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ORIGINAL RESEARCH

A computational method for solving stochastic Itô–Volterra integral equation with multi-stochastic terms

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Abstract

In this paper, a linear combination of quadratic modified hat functions is proposed to solve stochastic Itô–Volterra integral equation with multi-stochastic terms. All known and unknown functions are expanded in terms of modified hat functions and replaced in the original equation. The operational matrices are calculated and embedded in the equation to achieve a linear system of equations which gives the expansion coefficients of the solution. Also, under some conditions the error of the method is $O(h^3)$. The accuracy and reliability of the method are studied and compared with those of block pulse functions and generalized hat functions in some examples.

Keywords Modified hat functions · Stochastic operational matrix · Stochastic Itô–Volterra integral equation · Brownian motion

Introduction

Nowadays, modelling different problems in different issues of science leads to stochastic equations [1]. These equations arise in many fields of science such as mathematics and statistics [2–7], finance [8–10], physics [11–13], mechanics [14, 15], biology [16–18], and medicine [19, 20]. Whereas most of them do not have an exact solution, the role of numerical methods and finding a reliable and accurate numerical approximation have become highlighted [21].

In recent years, different orthogonal basic functions and polynomials have been used to find a numerical solution for integral equations such as block pulse functions [2, 21, 22], hat functions [23], hybrid functions [24, 25], wavelet methods [26–28], triangular functions [3, 29], and Bernstein polynomials [30]. In this paper, MHFs will be applied to find an approximate solution for the following stochastic Itô–Volterra integral equation with multi-stochastic terms,

$$X(t) = f(t) + \int_0^t \mu(s,t)X(s)\,\mathrm{d}s + \sum_{j=1}^{n} \int_0^t \sigma_j(s,t)X(s)\,dB_j(s),$$

where $t \in D = [0,T), X, f, \mu$ and $\sigma_j, j = 1,2,\ldots,n,$ for $s,t \in D$ are the stochastic processes defined on the same probability space (Ω, F, P) and X is unknown. Also $\int_0^t \sigma_j(s,t)X(s)\,dB_j(s)$, $j = 1,2,\ldots,n$ are Itô integrals and $B_1(t), B_2(t), \ldots B_n(t)$ are the Brownian motion processes [31, 32].

The paper is organized as follows: In "MHFs and their properties" section, the MHFs and their properties are described. In "Operational matrices" section, the operational matrices are found. In "Solving stochastic Itô–Volterra integral equation with multi-stochastic terms by the MHFs" section, the sets and operational matrices are applied in the above equation and the approximate solution is found. In "Error analysis" section, the error analysis of the present method is discussed. In the "Numerical examples" section, some numerical examples are solved by using this method. And finally, the last section concludes the paper.

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MHFs and their properties

In this section, we recall the definition and properties of modified hat functions [33]. Let $m \geq 2$ be an even integer and $h = \frac{T}{m}$. Also assume that the interval [0, T) is divided into $\frac{m}{2}$ equal subintervals $[ih, (i+2)h]$, $i = 0, 2, \dots, (m-2)$ and let X_m be the set of all continuous functions that are quadratic polynomials when restricted to each of the above subintervals. Because each element of X_m is completely determined by its values at the $(m+1)$ nodes $ih, i = 0, 1, \dots, m$, the dimension of X_m is $(m+1)$. Considering that $f \in \chi = C^3(D)$ can be approximated by its expansion with respect to the following set functions, $(m+1)$ set of MHFs are defined over D as

$$h_0(t) = \begin{cases} \frac{1}{2h^2}(t-h)(t-2h), & 0 \leq t \leq 2h \\ 0, & \text{otherwise.} \end{cases}$$

If i is odd and $1 \leq i \leq (m-1)$,

$$h_i(t) = \begin{cases} \frac{-1}{h^2}(t-(i-1)h)(t-(i+1)h), & (i-1)h \leq t \leq (i+1)h \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

If i is even and $2 \leq i \leq (m-2)$,

$$h_i(t) = \begin{cases} \frac{1}{2h^2}(t-(i-1)h)(t-(i-2)h), & (i-2)h \leq t \leq ih \\ \frac{1}{2h^2}(t-(i+1)h)(t-(i+2)h), & ih \leq t \leq (i+2)h \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_m(t) = \begin{cases} \frac{1}{2h^2}(t-(T-h))(t-(T-2h)), & T-2h \leq t \leq T \\ 0, & \text{otherwise.} \end{cases}$$

Properties of the MHFs

By considering the above definition, the following properties come as a result.

1) $h_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$

2) $h_i(t)h_j(t) = \begin{cases} 0, & i \text{ even and } |i-j| \geq 3 \\ 0, & i \text{ odd and } |i-j| \geq 2 \end{cases}.$

3) They are linearly independent.

4) $\sum_{i=0}^{m} h_i(t) = 1.$

Suppose

$$\mathbf{H}(t) = [h_0(t), h_1(t), \dots, h_m(t)]^T, \tag{2}$$

by applying the second property and considering definition (1), we obtain

5) $\mathbf{H}(t)\mathbf{H}^T(t) \simeq \begin{pmatrix} h_0(t) & 0 & \dots & 0 \\ 0 & h_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & h_m(t) \end{pmatrix}.$

6) $\mathbf{H}(t)\mathbf{H}(t)^T\mathbf{X} \simeq diag(\mathbf{X})\mathbf{H}(t),$

7) Let $\mathbf{A}$ be an $(m+1) \times (m+1)$ matrix and $\mathbf{H}(t)$ be the vector of $(m+1)$-MHFs defined in (2) then $\mathbf{H}(t)^T\mathbf{A}\mathbf{H}(t) \simeq \mathbf{H}(t)^T\tilde{\mathbf{A}}$, where $\tilde{\mathbf{A}}$ is a column vector with $(m+1)$ entries equal to the diagonal entries of the matrix $\mathbf{A}$.

Function approximation

An arbitrary real function f on D can be expanded by these functions as [34]

$$f(t) \simeq \sum_{i=0}^{m} f_i h_i(t) = \mathbf{F}^T\mathbf{H}(t) = \mathbf{H}^T(t)\mathbf{F}, \tag{3}$$

where $\mathbf{F} = [f_0, f_1, \dots, f_m]^T$ and $\mathbf{H}(t)$ is defined in relation (2) and the coefficients in (3) are given by $f_i = f(ih), i = 0, 1, \dots, m$.

Similarly, an arbitrary real function of two variables $g(s, t)$ on $D \times D$ can be expanded by these basic functions as

$$g(s, t) \simeq \mathbf{H}^T(s)\,\mathbf{G}\,\mathbf{I}(t), \tag{4}$$

where $\mathbf{H}(s), \mathbf{I}(t)$ are, respectively, (m_1+1)- and (m_2+1)-dimensional MHFs vectors. $\mathbf{G}$ is the $(m_1+1) \times (m_2+1)$ MHFs coefficient matrix with entries $G_{ij}, i = 0, 1, 2, \dots, m_1, j = 0, 1, 2, \dots, m_2$ and $G_{ij} = g(ih, jk)$, where $h = \frac{T}{m_1}$ and $k = \frac{T}{m_2}$. For convenience, we put $m_1 = m_2 = m$.

Operational matrices

In this section, we present both operational matrix of integrating the vector $\mathbf{H}(t)$, denoted by $\mathbf{P}$, and stochastic operational matrix of Itô integrating the vector $\mathbf{H}(t)$, denoted by $\mathbf{P}_s$. Therefore, by integrating the vector $\mathbf{H}(t)$ defined in (2), we have [34, 35]

$$\int_0^t \mathbf{H}(\tau)\,\mathrm{d}\tau = \mathbf{P}\mathbf{H}(t), \tag{5}$$

where $\mathbf{P}$ is the following $(m+1) \times (m+1)$ operational matrix of integration of MHFs

$$\mathbf{P}=\frac{h}{12}\begin{pmatrix} 0 & 5 & 4 & 4 & 4 & \dots & 4 & 4 & 4 & 4\\ 0 & 8 & 16 & 16 & 16 & \dots & 16 & 16 & 16 & 16\\ 0 & -1 & 4 & 9 & 8 & \dots & 8 & 8 & 8 & 8\\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 4 & 9 & 8\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 8 & 16\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 4 \end{pmatrix}.$$

Theorem 1 *Let* $\mathbf{H}(t)$ *be the vector defined in (2), the Itô integral of* $\mathbf{H}(t)$ *can be expressed as*

$$\int_0^t \mathbf{H}(\tau)\,dB(\tau)=\mathbf{P}_s\mathbf{H}(t), \tag{6}$$

where $\mathbf{P}_s$ *is the following* $(m+1)\times(m+1)$ *stochastic operational matrix of integration*

$$\left\{\begin{array}{llllllllll} 0 & \gamma_1 & \gamma_2 & \gamma_2 & \gamma_2 & \cdots & \gamma_2 & \gamma_2 & \gamma_2 & \gamma_2\\ 0 & B(h)+\theta_{1,1} & \theta_{2,1} & \theta_{2,1} & \theta_{2,1} & \dots & \theta_{2,1} & \theta_{2,1} & \theta_{2,1} & \theta_{2,1}\\ 0 & \eta_{1,2} & B(2h)+\eta_{2,2} & \eta_{3,2} & \eta_{4,2} & \dots & \eta_{4,2} & \eta_{4,2} & \eta_{4,2} & \eta_{4,2}\\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & 0 & 0 & 0 & 0 & \dots & \eta_{1,m-2} & B(T-2h)+\eta_{2,m-2} & \eta_{3,m-2} & \eta_{4,m-2}\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & B(T-h)+\theta_{1,m-1} & \theta_{2,m-1}\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \eta_{1,m} & B(T)+\eta_{2,m} \end{array}\right. \tag{7}$$

with

$$\gamma_1=-\int_0^h \frac{1}{2h^2}(2\tau-3h)B(\tau)\,\mathrm{d}\tau$$

$$\gamma_2=-\int_0^{2h} \frac{1}{2h^2}(2\tau-3h)B(\tau)\,\mathrm{d}\tau,$$

$$\theta_{1,i}=\int_{(i-1)h}^{ih} \frac{1}{h^2}(2\tau-2ih)B(\tau)\,\mathrm{d}\tau,$$

$$\theta_{2,i}=\int_{(i-1)h}^{(i+1)h} \frac{1}{h^2}(2\tau-2ih)B(\tau)\,\mathrm{d}\tau,$$

$$\eta_{1,i}=-\int_{(i-2)h}^{(i-1)h} \frac{1}{2h^2}(2\tau-(2i-3)h)B(\tau)\,\mathrm{d}\tau,$$

$$\eta_{2,i}=-\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2\tau-(2i-3)h)B(\tau)\,\mathrm{d}\tau,$$

$$\eta_{3,i}=-\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2\tau-(2i-3)h)B(\tau)\,\mathrm{d}\tau -\int_{ih}^{(i+1)h} \frac{1}{2h^2}(2\tau-(2i+3)h)B(\tau)\,\mathrm{d}\tau,$$

and

$$\eta_{4,i}=-\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2\tau-(2i-3)h)B(\tau)\,\mathrm{d}\tau -\int_{ih}^{(i+2)h} \frac{1}{2h^2}(2\tau-(2i+3)h)B(\tau)\,\mathrm{d}\tau.$$

Proof By considering definitions of $h_i(t), i=0,1,\dots,m$ and integrating by parts, we have

$$\begin{aligned}\int_0^t h_i(\tau)dB(\tau)&=h_i(t)B(t)-h_i(0)B(0)-\int_0^t h_i'(\tau)B(\tau)\mathrm{d}\tau\\ &=h_i(t)B(t)-\int_0^t h_i'(\tau)B(\tau)\mathrm{d}\tau,\end{aligned}$$

expanding $\int_0^t h_i(\tau)dB(\tau)$ in terms of MHFs yields

$$\int_0^t h_i(\tau)dB(\tau)\simeq\sum_{j=0}^{m} a_{ij}h_j(t)$$

and

$$a_{ij}=\int_0^{jh} h_i(\tau)dB(\tau),$$
$$=h_i(jh)B(jh)-\int_0^{jh} h_i'(\tau)B(\tau)\mathrm{d}\tau$$

so we obtain

$$a_{0j}=\begin{cases} 0, & j=0\\ -\int_0^h \frac{1}{2h^2}(2s-3h)B(s)\mathrm{d}s, & j=1\\ -\int_0^{2h} \frac{1}{2h^2}(2s-3h)B(s)\mathrm{d}s, & j\geq 2.\end{cases}$$

If i is odd and $1 \le i \le (m-1)$

$$a_{ij} = \begin{cases} 0, & j \le i-1 \\ B(ih) - \int_{(i-1)h}^{ih} \frac{-1}{h^2}(2s - 2ih)B(s)\mathrm{d}s, & j = i \\ -\int_{(i-1)h}^{(i+1)h} \frac{-1}{h^2}(2s - 2ih)B(s)\mathrm{d}s, & j \ge i+1. \end{cases}$$

If i is even and $2 \le i \le (m-2)$,

$$a_{ij} = \begin{cases} 0, & j \le i-2 \\ -\int_{(i-2)h}^{(i-1)h} \frac{1}{2h^2}(2s - (2i-3)h)B(s)\mathrm{d}s, & j = i-1 \\ B(ih) - \int_{(i-2)h}^{ih} \frac{1}{2h^2}(2s - (2i-3)h)B(s)\mathrm{d}s, & j = i \\ -\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2s - (2i-3)h)B(s)\mathrm{d}s & \\ -\int_{ih}^{(i+1)h} \frac{1}{2h^2}(2s - (2i+3)h)B(s)\mathrm{d}s, & j = i+1 \\ -\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2s - (2i-3)h)B(s)\mathrm{d}s & \\ -\int_{ih}^{(i+2)h} \frac{1}{2h^2}(2s - (2i+3)h)B(s)\mathrm{d}s, & j \ge i+2. \end{cases}$$

and

$$a_{mj} = \begin{cases} 0, & j \le m-2 \\ -\int_{(T-2h)}^{(T-h)} \frac{1}{2h^2}(2s - 2T + 3h)B(s)\mathrm{d}s, & j = m-1 \\ B(T) - \int_{(T-2h)}^{T} \frac{1}{2h^2}(2s - 2T + 3h)B(s)\mathrm{d}s, & j = m. \end{cases}$$

Putting the obtained components in the matrix form ends the proof. □

Solving stochastic Itô–Volterra integral equation with multi-stochastic terms by the MHFs

Our problem is to define the MHFs coefficients of X(t) in the following linear stochastic Itô–Volterra integral equation with several independent white noise sources,

$$X(t) = f(t) + \int_0^t \mu(s,t)X(s)\,\mathrm{d}s + \sum_{j=1}^{n} \int_0^t \sigma_j(s,t)X(s)\,dB_j(s),\ t \in D, \tag{8}$$

where X, f, μ and $\sigma_j, j = 1, 2, \dots, n$ for $s, t \in D$, are stochastic processes defined on the same probability space(Ω, F, P). Also $B_1(t), B_2(t), \dots, B_n(t)$ are Brownian motion processes, and $\int_0^t \sigma_j(s,t)\,dB_j(s)\,, j = 1, 2, \dots, n$ are the Itô integrals.

We replace $X(t), f(t), \mu(s,t)$ and $\sigma_j(s,t)\,, j = 1, 2, \dots, n$ by their approximations which are obtained by MHFs:

$$X(t) \simeq \mathbf{X}^T\mathbf{H}(t) = \mathbf{H}(t)^T\mathbf{X}, \tag{9}$$

$$f(t) \simeq \mathbf{F}^T\mathbf{H}(t) = \mathbf{H}(t)^T\mathbf{F}, \tag{10}$$

$$\mu(s,t) \simeq \quad \mathbf{H}(t)^T\mu^T\mathbf{H}(s) = \mathbf{H}(s)^T\mu\mathbf{H}(t), \tag{11}$$

$$\sigma_j(s,t) \simeq \mathbf{H}(t)^T\mathbf{\Delta}_j^T\mathbf{H}(s) = \mathbf{H}(s)^T\mathbf{\Delta}_j\mathbf{H}(t), \quad j = 1, 2, \dots, n, \tag{12}$$

where $\mathbf{X}$ and $\mathbf{F}$ are stochastic MHFs coefficient vectors and μ and $\mathbf{\Delta}_j\,, j = 1, 2, \dots, n$ are stochastic MHFs coefficient matrices. Substituting (9)–(12) in relation (8), we obtain

$$\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \left(\int_0^t \mathbf{H}(t)^T\mu^T\mathbf{H}(s)\mathbf{H}(s)^T\mathbf{X}\,\mathrm{d}s\right) + \sum_{j=1}^{n}\left(\int_0^t \mathbf{H}(t)^T\mathbf{\Delta_j}^T\mathbf{H}(s)\mathbf{H}(s)^T\mathbf{X}\,dB_j(s)\right). \tag{13}$$

Using the 6-th property in relation (13), we get

$$\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \mathbf{H}(t)^T\mu^T diag(\mathbf{X})\left(\int_0^t \mathbf{H}(s)\,\mathrm{d}s\right) + \sum_{j=1}^{n}\mathbf{H}(t)^T\mathbf{\Delta_j}^T diag(\mathbf{X})\left(\int_0^t \mathbf{H}(s)\,dB_j(s)\right). \tag{14}$$

Utilizing operational matrices defined in relations (5) and (6) in (14), we have

$$\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \mathbf{H}(t)^T\mu^T diag(\mathbf{X})\mathbf{P}\mathbf{H}(t) + \sum_{j=1}^{n}\mathbf{H}(t)^T\mathbf{\Delta_j}^T diag(\mathbf{X})\mathbf{P_s}\mathbf{H}(t). \tag{15}$$

Let $\mathbf{A} = \mu^T diag(\mathbf{X})\mathbf{P}$ and $\mathbf{B}_j = \mathbf{\Delta_j}^T diag(\mathbf{X})\mathbf{P_s}, j = 1, 2, \dots, n$. Applying property (7) in relation (15) yields

$$\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \mathbf{H}(t)^T\tilde{\mathbf{A}} + \sum_{j=1}^{n}\mathbf{H}(t)^T\tilde{\mathbf{B}}_j,$$

therefore, by using the third property and replacing $\simeq$ by $=$, we have

$$\mathbf{X} = \mathbf{F} + \tilde{\mathbf{A}} + \sum_{j=1}^{n}\tilde{\mathbf{B}}_j,$$

which is a linear system of equations that gives the approximation of X with the help of MHFs.

Error analysis

In this section, the error analysis is studied. We propose some conditions to show that the rate of convergence for this method is $O(h^3)$.

Theorem 2 [34] *Let* $t_j = jh, j = 0, 1, \dots, m, f \in \chi$ *and* f_m *be the MHFs expansion of* f *defined as* $f_m(t) = \sum_{j=0}^{m} f(t_j)h_j(t)$ *and also assume that* $e_m(t) = f(t) - f_m(t)$ *, for* $t \in D$ *, then we have*

$$\|e_m\| \le \frac{h^3}{9\sqrt{3}}\|f^{(3)}\|,$$

and hence $\|e_m\| = O(h^3)$ *. Where* $\|.\|$ *denotes the sup-norm for which any continuous function* f *is defined on the interval [0, T) by*

$$\|f\| = \sup_{t\in[0,T)} |f(t)|.$$

Theorem 3 [34] *Let* $s_i = t_i = ih, i = 0, 1, \dots, m$, $\mu \in C^3(D \times D)$ *and* $\mu_m(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{m} \mu(s_i, t_j)h_i(s)h_j(t)$, *be the MHFs expansion of* $\mu(s,t)$, *and also assume that* $e_m(s,t) = \mu(s,t) - \mu_m(s,t)$, *then we have*

$$\|e_m\| \le \frac{h^3}{9\sqrt{3}}\left(\|\mu_s^{(3)}\| + \|\mu_t^{(3)}\|\right) + \frac{h^6}{243}\|\mu_{s,t}^{(3+3)}\|,$$

and so $\|e_m\| = O(h^3)$.

Theorem 4 *Let* X *be the exact solution of* (8) *and* X_m *be the MHFs series approximate solution of* (8) *, and also assume that*

H_1 : $\|X\| \le \rho$,

H_2 : $\|\mu\| \le K$,

H_3 : $\|\sigma_j\| \le M_j, j = 1, 2, \dots, n$,

H_4 : $T(K + \gamma(h)) + \sum_{j=1}^{n}(M_j + \lambda_j(h))\|B_j\| < 1$,

then

$$\|X - X_m\| \le \frac{\Gamma(h) + T\rho\gamma(h) + \rho\sum_{j=1}^{n}\lambda_j(h)\|B_j\|}{1 - \left(T(K + \gamma(h)) + \sum_{j=1}^{n}(M_j + \lambda_j(h))\|B_j\|\right)},$$

and $\|X - X_m\| = O(h^3)$ *, where*

$$\Gamma(h) = \frac{h^3}{9\sqrt{3}}\|f^{(3)}\|,$$

$$\gamma(h) = \frac{h^3}{9\sqrt{3}}\left(\|\mu_s^{(3)}\| + \|\mu_t^{(3)}\|\right) + \frac{h^6}{243}\|\mu_{s,t}^{(3+3)}\|,$$

$$\lambda_j(h) = \frac{h^3}{9\sqrt{3}}\left(\|\sigma_{j_s}^{(3)}\| + \|\sigma_{j_t}^{(3)}\|\right) + \frac{h^6}{243}\|\sigma_{j_{s,t}}^{(3+3)}\|,$$

$j = 1, 2, \dots, n.$

Proof From relation (8), we have

$$X(t) - X_m(t) = f(t) - f_m(t) + \int_0^t \left(\mu(s,t)X(s) - \mu_m(s,t)X_m(s)\right) \mathrm{d}s + \sum_{j=1}^{n}\int_0^t \left(\sigma_j(s,t)X(s) - \sigma_{jm}(s,t)X_m(s)\right) dB_j(s),$$

now the following relation is concluded

$$|X(t) - X_m(t)| \le |f(t) - f_m(t)| + tN + \sum_{j=1}^{n}|B_j(t)|N_j, \tag{16}$$

where

$$N = \sup_{s,t\in[0,T)} |\mu(s,t)X(s) - \mu_m(s,t)X_m(s)|,$$

and

$$N_j = \sup_{s,t\in[0,T)} |\sigma_j(s,t)X(s) - \sigma_{jm}(s,t)X_m(s)|,$$

using Theorems 2 and 3, we also have

$$\begin{aligned} N &\le \|\mu\|\|X - X_m\| + \|\mu - \mu_m\|\left(\|X - X_m\| + \|X\|\right) \\ &\le \|X - X_m\|(K + \gamma(h)) + \gamma(h)\rho, \end{aligned} \tag{17}$$

and

$$\begin{aligned} N_j &\le \|\sigma_j\|\|X - X_m\| + \|\sigma_j - \sigma_{jm}\|(\|X - X_m\| + \|X\|) \\ &\le (M_j + \lambda_j(h))\|X - X_m\| + \lambda_j(h)\rho, \end{aligned} \tag{18}$$

Table 1 Numerical results for Example 1

$(m = 10)$				$(m = 40)$		
Nodes t_i	Errors of BPFs in [22]	Errors of GHFs in [23]	Errors of present method	Errors of BPFs in [22]	Errors of GHFs in [23]	Errors of Present method
0	$7.9e-5$	0	0	3.6e−5	0	0
0.1	5.4e−5	1.6e−5	8.9e−6	1.6e−5	1.3e−5	1.0e−5
0.2	1.4e−4	3.9e−5	1.5e−5	6.6e−5	3.4e−5	2.2e−5
0.3	1.1e−5	9.1e−5	3.8e−5	1.7e−6	8.2e−5	3.8e−5
0.4	1.9e−4	1.3e−4	3.7e−5	7.4e−5	1.2e−4	5.1e−5
0.5	7.2e−5	1.9e−4	6.7e−5	8.9e−5	1.7e−4	6.5e−5
0.6	7.1e−5	2.7e−4	6.2e−5	4.8e−5	2.5e−4	7.9e−5
0.7	1.0e−4	3.5e−4	9.0e−5	4.8e−5	3.2e−4	9.6e−5
0.8	9.7e−5	2.9e−4	8.2e−5	1.2e−4	2.5e−4	1.0e−4
0.9	5.7e−5	2.8e−4	1.0e−4	5.2e−5	2.4e−5	1.1e−4
1	1.0e−4	2.3e−4	9.4e−5	5.1e−3	1.9e−4	1.1e−4

j= 1, 2, ..., n.

By substituting (17) and (18) in relation (16), we obtain

$$\|X - X_m\| \le \Gamma(h) + T((K + \gamma(h))\|X - X_m\| + \rho\gamma(h))$$
$$+ \sum_{j=1}^{n} \|B_j\| \big((M_j + \lambda_j(h))\|X - X_m\| + \rho\lambda_j(h)\big),$$

and so

$$\|X - X_m\| \le \frac{\Gamma(h) + T\rho\gamma(h) + \rho \sum_{j=1}^{n} \lambda_j(h)\|B_j(t)\|}{1 - \left(T(K + \gamma(h)) + \sum_{j=1}^{n}(M_j + \lambda_j(h))\|B_j(t)\| \right)},$$

which means $\|X - X_m\| = O(h^3)$. Thus, the proof is complete. $\square$

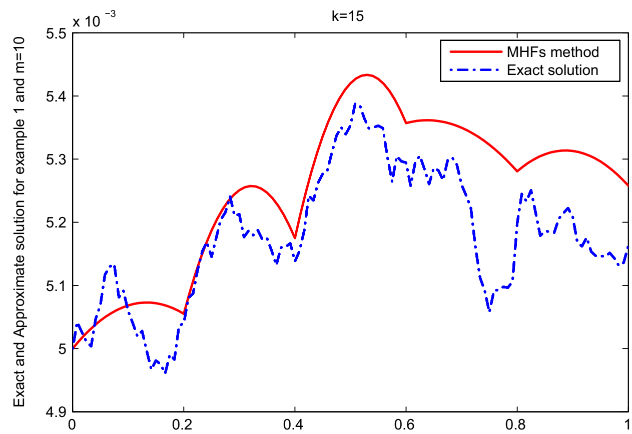

Fig. 1 Numerical results for Example 1 with $m = 10$

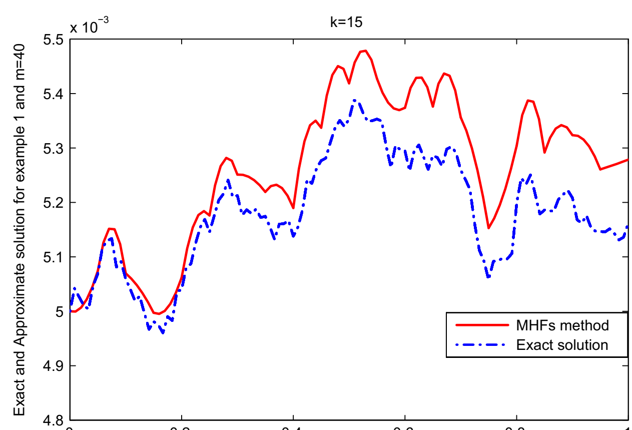

Fig. 2 Numerical results for Example 1 with $m = 40$

Numerical examples

In this section, we use our algorithm to solve stochastic Itô–Volterra integral equation with multi-stochastic terms stated in “Solving stochastic Itô–Volterra integral equation with multi-stochastic terms by the MHFs” section. In order to compare it with the method proposed in [22, 23], we consider some examples. The computations associated with the examples were performed using Matlab 7 and [36].

Example 1 Consider the following linear stochastic Itô–Volterra integral equation with multi-stochastic terms [22]

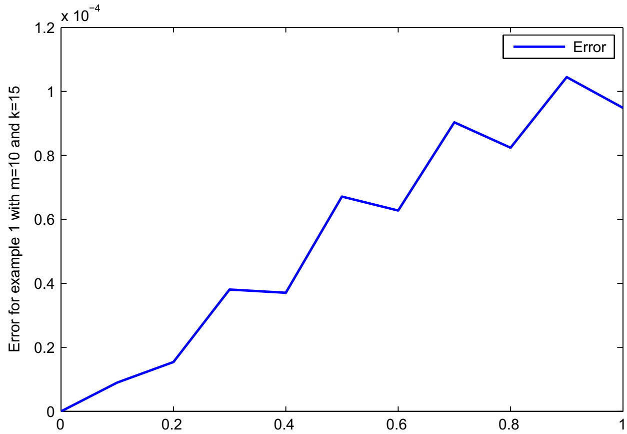

Fig. 3 Error curve of the method for Example 1 with $m = 10$

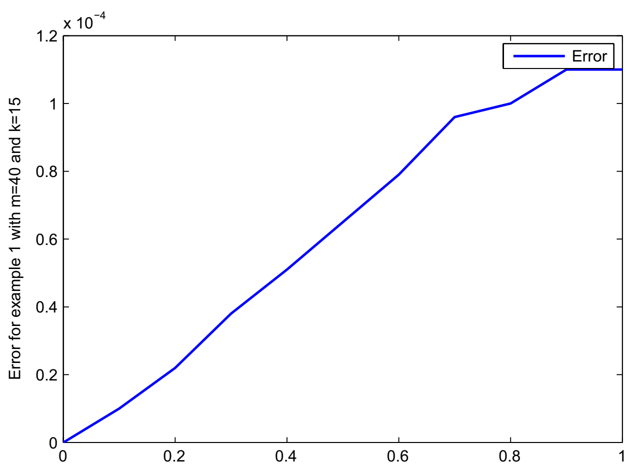

Fig. 4 Error curve of the method for Example 1 with $m = 40$

$$X(t) = X_0 + \int_0^t rX(s)\,\mathrm{d}s + \sum_{j=1}^{n} \int_0^t \alpha_j X(s) dB_j(s), \; s, t \in [0, 1),$$

with the exact solution

$$X(t) = X_0 e^{(r - \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2)t + \sum_{j=1}^{n} \alpha_j B_j(t)},$$

for $0 \le t < 1$ where X is the unknown stochastic process, defined on the probability space (Ω, F, P) and $B_1(t), B_2(t), \ldots, B_n(t)$ are the Brownian motion processes. The numerical results for $X_0 = \frac{1}{200}, r = \frac{1}{20}, \alpha_1 = \frac{1}{50}, \alpha_2 = \frac{2}{50}, \alpha_3 = \frac{4}{50}, \alpha_4 = \frac{9}{50}$ are shown in Table 1. Also curves in Figs. 1 and 2 show the exact and approximate solutions computed by this method for $m = 10$ and $m = 40$. Figures 3 and 4 represent the errors of the present method.

Example 2 Let [22]

$$X(t) = X_0 + \int_0^t r(s)X(s)\,\mathrm{d}s + \sum_{j=1}^{n} \int_0^t \alpha_j(s) X(s) dB_j(s), \; s, t \in [0, 1),$$

be a linear stochastic Itô–Volterra integral equation with multi-stochastic terms with the exact solution

Table 2 Numerical results for Example 2

$(m = 10)$				$(m = 40)$		
Nodes t_i	Errors of BPFs in [22]	Errors of GHFs in [23]	Errors of present method	Errors of BPFs in [22]	Errors of GHFs in [23]	Errors of Present method
0	$4.3e-4$	0	0	1.0e−4	0	0
0.1	7.5e−4	1.2e−4	1.8e−4	3.3e−4	1.2e−4	1.3e−4
0.2	9.5e−5	3.2e−4	1.7e−4	4.1e−4	2.2e−4	4.3e−5
0.3	9.5e−4	7.6e−4	4.4e−4	3.1e−4	4.7e−4	1.0e−4
0.4	3.7e−3	5.6e−3	1.2e−3	9.7e−4	2.6e−3	5.1e−4
0.5	4.2e−3	3.4e−2	2.2e−3	1.6e−3	1.1e−2	8.0e−4
0.6	1.1e−3	5.3e−3	3.1e−3	8.3e−4	1.5e−1	1.4e−3
0.7	1.5e−3	5.7e−2	3.9e−3	1.5e−3	3.5e−2	2.1e−3
0.8	4.2e−4	6.5e−3	9.3e−3	8.2e−3	2.6e−2	5.2e−3
0.9	2.4e−2	2.9e−2	8.8e−3	1.0e−2	1.0e−2	5.0e−3
1	1.6e−2	6.9e−2	1.6e−2	1.1e−2	6.3e−1	1.1e−2

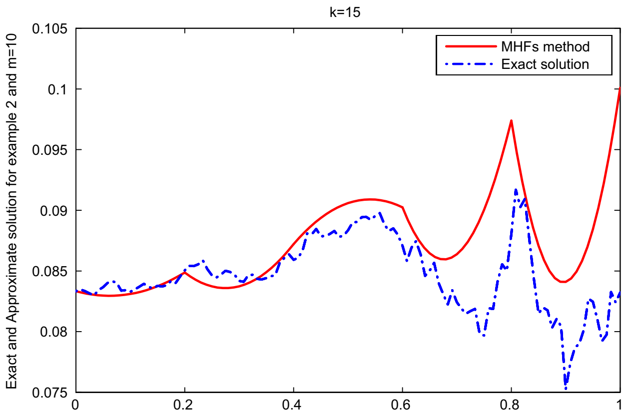

Fig. 5 Numerical results for Example 2 with $m = 10$

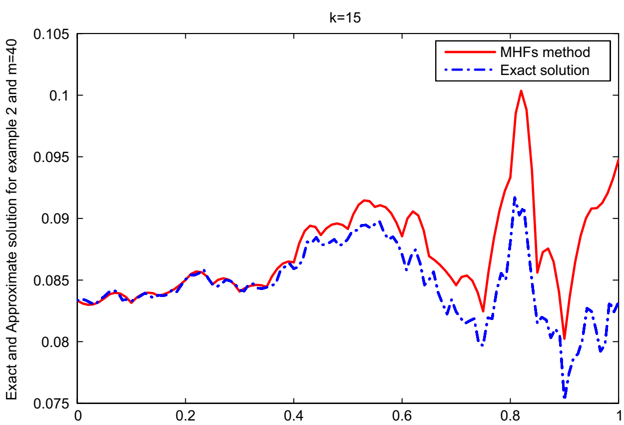

Fig. 6 Numerical results for Example 2 with $m = 40$

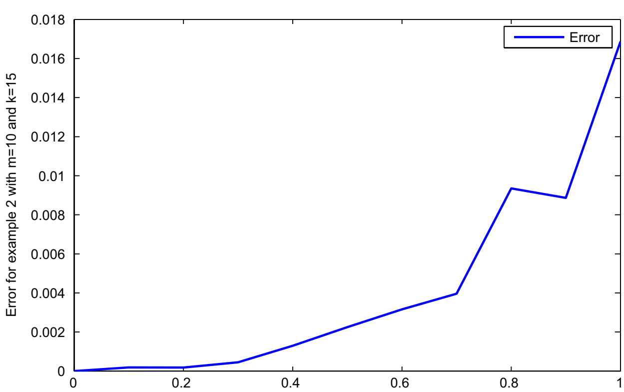

Fig. 7 Error curve of the method for Example 2 with $m = 10$

$$X(t) = X_0 e^{\left(\int_0^t r(s) - \frac{1}{2}\sum_{j=1}^n \alpha_j^2(s)\mathrm{d}s + \sum_{j=1}^n \int_0^t \alpha_j(s) dB_j(s)\right)}$$

for $0 \le t < 1$, where X is the unknown stochastic process defined on the probability space (Ω, F, P) and $B_1(t), B_2(t), \ldots, B_n(t)$ are the

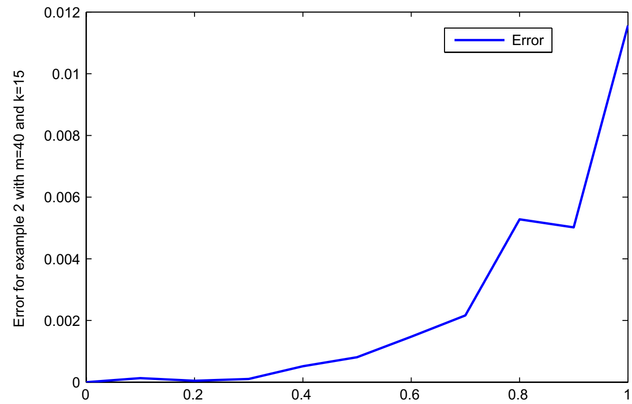

Fig. 8 Error curve of the method for Example 2 with $m = 40$

Brownian motion processes. The numerical results for $X_0 = \frac{1}{12}, r = \frac{1}{30}, \alpha_1 = \frac{1}{10}, \alpha_2(s) = s^2, \alpha_3(s) = \frac{\sin(s)}{3}$ are inserted in Table 2. Also curves in Figs. 5 and 6 show the exact and approximate solutions computed by this method for $m = 10$ and $m = 40$. Figures 7 and 8 represent the errors of the present method.

Conclusion

Finding an analytical exact solution for stochastic equations usually seems impossible. Therefore, it is convenient to use stochastic numerical methods to find some approximate solutions. The MHFs, as a simple and suitable basis, adopt to solve stochastic Itô–Volterra integral equations with multi-stochastic terms. With this choice, the vector and matrix coefficients are found easily. This method results in a linear system of equations that can be solved simply. Numerical results of the examples show that the MHFs tend to more accurate solutions than the BPFs and GHFs do.

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