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Local singular characteristics on $\mathbb{R}^2$

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Abstract

The singular set of a viscosity solution to a Hamilton–Jacobi equation is known to propagate, from any noncritical singular point, along singular characteristics which are curves satisfying certain differential inclusions. In the literature, different notions of singular characteristics were introduced. However, a general uniqueness criterion for singular characteristics, not restricted to mechanical systems or problems in one space dimension, is missing at the moment. In this paper, we prove that, for a Tonelli Hamiltonian on $\mathbb{R}^2$, two different notions of singular characteristics coincide up to a bi-Lipschitz reparameterization. As a significant consequence, we obtain a uniqueness result for the class of singular characteristics that was introduced by Khanin and Sobolevski in the paper [On dynamics of Lagrangian trajectories for Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.*, 219(2):861–885, 2016].

Keywords Hamilton–Jacobi equation · Viscosity solution · Singular characteristics

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1 Introduction

This paper is devoted to study the local propagation of singularities for viscosity solutions of the Hamilton–Jacobi equations

$$H(x, Du(x)) = 0, \quad x \in \mathbb{R}^n, \tag{HJ$_s$}$$

$$H(x, Du(x)) = 0, \quad x \in \Omega, \tag{HJ$_{\text{loc}}$}$$

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where H is a Tonelli Hamiltonian in (HJ$_s$) and H is of class C^1 and strictly convex in the p-variable in (HJ$_{\text{loc}}$). In (HJ$_s$), we assume that 0 on the right-hand side is Mañé's critical value. The existence of global weak KAM solutions of (HJ$_s$) was obtained in [12]. In (HJ$_{\text{loc}}$), we suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain.

Semiconcave functions are nonsmooth functions that play an important role in the study of (HJ$_s$) and (HJ$_{\text{loc}}$). For semiconcave viscosity solutions of Hamilton–Jacobi equations, Albano and the first author proved in [1] that singular arcs can be selected as generalized characteristics. Recall that a Lipschitz arc $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ is called a *generalized characteristic* starting from x for the pair (H, u) if it satisfies the following:

$$\begin{cases} \dot{\mathbf{x}}(s) \in \text{co}\, H_p\big(\mathbf{x}(s), D^+u(\mathbf{x}(s))\big) & \text{a.e. } s \in [0, \tau], \\ \dot{\mathbf{x}}(0) = x, \end{cases} \tag{1.1}$$

where co stands for the convex hull. If $x \in \text{Sing}\,(u)$—the singular set of u—then [1, Theorem 5] gives a sufficient condition for the existence of a generalized characteristic propagating the singularity of u locally.

The local structure of singular (generalized) characteristics was further investigated by the first author and Yu in [11], where *singular characteristics* were proved more regular near the starting point than the arcs constructed in [1]. Such additional properties will be crucial for the analysis we develop in this paper.

For any weak KAM solution u of (HJ$_s$), the class of *intrinsic singular* (generalized) *characteristics* was constructed in [4] by the authors of this paper, using the positive type Lax-Oleinik semi-group. Such a method allowed to construct global singular characteristics, which we now call *intrinsic*. Moreover, in [5,6] the "intrisic approach" turned out to be useful for pointing out topological properties of the cut locus of u, including homotopy equivalence to the complement of the Aubry set (see also [7] for applications to Dirichlet boundary value problems).

In spite of its success in capturing singular dynamics, it could be argued that the relaxation procedure in the original definition of generalized characteristics—that is, the presence of the convex hull in (1.1)—might cause a loss of information coming from the Hamiltonian dynamics behind. On the other hand, such a relaxation is necessary to ensure convexity of admissible velocities for the differential inclusion in (1.1), since the map $x \rightrightarrows H_p(x, D^+u(x))$ fails to be convex-valued, in general.

The most important example where the above relaxation is unnecessary is probably given by mechanical Hamiltonians of the form $H(x, p) = \frac{1}{2}\langle A(x)p, p\rangle + V(x)$, where $A(x)$ is a symmetric positive definite $n \times n$-matrix smoothly depending on x and $V(x)$ is a smooth function on $\mathbb{R}^n$. In this case, (1.1) reduces to the *generalized gradient system*

$$\begin{cases} \dot{\mathbf{x}}(t) \in A(\mathbf{x}(t))D^+u(\mathbf{x}(t)) & t > 0 \quad \text{a.e.} \\ \dot{\mathbf{x}}(0) = x, \end{cases} \tag{1.2}$$

the solution of which, unique for any initial datum, forms a Lipschitz semi-flow (see, e.g., [1–3,8,9]). Unfortunately, the argument that justifies such a uniqueness property cannot be adapted to general Hamiltonians (see [11,15]).

Recent significant progress in the attempt to develop a more restrictive notion of singular characteristics is due to Khanin and Sobolevski [13]. In this paper, we will call such curves *strict singular characteristic* but in the literature they are also refereed to as *broken characteristics*, see [16,17]. We now proceed to recall their definition: given a semiconcave solution u of (HJ$_{\text{loc}}$), a Lipschitz singular curve $\mathbf{x} : [0, T] \to \Omega$ is called a strict singular characteristic from $x \in \text{Sing}\,(u)$ if there exists a measurable selection $p(t) \in D^+u(\mathbf{x}(t))$ such that

$$\begin{cases} \dot{\mathbf{x}}(t) = H_p(\mathbf{x}(t), p(t)) \quad a.e.\ t \in [0, T], \\ \mathbf{x}(0) = x. \end{cases} \tag{1.3}$$

As already mentioned, the existence of strict singular characteristics (for a time dependent version of (HJ_{loc})) was proved in [13], where additional regularity properties of such curves were established, including the right-differentiability of $\mathbf{x}$ for every t, the right-continuity of $\dot{\mathbf{x}}$, and the fact that $p(\cdot) : [0, T] \to \mathbb{R}^n$ satisfies

$$H(\mathbf{x}(t), p(t)) = \min_{p \in D^+u(\mathbf{x}(t))} H(\mathbf{x}(t), p) \quad \forall t \in [0, T]. \tag{1.4}$$

In Appendix A, we give a proof of the existence and regularity of strict characteristics for solutions to (HJ_{loc}) for the reader's convenience.

In view of the above considerations, it is quite natural to raise the following questions:

(Q1) What is the relation between a strict singular characteristic, $\mathbf{x}$, and a singular characteristic, $\mathbf{y}$, from the same initial point?
(Q2) What kind of uniqueness result can be proved for singular characteristics? What about strict singular characteristics?

In this paper, we will answer the above questions in the two-dimensional case under the following additional conditions:

(A) $n = 2$ and $\mathbf{y}$ is Lipschitz;
(B) the singular initial point $x_0 = \mathbf{y}(0)$ of the singular characteristic $\mathbf{y}$ is not a critical point with respect the pair (H, u), i.e., $0 \notin H_p(x_0, D^+u(x_0))$;
(C) $\mathbf{y}$ is right differentiable at 0 and

$$\dot{\mathbf{y}}^+(0) = H(x_0, p_0),$$

where $p_0 = \arg\min\{H(x_0, p) : p \in D^+u(x_0)\}$;
(D) $\lim_{t\to 0^+} \text{ess sup}_{s\in[0,t]} |\dot{\mathbf{y}}(s) - \dot{\mathbf{y}}^+(0)| = 0$.

The meaning of conditions (A) is clear. Condition (B) ensures the fact that singular characteristics are not constant. The right differentiability of singular characteristics at 0 and the essential right continuity of $\dot{\mathbf{y}}$ at 0 are crucial properties to our approach. On the one hand, together with condition (B) they ensure that a singular characteristic is a genuine arc near $t = 0$. On the other hand, (D) is essential to construct the change of variable on which our uniqueness result is based. Notice that any strict singular characteristic $\mathbf{x}$ and the singular characteristic $\mathbf{y}$ given in [11] (see also Proposition 2.12) satisfy conditions (A)–(D) provided that the initial point is not critical. The intrinsic singular characteristic $\mathbf{z}$ constructed in [4] (see also Proposition 2.13) satisfies just conditions (A)–(C), in general.

The main results of this paper can be described as follows.

- For any pair of singular curves $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfying condition (A)-(D), there exists $\tau > 0$ and a bi-Lipschitz homeomorphism $\phi : [0, \tau] \to [0, \phi(\tau)]$ such that, $\mathbf{x}_1(\phi(t)) = \mathbf{x}_2(t)$ for all $t \in [0, \tau]$. In other words, the singular characteristic staring from a non-critical point x is unique up to a bi-Lipschitz reparametrization (Theorem 3.6).
- In particular, if $\mathbf{x}$ is a strict singular characteristic and $\mathbf{y}$ is a singular characteristic starting from the same noncritical initial point x, then there exists $\tau > 0$ and a bi-Lipschitz homeomorphism $\phi : [0, \tau] \to [0, \phi(\tau)]$ such that $\mathbf{y}(\phi(t)) = \mathbf{x}(t)$ for all $t \in [0, \tau]$ (Corollary 3.8).

- We have the following *uniqueness* property for strict singular characteristics: let
$$\mathbf{x}_j : [0, T] \to \Omega \quad (j = 1, 2)$$
be strict singular characteristics from the same noncritical initial point x. Then there exists $\tau \in (0, T]$ such that $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in [0, \tau]$. (Theorem 3.9)

Finally, we remark that the results of this paper cannot be applied to intrinsic singular characteristics because of the mentioned lack of condition (D). Extra techniques will have to be developed to cover such an important class.

The paper is organized as follows. In Sect. 2, we introduce necessary material on Hamilton–Jacobi equations, semiconcavity, and singular characteristics. In Sect. 3, we answer question (Q1)–(Q2) in the two-dimensional case. In the appendix, we give a detailed proof of the existence result for strict singular characteristics.

2 Hamilton–Jacobi equation and semiconcavity

In this section, we review some basic facts on semiconcave functions and Hamilton–Jacobi equations.

2.1 Semiconcave function

Let $\Omega \subset \mathbb{R}^n$ be a convex open set. We recall that a function $u : \Omega \to \mathbb{R}$ is *semiconcave* (with linear modulus) if there exists a constant $C > 0$ such that
$$\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq \frac{C}{2}\lambda(1 - \lambda)|x - y|^2 \tag{2.1}$$
for any $x, y \in \Omega$ and $\lambda \in [0, 1]$.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. For any $x \in \Omega$, the closed convex sets
$$D^-u(x) = \left\{p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|} \geq 0\right\},$$
$$D^+u(x) = \left\{p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|} \leq 0\right\}.$$
are called the *subdifferential* and *superdifferential* of u at x, respectively.

The following characterization of semiconcavity (with linear modulus) for a continuous function comes from proximal analysis.

Proposition 2.1 *Let $u : \Omega \to \mathbb{R}$ be a continuous function. If there exists a constant $C > 0$ such that, for any $x \in \Omega$, there exists $p \in \mathbb{R}^n$ such that*
$$u(y) \leq u(x) + \langle p, y - x\rangle + \frac{C}{2}|y - x|^2, \quad \forall y \in \Omega, \tag{2.2}$$
then u is semiconcave with constant C and $p \in D^+u(x)$. Conversely, if u is semiconcave in Ω with constant C, then (2.2) holds for any $x \in \Omega$ and $p \in D^+u(x)$.

Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz. We recall that a vector $p \in \mathbb{R}^n$ is called a *reachable* (or *limiting*) *gradient* of u at x if there exists a sequence $\{x_n\} \subset \Omega\backslash\{x\}$ such that u is differentiable at x_k for each $k \in \mathbb{N}$, and
$$\lim_{k\to\infty} x_k = x \quad \text{and} \quad \lim_{k\to\infty} Du(x_k) = p.$$

The set of all reachable gradients of u at x is denoted by $D^*u(x)$.

The following proposition concerns fundamental properties of semiconcave funtions and their gradients (see [10] for the proof).

Proposition 2.2 *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a semiconcave function and let $x \in \Omega$. Then the following properties hold.*

(a) *$D^+u(x)$ is a nonempty compact convex set in $\mathbb{R}^n$ and $D^*u(x) \subset \partial D^+u(x)$, where $\partial D^+u(x)$ denotes the topological boundary of $D^+u(x)$.*
(b) *The set-valued function $x \rightsquigarrow D^+u(x)$ is upper semicontinuous.*
(c) *If $D^+u(x)$ is a singleton, then u is differentiable at x. Moreover, if $D^+u(x)$ is a singleton for every point in Ω, then $u \in C^1(\Omega)$.*
(d) $D^+u(x) = \operatorname{co} D^*u(x)$.
(e) *If u is both semiconcave and semiconvex in Ω, then $u \in C^{1,1}(\Omega)$.*

Definition 2.3 Let $u : \Omega \to \mathbb{R}$ be a semiconcave function. $x \in \Omega$ is called a *singular point* of u if $D^+u(x)$ is not a singleton. The set of all singular points of u is denoted by $\mathrm{Sing}\,(u)$.

Definition 2.4 Let $k \in \{0, 1, \dots, n\}$ and let $C \subset \mathbb{R}^n$. C is called a *k-rectifiable* set if there exists a Lipschitz continuous function $f : \mathbb{R}^k \to \mathbb{R}^n$ such that $C \subset f(\mathbb{R}^k)$. C is called a *countably k-rectifiable* set if it is the union of a countable family of k-rectifiable sets.

Let us recall a result on the rectifiability of the singular set $\mathrm{Sing}\,(u)$ of a semiconcave function u in dimension two.

Proposition 2.5 [10] *Let $\Omega \subset \mathbb{R}^2$ be an open domain, $u : \Omega \to \mathbb{R}$ be a semiconcave function, and set*

$$\mathrm{Sing}_k(u) = \{x \in \mathrm{Sing}\,(u) : \dim(D^+u(x)) = k\}, \quad k = 0, 1, 2.$$

Then $\mathrm{Sing}_k(u)$ is countably $(2-k)$-rectifiable for $k = 0, 1, 2$. In particular, $\mathrm{Sing}_2(u)$ is countable.

2.2 Aspects of weak KAM theory

For any $x, y \in \mathbb{R}^n$ and $t > 0$, we denote by $\Gamma^t_{x,y}$ the set of all absolutely continuous curves ξ defined on $[0, t]$ such that $\xi(0) = x$ and $\xi(t) = y$. Define

$$A_t(x, y) = \inf_{\xi \in \Gamma^t_{x,y}} \int_0^t L(\xi(s), \dot{\xi}(s))\, ds, \quad x, y \in \mathbb{R}^n,\ t > 0. \tag{2.3}$$

We call $A_t(x, y)$ the *fundamental solution* for the Hamilton–Jacobi equation

$$D_t u(t, x) + H(x_0, D_x u(t, x)) = 0, \quad t > 0, x \in \mathbb{R}^n.$$

By classical results (Tonelli's theory), the infimum in (2.3) is a minimum. Each curve $\xi \in \Gamma^t_{x,y}$ attaining such a minimum is called a *minimal curve for $A_t(x, y)$*.

Definition 2.6 For each $u : \mathbb{R}^n \to \mathbb{R}$, let and $T_t u$ and $\breve{T}_t u$ be the *Lax-Oleinik evolution of negative and positive type* defined, respectively, by

$$\begin{aligned} T_t u(x) &= \inf_{y \in \mathbb{R}^n} \{u(y) + A_t(y, x)\}, \\ \breve{T}_t u(x) &= \sup_{y \in \mathbb{R}^n} \{u(y) - A_t(x, y)\}, \end{aligned} \quad (x \in \mathbb{R}^n, t > 0).$$

The following result is well-known.

Proposition 2.7 [12] *There exists a Lipschitz semiconcave viscosity solution of* (HJ$_s$)*. Moreover, such a solution u is a common fixed point of the semigroup $\{T_t\}$, i.e., $T_t u = u$ for all $t \geq 0$.*

Clearly, (HJ$_s$) has no unique solution and we call each solution, given as a fixed point of the semigroup $\{T_t\}$, a *weak KAM solution* of (HJ$_s$).

Definition 2.8 Let u be a continuous function on M. We say u is *L-dominated* if

$$u(\xi(b)) - u(\xi(a)) \leq \int_a^b L(\xi(s), \dot{\xi}(s))\, ds,$$

for all absolutely continuous curves $\xi : [a, b] \to \mathbb{R}^n$ $(a < b)$, with $\xi(a) = x$ and $\xi(b) = y$. We say such an absolutely continuous curve ξ is a *(u, L)-calibrated curve*, or a *u-calibrated curve* for short, if the equality holds in the inequality above. A curve $\xi : (-\infty, 0] \to \mathbb{R}^n$ is called a u-calibrated curve if it is u-calibrated on each compact sub-interval of $(-\infty, 0]$. In this case, we also say that ξ is a *backward calibrated curve* (with respect to u).

The following result explains the relation between the set of all reachable gradients and the set of all backward calibrated curves from x (see, e.g., [10] or [14] for the proof).

Proposition 2.9 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a weak KAM solution of* (HJ$_s$) *and let $x \in \mathbb{R}^n$. Then $p \in D^* u(x)$ if and only if there exists a unique C^2 curve $\xi : (-\infty, 0] \to \mathbb{R}^n$ with $\xi(0) = x$ and $p = L_v(x, \dot{\xi}(0))$, which is a backward calibrated curve with respect to u.*

2.3 Propagation of singularities

In this paper, we will discuss various types of singular arcs describing the propagation of singularities for Lipschitz semiconcave solutions of the Hamilton–Jacobi equations (HJ$_{\text{loc}}$) and (HJ$_s$).

Definition 2.10 x_0 is called a *critical point with respect to* (H, u) if $0 \in H_p(x_0, D^+ u(x))$.

Let u be a Lipschitz semiconcave viscosity solution of (HJ$_{\text{loc}}$) and $x \in \text{Sing}\,(u)$.

Definition 2.11 (a) A *singular characteristic from x_0* is a Lipschitz arc $\mathbf{x} : [0, \tau] \to \Omega$ $(\tau > 0)$ such that:

(1) $\mathbf{x}$ is a generalized characteristic with $\mathbf{x}(0) = x_0$,
(2) $\mathbf{x}(t) \in \text{Sing}\,(u)$ for all $t \in [0, \tau]$,
(3) $\dot{\mathbf{x}}^+(0) = H_p(x_0, p_0)$ where $p_0 = \arg\min\{H(x_0, p) : p \in D^+ u(x_0)\}$,
(4) $\lim_{t \to 0^+} \operatorname{ess\,sup}_{s \in [0,t]} |\dot{\mathbf{x}}(s) - \dot{\mathbf{x}}^+(0)| = 0$.

(b) A singular characteristic $\mathbf{x} : [0, T] \to \Omega$ from x_0 is called a *strict singular characteristic* if there exists a measurable selection $p(t) \in D^+ u(\mathbf{x}(t))$ such that

$$\begin{cases} \dot{\mathbf{x}}(t) = H_p(\mathbf{x}(t), p(t)) & a.e.\ t \in [0, T], \\ \mathbf{x}(0) = x_0. \end{cases}$$

The following existence of singular characteristic is due to [1,11].

Proposition 2.12 *Let u be a Lipschitz semiconcave solution of* (HJ_{loc}) *and* $x \in \mathrm{Sing}\,(u)$. *Then, there exists a singular characteristic* $\mathbf{y} : [0, T] \to \Omega$ *with* $\mathbf{y}(0) = x$.

Now, suppose u is a Lipschitz semiconcave weak KAM solution of (HJ_s). In [4], another singular curve for u is constructed as follows. First, it is shown that there exists $\lambda_0 > 0$ such that for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and any maximizer y for the function $u(\cdot) - A_t(x, \cdot)$, we have that $|y - x| \leq \lambda_0 t$. Then, taking $\lambda = \lambda_0 + 1$, one shows that there exists $t_0 > 0$ such that, if $t \in (0, t_0]$, then there exists a unique $y_{t,x} \in B(x, \lambda t)$ of $u(\cdot) - A_t(x, \cdot)$ such that

$$\breve{T}_t u(x) = u(y_{t,x}) - A_t(x, y_{t,x}). \tag{2.4}$$

Moreover, such a t_0 is such that $-A_t(x, \cdot)$ is concave with constant C_2/t and $C_1 - C_2/t < 0$ for $0 < t \leq t_0$. We now define the curve

$$\mathbf{z}(t) = \begin{cases} x, & t = 0, \\ y_{t,x}, & t \in (0, t_0]. \end{cases} \tag{2.5}$$

Proposition 2.13 [4] *Let the curve* $\mathbf{z}$ *be defined in* (2.5). *Then, the following holds:*

(1) $\mathbf{z}$ *is Lipschitz,*
(2) *if* $x \in \mathrm{Sing}\,(u)$ *then* $\mathbf{z}(t) \in \mathrm{Sing}\,(u)$ *for all* $t \in [0, t_0]$,
(3) $\dot{\mathbf{z}}^+(0)$ *exists and*

$$\dot{\mathbf{z}}^+(0) = H_p(x, p_0)$$

where $p_0 = \arg\min\{H(x, p) : p \in D^+u(x)\}$.

Definition 2.14 The Lipschitz arc $\mathbf{z}$ defined in (2.5) is called the *intrinsic characteristic* from $x \in \mathrm{Sing}\,(u)$.

3 Singular characteristic on $\mathbb{R}^2$

We now return to questions (Q1) and (Q2) from the Introduction. So far, we have introduced three kinds of singular arcs issuing from a point $x_0 \in \mathrm{Sing}\,(u)$, namely

- strict singular characteristics, that is, solutions to (1.3),
- singular characteristics, introduced in Definition 2.11, and
- the intrinsic singular characteristic $\mathbf{z}$ given by Proposition 2.13.

In this section, we will compare the first two notions of characteristics when $\Omega \subset \mathbb{R}^2$.

We begin by introducing the following class of Lipschitz arcs.

Definition 3.1 Given $T > 0$, we denote by $\mathrm{Lip}_0(0, T; \Omega)$ the class of all Lipschitz arcs $\mathbf{x} : [0, T] \to \Omega$ such that the right derivative

$$\dot{\mathbf{x}}^+(0) = \lim_{t \downarrow 0} \frac{\mathbf{x}(t) - \mathbf{x}(0)}{t}$$

does exist and satisfies

$$\lim_{t \to 0^+} \operatorname*{ess\,sup}_{s \in [0,t]} |\dot{\mathbf{x}}(s) - \dot{\mathbf{x}}^+(0)| = 0. \tag{3.1}$$

For any $\mathbf{x} \in \mathrm{Lip}_0(0, T; \Omega)$ we set

$$\omega_{\mathbf{x}}(t) := \operatorname*{ess\,sup}_{s \in [0,t]} |\dot{\mathbf{x}}(s) - \dot{\mathbf{x}}^+(0)|. \tag{3.2}$$

Owing to (3.1), we have that $\omega_{\mathbf{x}}(t) \to 0$ as $t \downarrow 0$.

Lemma 3.2 *Let* $\mathbf{x} \in \mathrm{Lip}_0(0, T; \Omega)$ *be such that* $\dot{\mathbf{x}}^+(0) \neq 0$. *Then,*

$$\big||\mathbf{x}(t_1) - \mathbf{x}(t_0)| - |t_1 - t_0| \cdot |\dot{\mathbf{x}}^+(0)|\big| \leq |t_1 - t_0|\omega_{\mathbf{x}}(t_1 \vee t_0) \quad \forall t_0, t_1 \in [0, T] \tag{3.3}$$

and $\mathbf{x}$ *is injective on some interval* $[0, T_0]$ *with* $0 < T_0 < T$.

Proof Observe that, for any $0 \leq t_0 \leq t_1 \leq T$, the identity

$$\mathbf{x}(t_1) - \mathbf{x}(t_0) = \int_{t_0}^{t_1} \dot{\mathbf{x}}(t)\, dt = (t_1 - t_0)\dot{\mathbf{x}}^+(0) + \int_{t_0}^{t_1} (\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^+(0))\, dt$$

immediately gives (3.3). In turn, (3.3) implies that, if $\mathbf{x}(t_1) - \mathbf{x}(t_0) = 0$, then

$$|t_1 - t_0| \cdot |\dot{\mathbf{x}}^+(0)| \leq |t_1 - t_0|\omega_{\mathbf{x}}(t_1)$$

Since $\dot{\mathbf{x}}^+(0) \neq 0$, we conclude that $t_1 = t_0$ if $t_0, t_1 \in [0, T_0]$ with T_0 sufficiently small.

Let $x \in \mathbb{R}^2$ and let $\theta \in \mathbb{R}^2$ be a unit vector. For any $\rho \in (0, 1)$ let us consider the cone

$$C_\rho(x, \theta) = \big\{y \in \mathbb{R}^2 \;\big|\; |\langle y - x, \theta\rangle| \geq \rho|y - x|\big\} \tag{3.4}$$

with vertex in x, amplitude ρ, and axis θ. Clearly, $C_\rho(x, \theta)$ is given by the union of the two cones

$$C_\rho^+(x, \theta) = \big\{y \in \mathbb{R}^2 \;\big|\; \langle y - x, \theta\rangle \geq \rho|y - x|\big\}$$

and

$$C_\rho^-(x, \theta) = \big\{y \in \mathbb{R}^2 \;\big|\; \langle y - x, \theta\rangle \leq -\rho|y - x|\big\},$$

which intersect each other only at x.

Lemma 3.3 *Let* $\mathbf{x}_j \in \mathrm{Lip}_0(0, T; \Omega)$ *(*$j = 1, 2$*) be such that*

(i) $\mathbf{x}_1(0) = \mathbf{x}_2(0) =: x_0$,
(ii) $\dot{\mathbf{x}}_1^+(0) = \dot{\mathbf{x}}_2^+(0)$, *and*
(iii) $\dot{\mathbf{x}}_j(s) \neq 0$ *(*$j = 1, 2$*) for a.e.* $s \in [0, T]$.

Define

$$\theta_1(s) = \frac{\dot{\mathbf{x}}_1(t)}{|\dot{\mathbf{x}}_1(t)|} \quad (s \in [0, T] \text{ a.e.}) \tag{3.5}$$

and fix $\rho \in (0, 1)$. *Then the following holds true:*

(a) *there exists* $s_\rho \in (0, T]$ *such that* $x_0 \in C_\rho^-(\mathbf{x}_1(s), \theta_1(s))$ *for a.e.* $s \in [0, s_\rho]$;
(b) *there exists* $\tau_\rho \in (0, T]$ *such that for all* $t \in (0, \tau_\rho]$ *there exists* $\sigma_\rho(t) \in (0, T]$ *such that*

$$|\mathbf{x}_2(t) - \mathbf{x}_1(s)| \leq \frac{1 + \rho}{2\rho} t|\dot{\mathbf{x}}_1^+(0)| \quad \forall s \in [0, \sigma_\rho(t)] \tag{3.6}$$

$$\mathbf{x}_2(t) \in C_\rho^+(\mathbf{x}_1(s), \theta_1(s)) \quad \text{for a.e. } s \in [0, \sigma_\rho(t)]. \tag{3.7}$$

Proof Hereafter, we denote by $o_i(s)$ $(i \in \mathbb{N})$ any (scalar- or vector-valued) function such that

$$\lim_{s \to 0^+} \frac{o_i(s)}{s} = 0.$$

In view of (3.3) we conclude that

$$|x_0 - \mathbf{x}_1(s)| = s|\dot{\mathbf{x}}_1^+(0)| + o_1(s) \quad \forall s \in [0, T]. \tag{3.8}$$

Moreover, setting $\theta_1(0) = \dot{\mathbf{x}}_1^+(0)/|\dot{\mathbf{x}}_1^+(0)|$, for a.e. $s \in [0, T]$ we have that

$$\begin{aligned}\langle x_0 - \mathbf{x}_1(s), \theta_1(s)\rangle &= \langle x_0 - \mathbf{x}_1(s), \theta_1(0)\rangle + \langle x_0 - \mathbf{x}_1(s), \theta_1(s) - \theta_1(0)\rangle \\ &= -s|\dot{\mathbf{x}}_1^+(0)| + o_2(s).\end{aligned} \tag{3.9}$$

Now, having fixed $\rho \in (0, 1)$ let $s_\rho \in (0, T_1]$ be such that, for a.e. $s \in [0, s_\rho]$,

$$\frac{|o_1(s)|}{s} \le \frac{1-\rho}{2\rho}|\dot{\mathbf{x}}_1^+(0)| \quad \text{and} \quad \frac{|o_2(s)|}{s} \le \frac{1-\rho}{2}|\dot{\mathbf{x}}_1^+(0)|.$$

Then $|x_0 - \mathbf{x}_1(s)| \le \frac{1+\rho}{2\rho} s|\dot{\mathbf{x}}_1^+(0)|$ by (3.8). From (3.9) it follows that

$$\langle x_0 - \mathbf{x}_1(s), \theta_1(s)\rangle \le -\frac{1+\rho}{2} s|\dot{\mathbf{x}}_1^+(0)| \le -\rho|x_0 - \mathbf{x}_1(s)| \quad (s \in [0, s_\rho] \text{ a.e.})$$

and (a) follows.

The proof of (b) is similar: since $\dot{\mathbf{x}}_2^+(0) = \dot{\mathbf{x}}_1^+(0)$ by condition (ii), for all $t \in [0, T]$ and $s \in [0, T]$ we have that

$$\mathbf{x}_2(t) - \mathbf{x}_1(s) = (t - s)\dot{\mathbf{x}}_1^+(0) + o_3(t) + o_3(s). \tag{3.10}$$

Hence, for all $s, t \in (0, T]$ we deduce that

$$\left|\frac{\mathbf{x}_2(t) - \mathbf{x}_1(s)}{t|\dot{\mathbf{x}}_1^+(0)|} - \frac{\dot{\mathbf{x}}_1^+(0)}{|\dot{\mathbf{x}}_1^+(0)|}\right| \le \frac{s}{t} + \frac{|o_3(t)| + |o_3(s)|}{t|\dot{\mathbf{x}}_1^+(0)|}.$$

So,

$$|\mathbf{x}_2(t) - \mathbf{x}_1(s)| \le t|\dot{\mathbf{x}}_1^+(0)|\Big(1 + \frac{s}{t} + \frac{|o_3(t)| + |o_3(s)|}{t|\dot{\mathbf{x}}_1^+(0)|}\Big). \tag{3.11}$$

Next, take the scalar product of each side of (3.10) with $\theta_1(s)$ to obtain

$$\begin{aligned}\langle \mathbf{x}_2(t) - \mathbf{x}_1(s), \theta_1(s)\rangle &= t\langle \dot{\mathbf{x}}_1^+(0), \theta_1(s)\rangle - \langle s\dot{\mathbf{x}}_1^+(0) - o_3(t) - o_3(s), \theta_1(s)\rangle \\ &= t|\dot{\mathbf{x}}_1^+(0)| + t\langle \dot{\mathbf{x}}_1^+(0), \theta_1(s) - \theta_1(0)\rangle - \langle s\dot{\mathbf{x}}_1^+(0) - o_3(t) - o_3(s), \theta_1(s)\rangle\end{aligned} \tag{3.12}$$

for all $t \in [0, T]$ and a.e. $s \in [0, T]$.

Once again, having fixed $\rho \in (0, 1)$, we can find $\tau_\rho \in (0, T]$ satisfying the following: for all $t \in (0, \tau_\rho]$ there exists $\sigma_\rho(t) \in (0, T]$ such that

$$t|\langle \dot{\mathbf{x}}_1^+(0), \theta_1(s) - \theta_1(0)\rangle| + |\langle s\dot{\mathbf{x}}_1^+(0) - o_3(t) - o_3(s), \theta_1(s)\rangle| \le \frac{1-\rho}{2} t|\dot{\mathbf{x}}_1^+(0)|$$

and

$$1 + \frac{s}{t} + \frac{|o_3(t)| + |o_3(s)|}{t|\dot{\mathbf{x}}_1^+(0)|} \le \frac{1+\rho}{2\rho}$$

for all $t \in [0, \tau_\rho]$ and a.e. $s \in [0, \sigma_\rho(t)]$. Then, (3.11) leads directly to (3.6). Moreover, returning to (3.12), for all $t \in [0, \tau_\rho]$ and a.e. $s \in [0, \sigma_\rho(t)]$ we conclude that

$$\langle \mathbf{x}_2(t) - \mathbf{x}_1(s), \theta_1(s)\rangle \ge t|\dot{\mathbf{x}}_1^+(0)| - \frac{1-\rho}{2} t|\dot{\mathbf{x}}_1^+(0)| = \frac{1+\rho}{2} t|\dot{\mathbf{x}}_1^+(0)| \ge \rho|\mathbf{x}_2(t) - \mathbf{x}_1(s)|,$$

where we have used (3.11) to deduce the last inequality. Hence, (3.7) follows.

Given a semiconcave solution u of (HJ_{loc}), we hereafter concentrate on singular arcs for u, that is, arcs $\mathbf{x} \in \mathrm{Lip}_0(0, T; \Omega)$ such that $\mathbf{x}(t) \in \mathrm{Sing}\,(u)$ for all $t \in [0, T]$. We denote such a subset of $\mathrm{Lip}_0(0, T; \Omega)$ by $\mathrm{Lip}_0^u(0, T; \Omega)$.

Lemma 3.4 *Let u be a semiconcave solution of* (HJ$_{\text{loc}}$) *and let* $\mathbf{x} \in \text{Lip}_0^u(0,T;\Omega)$ *be such that* $\dot{\mathbf{x}}^+(0) \neq 0$. *Then there exists* $T_0 \in (0,T]$ *such that the set*

$$S_{\mathbf{x}} = \left\{ s \in [0,T_0] \,\middle|\, D^+u(\mathbf{x}(s)) = [p_s^1, p_s^2] \text{ with } p_s^1, p_s^2 \in D^*u(\mathbf{x}(s)),\ p_s^1 \neq p_s^2 \right\}.$$

has full measure in $[0,T_0]$. *Moreover,* $\lim_{s\to 0^+} p_s^i = p_0^i$ *with* $p_0^i \in D^*u(x_0)$ $(i=1,2)$ *and*

$$\langle \dot{\mathbf{x}}(s), p_s^2 - p_s^1 \rangle = 0 \quad \text{for a.e. } s \in [0,T_0] \tag{3.13}$$

Proof The structure of the superdifferential of u along $\mathbf{x}$ is described by Proposition 2.5 and Proposition 3.3.15 in [10].

Lemma 3.5 *Let u be a semiconcave solution of* (HJ$_{\text{loc}}$) *and let* $x_0 \in \text{Sing}\,(u)$ *be such that* $0 \notin H_p(x_0, D^+u(x_0))$. *Let* $\mathbf{x} \in \text{Lip}_0^u(0,T;\Omega)$ *be such that* $\mathbf{x}(0) = x_0$ *and*

$$\dot{\mathbf{x}}^+(0) = H_p(x_0, p_0) \quad \text{where } p_0 = \arg\min\{H(x_0,p) : p \in D^+u(x_0)\}.$$

Let $T_0 \in (0,T]$ *be given by Lemma 3.4 and, for every* $s \in S_{\mathbf{x}}$, *let* ξ_s^1 *and* ξ_s^2 *be backward calibrated curves on* $(-\infty, 0]$ *satisfying*

$$\xi_s^i(0) = \mathbf{x}(s) \quad \text{and} \quad \dot{\xi}_s^i(0) = H_p(\mathbf{x}(s), p_s^i) \quad (i=1,2) \tag{3.14}$$

Then there exist constants $r_1 > 0$, $s_1 \in (0,T_0]$, *and* $\delta \in (0,1)$ *and such that*

$$|\mathbf{x}(s) - \xi_s^i(-r)| \geq \delta r \quad (i=1,2) \tag{3.15}$$

and, for all $s \in [0,s_1] \cap S_{\mathbf{x}}$ *and* $r \in [0,r_1]$,

$$\xi_s^1(-r) \in C_\delta^+(\mathbf{x}(s), \theta_2(s)) \quad \text{and} \quad \xi_s^2(-r) \in C_\delta^-(\mathbf{x}(s), \theta_2(s)) \tag{3.16}$$

where

$$\theta_2(s) = \frac{p_s^2 - p_s^1}{|p_s^2 - p_s^1|} \quad (s \in S_{\mathbf{x}}).$$

Proof The existence of backward calibrated curves satisfying (3.14) follows from Proposition 2.9. Moreover, for all $r \geq 0$ we have that

$$\mathbf{x}(s) - \xi_s^i(-r) = \xi_s^i(0) - \xi_s^i(-r) = r\dot{\xi}^i(0) + o(r) = rH_p(\mathbf{x}(s), p_s^i) + o(r) \quad (i=1,2) \tag{3.17}$$

where $\lim_{r\to 0^+} o(r)/r = 0$ uniformly with respect to $s \in S_{\mathbf{x}}$.

Now, observe that, since x_0 is not a critical point with respect to (u,H), by possibly reducing T_0 we have that $\mathbf{x}(s)$ is also not a critical point for all $s \in [0,T_0]$ due to the upper-semicontinuity of the set-valued map $s \rightrightarrows H_p(\mathbf{x}(s), D^+u(\mathbf{x}(s)))$. So, for some $r_0 > 0$, $s_0 \in (0,T_1]$, and $\delta_0 \in (0,1)$, we deduce that

$$\frac{r}{\delta_0} \geq |\mathbf{x}(s) - \xi_s^i(-r)| = r|H_p(\mathbf{x}(s), p_s^i)| + o(r) \geq \delta_0 r \quad (i=1,2) \tag{3.18}$$

for all $s \in [0,s_0] \cap S_{\mathbf{x}}$ and $r \in [0,r_0]$. This proves (3.15).

Next, recall that $H(x_0, p_0^i) = 0$ because $p_0^i \in D^*u(x_0)$ $(i=1,2)$. So, by the strict convexity of $H(x_0,\cdot)$, we deduce that there exists $\nu > 0$ such that

$$\langle H_p(x_0,p_0^2), p_0^2 - p_0^1 \rangle \geq H(x_0,p_0^2) - H(x_0,p_0^1) + \nu|p_0^2 - p_0^1|^2 = \nu|p_0^2 - p_0^1|^2 > 0$$
$$\langle H_p(x_0,p_0^1), p_0^2 - p_0^1 \rangle \leq H(x_0,p_0^2) - H(x_0,p_0^1) - \nu|p_0^2 - p_0^1|^2 = -\nu|p_0^2 - p_0^1|^2 < 0$$

Hence, the upper-semicontinuity of the set-valued map $s \rightrightarrows H_p(\mathbf{x}(s), D^+u(\mathbf{x}(s)))$ ensures the existence of numbers $\delta_1 \in (0,1)$ and $s_1 \in (0, s_0]$ such that

$$\langle H_p(\mathbf{x}(s), p_s^2), \theta_2(s)\rangle \geq \delta_1, \quad \langle H_p(\mathbf{x}(s), p_s^1), \theta_2(s)\rangle \leq -\delta_1 \quad \forall s \in [0, s_1] \cap S. \tag{3.19}$$

Therefore, combining (3.17) and (3.19), we conclude that, after possibly replacing r_0 by a smaller nummber $r_1 > 0$,

$$\langle \xi_s(-r) - \mathbf{x}(s), \theta_2(s)\rangle = -r\langle H_p(\mathbf{x}(s), p_s^1), \theta_2(s)\rangle + o(r) \geq r\delta_1 + o(r) \geq r\frac{\delta_1}{2}$$

for all $s \in [0, s_1] \cap S_{\mathbf{x}}$ and $r \in [0, r_1]$. By (3.18) and the above inequality we have that $\xi_s^1(-r) \in C_\delta^+(\mathbf{x}(s), \theta_2(s))$ with $\delta = \delta_0\delta_1/2$.

The analogous statement for ξ_s^2 in (3.16) can be proved by a similar argument.

We are now ready to state our main result, which ensures that singular curves coincide up to a bi-Lipschitz reparameterization, at least when x is not a critical point.

Theorem 3.6 *Let u be a semiconcave solution of* ($\mathrm{HJ_{loc}}$) *and let $x_0 \in \mathrm{Sing}\,(u)$ be such that $0 \notin H_p(x_0, D^+u(x_0))$. Let $\mathbf{x}_j \in \mathrm{Lip}_0^u(0, T; \Omega)$ $(j = 1, 2)$ be such that $\mathbf{x}_j(0) = x_0$ and*

$$\dot{\mathbf{x}}_j^+(0) = H_p(x_0, p_0) \text{ where } p_0 = \arg\min\{H(x_0, p) : p \in D^+u(x_0)\}.$$

Then, there exists $\sigma \in (0, T]$ such that there exists a unique bi-Lipschitz homeomorphism

$$\phi : [0, \sigma] \to [0, \phi(\sigma)] \subset [0, T_2]$$

satisfying $\mathbf{x}_1(s) = \mathbf{x}_2(\phi(s))$ for all $s \in [0, \sigma]$.

We begin the proof with the following lemma.

Lemma 3.7 *Under all assumptions of Theorem 3.6, there exists $\sigma \in (0, T]$ such that for all $s \in [0, \sigma]$ there exists a unique $t_s \in [0, T]$ satisfying $\mathbf{x}_2(t_s) = \mathbf{x}_1(s)$.*

Proof First, reduce $T > 0$ in order to ensure that $\mathbf{x}_1$ and $\mathbf{x}_2$ are both injective on $[0, T]$ and satisfy $\dot{\mathbf{x}}_j(s) \neq 0$ for a.e. $s \in [0, T]$ $(j = 1, 2)$.

Then, observe that Lemma 3.5, applied to $\mathbf{x} = \mathbf{x}_1$, ensures the existence of $r_1 > 0$, $s_1 \in (0, T]$, and $\delta \in (0, 1)$ such that for a.e. $s \in [0, s_1]$ one can find backward calibrated curves ξ_s^1 and ξ_s^2 on $(-\infty, 0]$ satisfying (3.14), (3.15), and (3.16) for all $r \in [0, r_1]$.

Next, choose

$$\rho = \frac{1 + \sqrt{1 - \delta^2}}{2} \in \left(\sqrt{1 - \delta^2}, 1\right)$$

in Lemma 3.3 and let s_ρ, τ_ρ, and $\sigma_\rho(\cdot)$ be such that

(i) $x_0 \in C_\rho^-(\mathbf{x}_1(s), \theta_1(s))$ for a.e. $s \in [0, s_\rho]$,
(ii) $\mathbf{x}_2(t) \in C_\rho^+(\mathbf{x}_1(s), \theta_1(s))$ for all $t \in [0, \tau_r]$ and a.e. $s \in [0, \sigma_\rho(t)]$,
(iii) $|\mathbf{x}_2(t) - \mathbf{x}_1(s)| \leq \frac{1+\rho}{2\rho} t|\dot{\mathbf{x}}_1^+(0)|$ for all $t \in [0, \tau_r]$ and all $s \in [0, \sigma_\rho(t)]$.

By possibly reducing τ_ρ, without loss of generality we can suppose that (Fig. 1)

$$\frac{1+\rho}{2\rho}\tau_\rho|\dot{\mathbf{x}}_1^+(0)| < \delta r_1. \tag{3.20}$$

Then, recalling that

$$\theta_1(s) = \frac{\dot{\mathbf{x}}_1(s)}{|\dot{\mathbf{x}}_1(s)|} \quad \text{and} \quad \theta_2(s) = \frac{p_s^2 - p_s^1}{|p_s^2 - p_s^1|} \quad (s \in [0, T] \text{ a.e.})$$

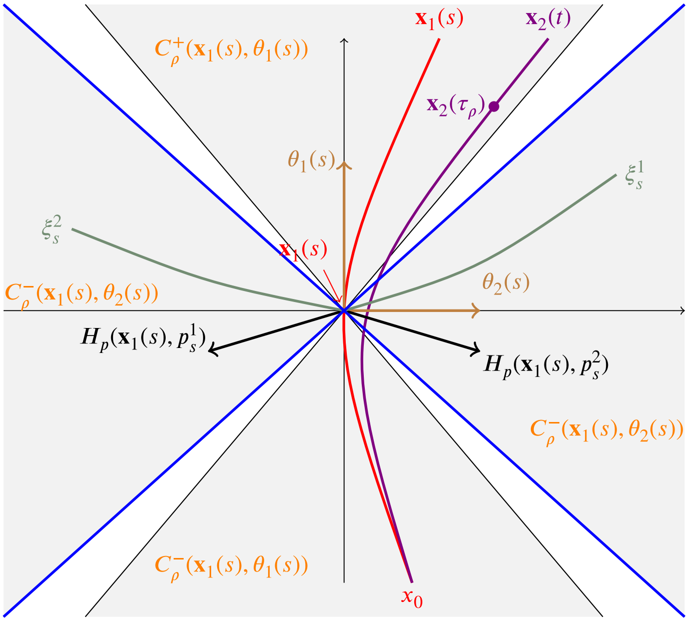

Fig. 1 The illustration of various objects near $\mathbf{x}_1(s)$ for sufficiently small $s > 0$

are orthogonal unit vectors, we claim that, for a.e. $0 \leq s \leq s_1 \wedge \sigma_\rho(\tau_\rho)$,

$$C_\rho(\mathbf{x}_1(s), \theta_1(s)) \bigcap C_\delta(\mathbf{x}_1(s), \theta_2(s)) = \{\mathbf{x}_1(s)\}.$$

Indeed, for any $x \in C_\rho(\mathbf{x}_1(s), \theta_1(s)) \cap C_\delta(\mathbf{x}_1(s), \theta_2(s))$ we have that

$$\begin{aligned} |x - \mathbf{x}_1(s)|^2 &= \langle x - \mathbf{x}_1(s), \theta_1(s)\rangle^2 + \langle x - \mathbf{x}_1(s), \theta_2(s)\rangle^2 \\ &\geq (\rho^2 + \delta^2)|x - \mathbf{x}_1(s)|^2. \end{aligned}$$

This yields $x = \mathbf{x}_1(s)$ because $\rho^2 + \delta^2 > 1$.

Now, define $\sigma = \min\{s_1, s_\rho, \sigma_\rho(\tau_\rho)\}$ and fix $s \in [0, \sigma]$ in the set of full measure on which (i) is satisfied together with (ii) and (iii), that is,

$$\mathbf{x}_2(\tau_\rho) \in C_\rho^+(\mathbf{x}_1(s), \theta_1(s)) \quad \text{and} \quad |\mathbf{x}_2(\tau_\rho) - \mathbf{x}_1(s)| < \delta r_1$$

where (3.20) has also been taken into account. By possibly reducing σ, we also have that $|\mathbf{x}_2(t) - \mathbf{x}_1(s)| < \delta r_1$ for all $t \in [0, \tau_\rho]$. So, the arc $\mathbf{x}_2$, restricted to $[0, \tau_\rho]$, connects the point $\mathbf{x}_2(\tau_\rho)$ of the cone $C_\rho^+(\mathbf{x}_1(s), \theta_1(s))$ with $x_0 \in C_\rho^-(\mathbf{x}_1(s), \theta_1(s))$, remaining in the open ball of radius δr_1 centered at $\mathbf{x}_1(s)$. Thus, in view of (3.15) and (3.16), $\mathbf{x}_2$ must intersect at least one of the two calibrated curves ξ_s^1 and ξ_s^2.[1] However, this can happen only at $\xi_s^1(0) = \mathbf{x}_1(s) = \xi_s^2(0)$, because u is smooth at all points $\xi_s^2(-r)$ with $0 < r < \infty$, whereas $\mathbf{x}_2$ is a singular arc. Finally, such an intersection occurs at a unique time t_s owing to Lemma 3.2.

To complete the proof we observe that $\mathbf{x}_2(t_s) = \mathbf{x}_1(s)$ for all $s \in [0, \sigma]$, not just on a set of full measure. This fact can be easily justified by an approximation argument.

[1] This is the point where our reasoning requires to be in dimension 2.

We are now in a position to prove our main result.

Proof of Theorem 3.6 Let $\sigma \in (0, T]$ be given by Lemma 3.7. Then for each $s \in [0, \sigma]$ there exists a unique $\phi(s) := t_s \in [0, T_1]$ with $\mathbf{x}_2(\phi(s)) = \mathbf{x}_1(s)$.

Recalling that, thanks to Lemma 3.2, both $\mathbf{x}_1(\cdot)$ and $\mathbf{x}_2(\cdot)$ can be assumed to be injective on $[0, \sigma]$ and $[0, \phi(\sigma)]$, respectively, we proceed to show that ϕ is also an injection. Observe that, for any $0 \le s_0, s_1 \le \sigma$,

$$\begin{aligned}\mathbf{x}_2(\phi(s_1)) - \mathbf{x}_2(\phi(s_0)) &= \int_{\phi(s_0)}^{\phi(s_1)} \dot{\mathbf{x}}_2(t)\,dt\\ &= \int_{\phi(s_0)}^{\phi(s_1)} (\dot{\mathbf{x}}_2(t) - \dot{\mathbf{x}}_2^+(0))\,dt + (\phi(s_1) - \phi(s_0))\dot{\mathbf{x}}_2^+(0).\end{aligned}$$

Therefore,

$$|\mathbf{x}_2(\phi(s_1)) - \mathbf{x}_2(\phi(s_0)) - (\phi(s_1) - \phi(s_0))\dot{\mathbf{x}}_2^+(0)| \le \omega_{\mathbf{x}_2}(\phi(s_1) \vee \phi(s_0))|\phi(s_1) - \phi(s_0)|,$$

where $\omega_{\mathbf{x}_2}$ is given by (3.2). Thus, returning to $\mathbf{x}_1 = \mathbf{x}_2 \circ \phi$ we derive

$$\begin{aligned}|\mathbf{x}_1(s_1) - \mathbf{x}_1(s_0)| &\ge |\phi(s_1) - \phi(s_0)|\big(|\dot{\mathbf{x}}_2^+(0)| - \omega_{\mathbf{x}_2}(\phi(s_1) \vee \phi(s_0))|\big),\\ |\mathbf{x}_1(s_1) - \mathbf{x}_1(s_0)| &\le |\phi(s_1) - \phi(s_0)|\big(|\dot{\mathbf{x}}_2^+(0)| + \omega_{\mathbf{x}_2}(\phi(s_1) \vee \phi(s_0))|\big).\end{aligned} \tag{3.21}$$

Notice that (3.21) leads to

$$|\phi(s_1) - \phi(s_0)| \ge \frac{|\mathbf{x}_1(s_1) - \mathbf{x}_1(s_0)|}{|\dot{\mathbf{x}}_2^+(0)| + \omega_{\mathbf{x}_2}(\phi(s_1) \vee \phi(s_0))|} \tag{3.22}$$

and this implies that ϕ is injective as so is $\mathbf{x}_1$.

Next, we prove that ϕ is continuous on $[0, \sigma]$, or the graph of ϕ is closed. Let $s_j \to \bar{s}$ be any sequence such that $\phi(s_j) \to \bar{t}$ as $j \to \infty$. Then

$$\mathbf{x}_1(s_j) \to \mathbf{x}_1(\bar{s}) \quad \text{and} \quad \mathbf{x}_2(\phi(s_j)) = \mathbf{x}_1(s_j) \to \mathbf{x}_2(\bar{t}) \quad \text{as} \quad j \to \infty.$$

So, $\mathbf{x}_2(\phi(\bar{s})) = \mathbf{x}_1(\bar{s}) = \mathbf{x}_2(\bar{t})$. Since $\mathbf{x}_2(\cdot)$ is injective, it follows that $\bar{t} = \phi(\bar{s})$.

Being continuous, ϕ is a homeomorphism. It remains to prove that ϕ is bi-Lipschitz. The continuity of ϕ at 0 ensures that, after possibly reducing σ,

$$\omega_{\mathbf{x}_1}(\phi(s_1)),\ \omega_{\mathbf{x}_2}(\phi(s_2)) \le \frac{|\dot{\mathbf{x}}_2^+(0)|}{2} = \frac{|\dot{\mathbf{x}}_1^+(0)|}{2} \tag{3.23}$$

for all $s_0, s_1 \in [0, \sigma]$. Thus, by (3.21) we have that

$$|\phi(s_1) - \phi(s_0)| \le \frac{|\mathbf{x}_1(s_1) - \mathbf{x}_1(s_0)|}{|\dot{\mathbf{x}}_2^+(0)| - \omega_{\mathbf{x}_2}(\phi(s_1) \vee \phi(s_0))|} \le \frac{2\,\mathrm{Lip}\,(\mathbf{x}_1)}{|\dot{\mathbf{x}}_1^+(0)|} \cdot |s_1 - s_0|$$

for all $s \in [0, \sigma]$ and $t \in [0, \phi(\sigma)]$. So, ϕ is Lipschitz on $[0, \sigma]$. The fact that ϕ^{-1} is also Lipschitz follows by a similar argument. Indeed, writing (3.22) for $t_i = \phi(s_i)$ and appealing to Lemma 3.2 and (3.23) once again we obtain

$$\begin{aligned}|t_1 - t_0| &\ge \frac{|\mathbf{x}_2(t_1) - \mathbf{x}_2(t_0)|}{|\dot{\mathbf{x}}_2^+(0)| + \omega_{\mathbf{x}_2}(t_1 \vee t_0)} = \frac{|\mathbf{x}_1(s_1) - \mathbf{x}_1(s_0)|}{|\dot{\mathbf{x}}_2^+(0)| + \omega_{\mathbf{x}_2}(t_1 \vee t_0)}\\ &\ge \frac{|\dot{\mathbf{x}}_1^+(0)| - \omega_{\mathbf{x}_1}(s_1 \vee s_0)}{|\dot{\mathbf{x}}_2^+(0)| + \omega_{\mathbf{x}_2}(t_1 \vee t_0)} \cdot |s_1 - s_0| \ge \frac{1}{3} \cdot |s_1 - s_0|\end{aligned}$$

The proof is completed noting that ϕ is unique due to the injectivity of $\mathbf{x}_1$ and $\mathbf{x}_2$.

Corollary 3.8 *Let* $\mathbf{x}$ *be a strict singular characteristic as in* (1.3) *and let* $\mathbf{y}$ *be any singular characteristic as in Proposition* 2.12*. If* x_0 *is not a critical point with respect to* (H, u)*, then there exists* $\sigma > 0$ *and a bi-Lipschitz homeomorphism* $\phi : [0, \sigma] \to [0, \phi(\sigma)]$ *such that* $\mathbf{y}(\phi(s)) = \mathbf{x}(s)$ *for all* $s \in [0, \sigma]$.

For strict singular characteristics, uniqueness holds without reparameterization as we show next.

Theorem 3.9 *Let* u *be a semiconcave solution of* ($\mathrm{HJ}_{\mathrm{loc}}$) *and let* $x_0 \in \mathrm{Sing}\,(u)$ *be such that* $0 \notin H_p(x_0, D^+u(x_0))$*. Let* $\mathbf{x}_j : [0, T] \to \Omega$ $(j = 1, 2)$ *be strict singular characteristics with initial point* x_0*. Then there exists* $\tau \in (0, T]$ *such that* $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ *for all* $t \in [0, \tau]$.

Proof By Theorem 3.6 there exists a bi-Lipschitz homeomorphism $\phi : [0, \tau_1] \to [0, \tau_2]$, with $0 \le \tau_j \le T$ $(j = 1, 2)$, such that

$$\mathbf{x}_1(t) = \mathbf{x}_2(\phi(t)) \quad \forall t \in [0, \tau_1]. \tag{3.24}$$

Moreover, since $\mathbf{x}_1$ and $\mathbf{x}_2$ are strict characteristics we have that

$$\begin{cases} \dot{\mathbf{x}}_j^+(t) = H_p(\mathbf{x}_j(t), p_j(t)) \\ H(\mathbf{x}_j(t), p_j(t)) = \min_{p \in D^+u(\mathbf{x}_j(t))} H(\mathbf{x}_j(t), p) \end{cases} \qquad \forall t \in [0, \tau_j]\,(j = 1, 2)$$

Therefore,

$$H_p(\mathbf{x}_1(t), p_1(t)) = \phi'(t) H_p(\mathbf{x}_2(\phi(t)), p_2(\phi(t))) \quad (t \in [0, \tau_1])$$

where, in addition to (3.24), we have that

$$p_2(\phi(t))) = \arg \min_{p \in D^+u(\mathbf{x}_2(\phi(t))} H(\mathbf{x}_2(\phi(t)), p) = \arg \min_{p \in D^+u(\mathbf{x}_1(t)} H(\mathbf{x}_1(t), p) = p_1(t).$$

So, $H_p(\mathbf{x}_1(t), p_1(t)) = \phi'(t) H_p(\mathbf{x}_1(t), p_1(t))$ for all $t \in [0, \tau_1]$. Since $0 \notin H_p(x_0, D^+u(x_0))$, we conclude that $\phi'(t) = 1$, or $\phi(t) = t$, on some interval $0 \le t \le \tau \le \tau$.

Theorems 3.6 and 3.9 establish a connection between the absence of critical points and uniqueness of strict singular characteristics. In this direction, we also have the following global result.

Corollary 3.10 *Let* u *be a semiconcave solution of* ($\mathrm{HJ}_{\mathrm{loc}}$) *and let* $x_0 \in \mathrm{Sing}\,(u)$*. Let* $\mathbf{x}_j : [0, T] \to \Omega$ $(j = 1, 2)$ *be strict singular characteristics with initial point* x_0 *such that* $0 \notin H_p(\mathbf{x}_j(t), D^+u(\mathbf{x}_j(t)))$ *for all* $t \in [0, T]$*. Then* $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ *for all* $t \in [0, T]$.

Proof On account of Theorem 3.9 we have that

$$\mathcal{T} := \left\{\tau \in (0, T] \mid \mathbf{x}_1(t) = \mathbf{x}_2(t)\,,\ \forall t \in [0, \tau]\right\}$$

is a nonempty set. Let $\tau_0 = \sup \mathcal{T} = \max \mathcal{T}$. We claim that $\tau_0 = T$. For if $\tau_0 < T$, applying Theorem 3.9 with initial point $\mathbf{x}_1(\tau_0)$ we conclude that $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ on some intarval $\tau_0 \le t < \tau_0 + \delta$, contradicting the definition of τ_0.

Another well-known example where we have uniqueness of the generalized characteristic is the mechanical Hamiltonian

$$H(x, p) = \frac{1}{2}\langle A(x)p, p\rangle + V(x), \quad (x, p) \in \Omega \times \mathbb{R}^n, \tag{3.25}$$

with $A(x)$ is positive definite symmetric $n\times n$-matrix C^2-smooth in x and V a smooth function on Ω. More precisely, if $x \in \mathrm{Sing}\,(u)$, then there exists a unique Lipschitz arc $\mathbf{y}$ determined by $\dot{\mathbf{y}}^+(t) = A(\mathbf{y}(t))p(t)$, where $\mathbf{y}(0) = x$ and $p(t) = \arg\min_{p\in D^+u(\mathbf{y}(t))}\langle A(\mathbf{y}(t))p, p\rangle$. In this case, uniqueness follows from semiconcavity by an application of Gronwall's lemma (see, e.g., [2,10]) ensuring that, in addition, any generalized characteristic is strict. We now give another justification of such a property from the point of view of this section.

Corollary 3.11 *If H is a mechanical Hamiltonian as in (3.25), then the reparameterization ϕ in Theorem 3.6 is the identity.*

Proof We observe that, for almost all $t \geq 0$,

$$\dot{\mathbf{y}}(t) = A(\mathbf{y}(t))\{\lambda(t)p_0(t) + (1-\lambda(t))p_1(t)\}$$

where $\lambda(t) \in [0,1]$ and we can assume $D^+u(\mathbf{y}(t))$ is a segment, say $[p_1(t), p_0(t)]$, or $\{p_0(t), p_1(t)\} \in D^*u(\mathbf{y}(t))$. Notice that $\{p_0(t), p_1(t)\}$ is also the set of extremal points of the convex set $D^+u(\mathbf{y}(t))$.

Since $\mathbf{x}(t) = \mathbf{y}(\phi(t))$, differentiating we obtain that

$$\begin{aligned}\dot{\mathbf{x}}(t) = p(t) &= \phi'(t)\dot{\mathbf{y}}(\phi(t))\\ &= \phi'(t)A(\mathbf{y}(\phi(t)))\{\lambda(\phi(t))p_0(\phi(t)) + (1-\lambda(\phi(t)))p_1(\phi(t))\}\end{aligned}$$

with $D^+u(\mathbf{y}(\phi(t))) = [p_0(\phi(t)), p_1(\phi(t))]$, or $\{p_0(\phi(t)), p_1(\phi(t))\} \in D^*u(\mathbf{y}(\phi(t)))$.

Therefore, there exists a unique $\lambda_t \in [0,t]$ such that

$$p(t) = A(\mathbf{y}(\phi(t)))\{\lambda_t p_0(\phi(t)) + (1-\lambda_t)p_1(\phi(t))\}.$$

It follows that

$$\phi'(t) = \phi'(t)\{\lambda(\phi(t)) + (1-\lambda(\phi(t)))) = \lambda_t + (1-\lambda_t) = 1.$$

Thus, $\phi(t) \equiv t$ and this completes the proof.

Remark 3.12 Observe that our results apply in particular to solutions of (HJ_s).

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Compliance with ethical standards

Conflict s of interest On behalf of all authors, Piermarco Cannarsa states that there is no conflict of interest.

Appendix A: Existence of strict singular characteristics

In this appendix, we prove the following result which ensures the existence of strict singular characteristics mentioned in the Introduction.

We recall that

$$\dot{\mathbf{x}}^+(t) := \lim_{h\downarrow 0} \frac{\mathbf{x}(t+h)-\mathbf{x}(t)}{h} \quad (t \in [0,T))$$

denotes the right derivative of $\mathbf{x} : [0,T] \to \Omega$, whenever such a derivative exists.

Theorem A.1 *Let u be a semiconcave solution of* (HJ$_{\text{loc}}$). *If $x_0 \in \operatorname{Sing}(u)$ satisfies*

$$0 \notin \operatorname{co} H_p(x_0, D^+u(x_0)), \tag{A.1}$$

then there exists a Lipschitz singular arc $\mathbf{x} : [0,T] \to \Omega$ and a right-continuous selection $p(t) \in D^+u(\mathbf{x}(t))$ such that

$$\begin{cases} \dot{\mathbf{x}}^+(t) = H_p(\mathbf{x}(t), p(t)) & \forall t \in [0,T), \\ \mathbf{x}(0) = x_0. \end{cases} \tag{A.2}$$

and

$$H(\mathbf{x}(t), p(t)) = \min_{p\in D^+u(\mathbf{x}(t))} H(\mathbf{x}(t), p) \quad \forall t \in [0,T). \tag{A.3}$$

Remark A.2 The existence of strict singular characteristics for time dependent Hamilton–Jacobi equations was proved by Khanin and Sobolevski under the additional assumption that the solution u can be locally represented as the minimum of a compact family of smooth functions. Theorem A.1 adapts [13, Theorem 2] to stationary equations removing such an extra assumption.

Proof The proof, which uses ideas from [13], requires several intermediate steps.

Let $R_0 > 0$ be such that the closed ball $B(x_0, 2R_0)$ is contained in Ω. Take any sequence of smooth functions $u_m : B(x_0, 2R_0) \to \mathbb{R}$ such that

$$\begin{cases} (a) & u_m \overset{m\to\infty}{\longrightarrow} u \ \text{ uniformly on } B(x_0, R_0) \\ (b) & \max\{\|Du\|_\infty, \|Du_m\|_\infty\} \leq C_1 \\ (c) & D^2u_m \leq C_2 I \end{cases}$$

for some constants $C_1, C_2 > 0$. A sequence with the above properties can be constructed in several ways, for instance by using mollifiers like in [11,18]. In view of the above uniform bounds, there exists $T_0 > 0$ such that for any $m \geq 1$ the Cauchy problem

$$\begin{cases} \dot{\mathbf{x}}(t) = H_p(\mathbf{x}(t), Du_m(\mathbf{x}(t))), & t \in [0, T_0] \\ \mathbf{x}(0) = x_0 \end{cases} \tag{A.4}$$

has a unique solution $\mathbf{x}_m : [0, T_0] \to B(x_0, R_0)$. Moreover, by possibly taking a subsequence, we can assume that $\mathbf{x}_m$ converges uniformly on $[0, T_0]$ to some Lipschitz arc $\mathbf{x} : [0, T_0] \to B(x_0, R_0)$. We will show that, after possibly replacing T_0 by a smaller $T > 0$, such a limiting curve $\mathbf{x}$ has the required properties.

Lemma A.3 *For every $\bar{t} \in [0, T_0)$ and $\varepsilon > 0$ there exists and integer $m_\varepsilon \geq 1$ and a real number $\tau_\varepsilon \in (0, T_0 - \bar{t})$ such that*

$$\frac{\mathbf{x}_m(t) - \mathbf{x}_m(\bar{t})}{t - \bar{t}} \in \text{co } H_p\big(\mathbf{x}(\bar{t}), Du^+(\mathbf{x}(\bar{t}))\big) + \varepsilon B \quad \forall m \geq m_\varepsilon \,, \ \forall t \in [\bar{t}, \bar{t} + \tau_\varepsilon], \tag{A.5}$$

where B denotes the closed unit ball of $\mathbb{R}^2$, centered at the origin.

Proof We begin by showing that for every $\bar{t} \in [0, T_0)$ and $\varepsilon > 0$ there exist $m_\varepsilon \geq 1$ and $\tau_\varepsilon \in (0, T_0 - \bar{t})$ satisfying

$$\dot{\mathbf{x}}_m(t) \in H_p\big(\mathbf{x}(\bar{t}), Du^+(\mathbf{x}(\bar{t}))\big) + \varepsilon B, \quad t \in [\bar{t}, \bar{t} + \tau_\varepsilon] \ \text{ a.e.} \tag{A.6}$$

for all $m \geq m_\varepsilon$. We argue by contradiction: set $\Phi(\bar{t}) = H_p\big(\mathbf{x}(\bar{t}), Du^+(\mathbf{x}(\bar{t}))\big)$ and suppose there exist $\bar{t} \in [0, T_0)$, $\varepsilon > 0$, and sequences $m_k \to \infty$ and $t_k \downarrow \bar{t}$ such that

$$\begin{cases} (i) & \dot{\mathbf{x}}_{m_k}(t_k) \notin \Phi(\bar{t}) + \varepsilon B, \quad \forall k \geq 1 \\ (ii) & Du_{m_k}\big(\mathbf{x}_{m_k}(t_k)\big) \to \bar{p} \quad (k \to \infty) \end{cases}$$

where we have used bound (b) above to justify (ii). We claim that $\bar{p} \in D^+u\big(\mathbf{x}(\bar{t})\big)$. Indeed, in view of (c) above we have that, for all $k \geq 1$,

$$u_{m_k}\big(\mathbf{x}_{m_k}(t_k) + y\big) - u_{m_k}\big(\mathbf{x}_{m_k}(t_k)\big) - \big\langle Du_{m_k}\big(\mathbf{x}_{m_k}(t_k)\big), y\big\rangle \leq C_2|y|^2, \quad \forall |y| \leq R_0.$$

Hence, in the limit as $k \to \infty$, we get

$$u(\mathbf{x}(\bar{t}) + y) - u(\mathbf{x}(\bar{t})) - \langle \bar{p}, y\rangle \leq C_2|y|^2, \quad \forall |y| \leq R_0,$$

which in turn proves our claim. Thus, we conclude that

$$\dot{\mathbf{x}}_{m_k}(t_k) = H_p\big(\mathbf{x}_{m_k}(t_k), Du_{m_k}(\mathbf{x}_{m_k}(t_k))\big) \overset{k\to\infty}{\longrightarrow} H_p(\mathbf{x}(\bar{t}), \bar{p}) \in \Phi(\bar{t})$$

in contrast with (i). So, (A.6) is proved.

Finally, (A.5) can be derived from (A.6) by integration.

By appealing to the upper semi-continuity of D^+u and assumption (A.1) we conclude that there exists $T \in (0, T_0]$ such that

$$0 \notin \ \text{co } H_p\big(\mathbf{x}(t), Du^+(\mathbf{x}(t))\big) \quad \forall t \in [0, T]. \tag{A.7}$$

Now, fix any $\bar{t} \in [0, T)$ and let $\bar{v} \in \mathbb{R}^2$ be any vector such that

$$\lim_{j\to\infty} \frac{\mathbf{x}(\bar{t} + \tau_j) - \mathbf{x}(\bar{t})}{\tau_j} = \bar{v} \tag{A.8}$$

for some sequence $\tau_j \searrow 0$ ($j \to \infty$). Observe that $\bar{v} \in \ \text{co } H_p\big(\mathbf{x}(\bar{t}), Du^+(\mathbf{x}(\bar{t}))\big)$ in view of Lemma A.3. So, $\bar{v} \neq 0$ owing to (A.7). Set $\bar{x} = \mathbf{x}(\bar{t})$ and define

$$\begin{aligned} &\bar{p} \in \mathbb{R}^2 \quad \text{by } \bar{v} = H_p(\bar{x}, \bar{p}) \quad (\text{or} \quad \bar{p} = L_v(\bar{x}, \bar{v})) \\ &F_{\bar{v}}(\bar{x}) = \big\{p^* \in D^+u(\bar{x}) : \langle p^*, \bar{v}\rangle = \min_{p \in D^+u(\bar{x})} \langle p, \bar{v}\rangle\big\}. \end{aligned}$$

Notice that $F_{\bar{v}}(\bar{x})$ is the exposed face of the convex set $D^+u(\bar{x})$ in the direction $\bar{v}$ (see, for instance, [10]). The following lemma identifies $\bar{p}$ (hence $\bar{v}$) uniquely.

Lemma A.4 *Suppose* $\bar{p} \in F_{\bar{v}}(\bar{x})$. *Then* $\bar{p}$ *is the unique element in* $D^+u(\bar{x})$ *such that*

$$H(\bar{x}, \bar{p}) = \min_{p \in D^+u(\bar{x})} H(\bar{x}, p). \tag{A.9}$$

Proof Since $\bar{p} \in F_{\bar{v}}(\bar{x})$, we have that

$$\langle \bar{p}, \bar{v} \rangle = \langle \bar{p}, H_p(\bar{x}, \bar{p}) \rangle = \min_{p \in D^+u(\bar{x})} \langle p, H_p(\bar{x}, \bar{p}) \rangle.$$

Therefore, by convexity we conclude that

$$0 \leq \langle H_p(\bar{x}, \bar{p}), p - \bar{p} \rangle \leq H(\bar{x}, p) - H(\bar{x}, \bar{p}), \quad \forall p \in D^+u(\bar{x}).$$

Since H is strictly convex in p, $\bar{p}$ is the unique element in $D^+u(\bar{x})$ satisfying (A.9).

Notice that the above lemma yields the existence of the right-derivative $\dot{\mathbf{x}}^+(\bar{t})$ as soon as one shows that $\bar{p} \in F_{\bar{v}}(\bar{x})$ for any $\bar{v}$ satisfying (A.8).

Next, to show that $\bar{p} \in F_{\bar{v}}(\bar{x})$, we proceed by contradiction assuming that

$$\bar{p} \notin F_{\bar{v}}(\bar{x}). \tag{A.10}$$

Let us define functions $\alpha, \beta : D^+u(\bar{x}) \to \mathbb{R}$ by

$$\alpha(p) = \langle p, \bar{v} \rangle - \frac{\partial u}{\partial \bar{v}}(\bar{x}), \quad \beta(x, p) = \langle p - \bar{p}, H_p(x, p) - H_p(x, \bar{p}) \rangle \quad \forall p \in D^+u(\bar{x})$$

where we have set $\frac{\partial u}{\partial \bar{v}}(\bar{x}) = \lim_{\lambda \to 0^+} \frac{u(\bar{x}+\lambda\bar{v}) - u(\bar{x})}{\lambda}$. Recall that, since u is semiconcave,

$$\frac{\partial u}{\partial \bar{v}}(\bar{x}) = \min_{p \in D^+u(\bar{x})} \langle p, \bar{v} \rangle \tag{A.11}$$

(see, for instance, [10]). The following simple lemma is crucial for the proof.

Lemma A.5 *If* $\bar{p} \notin F_{\bar{v}}(\bar{x})$, *then*

$$\mu := \min_{p \in D^+u(\bar{x})} \{\alpha(p) + \beta(\bar{x}, p)\} > 0.$$

Proof Observe first that $\beta(x, p) \geq 0$ by convexity and $\alpha(p) \geq 0$ for all $p \in D^+u(\bar{x})$ by (A.11). Since we suppose $\bar{p} \notin F_{\bar{v}}(\bar{x})$, just two cases are possible.

(1) If $\bar{p} \notin D^+u(\bar{x})$, then $p \neq \bar{p}$ for all $p \in D^+u(\bar{x})$. So $\beta(\bar{x}, p) > 0$ by strict convexity.
(2) If $\bar{p} \in D^+u(\bar{x}) \setminus F_{\bar{v}}(\bar{x})$, then $\alpha(p) > 0$.

In conclusion,

$$M(p) := \alpha(p) + \beta(\bar{x}, p) > 0, \quad \forall p \in D^+u(\bar{x}).$$

Since M is continuous and $D^+u(\bar{x})$ is compact, the conclusion follows.

For any $\varepsilon > 0$ set

$$F^{\varepsilon}_{\bar{v}}(\bar{x}) = F_{\bar{v}}(\bar{x}) + \varepsilon B \quad \text{and} \quad V_{\varepsilon} = D^+u(\bar{x}) + \varepsilon B.$$

Now, let us fix $\varepsilon = \varepsilon(\bar{v}, \mu) > 0$ such that

$$\bar{p} \notin F^{\varepsilon}_{\bar{v}}(\bar{x}) \quad \text{and} \quad \min_{p \in V_{\varepsilon}} \{\alpha(p) + \beta(\bar{x}, p)\} \geq \frac{2}{3}\mu. \tag{A.12}$$

Let $0 < R \leq R_0$ be such that

$$D^+u(x) \subset V_{\varepsilon/2} \quad \forall x \in B(\bar{x}, R).$$

Consider the line segment

$$\gamma(t) := \bar{x} + (t - \bar{t})\bar{v} \quad (t \in [\bar{t}, T])$$

and fix $q \in (0, 1)$. After possible reducing T, we can assume that

$$|\gamma(t) - \bar{x}| \leq qR \quad \text{and} \quad |\mathbf{x}(t) - \bar{x}| \leq qR \quad \forall t \in [\bar{t}, T].$$

Consequently, there exists $\bar{m} \in \mathbb{N}$ such that for all $m \geq \bar{m}$ we have

(i) $Du_m(x) \in V_\varepsilon$ for all $x \in B(\bar{x}, R)$;
(ii) $\mathbf{x}_m(t) \in B(\bar{x}, R)$ for all $t \in [\bar{t}, T]$.

Moreover, by cutting T down to size, we can have the following property satisfied:

(iii) for any $t \in [\bar{t}, T]$ there exists $m(t) \geq \bar{m}$ such that

$$d_{F_{\bar{v}}(\bar{x})}(Du_m(\gamma(t))) < \varepsilon, \quad \forall m \geq m(t). \tag{A.13}$$

We observe that (iii) is a consequence of Proposition 3.3.15 in [10] since $\bar{v} \neq 0$.

For $0 < \delta$ to be chosen later on, we define

$$\begin{aligned} K_\delta &= \bigcup_{\bar{t} \leq t \leq T} B(\gamma(t), \delta(t - \bar{t})) \\ &= \{x \in \mathbb{R}^n : \text{there exists } t \in [\bar{t}, T] \text{ such that } |x - \gamma(t)| \leq \delta(t - \bar{t})\}. \end{aligned}$$

Lemma A.6 *Let $\varepsilon > 0$ and $m(\cdot)$ be fixed so that (A.12) and (A.13) hold true. If $\bar{p} \notin F_{\bar{v}}(\bar{x})$, then there exists $\delta > 0$ such that for all j sufficiently large, $\mathbf{x}_m(t) \notin K_\delta$ for all $t \in (\bar{t}+3\tau_j, T)$ and m sufficiently large.*

Proof Throughout this proof $j \in \mathbb{N}$ is supposed to be so large that $\tau_j < (T - \bar{t})/3$. Moreover, in order to simplify the notation, abbreviate τ for τ_j and we assume $\bar{t} = 0$.

For all $t \in (3\tau, T)$ we have that

$$\begin{aligned} &\frac{d}{dt}\big(u_m(\mathbf{x}_m(t)) - \langle \bar{p}, \mathbf{x}_m(t)\rangle\big) \\ &\quad = \big\langle Du_m(\mathbf{x}_m(t)) - \bar{p}, \dot{\mathbf{x}}_m(t)\big\rangle = \big\langle H_p(\mathbf{x}_m(t), Du_m(\mathbf{x}_m(t))), Du_m(\mathbf{x}_m(t)) - \bar{p}\big\rangle. \end{aligned}$$

Therefore, by integrating on (τ, t),

$$\begin{aligned} &u_m(\mathbf{x}_m(t)) - \langle \bar{p}, \mathbf{x}_m(t)\rangle - u_m(\mathbf{x}_m(\tau)) + \langle \bar{p}, \mathbf{x}_m(\tau)\rangle \\ &\quad = \int_\tau^t \big\langle H_p(\mathbf{x}_m(s), Du_m(\mathbf{x}_m(s))), Du_m(\mathbf{x}_m(s)) - \bar{p}\big\rangle ds. \end{aligned} \tag{A.14}$$

Similarly,

$$\frac{d}{dt}\big(u_m(\gamma(t))) - \langle \bar{p}, \gamma(t)\rangle\big) = \langle Du_m(\gamma(t)) - \bar{p}, \bar{v}\rangle.$$

So, (iii) and Lebesgue's theorem ensure that

$$\begin{aligned} &u_m(\gamma(t)) - \langle \bar{p}, \gamma(t)\rangle - u_m(\gamma(\tau)) + \langle \bar{p}, \gamma(\tau)\rangle \\ &\quad = \int_\tau^t \langle Du_m(\gamma(s)) - \bar{p}, \bar{v}\rangle\, ds \leq \left(\frac{\partial u}{\partial \bar{v}}(\bar{x}) - \langle \bar{p}, \bar{v}\rangle + \varepsilon|\bar{v}|\right)(t - \tau). \end{aligned} \tag{A.15}$$

Therefore, by (A.14) and (A.15) we obtain

$$\begin{aligned}
&u_m(\mathbf{x}_m(t)) - \langle \bar{p}, \mathbf{x}_m(t)\rangle - u_m(\mathbf{x}_m(\tau)) + \langle \bar{p}, \mathbf{x}_m(\tau)\rangle \\
&\quad - \big(u_m(\gamma(t)) - \langle \bar{p}, \gamma(t)\rangle - u_m(\gamma(\tau)) + \langle \bar{p}, \gamma(\tau)\rangle\big) \\
&\quad \geq \int_\tau^t \left\{ \big\langle H_p(\mathbf{x}_m(s), Du_m(\mathbf{x}_m(s))), Du_m(\mathbf{x}_m(s)) - \bar{p}\big\rangle - \left(\frac{\partial u}{\partial \bar{v}}(\bar{x}) - \langle \bar{p}, \bar{v}\rangle + \varepsilon|\bar{v}|\right)\right\} ds
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&u_m(\mathbf{x}_m(t)) - \langle \bar{p}, \mathbf{x}_m(t)\rangle - u_m(\mathbf{x}_m(\tau)) + \langle \bar{p}, \mathbf{x}_m(\tau)\rangle \\
&\quad -\big(u_m(\gamma(t)) - \langle \bar{p}, \gamma(t)\rangle - u_m(\gamma(\tau)) + \langle \bar{p}, \gamma(\tau)\rangle\big) \\
&\geq \int_\tau^t \big\langle H_p(\mathbf{x}_m(s), Du_m(\mathbf{x}_m(s))) - \bar{v}, Du_m(\mathbf{x}_m(s)) - \bar{p}\big\rangle ds \\
&\quad + \int_\tau^t \left(\langle Du_m(\mathbf{x}_m(s)), \bar{v}\rangle - \frac{\partial u}{\partial \bar{v}}(\bar{x}) - \varepsilon|\bar{v}|\right) ds \\
&= \int_\tau^t \big\langle H_p(\mathbf{x}_m(s), Du_m(\mathbf{x}_m(s))) - H_p(\mathbf{x}_m(s), \bar{p}), Du_m(\mathbf{x}_m(s)) - \bar{p}\big\rangle ds \\
&\quad + \int_\tau^t \big\langle H_p(\mathbf{x}_m(s), \bar{p}) - H_p(\bar{x}, \bar{p}), Du_m(\mathbf{x}_m(s)) - \bar{p}\big\rangle\, ds \\
&\quad + \int_\tau^t \left(\langle Du_m(\mathbf{x}_m(s)), \bar{v}\rangle - \frac{\partial u}{\partial \bar{v}}(\bar{x}) - \varepsilon|\bar{v}|\right) ds \\
&\geq \int_\tau^t \big\{\alpha(Du_m(\mathbf{x}_m(s))) + \beta(\mathbf{x}_m(s), Du_m(\mathbf{x}_m(s))) - \varepsilon|\bar{v}|\big\} ds \\
&\quad + \int_\tau^t \big\langle H_p(\mathbf{x}_m(s), \bar{p}) - H_p(\bar{x}, \bar{p}), Du_m(\mathbf{x}_m(s)) - \bar{p}\big\rangle\, ds.
\end{aligned} \tag{A.16}$$

Now, observe the following:

$$\begin{aligned}
|u_m(\gamma(t)) - \langle \bar{p}, \gamma(t)\rangle - u_m(\gamma(\tau)) + \langle \bar{p}, \gamma(\tau)\rangle| &\leq (C_1 + |\bar{p}|)|\gamma(\tau) - \mathbf{x}_m(\tau)| \\
&\leq (C_1 + |\bar{p}|)(|\gamma(\tau) - \mathbf{x}(\tau)| + |\mathbf{x}(\tau) - \mathbf{x}_m(\tau)|)
\end{aligned}$$

where we recall that $C_1 \geq \|Du_m\|_\infty$.

Next, we fix $\tau = \tau_j$ with j large enough so that

$$|\gamma(\tau) - \mathbf{x}(\tau)| \leq \frac{\delta\tau}{2} \quad \text{and} \quad m \gg 1 \text{ so that } |\mathbf{x}(\tau) - \mathbf{x}_m(\tau)| \leq \frac{\delta\tau}{2}.$$

Then

$$|u_m(\gamma(\tau)) - \langle \bar{p}, \gamma(\tau)\rangle - u_m(\mathbf{x}_m(\tau)) + \langle \bar{p}, \mathbf{x}_m(\tau)\rangle| \leq (C_1 + |\bar{p}|)\delta\tau. \tag{A.17}$$

Since $Du_m(\mathbf{x}_m(s)) \in V_\varepsilon$ for all $s \in [0, \tau]$, by (A.12) we have that

$$\int_\tau^t \Big\{\alpha(Du_m(\mathbf{x}_m(s))) + \beta(\mathbf{x}_m(s), Du_m(\mathbf{x}_m(s)))\Big\} ds \geq \frac{2}{3}\mu(t - \tau). \tag{A.18}$$

We also have that, after cutting down on $T > 0$,

$$\begin{aligned}
&\langle H_p(\mathbf{x}_m(s), \bar{p}) - H_p(\bar{x}, \bar{p}), Du_m(\mathbf{x}_m(s)) - \bar{p}\rangle \\
&\quad \geq -(C_1 + |\bar{p}|) \cdot C_2'|\mathbf{x}_m(s) - \bar{x}| \\
&\quad \geq -\varepsilon C_2'(C_1 + |\bar{p}|)
\end{aligned} \tag{A.19}$$

So, by (A.16), (A.17), (A.18) and (A.19) we conclude that

$$u_m(\mathbf{x}_m(t)) - \langle \bar{p}, \mathbf{x}_m(t)\rangle - u_m(\gamma(t)) + \langle \bar{p}, \gamma(t)\rangle \geq \left(\frac{2}{3}\mu - \varepsilon(|\bar{v}| + C_2'(C_1 + |\bar{p}|))\right)(t-\tau) - \delta\tau(C_1 + |\bar{p}|).$$

On the other hand,

$$|u_m(\mathbf{x}_m(t)) - \langle \bar{p}, \mathbf{x}_m(t)\rangle - u_m(\gamma(t)) + \langle \bar{p}, \gamma(t)\rangle| \leq (C_1 + |\bar{p}|)|\mathbf{x}_m(t) - \gamma(t)|.$$

Therefore,

$$|\mathbf{x}_m(t) - \gamma(t)| \geq \frac{2\mu/3 - \varepsilon(|\bar{v}| + C_2'(C_1 + |\bar{p}|))}{C_1 + |\bar{p}|}(t-\tau) - \delta\tau. \tag{A.20}$$

We now take $0 \leq \varepsilon(|\bar{v}| + C_2'(C_1 + |\bar{p}|))| < \frac{\mu}{3}$ to obtain

$$|\mathbf{x}_m(t) - \gamma(t)| \geq \frac{\mu(t-\tau)}{3(C_1 + |\bar{p}|)} - \delta\tau$$

and look for $t < T$ such that

$$\frac{\mu(t-\tau)}{3(C_1 + |\bar{p}|)} - \delta\tau \geq 2\delta t, \tag{A.21}$$

or

$$t \geq \frac{3(C_1 + |\bar{p}|)\delta + \mu}{\mu - 6(C_1 + |\bar{p}|)\delta} \cdot \tau.$$

So, taking $0 < \delta \leq \frac{\mu}{12(C_1 + |\bar{p}|)}$, we have that

$$\frac{3(C_1 + |\bar{p}|)\delta + \mu}{\mu - 6\delta(C_1 + |\bar{p}|)} \leq 3.$$

Finally, $\frac{\delta}{\mu} \leq \frac{1}{12(C_1 + |\bar{p}|)}$ gives that (A.21) holds for all $t \in [3\tau, T]$. □

To complete the proof it suffices to note that Lemmas A.5 and A.6 ensure that assuming (A.10) leads to a contradiction. Indeed,

$$|\mathbf{x}_m(t) - \gamma(t)| \geq 2\delta t, \quad \forall t \in [3\tau_j, T], \forall j \gg 1$$

implies that $\mathbf{x}(t) \notin K_\delta$ for all $t \in [3\tau_j, T]$. On the other hand, $\mathbf{x}(\tau_i) \in K_\delta$ for $i \gg 1$ and, for any fixed i, $\tau_i \in [3\tau_j, T]$ for j sufficiently large.

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