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ORIGINAL RESEARCH

Using of Bernstein spectral Galerkin method for solving of weakly singular Volterra–Fredholm integral equations

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Abstract

In this work, we applied a new method for solving the linear weakly singular mixed Volterra–Fredholm integral equations. We now begin the theoretical study with acquirement of the variational form; in addition, we are using Bernstein spectral Galerkin method to be approximate to my problems. We estimate the error of the method by proved some theorems. Moreover, in the final section, we solved some numerical examples.

Keywords Linear weakly singular Volterra–Fredholm integral equation · Finite-element method · Error estimation

Mathematics Subject Classification 45D05 · 65L60

Introduction

One of the best subjects in the numerical analysis is a finite-element method (FEM). We used (FEM) to solve problems in mathematical physics, integral equations, and engineering, such as electromagnetic potential, fluid flow, radiation heats transfer, laminar boundary-layer theory and mass transport, Abel integral equations, and problem of mechanics or physics [3–5, 7, 18, 20]. For approximating of singular or weakly singular integral equations, there are several numerical method's existences. For example, Baratella and Orsi [8] were introduced weakly singular for Volterra, and discussed on operational matrix method with block-pulse functions by Babolian and Salimi [6]. Furthermore, some author works to be approximate to Abel integral equations, for example, Garza in [12] and Hall in [13] used the wavelet method, Legendre wavelets approximation by Yousefi in [19], Gauss–Jacobi quadrature rule by Fettis in [10], and Piessens and Verbaeten in [16, 17] with Chebyshev polynomials of the first kind.

In this paper, we use (FEM) and Bernstein polynomials to acquire an approximate solution for linear weakly singular mixed Volterra–Fredholm integral equation as follows:

$$u(x) = f(x) + \int_a^x \frac{W_1(x,t)}{(x-t)^{\alpha}} u(t)\mathrm{d}t + \int_a^b \frac{W_2(x,t)}{(x-t)^{\beta}} u(t)\mathrm{d}t, \quad 0 < \alpha,\ \beta < 1 \tag{1}$$

that $f(x)$, $W_1(x,t)$, and $W_2(x,t)$ are known continuous functions, and $u(x)$ is the unknown function.

Bernstein polynomials and their properties

On the interval $[a, b]$, the $m+1$ Bernstein basis polynomials (BPs) of degree m are defined as [1]

$$B_{i,m}(t) = \binom{m}{i} \frac{(t-a)^i (b-t)^{m-i}}{(b-a)^m}, \quad i = 0, 1, \ldots, m. \tag{2}$$

In addition, for $i < 0$ or $i > m$, we have

$$B_{i,m}(t) = 0.$$

The ten first Bernstein basis polynomials are

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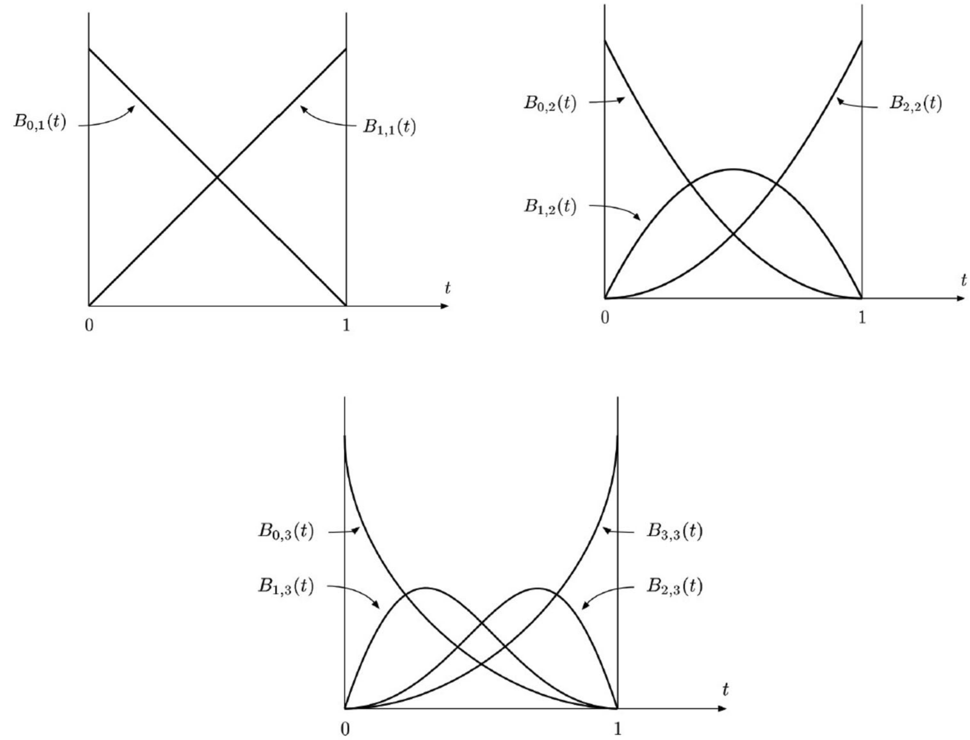

Fig. 1 Bernstein basis polynomials of degrees 1, 2, and 3

$$
\begin{aligned}
&B_{0,0}(t) = 1,\\
&B_{0,1}(t) = (1-t), \quad B_{1,1}(t) = t,\\
&B_{0,2}(t) = (1-t)^2, \quad B_{1,2}(t) = 2t(1-t), \quad B_{2,2}(t) = t^2,\\
&B_{0,3}(t) = (1-t)^3, \quad B_{1,3}(t) = 3t(1-t)^2, \quad B_{2,3}(t) = 3t^2(1-t), \quad B_{3,3}(t) = t^3.
\end{aligned}
$$

Therefore, we can be plotted this ten first Bernstein basis polynomials on the unit square as follows (Fig. 1):

Implementation of the Bernstein–Galerkin method

In this section with using of Galerkin method, we find an approximate solution for Eq. (1). For this purpose, we obtain weak and variational form.

If $\Omega = [a, b] \subset \mathbb{R}$ is an infinite dimensional space, and

$$\mathbb{V} = H^0(\Omega) = L_2(\Omega).$$

We let $B : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is a bilinear form and $L : \mathbb{V} \to \mathbb{R}$ is a linear functional; then for all arbitrary functions $v(x) \in \mathbb{V}$, we have $B(u, v) = L(v)$, where

$$
\begin{aligned}
B(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x - \int_\Omega v(x)\left(\int_a^x \frac{W_1(x,t)\,u(t)}{(x-t)^\alpha}\mathrm{d}t\right)\mathrm{d}x \\
- \int_\Omega v(x)\left(\int_a^b \frac{W_2(x,t)\,u(t)}{(x-t)^\beta}\mathrm{d}t\right)\mathrm{d}x,
\end{aligned}
\tag{3}
$$

and

$$L(v) = \int_\Omega f(x)v(x)\mathrm{d}x,$$

thus

$$
\begin{aligned}
B(\lambda_1 u + \lambda_2 w, v) = &\int_\Omega (\lambda_1 u(x) + \lambda_2 w(x))v(x)\mathrm{d}x \\
&- \int_\Omega v(x) \int_a^x \frac{W_1(x,t)(\lambda_1 u(t) + \lambda_2 w(t))}{(x-t)^\alpha}\mathrm{d}t\mathrm{d}x \\
&- \int_\Omega v(x) \int_a^b \frac{W_2(x,t)(\lambda_1 u(t) + \lambda_2 w(t))}{(x-t)^\beta}\mathrm{d}t\mathrm{d}x,
\end{aligned}
$$

then, B is a bilinear form:

$$B(\lambda_1 u + \lambda_2 w, v) = \lambda_1 B(u, v) + \lambda_2 B(w, v).$$

We consider

$$u_h(x) = \sum_{i=1}^n a_i \phi_i(x), \quad v_h(x) = \phi_j(x), \quad u_h(x), v_h(x) \in \mathbb{V}_h \tag{4}$$

that $V_h = span\{\phi_1, \phi_2, \ldots, \phi_n\}$ is a subspace of $\mathbb{V}$, and $\{\phi_i\}_{i=1}^n$ are a set of Bernstein polynomial functions of degree at most m in each subinterval. Hence, by substituting (4) in variational formulation, we have

$$\sum_{i=1}^{n} a_i \Big\{ \int_\Omega \phi_i(x)\phi_j(x)\mathrm{d}x - \int_\Omega \phi_j(x)\left(\int_0^x \frac{W_1(x,t)}{(x-t)^\alpha} \phi_i(t)\,\mathrm{d}t \right)\mathrm{d}x - \int_\Omega \phi_j(x)\left(\int_a^b \frac{W_2(x,t)}{(x-t)^\beta} \phi_i(t)\,\mathrm{d}t \right)\mathrm{d}x \Big\} - \int_\Omega g(x)\phi_j(x)\mathrm{d}x = 0. \tag{5}$$

Now, for $i, j = 1, 2, \ldots, n$, we define

$$C_{i,j} = \int_\Omega \phi_i(x)\phi_j(x)\mathrm{d}x - \int_\Omega \phi_j(x) \int_0^x \frac{W_1(x,t)}{(x-t)^\alpha} \phi_i(t)\,\mathrm{d}t\,\mathrm{d}x - \int_\Omega \phi_j(x) \int_a^b \frac{W_2(x,t)}{(x-t)^\beta} \phi_i(t)\,\mathrm{d}t\,\mathrm{d}x, \tag{6}$$

and

$$G_j = \int_\Omega g(x)\phi_j(x)\mathrm{d}x, \qquad j = 1, 2, \ldots, n \tag{7}$$

thus

$$\sum_{i=1}^{n} C_{ij} a_i = G_j, \qquad j = 1, 2, \ldots, n. \tag{8}$$

From system (8), we have

$$C^T \mathbf{A} = \mathbf{G} \tag{9}$$

that

$$\mathbf{A} = [a_1, a_2, \ldots, a_n]^T, \quad \mathbf{G} = [G_1, G_2, \ldots, G_n]^T, \quad C = [C_{ij}], \quad i, j = 1, 2, \ldots, n.$$

By solving of the system (9), we can obtain approximate solution of Eq. (1).

Error analysis

In this section, using the theorem, we get an upper bound for the error of our method, and we proved that the order of convergence is a $O(h^\zeta)$. For this purpose, suppose that $\mathbb{V}$ and B are a Hilbert space and symmetric, respectively.

Definition 1 If B is a $\mathbb{V}$-elliptic bilinear form, then an inner product energy is a $(.,.) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and the energy norm as

$$||u||_E^2 = (u,u)_B = B(u,u).$$

Definition 2 For operator $\Pi : \mathbb{V} \to \mathbb{V}_h$, projection operators as

$$\Pi u = \tilde{u}_h = \sum_{i=1}^{n} \tilde{a}_i \phi_i(x).$$

Theorem 1 *Let $\alpha > 0$, then bilinear form B, defined by (3) is a $\mathbb{V}$-ellipticity and Eq. (1) has a unique solution, and order of convergence is a $O(h^\zeta)$.*

Proof From Eq. (3), we have

$$|B(u,v)| = \left| \int_\Omega u(x)v(x)\mathrm{d}x - \int_\Omega v(x) \int_a^x \frac{W_1(x,t)u(t)}{(x-t)^\alpha}\mathrm{d}t\mathrm{d}x - \int_\Omega v(x) \int_a^b \frac{W_2(x,t)u(t)}{(x-t)^\beta}\mathrm{d}t\mathrm{d}x \right|,$$

with using of the Cauchy–Schwarz inequality and L_2-norm, we have

$$\begin{aligned}
|B(u,v)| &\le ||u||_{L_2(\Omega)}||v||_{L_2(\Omega)} + W_1\left| \int_a^b v(x) \int_a^x \frac{u(t)}{(x-t)^\alpha}\mathrm{d}t\mathrm{d}x \right| + W_2\left| \int_a^b v(x) \int_a^b \frac{u(t)}{(x-t)^\beta}\mathrm{d}t\mathrm{d}x \right| \\
&= ||u||_{L_2(\Omega)}||v||_{L_2(\Omega)} + W_1\left| \int_a^b v(x)u(\eta_x) \int_a^x \frac{1}{(x-t)^\alpha}\mathrm{d}t\mathrm{d}x \right| + W_2\left| \int_a^b v(x)u(\zeta_x) \int_a^b \frac{1}{(x-t)^\beta}\mathrm{d}t\mathrm{d}x \right| \\
&\le ||u||_{L_2(\Omega)}||v||_{L_2(\Omega)} + W_1 \left| \int_a^b v(x)u(\eta_x) \frac{1}{1-\alpha}(x-t)^{1-\alpha} \Big|_{t=a}^{t=x} \mathrm{d}x \right| + W_2\left| \int_a^b v(x)u(\zeta_x) \frac{1}{1-\beta}(x-t)^{1-\beta} \Big|_{t=a}^{t=b} \mathrm{d}x \right| \\
&\le ||u||_{L_2(\Omega)}||v||_{L_2(\Omega)} + \frac{W_1(b-a)^{1-\alpha}}{1-\alpha}\left| \int_a^b v(x)u(\eta_x)\mathrm{d}x \right| + \frac{W_2(b-a)^{1-\beta}}{1-\beta}\left| \int_a^b v(x)u(\zeta_x)\mathrm{d}x \right| \\
&\le \left(1 + \frac{W_1(b-a)^{1-\alpha}}{1-\alpha} + \frac{W_2(b-a)^{1-\beta}}{1-\beta} \right) ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)},
\end{aligned}$$

where

$$\begin{aligned}
W_1 &= \max|W_1(x,t)|, \quad x \in [a,b], \quad \text{and} \quad t \in [a,x], \\
W_2 &= \max|W_2(x,t)|, \quad x \in [a,b], \quad \text{and} \quad t \in [a,b],
\end{aligned} \tag{10}$$

then B is a continuous. Furthermore, we proved V-ellipticity of B; for this purpose, we have

$$B(v,v)=\int_{\Omega}v(x)v(x)\mathrm{d}x-\int_{\Omega}v(x)\int_{a}^{x}\frac{W_1(x,t)v(t)}{(x-t)^{\alpha}}\mathrm{d}t\mathrm{d}x$$
$$-\int_{\Omega}v(x)\int_{a}^{b}\frac{W_2(x,t)v(t)}{(x-t)^{\beta}}\mathrm{d}t\mathrm{d}x$$
$$\geq ||v||^2_{L_2(\Omega)}-W_1\left(\frac{(b-a)^{1-\alpha}}{1-\alpha}\right)||v||^2_{L_2}$$
$$-W_2\left(\frac{(b-a)^{1-\beta}}{1-\beta}\right)||v||^2_{L_2}=(\eta)||v||^2_{L_2(\Omega)}, \tag{11}$$

then

$$B(v,v)\geq(\eta)||v||^2_{L_2(\Omega)}, \tag{12}$$

where

$$\eta=1-W_1\left(\frac{(b-a)^{1-\alpha}}{1-\alpha}\right)-W_2\left(\frac{(b-a)^{1-\beta}}{1-\beta}\right),$$

thus B is a V-ellipticity; therefore, using of Lax–Milgram theorem and V-ellipticity of B, Eq. (1) has a unique solution. Suppose u_h is an approximate solution, so we have

$$B(u,v_h)=L(v_h), \qquad \forall v_h\in V_h, \tag{13}$$

and

$$B(u_h,v_h)=L(v_h), \quad \forall v_h\in V_h. \tag{14}$$

If $e=u-u_h$ that u are an exact solution of Eq. (1), then

$$B(e,v_h)=0, \quad \forall v_h\in V_h. \tag{15}$$

By Schwartz's inequality, and relation between energy norm and inner product, we have

$$|B(v,w)|\leq||v||_E||w||_E, \qquad \forall v,w\in V. \tag{16}$$

Using (15), we have

$$(e,v_h)_B=B(e,v_h)=0.$$

Therefore, e is an orthogonal for any v_h. Using

$$||u-u_h||_E=min\{||u-v_h||_E;v_h\in V_h\},$$

and Cea's Lemma [9], for each particular $\tilde{v}_h$ in V_h, we have

$$||u-v_h||_V\leq\frac{M}{\eta}\inf||u-v_h||_V, \quad v_h\in V_v,$$

where

$$M=\left(1+\frac{W_1(b-a)^{1-\alpha}}{1-\alpha}+\frac{W_2(b-a)^{1-\beta}}{1-\beta}\right).$$

Since

$$\inf||u-v_h||_V\leq||u-\tilde{v}_h||_V,$$

if $\tilde{v}_h$ is equal to $\tilde{u}_h$, then

$$||u-u_h||_V\leq\frac{M}{\eta}||u-\tilde{u}_h||_V.$$

If we get an upper bounded for the interpolation error, we have

$$||u-\tilde{u}_h||_V\leq ch^{\zeta}, \quad \zeta>0,$$

that c is not dependent of h; therefore

$$||u-u_h||_V\leq\frac{cM}{\eta}h^{\zeta}.$$

Thus, $h\to 0$, and the order of convergence is a $O(h^{\zeta})$, □

Numerical examples

Example 1 Consider the linear weakly singular Volterra–Fredholm integral equation:

$$u(x)-\int_0^x\frac{1}{(x-t)^{\frac{1}{4}}}u(t)\mathrm{d}t-\int_0^1\frac{1}{(x-t)^{\frac{1}{4}}}u(t)\mathrm{d}t=f(x),$$
$$0<x\leq 1$$

that

$$f(x)=\frac{1}{1155}\left(-512x^3-256x^2+144x+112\right)(x-1)^{\frac{3}{4}}$$
$$-x^3+x^2-\frac{256}{231}x^{\frac{11}{4}}+\frac{1024}{1155}x^{\frac{15}{4}},$$

and the exact solution is a $u(x)=x^2(1-x)$.

With using Bernstein basis polynomials of degree 2, and $M=5$, the results of obtained are presented in Table 1 and Fig. 2.

Example 2 In this example, we consider

$$u(x)-\int_0^x\frac{1}{(x-t)^{\frac{1}{3}}}u(t)\mathrm{d}t-\int_0^1\frac{1}{(x-t)^{\frac{1}{2}}}u(t)\mathrm{d}t=f(x),$$
$$0<x\leq 1$$

that

$$f(x)=x-\frac{9}{10}x^{\frac{5}{3}}-\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{3}\sqrt{x-1}+\frac{4}{3}x\sqrt{x-1}.$$

In addition, the exact solution is $u(x)=x$.

With using of Bernstein basis polynomials of degree 2, and $M=5$, the results of obtained are presented in Table 2 and Fig. 3.

Example 3 Consider the equation:

Table 1 Numerical results for Example 1

t	Exact solution	Present method	Finite-element method	RBF method
0.1	0.0009000	0.0091050	0.007818	0.016214
0.2	0.0320000	0.0318483	0.031339	0.039239
0.3	0.0630000	0.0630459	0.062834	0.069327
0.4	0.0960000	0.0959182	0.096335	0.100756
0.5	0.1250000	0.1250346	0.125836	0.127732
0.6	0.1440000	0.1439448	0.145334	0.144997
0.7	0.1470000	0.1470188	0.148841	0.147017
0.8	0.1280000	0.1280360	0.130317	0.127756
0.9	0.8100000	0.0809419	0.083896	0.082547
1	0.0000000	0.0005053	0.003101	0.010865

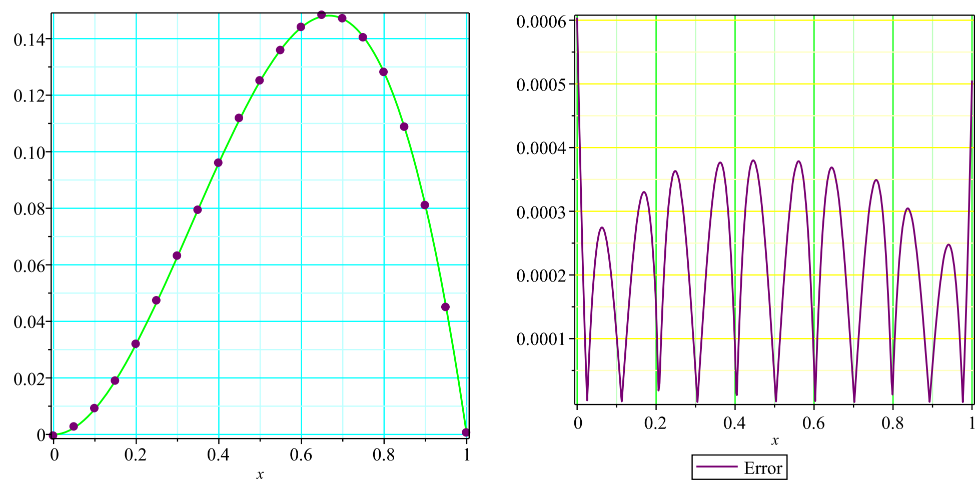

Fig. 2 Diagrams of exact and numerical solutions and graph of error for Example 1

Table 2 Numerical results for Example 2

t	Exact solution	Present method	Finite-element method	RBF method
0.1	0.1000000	0.0999999	0.0999997	0.0999999
0.2	0.2000000	0.2000000	0.1999999	0.1999999
0.3	0.3000000	0.3000000	0.2999999	0.2999999
0.4	0.4000000	0.4000000	0.3999999	0.3999998
0.5	0.5000000	0.5000000	0.4999999	0.4999995
0.6	0.6000000	0.6000000	0.5999999	0.5999991
0.7	0.7000000	0.7000000	0.6999999	0.6999983
0.8	0.8000000	0.7999999	0.7999999	0.7999964
0.9	0.9000000	0.9000000	0.9000000	0.8999921
1	1.0000000	1.0000000	0.9999998	0.9999835

$$u(x) - \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}} u(t)\mathrm{d}t - \int_0^1 \frac{1}{(x-t)^{\frac{1}{2}}} u(t)\mathrm{d}t = f(x),$$
$$0 < x \leq 1,$$

that

$$f(x) = \exp(x)\left(\mathrm{erf}(\sqrt{x-1})\sqrt{\pi} - 2\mathrm{erf}(\sqrt{x})\sqrt{\pi} + 1\right),$$

and the exact solution is $u(x) = \exp(x)$.

With using of Bernstein basis polynomials of degree 2, and $M = 5$, the results of obtained are presented in Table 3 and Fig. 4.

Conclusions

In this paper, we used of Galerkin method and Bernstein polynomials to solving one of the most important linear weakly singular Volterra–Fredholm integral equation, with

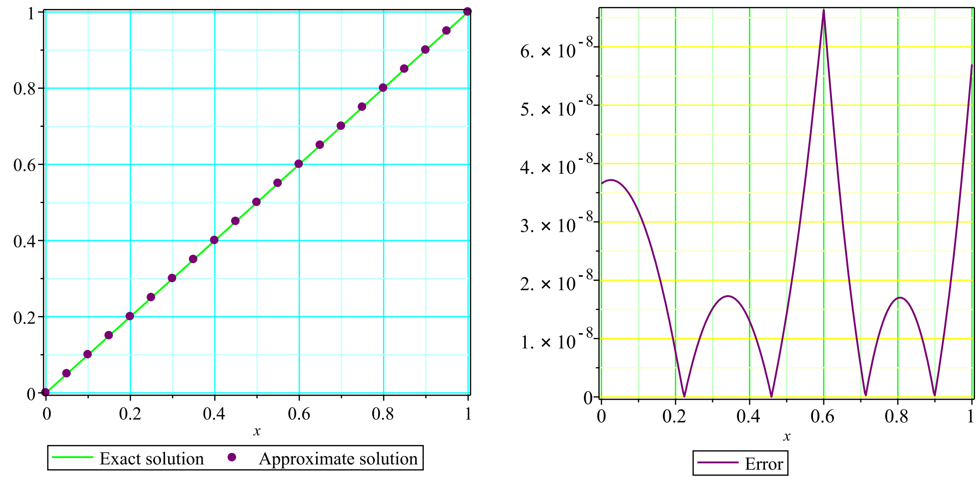

Fig. 3 Diagrams of exact and numerical solutions and graph of error for Example 2

Table 3 Numerical results for Example 3

t	Exact solution	Present method	Finite-element method	RBF method
0.1	1.1051709	1.1051108	1.1042527	1.1051602
0.2	1.2214027	1.2215436	1.2203756	1.2214104
0.3	1.3498588	1.3497954	1.3487213	1.3498715
0.4	1.4918246	1.4919761	1.4905784	1.4918240
0.5	1.6487212	1.6486511	1.6473242	1.6487057
0.6	1.8221188	1.8222743	1.8206197	1.8221042
0.7	2.0137527	2.0136890	2.0120126	2.0137483
0.8	2.2255409	2.2256379	2.2237996	2.2254962
0.9	2.4596031	2.4595848	2.4573309	2.4593229
1	2.7182818	2.7181290	2.7164838	2.7173060

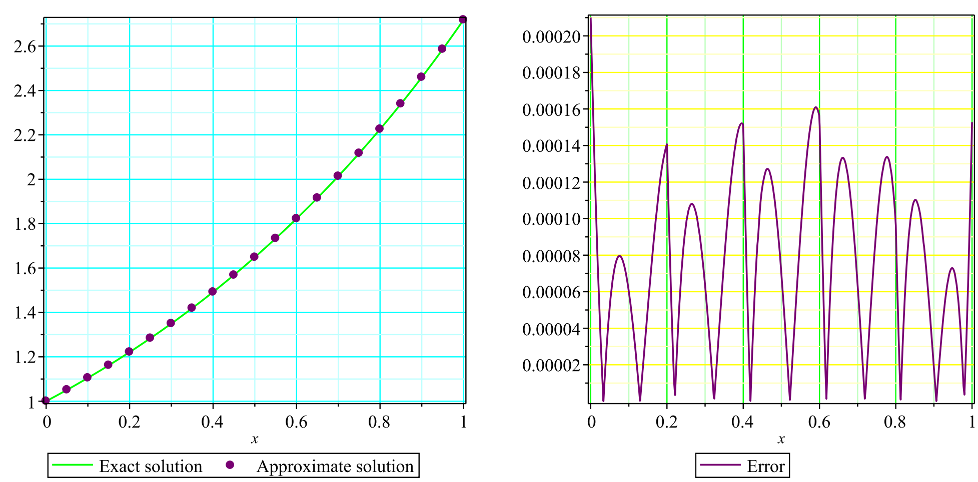

Fig. 4 Diagrams of exact and numerical solutions and graph of error for Example 3

using of Bernstein basis polynomials. We now begin the theoretical study with acquire of the variational form of Eq. (1), and with using of the system (9), we can obtain approximate solution. In section error analysis, we proved that B is a $\mathbb{V}$-ellipticity and Eq. (1) has a unique solution, and order of convergence is a $O(h^{\zeta})$. In section Numerical Examples, we have solved three problems considered, the results obtained are presented in Tables 1, 2, and 3, and Figs. 2, 3, and 4, the comparison of results confirms the better accuracy with this method.

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