

6 Proof of Theorem 2

The proof of Theorem 2 follows from similar steps, techniques, and arguments as given in the proof of Theorem 1. So, we do not give the details here. Below, we give the solutions to the Diophantine equation (9) for the exceptional d cases stated in Theorem 2.

For the $+4$ case:

$$
\begin{aligned}
(d=3)\, X_1 &= 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5,\\
X_2 &= 14 = P_{11} + P_5 = P_{10} + P_8, \quad X_3 = 52 = P_{16} + P_6;\\
(d=5)\, X_1 &= 3 = P_6 + P_0 = P_5 + P_3, \quad X_2 = 7 = P_9 + P_0 = P_7 + P_6,\\
X_3 &= 18 = P_{12} + P_5;\\
(d=21)\, X_1 &= 5 = P_8 + P_0 = P_7 + P_3 = P_6 + P_5,\\
X_2 &= 23 = P_{13} + P_5 = P_{12} + P_9, \quad X_3 = 2525 = P_{30} + P_{11}.
\end{aligned}
$$

For the -4 case:

$$
\begin{aligned}
(d=2)\, X_1 &= 2 = P_5 + P_0 = P_3 + P_3, \quad X_2 = 14 = P_{11} + P_5 = P_{10} + P_8;\\
(d=5)\, X_1 &= 1 = P_3 + P_0, \quad X_2 = 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5,\\
X_3 &= 11 = P_{10} + P_5 = P_9 + P_7, \quad X_4 = 29 = P_{14} + P_3.
\end{aligned}
$$

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