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Fractional integral inequalities for generalized-m-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings via an extended generalized Mittag–Leffler function

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Abstract The authors discover a new identity concerning differentiable mappings defined on **m**-invex set via general fractional integrals. Using the obtained identity as an auxiliary result, some fractional integral inequalities for generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings by involving an extended generalized Mittag–Leffler function are presented. It is pointed out that some new special cases can be deduced from main results. Also these inequalities have some connections with known integral inequalities. At the end, some applications to special means for different positive real numbers are provided as well.

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1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality (1.1) is also known as trapezium inequality.

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1.1) through various classes of convex functions interested readers are referred to [2,4–23,25–31,33–36,41,44,45].

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Recently in [1], Andriĉ et al. defined an extended generalized Mittag–Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\cdot\,;p)$ as follows.

Definition 1.2 Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined by

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{+\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.2}$$

where β_p is the generalized beta function defined by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\,\mathrm{d}t \tag{1.3}$$

and $(c)_{nk}$ is the Pochhammer symbol defined as $\dfrac{\Gamma(c+nk)}{\Gamma(c)}$.

In [1], properties of the extended generalized Mittag–Leffler function are studied in details, and it is given that $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is absolutely convergent for $k < \delta + \Re(\mu)$. For other recent results see [15,37,38,46]. Let S be the sum of series of absolute terms of $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$, then we have $\left|E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)\right| \leq S$. We will use this property of extended generalized Mittag–Leffler function in sequel.

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.3 [43] A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in S$ for every $x, y \in S$ and $t \in [0,1]$.

The invex set S is also termed an η-connected set.

Definition 1.4 [27] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non–negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.4}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 1.4, f becomes a preinvex function [32]. If the mapping $\eta(y,x) = y - x$ in Definition 1.4, then the non–negative function f reduces to h-convex mappings [40].

Definition 1.5 [42] Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s \in (0,1]$, if for every $x, y \in S$ and $t \in [0,1]$,

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^s f(x) + t^s f(y). \tag{1.5}$$

Definition 1.6 [30] A function $f : K \longrightarrow \mathbb{R}$ is said to be s-Godunova-Levin-Dragomir-preinvex of second kind, if

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^{-s} f(x) + t^{-s} f(y), \tag{1.6}$$

for each $x, y \in K$, $t \in (0,1)$ and $s \in (0,1]$.

Definition 1.7 [39] A non–negative function $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be tgs-convex on K if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.7}$$

grips for all $x, y \in K$ and $t \in (0,1)$.

Definition 1.8 [24] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex, if it is non–negative and $\forall\, x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.8}$$

The concept of η-convex functions (at the beginning was named by φ-convex functions), considered in [14], has been introduced as the following.

Definition 1.9 Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \longrightarrow \mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$ is called convex with respect to η (briefly η-convex), if

$$f\big(\lambda x + (1-\lambda)y\big) \leq f(y) + \lambda\eta(f(x), f(y)), \tag{1.9}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. For more results about η-convex functions, see [8,9,13,14].

Definition 1.10 [2] Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \longrightarrow \mathbb{R}$. Consider $f : I \longrightarrow \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \longrightarrow \mathbb{R}$. The function f is said to be (η_1, η_2)-convex if

$$f\big(x + \lambda\eta_1(y, x)\big) \leq f(x) + \lambda\eta_2(f(y), f(x)), \tag{1.10}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Motivated by the above literatures, the main objective of this paper is to establish in Sect. 2, some new fractional integral inequalities for generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings by involving an extended generalized Mittag–Leffler function. It is pointed out that some new special cases will be deduced from main results. Also we will see that these inequalities have some connections with known integral inequalities. In Sect. 3, some applications to special means for different positive real numbers will be given.

2 Main results

The following definitions will be used in this section.

Definition 2.1 Let $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as **m**-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi\eta(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0, 1]$.

Remark 2.2 In Definition 2.1, under certain conditions, the mapping $\eta(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0, 1]$ could reduce to $\eta(y, mx)$. For example when $\mathbf{m}(t) = m$ for all $t \in [0, 1]$, then the **m**-invex set degenerates an m-invex set on K. In addition, taking $\mathbf{m}(t) \equiv 1$, $\forall t \in [0, 1]$ in Definition 2.1, then we get Definition 1.3.

We next introduce the concept of generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings.

Definition 2.3 Let $K \subseteq \mathbb{R}$ be an closed **m**-invex set with respect to the mapping $\eta_1 : K \times K \longrightarrow \mathbb{R}$ and $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\theta : I \longrightarrow \mathbb{R}$ are continuous. Consider $f : K \longrightarrow (0, +\infty)$ and $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. The mapping f is said to be generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex if

$$\begin{aligned} &f\big(\mathbf{m}(t)\theta(x) + \xi\eta_1(\theta(y), \mathbf{m}(t)\theta(x))\big) \\ &\quad\leq \big[\mathbf{m}(\xi)h_1^p(\xi)f^r(x) + h_2^q(\xi)\eta_2(f^r(y), f^r(x))\big]^{\frac{1}{r}}, \end{aligned} \tag{2.1}$$

holds for all $x, y \in I$, $r \neq 0$, $t, \xi \in [0, 1]$ and any fixed $p, q > -1$.

Remark 2.4 In Definition 2.3, if we choose $\mathbf{m}(t) = m$, $\forall t \in [0, 1]$ and $p = q = 1$, then we get Definition 2.3 in [19]. In addition, in Definition 2.3, if we choose $\mathbf{m} = p = q = r = 1$ and $\theta(x) = x$, $\forall x \in I$ then we get Definition 1.10.

Remark 2.5 In Definition 2.3, if we choose $\mathbf{m} = p = q = r = 1$, $h_1(t) = 1$, $h_2(t) = t$, $\eta_1(\theta(y), \mathbf{m}(t)\theta(x)) = \theta(y) - \mathbf{m}(t)\theta(x)$, $\eta_2(f^r(y), f^r(x)) = \eta(f^r(y), f^r(x))$ and $\theta(x) = x$, $\forall x \in I$, then we get Definition 1.9. Also, in Definition 2.3, if we choose $\mathbf{m} = p = q = r = 1$, $h_1(t) = 1$, $h_2(t) = t$ and $\theta(x) = x$, $\forall x \in I$, then we get Definition 1.10. Under some suitable choices as we done above, we can get also the Definitions 1.5 and 1.6.

Remark 2.6 Let us discuss some special cases in Definition 2.3 as follows.

(I) Taking $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we get generalized-**m**-$((h^p(1-t), h^q(t)); (\eta_1, \eta_2))$-convex mappings.
(II) Taking $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0, 1]$, then we get generalized-**m**-$(((1-t)^{sp}, t^{sq}); (\eta_1, \eta_2))$-Breckner-convex mappings.
(III) Taking $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized-**m**-$(((1-t)^{-sp}, t^{-sq}); (\eta_1, \eta_2))$-Godunova–Levin–Dragomir-convex mappings.
(IV) Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized-**m**-$((t(1-t))^{sp}, (t(1-t))^{sq}); (\eta_1, \eta_2))$-convex mappings.
(V) Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized-**m** $\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^p, \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^q\right);\right.$ $\left.(\eta_1, \eta_2)\right)$-convex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Let see the following example of a generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mapping which is not convex.

Example 2.7 Let take $\mathbf{m} = r = \frac{1}{2}$, $h_1(t) = t^l$, $h_2(t) = t^s$ for all $l, s \in [0, 1]$, any fixed $p, q \geq 1$ and θ an identity function. Consider the function $f : [0, +\infty) \longrightarrow [0, +\infty)$ by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 2, & x > 1. \end{cases}$$

Define two bifunctions $\eta_1 : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$ and $\eta_2 : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ by

$$\eta_1(x, y) = \begin{cases} -y, & 0 \leq y \leq 1; \\ x + y, & y > 1, \end{cases}$$

and

$$\eta_2(x, y) = \begin{cases} x + y, & x \leq y; \\ 2(x + y), & x > y. \end{cases}$$

Then f is generalized $\frac{1}{2}$-$((t^{lp}, t^{sq}); (\eta_1, \eta_2))$-convex mapping. But f is not preinvex with respect to η_1 and also it is not convex (consider $x = 0$, $y = 2$ and $t \in (0, 1]$).

For establishing our main results we need to prove the following lemma.

Lemma 2.8 *Let $\theta : I \longrightarrow \mathbb{R}$ and $g : K \longrightarrow \mathbb{R}$ are continuous functions and $\boldsymbol{m} : [0, 1] \longrightarrow (0, 1]$. Suppose $K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + \Xi] \subseteq \mathbb{R}$ be an closed $\boldsymbol{m}$-invex subset with respect to $\Psi : K \times K \longrightarrow \mathbb{R}$ for $\Xi = \Psi(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ and $\forall\, t \in [0, 1]$. Assume that $f : K \longrightarrow \mathbb{R}$ be a differentiable mapping on K°. If $f' \in L(K)$, then the following equality for extended generalized fractional integral operators holds:*

$$\begin{aligned}
&\left(\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu}; p)\mathrm{d}s\right)^{\nu} \times \Big[f(\boldsymbol{m}(t)\theta(a)) + f(\boldsymbol{m}(t)\theta(a) + \Xi)\Big] \\
&\quad -\nu \int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi} \left(\int_{\boldsymbol{m}(t)\theta(a)}^{\xi} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu}; p)\mathrm{d}s\right)^{\nu-1} g(\xi)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega \xi^{\mu}; p)f(\xi)\mathrm{d}\xi \\
&\quad -\nu \int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi} \left(\int_{\xi}^{\boldsymbol{m}(t)\theta(a)+\Xi} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu}; p)\mathrm{d}s\right)^{\nu-1} g(\xi)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega \xi^{\mu}; p)f(\xi)\mathrm{d}\xi \\
&= \int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi} \left(\int_{\boldsymbol{m}(t)\theta(a)}^{\xi} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu}; p)\mathrm{d}s\right)^{\nu} f'(\xi)\mathrm{d}\xi \\
&\quad - \int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi} \left(\int_{\xi}^{\boldsymbol{m}(t)\theta(a)+\Xi} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu}; p)\mathrm{d}s\right)^{\nu} f'(\xi)\mathrm{d}\xi.
\end{aligned} \tag{2.2}$$

We denote

$$T_{f,g}(\Psi,\theta,\boldsymbol{m};a,b)=\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi}\left(\int_{\boldsymbol{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu}f'(\xi)\mathrm{d}\xi$$
$$-\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi}\left(\int_{\xi}^{\boldsymbol{m}(t)\theta(a)+\Xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu}f'(\xi)\mathrm{d}\xi. \tag{2.3}$$

Proof Integrating by parts Eq. (2.3), we get

$$\begin{aligned}
&T_{f,g}(\Psi,\theta,\mathbf{m};a,b)\\
&=\left(\int_{\mathbf{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu}f(\xi)\Big|_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}\\
&\quad-\nu\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}\left(\int_{\mathbf{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu-1}g(\xi)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega\xi^{\mu};p)f(\xi)\mathrm{d}\xi\\
&\quad-\left(\int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu}f(\xi)\Big|_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}\\
&\quad-\nu\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}\left(\int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu-1}g(\xi)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega\xi^{\mu};p)f(\xi)\mathrm{d}\xi\\
&=\left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu}\times\left[f(\mathbf{m}(t)\theta(a))+f(\mathbf{m}(t)\theta(a)+\Xi)\right]\\
&\quad-\nu\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}\left(\int_{\mathbf{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu-1}g(\xi)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega\xi^{\mu};p)f(\xi)\mathrm{d}\xi\\
&\quad-\nu\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi}\left(\int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right)^{\nu-1}g(\xi)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega\xi^{\mu};p)f(\xi)\mathrm{d}\xi.
\end{aligned}$$

The proof of Lemma 2.8 is completed. □

Remark 2.9 Using Lemma 2.8, for different values of parameters in extended Mittag–Leffler function where $g(s)\equiv 1$, $\forall s\in K$, we get several integral identities of Hermite–Hadamard type.

Using Lemma 2.8, we now state the following theorems for the corresponding version for power of first derivative.

Theorem 2.10 *Let* $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$, $\theta:I\longrightarrow\mathbb{R}$ *and* $g:K\longrightarrow\mathbb{R}$ *are continuous functions and* $\boldsymbol{m}:[0,1]\longrightarrow(0,1]$. *Suppose* $K=[\boldsymbol{m}(t)\theta(a),\boldsymbol{m}(t)\theta(a)+\Xi_1]\subseteq\mathbb{R}$ *be an closed* $\boldsymbol{m}$*-invex subset with respect to* $\Psi_1:K\times K\longrightarrow\mathbb{R}$ *for* $\Xi_1=\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))>0$, $\forall t\in[0,1]$ *and* $\Psi_2:f(K)\times f(K)\longrightarrow\mathbb{R}$. *Assume that* $f:K\longrightarrow(0,+\infty)$ *be a differentiable mapping on* K° *such that* $f'\in L(K)$. *If* $(f'(x))^q$ *is positive generalized-*$\boldsymbol{m}$*-*$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$*-convex mapping,* $0<r\le 1$, $p_1,p_2>-1$, $k<\delta+\Re(\mu)$, $q>1$, $p^{-1}+q^{-1}=1$ *and* $\|g\|_\infty=\sup_{s\in K}|g(s)|$, *the following inequality for extended generalized fractional integral operators holds:*

$$\left|T_{f,g}(\Psi_1,\theta,\boldsymbol{m};a,b)\right|\le\frac{2\|g\|_\infty^{\nu}S^{\nu}\Psi_1^{\nu+1}(\theta(b),\boldsymbol{m}(t)\theta(a))}{\sqrt[p]{p\nu+1}}$$
$$\times\sqrt[rq]{(f'(a))^{rq}I^r(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)I^r(h_2(\xi))}, \tag{2.4}$$

where

$$I(h_1(\xi))=\int_0^1\boldsymbol{m}^{\frac{1}{r}}(\xi)h_1^{\frac{p_1}{r}}(\xi)\mathrm{d}\xi,\quad I(h_2(\xi))=\int_0^1 h_2^{\frac{p_2}{r}}(\xi)\mathrm{d}\xi.$$

Proof From Lemma 2.8, positive generalized-**m**-$((h_1^{p_1}, h_2^{p_2}); (\Psi_1, \Psi_2))$-convexity of $(f'(x))^q$, Hölder inequality, Minkowski inequality, absolute convergence of extended Mittag–Leffler function, properties of the modulus, the fact $g(s) \leq \|g\|_\infty$, $\forall s \in K$ and changing the variable $u = \mathbf{m}(t)\theta(a) + \xi \Xi_1$, $\forall t \in [0, 1]$, we have

$$
\begin{aligned}
\left|T_{f,g}(\Psi_1, \theta, \mathbf{m}; a, b)\right| &\leq \int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} \left| \int_{\mathbf{m}(t)\theta(a)}^{\xi} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p}) \mathrm{d}s \right|^\nu |f'(\xi)| \mathrm{d}\xi \\
&\quad + \int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} \left| \int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi_1} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p}) \mathrm{d}s \right|^\nu |f'(\xi)| \mathrm{d}\xi \\
&\leq \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} \left| \int_{\mathbf{m}(t)\theta(a)}^{\xi} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p}) \mathrm{d}s \right|^{p\nu} \mathrm{d}\xi \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} (f'(\xi))^q \mathrm{d}\xi \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} \left| \int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi_1} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p}) \mathrm{d}s \right|^{p\nu} \mathrm{d}\xi \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} (f'(\xi))^q \mathrm{d}\xi \right)^{\frac{1}{q}} \\
&\leq \|g\|_\infty^\nu S^\nu \times \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} (f'(\xi))^q \mathrm{d}\xi \right)^{\frac{1}{q}} \\
&\quad \times \left\{ \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} (\xi - \mathbf{m}(t)\theta(a))^{p\nu} \mathrm{d}\xi \right)^{\frac{1}{p}} + \left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1} (\mathbf{m}(t)\theta(a) + \Xi_1 - \xi)^{p\nu} \mathrm{d}\xi \right)^{\frac{1}{p}} \right\} \\
&= \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{\sqrt[p]{p\nu+1}} \times \left(\int_0^1 \left(f'(\mathbf{m}(t)\theta(a) + \xi \Xi_1) \right)^q \mathrm{d}\xi \right)^{\frac{1}{q}} \\
&\leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{\sqrt[p]{p\nu+1}} \\
&\quad \times \left(\int_0^1 \left[\mathbf{m}(\xi) h_1^{p_1}(\xi)(f'(a))^{rq} + h_2^{p_2}(\xi)\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) \right]^{\frac{1}{r}} \mathrm{d}\xi \right)^{\frac{1}{q}} \\
&\leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{\sqrt[p]{p\nu+1}} \\
&\quad \times \left[\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q h_1^{\frac{p_1}{r}}(\xi) \mathrm{d}\xi \right)^r + \left(\int_0^1 \Psi_2^{\frac{1}{r}}\left((f'(b))^{rq}, (f'(a))^{rq}\right) h_2^{\frac{p_2}{r}}(\xi) \mathrm{d}\xi \right)^r \right]^{\frac{1}{rq}} \\
&= \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{\sqrt[p]{p\nu+1}} \\
&\quad \times \sqrt[rq]{(f'(a))^{rq} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h_2(\xi))}.
\end{aligned}
$$

The proof of Theorem 2.10 is completed. □

Remark 2.11 In Theorem 2.10, for $h_1(t) = t$, $h_2(t) = 1 - t$, $r = 1$, if we choose $\Xi_1 = \theta(b) - \mathbf{m}(t)\theta(a)$, where $\mathbf{m}(t) \equiv 1$, $\forall t \in [0, 1]$, $\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) = (f'(b))^{rq}$ and $\theta(x) = x$, $\forall x \in I$, then

1. If we put $\omega = \overline{p} = 0$, we get [34, Theorem 7].
2. If we put $\omega = \overline{p} = 0$ along with $\nu = \frac{\alpha}{k}$, we get [12, Theorem 2.5].
3. If we put $g(s) = 1$ and $\omega = \overline{p} = 0$, we get [10, Theorem 2.3].
4. If we put $\omega = \overline{p} = 0$ and $\nu = 1$, we get [10, Corollary 3].

Remark 2.12 In Theorem 2.10, for $h_1(t) = t$, $h_2(t) = 1 - t$, $r = 1$, $\overline{p} = 0$, if we choose $\Xi_1 = \theta(b) - \mathbf{m}(t)\theta(a)$, where $\mathbf{m}(t) \equiv 1$, $\forall t \in [0, 1]$, $\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) = (f'(b))^{rq}$ and $\theta(x) = x$, $\forall x \in I$, we get [11, Corollary 3.8].

We point out some special cases of Theorem 2.10.

Corollary 2.13 *In Theorem 2.10 for $p = q = 2$, we get*

$$\left|T_{f,g}(\Psi_1, \theta, \boldsymbol{m}; a, b)\right| \leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \boldsymbol{m}(t)\theta(a))}{\sqrt{2\nu+1}} \times \sqrt[2r]{(f'(a))^{2r} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{2r}, (f'(a))^{2r}\right) I^r(h_2(\xi))}. \tag{2.5}$$

Corollary 2.14 *In Theorem 2.10 for $g(s) \equiv 1$, we have*

$$\begin{aligned} \left|T_f(\Psi_1, \theta, \boldsymbol{m}; a, b)\right| = \Bigg| &\left(\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi_1} E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p})\mathrm{d}s\right)^\nu \\ &\times\left[f(\boldsymbol{m}(t)\theta(a)) + f(\boldsymbol{m}(t)\theta(a) + \Xi_1)\right] \\ &-\nu\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi_1}\left(\int_{\boldsymbol{m}(t)\theta(a)}^{\xi} E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p})\mathrm{d}s\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega \xi^\mu; \overline{p}) f(\xi)\mathrm{d}\xi \\ &-\nu\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Xi_1}\left(\int_{\xi}^{\boldsymbol{m}(t)\theta(a)+\Xi_1} E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^\mu; \overline{p})\mathrm{d}s\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega \xi^\mu; \overline{p}) f(\xi)\mathrm{d}\xi\Bigg| \\ \leq &\frac{2S^\nu \Psi_1^{\nu+1}(\theta(b), \boldsymbol{m}(t)\theta(a))}{\sqrt[p]{p\nu+1}} \\ &\times \sqrt[rq]{(f'(a))^{rq} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h_2(\xi))}. \end{aligned} \tag{2.6}$$

Corollary 2.15 *In Theorem 2.10 for $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we obtain*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{\sqrt[p]{p\nu+1}} \times \sqrt[rq]{m(f'(a))^{rq} I^r(h(1-\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h(\xi))}. \tag{2.7}$$

Corollary 2.16 *In Corollary 2.15 for $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$, we get*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{\sqrt[p]{p\nu+1}} \times \sqrt[rq]{m(f'(a))^{rq}\left(\frac{r}{r+sp_1}\right)^r + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{r}{r+sp_2}\right)^r}. \tag{2.8}$$

Corollary 2.17 *In Corollary 2.15 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $r > s \cdot \max\{p_1, p_2\}$, we have*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{\sqrt[p]{p\nu+1}} \times \sqrt[rq]{m(f'(a))^{rq}\left(\frac{r}{r-sp_1}\right)^r + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{r}{r-sp_2}\right)^r}. \tag{2.9}$$

Corollary 2.18 *In Theorem 2.10 for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we obtain*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \leq \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{\sqrt[p]{p\nu+1}} \times \sqrt[rq]{m(f'(a))^{rq}\beta^r\left(1+\frac{p_1}{r}, 1+\frac{p_1}{r}\right) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\beta^r\left(1+\frac{p_2}{r}, 1+\frac{p_2}{r}\right)}. \tag{2.10}$$

Corollary 2.19 *In Corollary 2.15 for* $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r>\frac{1}{2}\cdot\max\{p_1,p_2\}$, *we get*

$$\left|T_{f,g}(\Psi_1,\theta,m;a,b)\right|\leq\frac{2\|g\|_\infty^{\nu}S^{\nu}\Psi_1^{\nu+1}(\theta(b),m\theta(a))}{\sqrt[p]{p\nu+1}}\times\Bigg[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^{r}\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^{r}\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}. \quad (2.11)$$

Theorem 2.20 *Let* $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$, $\theta:I\longrightarrow\mathbb{R}$ *and* $g:K\longrightarrow\mathbb{R}$ *are continuous functions and* $\boldsymbol{m}:[0,1]\longrightarrow(0,1]$. *Suppose* $K=[\boldsymbol{m}(t)\theta(a),\boldsymbol{m}(t)\theta(a)+\Xi_1]\subseteq\mathbb{R}$ *be an closed* $\boldsymbol{m}$*-invex subset with respect to* $\Psi_1:K\times K\longrightarrow\mathbb{R}$ *for* $\Xi_1=\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))>0$, $\forall t\in[0,1]$ *and* $\Psi_2:f(K)\times f(K)\longrightarrow\mathbb{R}$. *Assume that* $f:K\longrightarrow(0,+\infty)$ *be a differentiable mapping on* K° *such that* $f'\in L(K)$. *If* $(f'(x))^q$ *is positive generalized-*$\boldsymbol{m}$*-*$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$*-convex mapping,* $0<r\leq 1$, $p_1,p_2>-1$, $k<\delta+\Re(\mu)$, $q\geq 1$ *and* $\|g\|_\infty=\sup_{s\in K}|g(s)|$, *the following inequality for extended generalized fractional integral operators holds:*

$$\left|T_{f,g}(\Psi_1,\theta,\boldsymbol{m};a,b)\right|\leq\frac{\|g\|_\infty^{\nu}S^{\nu}\Psi_1^{\nu+1}(\theta(b),\boldsymbol{m}(t)\theta(a))}{(\nu+1)^{1-\frac{1}{q}}}\times\Bigg\{\sqrt[rq]{(f'(a))^{rq}F^{r}(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)F^{r}(h_2(\xi))}+\sqrt[rq]{(f'(a))^{rq}G^{r}(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)G^{r}(h_2(\xi))}\Bigg\}, \quad (2.12)$$

where

$$F(h_1(\xi))=\int_0^1\boldsymbol{m}^{\frac{1}{r}}(\xi)\xi^{\nu}h_1^{\frac{p_1}{r}}(\xi)\mathrm{d}\xi;\quad F(h_2(\xi))=\int_0^1\xi^{\nu}h_2^{\frac{p_2}{r}}(\xi)\mathrm{d}\xi$$

and

$$G(h_1(\xi))=\int_0^1\boldsymbol{m}^{\frac{1}{r}}(\xi)(1-\xi)^{\nu}h_1^{\frac{p_1}{r}}(\xi)\mathrm{d}\xi;\quad G(h_2(\xi))=\int_0^1(1-\xi)^{\nu}h_2^{\frac{p_2}{r}}(\xi)\mathrm{d}\xi.$$

Proof From Lemma 2.8, positive generalized-$\mathbf{m}$-$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$-convexity of $(f'(x))^q$, the well-known power mean inequality, Minkowski inequality, absolute convergence of extended Mittag–Leffler function, properties of the modulus, the fact $g(s)\leq\|g\|_\infty$, $\forall s\in K$ and changing the variable $u=\mathbf{m}(t)\theta(a)+\xi\,\Xi_1$, $\forall t\in[0,1]$, we have

$$\begin{aligned}\left|T_{f,g}(\Psi_1,\theta,\mathbf{m};a,b)\right|&\leq\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1}\left|\int_{\mathbf{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right|^{\nu}|f'(\xi)|\mathrm{d}\xi\\&\quad+\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1}\left|\int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi_1}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right|^{\nu}|f'(\xi)|\mathrm{d}\xi\\&\leq\left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1}\left|\int_{\mathbf{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right|^{\nu}\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\&\quad\times\left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1}\left|\int_{\mathbf{m}(t)\theta(a)}^{\xi}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right|^{\nu}(f'(\xi))^{q}\mathrm{d}\xi\right)^{\frac{1}{q}}\\&\quad+\left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1}\left|\int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi_1}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right|^{\nu}\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\&\quad\times\left(\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Xi_1}\left|\int_{\xi}^{\mathbf{m}(t)\theta(a)+\Xi_1}g(s)E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)\mathrm{d}s\right|^{\nu}(f'(\xi))^{q}\mathrm{d}\xi\right)^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{(\nu+1)^{1-\frac{1}{q}}}$$
$$\times \left\{ \left[\int_0^1 \xi^\nu \left(f'(\mathbf{m}(t)\theta(a) + \xi\,\Xi_1) \right)^q \mathrm{d}\xi \right]^{\frac{1}{q}} + \left[\int_0^1 (1-\xi)^\nu \left(f'(\mathbf{m}(t)\theta(a) + \xi\,\Xi_1) \right)^q \mathrm{d}\xi \right]^{\frac{1}{q}} \right\}$$
$$\leq \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{(\nu+1)^{1-\frac{1}{q}}}$$
$$\times \left\{ \left[\int_0^1 \xi^\nu \left[\mathbf{m}(\xi) h_1^{p_1}(\xi)(f'(a))^{rq} + h_2^{p_2}(\xi)\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) \right]^{\frac{1}{r}} \mathrm{d}\xi \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\int_0^1 (1-\xi)^\nu \left[\mathbf{m}(\xi) h_1^{p_1}(\xi)(f'(a))^{rq} + h_2^{p_2}(\xi)\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) \right]^{\frac{1}{r}} \mathrm{d}\xi \right]^{\frac{1}{q}} \right\}$$
$$\leq \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{(\nu+1)^{1-\frac{1}{q}}}$$
$$\times \left\{ \left[\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q \xi^\nu h_1^{\frac{p_1}{r}}(\xi)\mathrm{d}\xi \right)^r + \left(\int_0^1 \Psi_2^{\frac{1}{r}}\left((f'(b))^{rq}, (f'(a))^{rq}\right) \xi^\nu h_2^{\frac{p_2}{r}}(\xi)\mathrm{d}\xi \right)^r \right]^{\frac{1}{rq}} \right.$$
$$+ \left[\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q (1-\xi)^\nu h_1^{\frac{p_1}{r}}(\xi)\mathrm{d}\xi \right)^r \right.$$
$$\left. \left. + \left(\int_0^1 \Psi_2^{\frac{1}{r}}\left((f'(b))^{rq}, (f'(a))^{rq}\right)(1-\xi)^\nu h_2^{\frac{p_2}{r}}(\xi)\mathrm{d}\xi \right)^r \right]^{\frac{1}{rq}} \right\}$$
$$= \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \mathbf{m}(t)\theta(a))}{(\nu+1)^{1-\frac{1}{q}}}$$
$$\times \left\{ \sqrt[rq]{(f'(a))^{rq} F^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) F^r(h_2(\xi))} \right.$$
$$\left. + \sqrt[rq]{(f'(a))^{rq} G^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) G^r(h_2(\xi))} \right\}.$$

The proof of Theorem 2.20 is completed. □

We point out some special cases of Theorem 2.20.

Corollary 2.21 *In Theorem 2.20 for $q = 1$, we get*

$$\left| T_{f,g}(\Psi_1, \theta, \boldsymbol{m}; a, b) \right| \leq \|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), \boldsymbol{m}(t)\theta(a))$$
$$\times \left\{ \sqrt[r]{(f'(a))^r F^r(h_1(\xi)) + \Psi_2\left((f'(b))^r, (f'(a))^r\right) F^r(h_2(\xi))} \right.$$
$$\left. + \sqrt[r]{(f'(a))^r G^r(h_1(\xi)) + \Psi_2\left((f'(b))^r, (f'(a))^r\right) G^r(h_2(\xi))} \right\}. \quad (2.13)$$

Corollary 2.22 *In Theorem 2.20 for $g(s) \equiv 1$, we have*

$$\left| T_f(\Psi_1, \theta, \boldsymbol{m}; a, b) \right| \leq \frac{S^\nu \Psi_1^{\nu+1}(\theta(b), \boldsymbol{m}(t)\theta(a))}{(\nu+1)^{1-\frac{1}{q}}}$$
$$\times \left\{ \sqrt[rq]{(f'(a))^{rq} F^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) F^r(h_2(\xi))} \right.$$
$$\left. + \sqrt[rq]{(f'(a))^{rq} G^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) G^r(h_2(\xi))} \right\}. \quad (2.14)$$

Corollary 2.23 *In Theorem 2.20 for* $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ *and* $\boldsymbol{m}(t) = m \in (0, 1]$ *for all* $t \in [0, 1]$, *we obtain*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \le \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{(\nu+1)^{1-\frac{1}{q}}} \times\left\{ \sqrt[rq]{m(f'(a))^{rq} F^r(h(1-\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) F^r(h(\xi))} + \sqrt[rq]{m(f'(a))^{rq} G^r(h(1-\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) G^r(h(\xi))}\right\}. \tag{2.15}$$

Corollary 2.24 *In Corollary 2.23 for* $h_1(t) = (1-t)^s$ *and* $h_2(t) = t^s$, *we get*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \le \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{(\nu+1)^{1-\frac{1}{q}}} \times\left\{ \sqrt[rq]{m(f'(a))^{rq}\beta^r\left(\frac{sp_1}{r}+1, \nu+1\right) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{1}{\frac{sp_2}{r}+\nu+1}\right)^r} + \sqrt[rq]{m(f'(a))^{rq}\left(\frac{1}{\frac{sp_1}{r}+\nu+1}\right)^r + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\beta^r\left(\frac{sp_2}{r}+1, \nu+1\right)}\right\}. \tag{2.16}$$

Corollary 2.25 *In Corollary 2.23 for* $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *and* $r > s \cdot \max\{p_1, p_2\}$, *we have*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \le \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{(\nu+1)^{1-\frac{1}{q}}} \times\left\{ \sqrt[rq]{m(f'(a))^{rq}\beta^r\left(1-\frac{sp_1}{r}, \nu+1\right) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{1}{1+\nu-\frac{sp_2}{r}}\right)^r} + \sqrt[rq]{m(f'(a))^{rq}\left(\frac{1}{1+\nu-\frac{sp_1}{r}}\right)^r + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\beta^r\left(1-\frac{sp_2}{r}, \nu+1\right)}\right\}. \tag{2.17}$$

Corollary 2.26 *In Theorem 2.20 for* $h_1(t) = h_2(t) = t(1-t)$ *and* $\boldsymbol{m}(t) = m \in (0, 1]$ *for all* $t \in [0, 1]$, *we obtain*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \le \frac{2\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{(\nu+1)^{1-\frac{1}{q}}} \times \sqrt[rq]{m(f'(a))^{rq}\beta^r\left(\frac{p_1}{r}+\nu+1, \frac{p_1}{r}+1\right) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\beta^r\left(\frac{p_2}{r}+\nu+1, \frac{p_2}{r}+1\right)}. \tag{2.18}$$

Corollary 2.27 *In Corollary 2.23 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r > \frac{1}{2} \cdot \max\{p_1, p_2\}$, *we get*

$$\left|T_{f,g}(\Psi_1, \theta, m; a, b)\right| \le \frac{\|g\|_\infty^\nu S^\nu \Psi_1^{\nu+1}(\theta(b), m\theta(a))}{(\nu+1)^{1-\frac{1}{q}}} \times\left[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r\left(\nu-\frac{p_1}{2r}+1, 1+\frac{p_1}{2r}\right)\right.$$

$$+\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r\left(\nu+\frac{p_2}{2r}+1, 1-\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r\left(\nu+\frac{p_1}{2r}+1, 1-\frac{p_1}{2r}\right)$$

$$+\Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r\left(\nu-\frac{p_2}{2r}+1, 1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}. \quad (2.19)$$

Remark 2.28 By taking particular values of parameters used in extended Mittag–Leffler function in above Theorems 2.10 and 2.20, several fractional integral inequalities can be obtained.

Remark 2.29 In addition, applying our Theorems 2.10 and 2.20, for $0 < f'(x) \leq L$, for all $x \in I$, we can get some new general fractional integral inequalities.

3 Applications to special means

Definition 3.1 [3] A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

Let consider some special means for arbitrary different positive real numbers $a < b$ as follows: The arithmetic mean $A = A(a, b)$; The geometric mean $G = G(a, b)$; The harmonic mean $H = H(a, b)$; The power mean $P_r = P_r(a, b)$; The identric mean $I = I(a, b)$; The logarithmic mean $L = L(a, b)$; The generalized log–mean $L_p = L_p(a, b)$; The weighted p-power mean $M = M_p$. Let consider continuous functions $\theta : I \longrightarrow K$, $g : K \longrightarrow \mathbb{R}$, $\Psi_1 : K \times K \longrightarrow \mathbb{R}$, $\Psi_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$ and $\overline{M} = M(\theta(a), \theta(b)) : [\theta(a), \theta(a) + \Psi_1(\theta(b), \theta(a))] \times [\theta(a), \theta(a) + \Psi_1(\theta(b), \theta(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means. Therefore, one can obtain various inequalities using the results of Section 2 for these means as follows. Replace $\Psi_1(\theta(y), \mathbf{m}(t)\theta(x))$ with $\Psi_1(\theta(y), \theta(x))$ where $\mathbf{m}(t) \equiv 1$, for all $t \in [0, 1]$ and setting $\Psi_1(\theta(y), \theta(x)) = M(\theta(x), \theta(y))$ for all $x, y \in I$, in (2.4) and (2.12), one can obtain the following interesting inequalities involving means:

$$\left|T_{f,g}(M(\cdot\,,\cdot), \theta, 1; a, b)\right| \leq \frac{2\|g\|_\infty^\nu S^\nu \overline{M}^{\nu+1}}{\sqrt[p]{p\nu+1}} \times \sqrt[rq]{(f'(a))^{rq} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h_2(\xi))}, \quad (3.1)$$

$$\left|T_{f,g}(M(\cdot\,,\cdot), \theta, 1; a, b)\right| \leq \frac{\|g\|_\infty^\nu S^\nu \overline{M}^{\nu+1}}{(\nu+1)^{1-\frac{1}{q}}} \times \left\{\sqrt[rq]{(f'(a))^{rq} F^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) F^r(h_2(\xi))} + \sqrt[rq]{(f'(a))^{rq} G^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) G^r(h_2(\xi))}\right\}. \quad (3.2)$$

Letting $\overline{M} = A, G, H, P_r, I, L, L_p, M_p$ in (3.1) and (3.2), we get the inequalities involving means for a particular choices of $(f'(x))^q$ that are positive generalized-1-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mappings.

Remark 3.2 In addition, applying our Theorems 2.10 and 2.20 for appropriate choices of functions h_1 and h_2 (see remark 2.6) such that $(f'(x))^q$ to be positive generalized-1-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mappings (see examples: $f(x) = x^{\alpha}$, where $\alpha > 1, \forall x > 0$; $f(x) = -\frac{1}{x}, \forall x \neq 0$; $f(x) = e^{x}, \forall x \in \mathbb{R}$; $f(x) = \ln x, \forall x > 0$; etc.), we can deduce some new inequalities using above special means. We omit their proof and the details are left to the interested reader.

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