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A Zygmund-type integral inequality for polynomials

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Abstract Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < 1$. Then it was proved by Hans and Lal (Anal Math 40:105–115, 2014) that

$$\left|z^s P^{(s)} + \beta \frac{n_s}{2^s} P(z)\right| \le \frac{n_s}{2}\left(\left|1 + \frac{\beta}{2^s}\right| + \left|\frac{\beta}{2^s}\right|\right)\max_{|z|=1}|P(z)|,$$

for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $1 \le s \le n$ and $|z| = 1$.
The L^γ analog of the above inequality was recently given by Gulzar (Anal Math 42:339–352, 2016) who under the same hypothesis proved

$$\left\{\int_0^{2\pi} \left|e^{is\theta} P^{(s)}(e^{i\theta}) + \beta \frac{n_s}{2^s} P(e^{i\theta})\right|^\gamma \mathrm{d}\theta\right\}^{\frac{1}{\gamma}}$$
$$\le n_s \left\{\int_0^{2\pi} \left|\left(1 + \frac{\beta}{2^s}\right)e^{i\alpha} + \frac{\beta}{2^s}\right|^\gamma \mathrm{d}\alpha\right\}^{\frac{1}{\gamma}} \frac{\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma \mathrm{d}\theta\right\}^{\frac{1}{\gamma}}}{\left\{\int_0^{2\pi} \left|1 + e^{i\alpha}\right|^\gamma \mathrm{d}\alpha\right\}^{\frac{1}{\gamma}}},$$

where $n_s = n(n-1)\ldots(n-s+1)$ and $0 \le \gamma < \infty$.
In this paper, we generalize this and some other related results.

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1 Introduction

Let $\mathbb{P}_n$ be the class of polynomials $P(z) = \sum_{v=0}^{n} a_v z^v$ of degree n and $P^{(s)}(z)$ be its sth derivative. For $P \in \mathbb{P}_n$, we have

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| \tag{1.1}$$

and for every $\gamma \ge 1$,

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$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\right\}^{\frac{1}{\gamma}} \le n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\right\}^{\frac{1}{\gamma}}. \tag{1.2}$$

The inequality (1.1) is a classical result of Bernstein [10], whereas the inequality (1.2) is due to Zygmund [13] who proved it for all trigonometric polynomials of degree n and not only for those of the form $P(e^{i\theta})$. Arestov [1] proved that (1.2) remains true for $0 < \gamma < 1$ as well. If we let $\gamma \to \infty$ in (1.2), we get (1.1).

The above two inequalities (1.1) and (1.2) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be, respectively, replaced by

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\max_{|z|=1}|P(z)| \tag{1.3}$$

and

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\right\}^{\frac{1}{\gamma}} \le nC_\gamma\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\right\}^{\frac{1}{\gamma}}, \tag{1.4}$$

where

$$C_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\alpha}\right|^{\gamma}\mathrm{d}\alpha\right\}^{\frac{-1}{\gamma}}. \tag{1.5}$$

The inequality (1.3) was conjectured by Erdös and later proved by Lax [9], whereas (1.4) was proved by De-Bruijn [4] for $\gamma \ge 1$. Further, Rahman and Schmeisser [11] have shown that (1.4) holds for $0 < \gamma < 1$ as well. If we let $\gamma \to \infty$ in inequality (1.4), we get (1.3).

As an extension of (1.3), Jain [7] proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right| \le \frac{n}{2}\left(\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right)\max_{|z|=1}|P(z)|, \tag{1.6}$$

for $|z| = 1$ and for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$.

In 2000, Jain [8] further improved (1.6) by obtaining under the same hypothesis that

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right| \le \frac{n}{2}\left\{\left(\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right)\max_{|z|=1}|P(z)| - \left(\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|P(z)|\right\}, \tag{1.7}$$

for $|z| = 1$ and for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$.

Recently, Hans and Lal [6] generalized (1.6) and (1.7) for the sth derivative of polynomials and proved the following results.

Theorem A *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $1 \le s \le n$ and $|z| = 1$,*

$$\left|z^sP^{(s)}(z)+\beta\frac{n_s}{2}P(z)\right| \le \frac{n_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|+\left|\frac{\beta}{2^s}\right|\right)\max_{|z|=1}|P(z)|. \tag{1.8}$$

Theorem B *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $1 \le s \le n$ and $|z| = 1$,*

$$\left|z^sP^{(s)}(z)+\beta\frac{n_s}{2}P(z)\right| \le \frac{n_s}{2}\left\{\left(\left|1+\frac{\beta}{2^s}\right|+\left|\frac{\beta}{2^s}\right|\right)\max_{|z|=1}|P(z)| - \left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\min_{|z|=1}|P(z)|\right\}. \tag{1.9}$$

The above inequalities (1.8) and (1.9) were further improved and generalized by Zireh [12]. More recently, Gulzar [5] obtained an L^γ analogue of Theorem A by proving the following result.

Theorem C *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$ and $0 \leq \gamma < \infty$,*

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \beta \frac{n_s}{2^s} P(e^{i\theta}) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}} \leq n_s E_\gamma \left\{ \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2^s}\right) e^{i\alpha} + \frac{\beta}{2^s} \right|^\gamma \mathrm{d}\alpha \right\}^{\frac{1}{\gamma}} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}, \tag{1.10}$$

where

$$E_\gamma = \left\{ \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma \mathrm{d}\alpha \right\}^{\frac{-1}{\gamma}}. \tag{1.11}$$

The result is best possible and equality in (1.10) holds for $P(z) = az^n + b$ with $|a| = |b| = 1$.

2 Main results

The main aim of this paper is to prove an L^γ analog of Theorem B and thereby to obtain a generalization of Theorem C. More precisely, we prove

Theorem 2.1 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for any $\beta, \delta \in \mathbb{C}$ with $|\beta| \leq 1, |\delta| \leq 1, 1 \leq s \leq n$ and $0 \leq \gamma < \infty$,*

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \beta \frac{n_s}{2^s} P(e^{i\theta}) + \delta m \frac{n_s}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}} \leq n_s E_\gamma \left\{ \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2^s}\right) e^{i\alpha} + \frac{\beta}{2^s} \right|^\gamma \mathrm{d}\alpha \right\}^{\frac{1}{\gamma}} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}, \tag{2.1}$$

where here and throughout $m = \min_{|z|=1} |P(z)|$ and E_γ is defined by (1.11).

The result is best possible and equality in (2.1) holds for $P(z) = az^n + b$ with $|a| = |b| = 1$.

Now, we present and discuss some consequences of this result. First, we point out that inequalities involving polynomials in the Chebyshev norm on the unit circle in the complex plane are a special case of the polynomial inequalities involving the integral norm. For example, if we let $\gamma \to \infty$ in (2.1) and choose the argument of δ suitably with $|\delta| = 1$, we get (1.9).

Remark 2.2 For $\delta = 0$, Theorem 2.1 reduces to Theorem C. If we take $s = 1$ in (2.1), we get the following result which provides an L^γ analogue of (1.7).

Corollary 2.3 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta, \delta \in \mathbb{C}$ with $|\beta| \leq 1, |\delta| \leq 1$ and $0 \leq \gamma < \infty$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} P'(e^{i\theta}) + \frac{\beta n}{2} P(e^{i\theta}) + \frac{\delta m n}{2} \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}} \leq n E_\gamma \left\{ \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2}\right) e^{i\alpha} + \frac{\beta}{2} \right|^\gamma \mathrm{d}\alpha \right\}^{\frac{1}{\gamma}} \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}, \tag{2.2}$$

where E_γ is defined by (1.11).

Remark 2.4 Inequality (1.7) can be obtained by letting $\gamma \to \infty$ and by choosing the argument of δ suitably with $|\delta| = 1$ in (2.2).

Several other interesting results easily follow from Theorem 2.1. Here, we mention a few of these. Taking $\beta = 0$ in (2.1), we immediately get the following result.

Corollary 2.5 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\delta| \leq 1$, $1 \leq s \leq n$, and $0 \leq \gamma < \infty$,*

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\delta m n_s}{2} \right|^{\gamma} \mathrm{d}\theta \right\}^{\frac{1}{\gamma}} \leq n_s C_\gamma \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^{\gamma} \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}, \tag{2.3}$$

where C_γ is defined in (1.5).

For $s = 1$ and $\delta = 0$, inequality (2.3) reduces to inequality (1.4). The following corollary which is a refinement as well as a generalization of (1.3) is obtained by letting $\gamma \to \infty$ and by choosing the argument of δ with $|\delta| = 1$ suitably in (2.3).

Corollary 2.6 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $1 \leq s \leq n$, we have*

$$\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n_s}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}.$$

Remark 2.7 For $s = 1$, Corollary 2.6 reduces to a result of Aziz and Dawood [2].

For the proof of Theorem 2.1, we need the following lemmas.

3 Lemmas

Lemma 3.1 *Let $F \in \mathbb{P}_n$ and $F(z)$ has all its zeros in $|z| \leq 1$. If $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \ \textit{for}\ |z| = 1,$$

then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\left| z^s P^{(s)}(z) + \frac{\beta n_s}{2^s} P(z) \right| \leq \left| z^s F^{(s)}(z) + \frac{\beta n_s}{2^s} F(z) \right|, \ \textit{for}\ |z| \geq 1.$$

The above lemma is due to Hans and Lal [6].

By applying Lemma 3.1 to polynomials $P(z)$ and $z^n \min_{|z|=1} |P(z)|$, we get the following result.

Lemma 3.2 *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $1 \leq s \leq n$,*

$$\left| z^s P^{(s)}(z) + \frac{\beta n_s}{2^s} P(z) \right| \geq n_s |z|^n \left| 1 + \frac{\beta}{2^s} \right| \min_{|z|=1} |P(z)|, \quad \textit{for}\ |z| \geq 1.$$

Lemma 3.3 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $1 \leq s \leq n$ and $|z| = 1$,*

$$\left| z^s P^{(s)}(z) + \frac{\beta n_s}{2^s} P(z) \right| \leq \left| z^s Q^{(s)}(z) + \frac{\beta n_s}{2^s} Q(z) \right| - n_s \left\{ \left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right\} m,$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Proof of Lemma 3.3. If $P(z)$ has a zero on $|z| = 1$, then $m = 0$ and the result follows by Lemma 3.1. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| > 1$ and so $m > 0$, we have $|\lambda m| < |P(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. It follows by Rouche's theorem that the polynomial $G(z) = P(z) - \lambda m$ has no zeros in $|z| < 1$. Therefore, the polynomial $H(z) = z^n \overline{G(\frac{1}{\bar{z}})} = Q(z) - m\bar{\lambda} z^n$ will have all its zeros in $|z| \leq 1$. Also $|G(z)| = |H(z)|$ for $|z| = 1$. On applying Lemma 3.1, we get for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $1 \leq s \leq n$ and $|z| \geq 1$,

$$\left| z^s G^{(s)}(z) + \frac{\beta n_s}{2^s} G(z) \right| \leq \left| z^s H^{(s)}(z) + \frac{\beta n_s}{2^s} H(z) \right|.$$

Equivalently

$$\begin{aligned}&\left|z^s P^{(s)}(z)+\frac{\beta n_s}{2^s}\big(P(z)-\lambda m\big)\right|\\&\quad\le\left|\Big(z^s Q^{(s)}(z)-\bar{\lambda} m n_s z^n\Big)+\frac{\beta n_s}{2^s}\big(Q(z)-\bar{\lambda} m z^n\big)\right|.\end{aligned}$$

This implies that

$$\begin{aligned}&\left|\Big(z^s P^{(s)}(z)+\frac{\beta n_s}{2^s}P(z)\Big)-\frac{\beta n_s}{2^s}\lambda m\right|\\&\quad\le\left|\Big(z^s Q^{(s)}(z)+\frac{\beta n_s}{2^s}Q(z)\Big)-\bar{\lambda} m n_s z^n\Big(1+\frac{\beta}{2^s}\Big)\right|.\end{aligned}\tag{3.1}$$

Since $Q(z)$ has all its zeros in $|z|\le 1$, therefore, by Lemma 3.2 we have for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$ and $|z|\ge 1$,

$$\begin{aligned}\left|z^s Q^{(s)}(z)+\frac{\beta n_s}{2^s}Q(z)\right|&\ge n_s|z|^n\left|1+\frac{\beta}{2^s}\right|\min_{|z|=1}|Q(z)|\\&=n_s|z|^n\left|1+\frac{\beta}{2^s}\right|m.\end{aligned}\tag{3.2}$$

Now choosing a suitable argument of λ in the right-hand side of (3.1), in view of (3.2), we get for $|z|=1$,

$$\begin{aligned}&\left|z^s P^{(s)}(z)+\frac{\beta n_s}{2^s}P(z)\right|-|\lambda| n_s\left|\frac{\beta}{2^s}\right|m\\&\quad\le\left|z^s Q^{(s)}(z)+\frac{\beta n_s}{2^s}Q(z)\right|-n_s\left|1+\frac{\beta}{2^s}\right||\lambda| m.\end{aligned}$$

Equivalently

$$\begin{aligned}&\left|z^s P^{(s)}(z)+\frac{\beta n_s}{2^s}P(z)\right|\\&\quad\le\left|z^s Q^{(s)}(z)+\frac{\beta n_s}{2^s}Q(z)\right|-n_s\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)|\lambda| m.\end{aligned}$$

Letting $|\lambda|\to 1$, we get for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}&\left|z^s P^{(s)}(z)+\frac{\beta n_s}{2^s}P(z)\right|\\&\quad\le\left|z^s Q^{(s)}(z)+\frac{\beta n_s}{2^s}Q(z)\right|-n_s\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)m,\end{aligned}$$

which completes the proof of the lemma. □

The following lemma is due to Aziz and Shah [3].

Lemma 3.4 *If A, B, C are non-negative real numbers such that $B+C\le A$, then for every real number α,*

$$\left|(A-C)+e^{i\alpha}(B+C)\right|\le\left|A+e^{i\alpha}B\right|.$$

Lemma 3.5 *If $P\in\mathbb{P}_n$ and $P(z)\ne 0$ in $|z|<1$ and $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ then, for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$, α real, $1\le s\le n$ and $\gamma>0$,*

$$\int_0^{2\pi}\left|\left(e^{is\theta}P^{(s)}(e^{i\theta})+\beta\frac{n_s}{2^s}P(e^{i\theta})\right)e^{i\alpha}+e^{in\theta}\overline{\left(e^{is\theta}Q^{(s)}(e^{i\theta})+\beta\frac{n_s}{2^s}Q(e^{i\theta})\right)}\right|\mathrm{d}\theta$$
$$\le n_s^{\gamma}\left|\left(1+\frac{\beta}{2^s}\right)e^{i\alpha}+\frac{\overline{\beta}}{2^s}\right|^{\gamma}\int_0^{2\pi}\left|P(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta.$$

The above lemma is due to Gulzar [5].

Lemma 3.6 *Let $a,\ b\in\mathbb{C}$ with $|b|\ge|a|$. Then for $r>0$ and γ real, we have*

$$\int_0^{2\pi}\left|a+e^{i\gamma}b\right|^r\mathrm{d}\gamma\ge|a|^r\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r\mathrm{d}\gamma. \tag{3.3}$$

Proof of Lemma 3.6. If $a=0$, then (3.3) is obvious. Henceforth, we assume that $a\ne0$. Now for every real γ and $t\ge1$, it can be easily verified that

$$|t+e^{i\gamma}|\ge|1+e^{i\gamma}|. \tag{3.4}$$

Hence, using (3.4), we get

$$\begin{aligned}\int_0^{2\pi}\left|1+e^{i\gamma}\frac{b}{a}\right|^r\mathrm{d}\gamma&=\int_0^{2\pi}\left|1+e^{i\gamma}\left|\frac{b}{a}\right|\right|^r\mathrm{d}\gamma\\&=\int_0^{2\pi}\left|\left|\frac{b}{a}\right|+e^{i\gamma}\right|^r\mathrm{d}\gamma\\&\ge\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r\mathrm{d}\gamma,\end{aligned}$$

which is equivalent to (3.3) and this completes the proof of the lemma. □

4 Proof of the Theorem

Proof of Theorem 2.1. Since $P(z)\ne0$ in $|z|<1$, therefore, by Lemma 3.3, for each θ, $0\le\theta<2\pi$, $\beta\in\mathbb{C}$ with $|\beta|\le1$ and $1\le s\le n$,

$$\begin{aligned}&\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|+\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\\&\quad\le\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|-\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right).\end{aligned} \tag{4.1}$$

Taking

$$A=\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|,$$
$$B=\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|$$

and

$$C=\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)$$

in Lemma 3.4, so that by (4.1),

$$B+C\le A-C\le A,$$

we get for every real α,

$$\left|\left\{\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|+\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\right\}e^{i\alpha}\right.$$
$$\left.+\left\{\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|-\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\right\}\right|$$
$$\leq\left|\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|e^{i\alpha}+\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|\right|.$$

This implies for each $\gamma>0$,

$$\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma}\mathrm{d}\theta\leq\int_0^{2\pi}\left|\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|e^{i\alpha}\right.$$
$$\left.+\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|\right|^{\gamma}\mathrm{d}\theta, \tag{4.2}$$

where

$$G(\theta)=\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|+\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)$$

and

$$F(\theta)=\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|-\frac{mn_s}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right).$$

Integrating both sides of(4.2) with respect to α from 0 to 2π, we get with the help of Lemma 3.5, for each $\gamma>0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma}\mathrm{d}\theta\mathrm{d}\alpha\leq\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|e^{i\alpha}\right.\right.$$
$$\left.\left.+\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}Q(e^{i\theta})\right|\right|^{\gamma}\mathrm{d}\alpha\right\}\mathrm{d}\theta$$
$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right|e^{i\alpha}\right.\right.$$
$$\left.\left.+\left|e^{in\theta}\overline{\left(e^{is\theta}Q^{(s)}(e^{i\theta})+\beta\frac{n_s}{2^s}Q(e^{i\theta})\right)}\right|\right|^{\gamma}\mathrm{d}\alpha\right\}\mathrm{d}\theta$$
$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n_s}{2^s}P(e^{i\theta})\right)e^{i\alpha}\right.\right.$$
$$\left.\left.+\ e^{in\theta}\overline{\left(e^{is\theta}Q^{(s)}(e^{i\theta})+\beta\frac{n_s}{2^s}Q(e^{i\theta})\right)}\right|^{\gamma}\mathrm{d}\theta\right\}\mathrm{d}\alpha$$
$$\leq n_s^{\gamma}\int_0^{2\pi}\left|\left(1+\frac{\beta}{2^s}\right)e^{i\alpha}+\frac{\overline{\beta}}{2^s}\right|^{\gamma}\mathrm{d}\alpha\int_0^{2\pi}\left|P(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta. \tag{4.3}$$

Since

$$\int_0^{2\pi}\left|\left(1+\frac{\beta}{2^s}\right)e^{i\alpha}+\frac{\overline{\beta}}{2^s}\right|^{\gamma}\mathrm{d}\alpha=\int_0^{2\pi}\left|\left|1+\frac{\beta}{2^s}\right|e^{i\alpha}+\left|\frac{\overline{\beta}}{2^s}\right|\right|^{\gamma}\mathrm{d}\alpha$$

$$= \int_0^{2\pi} \left| \left| 1 + \frac{\beta}{2^s} \right| e^{i\alpha} + \left| \frac{\beta}{2^s} \right| \right|^\gamma \mathrm{d}\alpha$$

$$= \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2^s} \right) e^{i\alpha} + \frac{\beta}{2^s} \right|^\gamma \mathrm{d}\alpha.$$

Using this in (4.3), we get for each $\gamma > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^\gamma \mathrm{d}\theta \mathrm{d}\alpha \le n_s^\gamma \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2^s} \right) e^{i\alpha} + \frac{\beta}{2^s} \right|^\gamma \mathrm{d}\alpha \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma \mathrm{d}\theta. \tag{4.4}$$

If we take

$$a = G(\theta),$$
$$b = F(\theta),$$

because $|b| \ge |a|$ from (4.1), we get from Lemma 3.6, that for each $\gamma > 0$

$$\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^\gamma \mathrm{d}\alpha \ge |G(\theta)|^\gamma \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma \mathrm{d}\alpha. \tag{4.5}$$

Integrating both sides of (4.5) with respect to θ from 0 to 2π, we get from (4.4), that for each $\gamma > 0$,

$$\left\{ \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma \mathrm{d}\alpha \int_0^{2\pi} \left(\left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n_s}{2^s} P(e^{i\theta}) \right| + \frac{m n_s}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right)^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}$$
$$\le n_s \left\{ \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2^s} \right) e^{i\alpha} + \frac{\beta}{2^s} \right|^\gamma \mathrm{d}\alpha \right\}^{\frac{1}{\gamma}} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}. \tag{4.6}$$

Now using the fact that for every $\delta \in \mathbb{C}$ with $|\delta| \le 1$,

$$\left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n_s}{2^s} P(e^{i\theta}) + \frac{\delta m n_s}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right|$$
$$\le \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n_s}{2^s} P(e^{i\theta}) \right| + \frac{m n_s}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right),$$

we get from (4.6) that for every $\gamma > 0$,

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n_s}{2^s} P(e^{i\theta}) + \frac{\delta m n_s}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}$$
$$\le n_s \left\{ \int_0^{2\pi} \left| \left(1 + \frac{\beta}{2^s} \right) e^{i\alpha} + \frac{\beta}{2^s} \right|^\gamma \mathrm{d}\alpha \right\}^{\frac{1}{\gamma}} \frac{\left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma \mathrm{d}\theta \right\}^{\frac{1}{\gamma}}}{\left\{ \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma \mathrm{d}\alpha \right\}^{\frac{1}{\gamma}}},$$

which proves Theorem 2.1 for $\gamma > 0$. To establish this result for $\gamma = 0$, we simply make $\gamma \to 0_+$. □

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