Arab. J. Math. (2019) 8:63–77
https://doi.org/10.1007/s40065-018-0211-y
Arabian Journal of Mathematics

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q-Difference equations for the 2-iterated q-Appell and mixed type q-Appell polynomials

Received: 2 August 2016 / Accepted: 4 June 2018 / Published online: 15 June 2018

Abstract In this article, the authors establish the recurrence relations and q-difference equations for the 2-iterated q-Appell polynomials. The recurrence relations and the q-difference equations for the 2-iterated q-Bernoulli polynomials, the q-Euler polynomials and the q-Genocchi polynomials are also derived. An analogous study of certain mixed type q-special polynomials is also presented.

Mathematics Subject Classification 33D45 · 33D99 · 33E20

الملخص

في هذا البحث، يقوم المؤلفون بإثبات علاقة ارتدادية وq-معادلة فروقية لكثيرات حدود q-أبيل التكرارية من نوع 2. لقد اشتقت العلاقات الارتدادية وq-المعادلات الفروقية لكثيرات حدود q-برنولي التكرارية من نوع 2 ولكثيرات حدود q-أولرو لكثيرات حدود q-جينوتشي. أيضا، قدّمت دراسة مشابهة لبعض كثيرات الحدود الq-خاصة من الصنف المختلط.

1 Introduction and preliminaries

The subject of q-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. The development of quantum groups and their applications in mathematics and physics has led to renewed interest in the subject of q-series. The recent interest in the subject

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is due to the fact that q-series has popped in such diverse areas as statistical mechanics, quantum groups, transcendental number theory, etc.

The definitions and notations of q-calculus reviewed here are taken from [3].

The q-analogue of the shifted factorial $(a)_n$ is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a), \quad n \in \mathbb{N}. \tag{1.1}$$

The q-analogues of a complex number a and of factorial function are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \in \mathbb{C} - \{1\}; \quad a \in \mathbb{C} \tag{1.2}$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n}, \quad q \neq 1; \quad n \in \mathbb{N}, \quad [0]_q! = 1, \quad q \in \mathbb{C}; \quad 0 < q < 1. \tag{1.3}$$

The Gauss q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad k = 0, 1, \ldots, n. \tag{1.4}$$

The q-exponential functions are defined as:

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}, \quad 0 < |q| < 1. \tag{1.5}$$

The q-derivative $D_q f$ of a function f at a point $0 \neq z \in \mathbb{C}$ is defined as:

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1. \tag{1.6}$$

Also, for any two arbitrary functions $f(z)$ and $g(z)$, the following relation for the q-derivative holds true:

$$D_{q,z}(f(z)g(z)) = f(z)D_{q,z}g(z) + g(qz)D_{q,z}f(z). \tag{1.7}$$

Al-Salaam [1] introduced the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n\geq 0}$ and studied some of its properties. The n-degree polynomials $A_{n,q}(x)$ are called q-Appell provided they satisfy the following q-differential equation:

$$D_{q,x}\ \{A_{n,q}(x)\} = [n]_q\ A_{n-1,q}(x), \quad n = 0, 1, 2, \ldots; \quad q \in \mathbb{C}; \quad 0 < q < 1. \tag{1.8}$$

The q-Appell polynomials $A_{n,q}(x)$ are also defined by means of the following generating function [1]:

$$A_q(t)e_q(xt) = \sum_{n=0}^{\infty}A_{n,q}(x)\frac{t^n}{[n]_q!}, \quad 0 < q < 1, \tag{1.9}$$

where

$$A_q(t) := \sum_{n=0}^{\infty}A_{n,q}\frac{t^n}{[n]_q!}, \quad A_{0,q} = 1; \quad A_q(t) \neq 0. \tag{1.10}$$

It is to be noted that $A_q(t)$ is an analytic function at $t = 0$ and

$$A_{n,q} := A_{n,q}(0) \tag{1.11}$$

are the q-Appell numbers.

Based on appropriate selection for the function $A_q(t)$, different members belonging to the family of q-Appell polynomials can be obtained. These members are mentioned in Table 1.

Table 1 Certain members belonging to the q-Appell family

S. no.	Name of the q-special polynomials and related number	$A_q(t)$	Generating function	Series definition
I.	q-Bernoulli polynomials and number [2,8]	$\left(\frac{t}{e_q(t)-1}\right)$	$\left(\frac{t}{e_q(t)-1}\right)e_q(xt)=\sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{t}{e_q(t)-1}\right)=\sum_{n=0}^{\infty}B_{n,q}\frac{t^n}{[n]_q!}$ $B_{n,q}:=B_{n,q}(0)$	$B_{n,q}(x)=\sum_{k=0}^{n}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q B_{k,q}x^{n-k}$
II.	q-Euler polynomials and number [8,19]	$\left(\frac{2}{e_q(t)+1}\right)$	$\left(\frac{2}{e_q(t)+1}\right)e_q(xt)=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{2}{e_q(t)+1}\right)=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{[n]_q!}$ $E_{n,q}:=E_{n,q}(0)$	$E_{n,q}(x)=\sum_{k=0}^{n}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q E_{k,q}x^{n-k}$
III.	q-Genocchi polynomials and number [11,19]	$\left(\frac{2t}{e_q(t)+1}\right)$	$\left(\frac{2t}{e_q(t)+1}\right)e_q(xt)=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{2t}{e_q(t)+1}\right)=\sum_{n=0}^{\infty}G_{n,q}\frac{t^n}{[n]_q!}$ $G_{n,q}:=G_{n,q}(0)$	$G_{n,q}(x)=\sum_{k=0}^{n}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q G_{k,q}x^{n-k}$

The q-Appell polynomials are the generalizations of the Appell polynomials $A_n(x)$ [4] which are determined by the power series expansion of the product $A(t)\mathrm{e}^{xt}$, that is

$$A(x,t):=A(t)\mathrm{e}^{xt}=\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!}. \tag{1.12}$$

The function $A(t)\mathrm{e}^{xt}$ is called generating function of the sequence of polynomials $A_n(x)$ and the function $A(t)$ is an analytic function at $t=0$ and

$$A_n:=A_n(0) \tag{1.13}$$

are the Appell numbers.

The set of all Appell sequences form an abelian group under the umbral composition of polynomial sequences. The Appell polynomial sequences are well studied from different aspects [4–7,10,25] due to their applications in various fields. One aspect of such study is to find recurrence relations and differential equations for the Appell sequences. For example, He and Ricci [10] established the finite order recurrence relations and differential equations for the Appell sequences using factorization method.

Recently, certain mixed special polynomial families related to the Appell sequences are studied in a systematic way, see for example, [13,15,18,24,32]. These polynomials are studied thoroughly due to their applications in various fields of mathematics, physics and engineering. The properties of these mixed special families lie within the properties of the parent polynomials. To find the differential, integro-differential and partial differential equations for a mixed special polynomial family [18] is a recent investigation [32]. The recurrence relations, differential equations and other results of these mixed type special polynomials can be used to solve the existing as well as new emerging problems in certain branches of science. Introducing a determinant form for the mixed special polynomials via operational and algebraic techniques is a new study, which has been taken into consideration and can be helpful for computation purposes. The technique of combining two sequences by means of umbral composition [27] is a systematic way of constructing mixed special sequences.

Khan and Raza [14] introduced and studied a composite family by combining two different sets of Appell sequences namely the 2-iterated Appell polynomial sequences $A_n^{[2]}(x)$, which are defined by means of the following generating relation:

$$A_1(t)\;A_2(t)\;\mathrm{e}^{xt}=\sum_{n=0}^{\infty}A_n^{[2]}(x)\frac{t^n}{n!}. \tag{1.14}$$

The set of all 2-iterated Appell sequences $A_n^{[2]}(x)$ also form an abelian group under the operation of umbral composition. With the help of determinant form of the 2-iterated Appell sequences $A_n^{[2]}(x)$ considered in [17], it may be possible to compute the coefficients or the value in a chosen point, for particular sequences of the 2-iterated Appell polynomial family, through an efficient and stable Gaussian algorithm. It can also be useful in finding the solution of general linear interpolation problem.

Khan and Riyasat [16] studied the differential and integral equations for the 2-iterated Appell polynomial sequences $A_n^{[2]}(x)$ and mentioned that the respective differential equations can be used to study the d-orthogonality property for these sequences, thus making these 2-iterated sequences important from different view point.

In 1985, Roman proposed an approach similar to the umbral approach under the area of nonclassical umbral calculus which is called q-umbral calculus [26,28]. By using q-analysis and q-umbral calculus, the q-polynomials are introduced and characterized by several authors, for this see [8,9,11,29–31].

The 2-iterated q-Appell polynomials (2IqAP) are introduced and studied by combining two different sets of q-Appell polynomials using the concept of q-umbral composition of polynomial sequences. The generating function for the 2-iterated q-special polynomial families is introduced using a different approach based on replacement techniques. The 2IqAP are defined by means of the following generating function [17]:

$$G_q(x,t) := A_q^{\mathrm{I}}(t)A_q^{\mathrm{II}}(t)e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}, \quad 0<q<1, \tag{1.15}$$

where

$$A_q^{\mathrm{I}}(t) := \sum_{n=0}^{\infty} A_{n,q}^{\mathrm{I}}\frac{t^n}{[n]_q!}; \quad A_{n,q}^{\mathrm{I}} := A_{n,q}^{\mathrm{I}}(0); \quad A_{0,q}^{\mathrm{I}} = 1; \quad A_q^{\mathrm{I}}(t) \neq 0 \tag{1.16}$$

and

$$A_q^{\mathrm{II}}(t) := \sum_{n=0}^{\infty} A_{n,q}^{\mathrm{II}}\frac{t^n}{[n]_q!}; \quad A_{n,q}^{\mathrm{II}} := A_{n,q}^{\mathrm{II}}(0); \quad A_{0,q}^{\mathrm{II}} = 1; \quad A_q^{\mathrm{II}}(t) \neq 0, \tag{1.17}$$

respectively. It is to be noted that $A_q^{\mathrm{I}}(t)$ and $A_q^{\mathrm{II}}(t)$ are analytic functions at $t=0$ and $A_{n,q}^{[2]} := A_{n,q}^{[2]}(0)$ are the 2-iterated q-Appell numbers.

The series definition for the 2IqAP $A_{n,q}^{[2]}(x)$ is given as:

$$A_{n,q}^{[2]}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,q}^{\mathrm{I}}\, A_{n-k,q}^{\mathrm{II}}(x). \tag{1.18}$$

where

$$A_{n,q}^{[2]}(0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,q}^{\mathrm{I}}\, A_{n-k,q}^{\mathrm{II}} \tag{1.19}$$

denotes the 2-iterated q-Appell numbers.

We recall that the set of all q-Appell sequences is closed under the operation of q-umbral composition of polynomial sequences. Under this operation the set of all q-Appell sequences is an abelian group and it can be seen by considering the fact that every q-Appell sequence is of the form

$$p_{n,q}(x) = \left(\sum_{k=0}^{\infty} \frac{c_{k,q}}{[k]_q!} D_q^k\right) x^n \tag{1.20}$$

and that umbral composition of q-Appell sequences corresponds to multiplication of these formal q-power series in the operator D_q. In view of above fact, it is remarked that the 2-iterated q-Appell polynomials $A_{n,q}^{[2]}(x)$ satisfy the following relation:

$$A_{n,q}^{[2]}(x) = \left(\sum_{k=0}^{\infty} \frac{A_{k,q}^{\mathrm{I}}}{[k]_q!} D_q^k\right) A_{n,q}^{\mathrm{II}}(x). \tag{1.21}$$

Again, if $p_{n,q}(x)$ and $q_{n,q}(x) = \sum_{k=0}^{n} q_{n,k;q}\ x^k$ are sequences of q-polynomials, then the q-umbral composition of $q_{n,q}(x)$ with $p_{n,q}(x)$ is defined to be the sequence

$$q_{n,q}(p_q(x)) = \sum_{k=0}^{n} q_{n,k;q}\ p_{k;q}(x), \tag{1.22}$$

which is equivalent to condition (1.18).

Since the generating function of the 2IqAP is of the form $A_q^\star(t)e_q(xt)$, with $A_q^\star(t)$ as the product of two similar functions of t. Therefore, the set of all 2IqAP sequences also form an abelian group under the operation of q-umbral composition. The determinant form of the 2IqAP introduced in [17] can also be used for computation purposes. That is by applying stable Gaussian algorithm, it may be possible to compute the coefficients or the value in a chosen point, for particular sequences of the 2IqAP family.

Since the generating function (1.15) of the 2IqAP sequences is the product of two functions $A_q^{\mathrm{I}}(t)$ and $A_q^{\mathrm{II}}(t)$, which shows that by making appropriate selection for the function $A_q^{\mathrm{I}}(t)$ and $A_q^{\mathrm{II}}(t)$, different members belonging to the family of the 2-iterated q-Appell polynomials can be obtained. By making the combinations of two same members of the q-Appell family in the 2-iterated q-Appell family, a new 2-iterated q-polynomial can be obtained. The generating function and series definition of these 2-iterated q-polynomials are given in Table 2.

By taking the combination of any two different members of the q-Appell family in the 2-iterated q-Appell family, a new mixed type q-special polynomial can be obtained. The generating function and series definition of these mixed type q-special polynomials are given in Table 3.

Table 2 Certain members belonging to the 2-iterated q-Appell family

S. no.	$A_q^{\mathrm{I}}(t) = A_q^{\mathrm{II}}(t)$	Notation and name of the resultant 2IqAP	Generating function	Series definition
I.	$\left(\frac{t}{e_q(t)-1}\right)$	$B_{n,q}^{[2]}(x)$:= 2-iterated q-Bernoulli polynomials (2IqBP)	$\left(\frac{t}{e_q(t)-1}\right)^2 e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}$	$B_{n,q}^{[2]}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q} B_{n-k,q}(x)$
II.	$\left(\frac{2}{e_q(t)+1}\right)$	$E_{n,q}^{[2]}(x)$:= 2-iterated q-Euler polynomials (2IqEP)	$\left(\frac{2}{e_q(t)+1}\right)^2 e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}$	$E_{n,q}^{[2]}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q} E_{n-k,q}(x)$
III.	$\left(\frac{2t}{e_q(t)+1}\right)$	$G_{n,q}^{[2]}(x)$:= 2-iterated q-Genocchi polynomials (2IqGP)	$\left(\frac{2t}{e_q(t)+1}\right)^2 e_q(xt) = \sum_{n=0}^{\infty} G_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}$	$G_{n,q}^{[2]}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q} G_{n-k,q}(x)$

Table 3 Certain mixed type q-special polynomials

S. no.	$A_q^{\mathrm{I}}(t)$; $A_q^{\mathrm{II}}(t)$	Notation and name of the mixed type q-special polynomials	Generating functions	Series definitions
I.	$\left(\frac{t}{e_q(t)-1}\right)$; $\left(\frac{2}{e_q(t)+1}\right)$	${}_BE_{n,q}(x)$:= q-Bernoulli–Euler polynomials (qBEP)	$\frac{2t}{(e_q(t)-1)(e_q(t)+1)} e_q(xt) = \sum_{n=0}^{\infty} {}_BE_{n,q}(x)\frac{t^n}{[n]_q!}$	${}_BE_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q} B_{n-k,q}(x)$
II.	$\left(\frac{t}{e_q(t)-1}\right)$; $\left(\frac{2t}{e_q(t)+1}\right)$	${}_BG_{n,q}(x)$:= q-Bernoulli–Genocchi polynomials (qBGP)	$\frac{2t^2}{(e_q(t)-1)(e_q(t)+1)} e_q(xt) = \sum_{n=0}^{\infty} {}_BG_{n,q}(x)\frac{t^n}{[n]_q!}$	${}_BG_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q} B_{n-k,q}(x)$
III.	$\left(\frac{2}{e_q(t)+1}\right)$; $\left(\frac{2t}{e_q(t)+1}\right)$	${}_EG_{n,q}(x)$:= q-Euler–Genocchi polynomials (qEGP)	$\left(\frac{2t^{1/2}}{e_q(t)+1}\right)^2 e_q(xt) = \sum_{n=0}^{\infty} {}_EG_{n,q}(x)\frac{t^n}{[n]_q!}$	${}_EG_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q} E_{n-k,q}(x)$

Table 4 Certain generalized members belonging to q-Appell family

S. no.	Name of the q-special polynomial	$A_q(t)$	Generating function	Series definition
I.	Generalized q-Bernoulli polynomials (GqBP) of order α [21]	$\left(\frac{t^m}{e_q(t)-T_{m-1,q}(t)}\right)^\alpha$	$\left(\frac{t^m}{e_q(t)-T_{m-1,q}(t)}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(x)\frac{t^n}{[n]_q!}$	$B_{n,q}^{[m-1,\alpha]}(x) = \sum_{k=0}^{n}\binom{n}{k}_q B_{k,q}^{[m-1,\alpha]}x^{n-k}$
II.	Generalized q-Euler polynomials (GqEP) order α [21]	$\left(\frac{2^m}{e_q(t)+T_{m-1,q}(t)}\right)^\alpha$	$\left(\frac{2^m}{e_q(t)+T_{m-1,q}(t)}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x)\frac{t^n}{[n]_q!}$	$E_{n,q}^{[m-1,\alpha]}(x) = \sum_{k=0}^{n}\binom{n}{k}_q E_{k,q}^{[m-1,\alpha]}x^{n-k}$
III.	Generalized q-Genocchi polynomials (GqGP) of order α [21]	$\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^\alpha$	$\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(x)\frac{t^n}{[n]_q!}$	$G_{n,q}^{[m-1,\alpha]}(x) = \sum_{k=0}^{n}\binom{n}{k}_q G_{k,q}^{[m-1,\alpha]}x^{n-k}$
IV.	Generalized q-Apostol Bernoulli polynomials (GqABP) of order α [22]	$\left(\frac{t^m}{\lambda e_q(t)-T_{m-1,q}(t)}\right)^\alpha$	$\left(\frac{t^m}{\lambda e_q(t)-T_{m-1,q}(t)}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{[n]_q!}$	$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(\lambda)\times x^{n-k}$
V.	Generalized q-Apostol Euler polynomials (GqAEP) of order α [22]	$\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^\alpha$	$\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{[n]_q!}$	$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(\lambda)\times x^{n-k}$
VI.	Generalized q-Apostol Genocchi polynomials (GqAGP) of order α [22]	$\left(\frac{2^m t^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^\alpha$	$\left(\frac{2^m t^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{[n]_q!}$	$\mathcal{G}_{n,q}^{[m-1,\alpha]}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_q \mathcal{G}_{k,q}^{[m-1,\alpha]}(\lambda)\times x^{n-k}$

$T_{m-1,q}(t) := \sum_{k=0}^{m-1}\frac{t^k}{[k]_q!}$

Hence, the generating function (1.15) in its product form gains special importance due to the fact by making the combinations of some other generalized members belonging to the q-Appell family, certain other new 2-iterated and mixed type q-special polynomials related to the 2IqAP family can be obtained. These generalized members are listed in Table 4.

The determinant forms related to the 2IqBP $B_{n,q}^{[2]}(x)$, 2IqEP $E_{n,q}^{[2]}(x)$, 2IqGP $G_{n,q}^{[2]}(x)$, qBEP ${}_BE_{n,q}(x)$, qBGP ${}_BG_{n,q}(x)$ and qEGP ${}_EG_{n,q}(x)$ are considered in [17]. The respective determinant forms can be useful in finding the solution of various general linear interpolation problems.

Also, the shapes of the 2IqBP $B_{n,q}^{[2]}(x)$, 2IqEP $E_{n,q}^{[2]}(x)$, 2IqGP $G_{n,q}^{[2]}(x)$, qBEP ${}_BE_{n,q}(x)$, qBGP ${}_BG_{n,q}(x)$ and qEGP ${}_EG_{n,q}(x)$ are displayed and the real and complex zeros of these polynomials are computed for index $n = 1, 2, 3, 4$ and $q = 1/2$ $(0 < q < 1)$ using Matlab in [17]. The distribution and structure of the zeros is also displayed. By finding the real zeros of these polynomials, the approximate solutions and spectral properties of these mixed q-special polynomials are also studied using "Matlab". Thus, making these polynomials important from another view point.

In the 21st century, the computing environment is making more and more rapid progress. Using computer, a realistic study of these new mixed q-special numbers and polynomials seems very interesting. Further, by using numerical investigations and computer experiments, we can observe an interesting phenomenon of "scattering" of the zeros and the regular lattice behavior of almost all of the real and complex zeros of mixed type special and q-special polynomials for higher values of n, i.e. $n > 4$ or for a fixed range of n i.e. $n = 1 - 50$ and for different values of q or for a range of q such that $0 < q < 1$. However, to this point there have been no such investigations for the mixed type special and q-special polynomials. Hence, this will gain extra importance to these new classes of mixed special polynomials.

Since the raising operators are not available for q-polynomials, although lowering operators always exist. Recently, Mahmudov [20] used the lowering operators to study the q-difference equations for the q-Appell polynomials $A_{n,q}(x)$. This provides motivation to establish the q-difference equations for the 2-iterated q-Appell and mixed type q-Appell polynomials.

The article is organized as follows. In Sect. 2, the recurrence relations and q-difference equations for the 2-iterated q-Appell polynomials are introduced. In Sect. 3, the recurrence relations and q-difference equations for the 2-iterated q-Bernoulli, 2-iterated q-Euler and 2-iterated q-Genocchi polynomials and certain mixed type q-special polynomials are also established.

2 Recurrence relations and q-difference equations

In this section, the recurrence relations and q-difference equations for the 2-iterated q-Appell polynomials are established. To derive the recurrence relation for the 2IqAP $A^{[2]}_{n,q}(x)$, the following result is proved:

Theorem 2.1 *For two different sets of q-Appell polynomials $A^{\mathrm{I}}_{n,q}(x)$ and $A^{\mathrm{II}}_{n,q}(x)$ and with $A^{\mathrm{I}}_{q}(t)$ and $A^{\mathrm{II}}_{q}(t)$ defined by Eqs. (1.16) and (1.17), assume that*

$$t\frac{D_{q,t}A^{\mathrm{I}}_{q}(t)}{A^{\mathrm{I}}_{q}(qt)}=\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_q!}, \tag{2.1}$$

$$t\frac{D_{q,t}A^{\mathrm{II}}_{q}(t)}{A^{\mathrm{II}}_{q}(qt)}=\sum_{n=0}^{\infty}\beta_n\frac{t^n}{[n]_q!} \tag{2.2}$$

and

$$\frac{A^{\mathrm{I}}_{q}(t)}{A^{\mathrm{I}}_{q}(qt)}=\sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!}, \tag{2.3}$$

respectively.

Then, the following linear homogeneous recurrence relation for the 2-iterated q-Appell polynomials $A^{[2]}_{n,q}(x)$ holds true:

(i)
$$[n]_qA^{[2]}_{n,q}(qx)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\alpha_k\,q^{n-k}A^{[2]}_{n-k,q}(x)+\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_q\beta_{k-s}\,\gamma_s\right)q^{n-k}A^{[2]}_{n-k,q}(x)$$
$$+x[n]_qq^nA^{[2]}_{n-1,q}(x). \tag{2.4}$$

(ii)
$$A^{[2]}_{n,q}(qx)=\frac{1}{[n]_q}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_qq^k\left(\alpha_{n-k}+\sum_{s=0}^{n-k}\binom{n-k}{s}_q\beta_{n-k-s}\gamma_s\right)A^{[2]}_{k,q}(x)+xq^nA^{[2]}_{n-1,q}(x),\quad n\geq 1. \tag{2.5}$$

(iii)
$$A^{[2]}_{n,q}(qx)=\frac{1}{[n]_q}(\alpha_0+\beta_0\,\gamma_0)q^nA^{[2]}_{n,q}(x)+q^n(x+\alpha_1q^{-1}+\beta_1\,\gamma_0q^{-1}+\beta_0\,\gamma_1q^{-1})A^{[2]}_{n-1,q}(x)$$
$$+\frac{1}{[n]_q}\sum_{k=0}^{n-2}\begin{bmatrix}n\\k\end{bmatrix}_q(\alpha_{n-k}+\sum_{s=0}^{n-k}\binom{n-k}{s}_q\beta_{n-k-s}\gamma_s)q^kA^{[2]}_{k,q}(x),\quad n\geq 1. \tag{2.6}$$

Proof (i) Differentiating generating function (1.15) k-times with respect to x and using the fact that

$$\frac{\partial^kG_q(x,t)}{\partial x^k}=t^kG_q(x,t), \tag{2.7}$$

it follows that

$$\sum_{n=0}^{\infty}A^{[2]}_{n,q}(x)\frac{t^{n+k}}{[n]_q!}=\sum_{n=0}^{\infty}D^k_{q,x}\{A^{[2]}_{n,q}(x)\}\frac{t^n}{[n]_q!}, \tag{2.8}$$

which on equating the coefficients of same powers of t gives

$$D^k_{q,x}A^{[2]}_{n,q}(x)=\frac{[n]_q!}{[n-k]_q!}A^{[2]}_{n-k,q}(x). \tag{2.9}$$

Since the operator $\Phi_{n,q}=\frac{1}{[n]_q}D_{q,x}$ satisfies the following operational relation:

$$\Phi_{n,q}A^{[2]}_{n,q}(x)=A^{[2]}_{n-1,q}(x). \tag{2.10}$$

Therefore, considering the lowering operator as:

$$\Phi_{n,q} = \frac{1}{[n]_q} D_{q,x}, \tag{2.11}$$

it follows that

$$A^{[2]}_{n-k,q}(x) = (\Phi_{n-k,q}.\Phi_{n-k+1,q} \dots \Phi_{n,q})\{A^{[2]}_{n,q}(x)\} = \frac{[n-k]_q!}{[n]_q!} D^k_{q,x}\{A^{[2]}_{n,q}(x)\}, \tag{2.12}$$

which is the k-times derivative operator for the 2IqAP $A^{[2]}_{n,q}(x)$.

Replacement of x by qx in generating function (1.15) and then differentiation of the resultant equation with respect to t using formula (1.7), gives

$$A^{\mathrm{I}}_q(qt)A^{\mathrm{II}}_q(qt)e_q(tqx)qx + D_{q,t}(A^{\mathrm{I}}_q(t)A^{\mathrm{II}}_q(t))e_q(tqx) = \sum_{n=0}^{\infty} A^{[2]}_{n+1,q}(qx)\frac{t^n}{[n]_q!}. \tag{2.13}$$

Further, use of formula (1.7) in Eq. (2.13) and then multiplication by t yields

$$\begin{aligned} &A^{\mathrm{I}}_q(qt)A^{\mathrm{II}}_q(qt)e_q(tqx)tqx + tD_{q,t}(A^{\mathrm{I}}_q(t))A^{\mathrm{II}}_q(qt)e_q(tqx) + tD_{q,t}(A^{\mathrm{II}}_q(t))A^{\mathrm{I}}_q(t)\\ &e_q(tqx) = \sum_{n=0}^{\infty}[n]_q A^{[2]}_{n,q}(qx)\frac{t^n}{[n]_q!}, \end{aligned} \tag{2.14}$$

which on simplifying and interchanging the sides becomes

$$\sum_{n=0}^{\infty}[n]_q A^{[2]}_{n,q}(qx)\frac{t^n}{[n]_q!} = A^{\mathrm{I}}_q(qt)A^{\mathrm{II}}_q(qt)e_q(tqx)\left[t\frac{D_{q,t}A^{\mathrm{I}}_q(t)}{A^{\mathrm{I}}_q(qt)} + t\frac{D_{q,t}A^{\mathrm{II}}_q(t)}{A^{\mathrm{II}}_q(qt)}\frac{A^{\mathrm{I}}_q(t)}{A^{\mathrm{I}}_q(qt)} + tqx\right]. \tag{2.15}$$

In view of assumptions (2.1)–(2.3) and Eq. (1.15) (with t replaced by qt), the above equation gives

$$\sum_{n=0}^{\infty}[n]_q A^{[2]}_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n A^{[2]}_{n,q}(x)\frac{t^n}{[n]_q!}\left[\sum_{k=0}^{\infty}\alpha_k\frac{t^k}{[k]_q!} + \sum_{k=0}^{\infty}\beta_k\frac{t^k}{[k]_q!}\sum_{s=0}^{\infty}\gamma_s\frac{t^s}{[s]_q!} + tqx\right], \tag{2.16}$$

which on rearranging the summations in the r.h.s. becomes

$$\begin{aligned} \sum_{n=0}^{\infty}[n]_q A^{[2]}_{n,q}(qx)\frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \alpha_k q^{n-k} A^{[2]}_{n-k,q}(x)\frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\left(\sum_{s=0}^{k}\begin{bmatrix} k\\ s\end{bmatrix}_q \beta_{k-s}\,\gamma_s\right)q^{n-k}\\ &\quad\times A^{[2]}_{n-k,q}(x)\frac{t^n}{[n]_q!} + x\sum_{n=0}^{\infty} q^n[n]_q A^{[2]}_{n-1,q}(x)\frac{t^n}{[n]_q!}. \end{aligned} \tag{2.17}$$

On equating the coefficients of same powers of t in both sides of the above equation, assertion (2.4) is proved.

(ii) Replacement of k by $n-k$ in the first two terms of the r.h.s. of Eq. (2.4), yields assertion (2.5).

(iii) Solving the summation for $k = n,\ n-1$ in the first term of the r.h.s. of Eq. (2.5) and then simplifying the resultant equation, assertion (2.6) is proved. □

Next, the q-difference equation for the 2IqAP $A^{[2]}_{n,q}(x)$ is derived by proving the following result:

Theorem 2.2 *The 2-iterated q-Appell polynomials* $A^{[2]}_{n,q}(x)$ *satisfy the following q-difference equation:*

$$\begin{aligned} &\left(\frac{1}{[n]_q!}\left(\alpha_n + \sum_{s=0}^{n}\begin{bmatrix} n\\ s\end{bmatrix}_q \beta_{n-s}\gamma_s\right)D^n_{q,x} + \frac{1}{[n-1]_q!}\left(\alpha_{n-1} + \sum_{s=0}^{n-1}\begin{bmatrix} n-1\\ s\end{bmatrix}_q \beta_{n-1-s}\gamma_s\right)D^{n-1}_{q,x} + \cdots\right.\\ &\left. + q^n(x + \alpha_1 q^{-1} + \beta_1\,\gamma_0 q^{-1} + \beta_0\,\gamma_1 q^{-1})D_{q,x} + q^n(\alpha_0 + \beta_0\,\gamma_0)\right)A^{[2]}_{n,q}(x) - [n]_q A^{[2]}_{n,q}(qx) = 0. \end{aligned} \tag{2.18}$$

Proof Using identity (2.12) in the r.h.s. of Eq. (2.4), it follows that

$$[n]_q A^{[2]}_{n,q}(qx) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{n-k} \left(\alpha_k + \sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix}_q \beta_{k-s}\gamma_s \right) \frac{[n-k]_q!}{[n]_q!} D^k_{q,x}\{A^{[2]}_{n,q}(x)\}$$
$$+x[n]_q q^n \frac{[n-1]_q!}{[n]_q!} D_{q,x}\{A^{[2]}_{n,q}(x)\}, \tag{2.19}$$

which on simplifying yields assertion (2.18). □

In the next section, the recurrence relations and q-difference equations for the 2-iterated q-Appell polynomials given in Table 2 and for the mixed type q-special polynomials given in Table 3 are established.

3 Examples

To derive the recurrence relations and q-difference equations for the 2-iterated q-Bernoulli, 2-iterated q-Euler and 2-iterated q-Genocchi polynomials, the following examples are considered:

Example 3.1 Taking $A^{\mathrm{I}}_q(t) = A^{\mathrm{II}}_q(t) = \left(\frac{t}{e_q(t)-1}\right)$ (that is when the 2IqAP $A^{[2]}_{n,q}(x)$ reduce to the 2IqBP $B^{[2]}_{n,q}(x)$) in Eqs. (2.1)–(2.3), so that

$$t\frac{D_{q,t}\frac{t}{e_q(t)-1}}{\frac{qt}{e_q(qt)-1}} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!} \tag{3.1}$$

and

$$\frac{\frac{t}{e_q(t)-1}}{\frac{qt}{e_q(qt)-1}} = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{[n]_q!}, \tag{3.2}$$

respectively.

From the generating function of the q-Bernoulli numbers (Table 1, I) and result [12, p. 6(24), (25)], it follows that

$$\alpha_n = \beta_n = \frac{-1}{q} B_{n,q}; \quad \alpha_0 = \beta_0 = 0; \quad \alpha_1 = \beta_1 = -\frac{1}{[2]_q} \tag{3.3}$$

and

$$\gamma_n = \frac{q-1}{q} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}, \quad n \geq 1; \quad \gamma_0 = 1, \tag{3.4}$$

respectively.

Substituting the values from Eqs. (3.3) and (3.4) in recurrence relation (2.6), the following linear homogeneous recurrence relation for the 2IqBP $B^{[2]}_{n,q}(x)$ is obtained:

$$B^{[2]}_{n,q}(qx) = \left(x - \frac{2}{[2]_q q}\right) q^n B^{[2]}_{n-1,q}(x) - \frac{1}{[n]_q} \sum_{k=0}^{n-2} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{1}{q} B_{n-k,q} + \sum_{s=0}^{n-k} \sum_{l=0}^{s} \begin{bmatrix} n-k \\ s \end{bmatrix}_q \begin{bmatrix} s \\ l \end{bmatrix}_q \right.$$
$$\left. \times \frac{q-1}{q^2} B_{n-k-s,q}\ B_{l,q} \right) q^k B^{[2]}_{k,q}(x), \quad n \geq 1. \tag{3.5}$$

Similarly, substitution of values from Eqs. (3.3) and (3.4) in Eq. (2.18) gives the following q-difference equation for the 2IqBP $B^{[2]}_{n,q}(x)$:

$$\left(\frac{1}{[n]_q!} \left(\frac{1}{q} B_{n,q} + \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q \sum_{l=0}^{s} \begin{bmatrix} l \\ s \end{bmatrix}_q \frac{q-1}{q^2} B_{n-s,q} B_{l,q} \right) D^n_{q,x} - \cdots - q^n \left(x - \frac{2}{[2]_q\, q} \right) D_{q,x} \right) B^{[2]}_{n,q}(x)$$
$$-[n]_q B^{[2]}_{n,q}(qx) = 0. \tag{3.6}$$

Example 3.2 Taking $A_q^{\mathrm{I}}(t) = A_q^{\mathrm{II}}(t) = \left(\frac{2}{e_q(t)+1}\right)$ (that is when the 2IqAP $A_{n,q}^{[2]}(x)$ reduce to the 2IqEP $E_{n,q}^{[2]}(x)$) in Eqs. (2.1)–(2.3), so that

$$t\frac{D_{q,t}\frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\beta_n\frac{t^n}{[n]_q!} \tag{3.7}$$

and

$$\frac{\frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!}, \tag{3.8}$$

respectively.

From the generating function of the q-Euler numbers (Table 1, II) and result [12, p. 8(32), (33)], it follows that

$$\alpha_n = \beta_n = \frac{1}{2}E_{n-1,q};\quad \alpha_0 = \beta_0 = 0;\quad \alpha_1 = \beta_1 = -\frac{1}{2} \tag{3.9}$$

and

$$\gamma_n = \frac{q-1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q},\quad n \geq 1;\quad \gamma_0 = \frac{q+1}{2}, \tag{3.10}$$

respectively.

Substituting the values from Eqs. (3.9) and (3.10) in Eq. (2.6), the following linear homogeneous recurrence relation for the 2IqEP $E_{n,q}^{[2]}(x)$ is obtained:

$$\begin{aligned} E_{n,q}^{[2]}(qx) = q^n\left(x - \frac{1}{2q} - \frac{q+1}{4q}\right)E_{n-1,q}^{[2]}(x) + \frac{1}{[n]_q}\sum_{k=0}^{n-2}\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{1}{2}E_{n-k-1,q} + \sum_{s=0}^{n-k}\sum_{l=0}^{s}\begin{bmatrix} n-k \\ s \end{bmatrix}_q\begin{bmatrix} s \\ l \end{bmatrix}_q\right. \\ \left.\times\frac{q-1}{4}E_{n-k-s-1,q}\ E_{l,q}\right)q^k E_{k,q}^{[2]}(x),\quad n \geq 1. \end{aligned} \tag{3.11}$$

Similarly, on substituting the values from Eqs. (3.9) and (3.10) in Eq. (2.18), the following q-difference equation for the 2IqEP $E_{n,q}^{[2]}(x)$ is obtained:

$$\begin{aligned} &\left(\frac{1}{[n]_q!}\left(\frac{1}{2}E_{n-1,q} + \sum_{s=0}^{n}\begin{bmatrix} n \\ s \end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix} l \\ s \end{bmatrix}_q\frac{q-1}{4}E_{n-s-1,q}E_{l,q}\right)D_{q,x}^n\right. \\ &\left.+\cdots + q^n\left(x - \frac{1}{2q} - \frac{q+1}{4q}\right)D_{q,x}\right)E_{n,q}^{[2]}(x) \\ &-[n]_q E_{n,q}^{[2]}(qx) = 0. \end{aligned} \tag{3.12}$$

Example 3.3 Taking $A_q^{\mathrm{I}}(t) = A_q^{\mathrm{II}}(t) = \left(\frac{2t}{e_q(t)+1}\right)$ (that is when the 2IqAP $A_{n,q}^{[2]}(x)$ reduce to the 2IqGP $G_{n,q}^{[2]}(x)$) in Eqs. (2.1)–(2.3), so that

$$t\frac{D_{q,t}\frac{2t}{e_q(t)+1}}{\frac{2qt}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\beta_n\frac{t^n}{[n]_q!} \tag{3.13}$$

and

$$\frac{\frac{2t}{e_q(t)+1}}{\frac{2qt}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!}, \tag{3.14}$$

respectively.

From the generating function of the q-Genocchi numbers (Table 1, III) and result [12, p. 9(37),(38)], it follows that

$$\alpha_n = \beta_n = \frac{1}{2q} G_{n,q}; \quad \alpha_0 = \beta_0 = \frac{1}{q}; \quad \alpha_1 = \beta_1 = -\frac{1}{q} \tag{3.15}$$

and

$$\gamma_n = \frac{q-1}{2q} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}, \quad n \geq 1; \quad \gamma_0 = \frac{1}{q}, \tag{3.16}$$

respectively.

Substituting the values from Eqs. (3.15) and (3.16) in Eq. (2.6), the following linear homogeneous recurrence relation for the 2IqGP $G^{[2]}_{n,q}(x)$ is obtained:

$$\begin{aligned} G^{[2]}_{n,q}(qx) &= \frac{1}{[n]_q}\Big(\frac{1}{q} + \frac{1}{q^2}\Big) q^n G^{[2]}_{n,q}(x) + q^n \left(x - \frac{1}{q^2} - \frac{1}{q^3} + \frac{(q-1)(2-q)}{2q^3(q+1)} \right) G^{[2]}_{n-1,q}(x) \\ &\quad + \frac{1}{[n]_q} \sum_{k=0}^{n-2} \begin{bmatrix} n \\ k \end{bmatrix}_q \Big(\frac{1}{2q} G_{n-k,q} + \sum_{s=0}^{n-k} \sum_{l=0}^{s} \begin{bmatrix} n-k \\ s \end{bmatrix}_q \begin{bmatrix} s \\ l \end{bmatrix}_q \frac{q-1}{4q^2} G_{n-k-s,q}\, G_{l,q} \Big) q^k G^{[2]}_{k,q}(x), \quad n \geq 1. \end{aligned} \tag{3.17}$$

Similarly, in view of Eqs. (3.15), (3.16) and (2.18), the following q-difference equation for the 2IqGP $G^{[2]}_{n,q}(x)$ is obtained:

$$\begin{aligned} &\Big(\frac{1}{[n]_q!} \Big(\frac{1}{2q} G_{n,q} + \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q \sum_{l=0}^{s} \begin{bmatrix} l \\ s \end{bmatrix}_q \frac{q-1}{4q^2} G_{n-s,q} G_{l,q} \Big) D^n_{q,x} + \cdots + q^n \Big(x - \frac{1}{q^2} - \frac{1}{q^3} + \frac{(q-1)(2-q)}{2q^3(q+1)} \Big) \\ &\quad D_{q,x} \Big) G^{[2]}_{n,q}(x) - [n]_q G^{[2]}_{n,q}(qx) = 0. \end{aligned} \tag{3.18}$$

Further, the recurrence relations and q-difference equations for certain mixed type q-special polynomials are derived by considering the following examples:

Example 3.4 Taking $A^{\mathrm{I}}_q(t) = \Big(\frac{t}{e_q(t)-1}\Big)$ and $A^{\mathrm{II}}_q(t) = \Big(\frac{2}{e_q(t)+1}\Big)$ (that is when the 2IqAP $A^{[2]}_{n,q}(x)$ reduce to the qBEP ${}_BE_{n,q}(x)$) in Eqs. (2.1)–(2.3), respectively, so that

$$t \frac{D_{q,t} \frac{t}{e_q(t)-1}}{\frac{qt}{e_q(qt)-1}} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!}, \tag{3.19}$$

$$t \frac{D_{q,t} \frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!} \tag{3.20}$$

and

$$\frac{\frac{t}{e_q(t)-1}}{\frac{qt}{e_q(qt)-1}} = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{[n]_q!}, \tag{3.21}$$

respectively.

In view of generating functions (Table 1, I, II), the above equations give

$$\alpha_n = \frac{-1}{q} B_{n,q}; \quad \alpha_0 = 0; \quad \alpha_1 = -\frac{1}{[2]_q}, \tag{3.22}$$

$$\beta_n = \frac{1}{2} E_{n-1,q}; \quad \beta_0 = 0; \quad \beta_1 = -\frac{1}{2} \tag{3.23}$$

and

$$\gamma_n = \frac{q-1}{q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}, \quad n \geq 1; \quad \gamma_0 = 1, \tag{3.24}$$

respectively.

Substitution of values from Eqs. (3.22)–(3.24) in Eq. (2.6) yields the following linear homogeneous recurrence relation for the qBEP ${}_BE_{n,q}(x)$:

$$\begin{aligned} {}_BE_{n,q}(qx) &= q^n\left(x - \frac{1}{[2]_q q} - \frac{1}{2q}\right){}_BE_{n-1,q}(x) - \frac{1}{[n]_q}\sum_{k=0}^{n-2}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{1}{q}B_{n-k,q} - \sum_{s=0}^{n-k}\sum_{l=0}^{s}\begin{bmatrix} n-k \\ s \end{bmatrix}_q \begin{bmatrix} s \\ l \end{bmatrix}_q\right. \\ &\quad \left.\times \frac{q-1}{2q}E_{n-k-s-1,q}\ B_{l,q}\right)q^k\, {}_BE_{k,q}(x), \quad n \geq 1. \end{aligned} \tag{3.25}$$

Similarly, substituting the values from Eqs. (3.22)–(3.24) in Eq. (2.18), the following q-difference equation for the qBEP ${}_BE_{n,q}(x)$ is obtained:

$$\begin{aligned} &\left(\frac{1}{[n]_q!}\left(\frac{1}{q}B_{n,q} - \sum_{s=0}^{n}\binom{n}{s}_q \sum_{l=0}^{s}\binom{l}{s}_q \frac{q-1}{2q}E_{n-s-1,q}B_{l,q}\right)\right. \\ &\left. D_{q,x}^n - \cdots - q^n\left(x - \frac{1}{[2]_q\, q} - \frac{1}{2q}\right)D_{q,x}\right){}_BE_{n,q}(x) \\ &-[n]_q\, {}_BE_{n,q}(qx) = 0. \end{aligned} \tag{3.26}$$

Example 3.5 Taking $A_q^{\mathrm{I}}(t) = \left(\frac{t}{e_q(t)-1}\right)$ and $A_q^{\mathrm{II}}(t) = \left(\frac{2t}{e_q(t)+1}\right)$ (that is when the 2IqAP $A_{n,q}^{[2]}(x)$ reduce to the qBGP ${}_BG_{n,q}(x)$) in Eqs. (2.1)–(2.3), respectively, so that

$$t\frac{D_{q,t}\frac{t}{e_q(t)-1}}{\frac{qt}{e_q(qt)-1}} = \sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_q!}, \tag{3.27}$$

$$t\frac{D_{q,t}\frac{2t}{e_q(t)+1}}{\frac{2qt}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\beta_n\frac{t^n}{[n]_q!} \tag{3.28}$$

and

$$\frac{\frac{t}{e_q(t)-1}}{\frac{qt}{e_q(qt)-1}} = \sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!}, \tag{3.29}$$

respectively.

In view of generating functions (Table 1, I, III), the above equations give

$$\alpha_n = \frac{-1}{q}B_{n,q}; \quad \alpha_0 = 0; \quad \alpha_1 = -\frac{1}{[2]_q}, \tag{3.30}$$

$$\beta_n = \frac{1}{2q}G_{n,q}; \quad \beta_0 = \frac{1}{q}; \quad \beta_1 = -\frac{1}{q} \tag{3.31}$$

and

$$\gamma_n = \frac{q-1}{q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}, \quad n \geq 1; \quad \gamma_0 = 1 \tag{3.32}$$

respectively.

Substituting the values from Eqs. (3.30)–(3.32) in Eq. (2.6), the following linear homogeneous recurrence relation for the qBGP ${}_BG_{n,q}(x)$ is obtained:

$$ {}_BG_{n,q}(qx) = \frac{q^{n-1}}{[n]_q}\,{}_BG_{n,q}(x) + q^n\left(x - \frac{1}{q[2]_q} - \frac{1}{q^2} + \frac{q-1}{q^3} - \frac{q-1}{q^3(q+1)}\right){}_BG_{n-1,q}(x) - \frac{1}{[n]_q}\sum_{k=0}^{n-2}\begin{bmatrix} n \\ k \end{bmatrix}_q $$
$$ \times\left(\frac{1}{q}B_{n-k,q} - \sum_{s=0}^{n-k}\sum_{l=0}^{s}\begin{bmatrix} n-k \\ s \end{bmatrix}_q\begin{bmatrix} s \\ l \end{bmatrix}_q\frac{q-1}{2q^2}G_{n-k-s,q}\,B_{l,q}\right)q^k\,{}_BG_{k,q}(x), \quad n \geq 1. \tag{3.33} $$

Similarly, using Eqs. (3.30)–(3.32) in Eq. (2.18), the following q-difference equation for the qBGP ${}_BG_{n,q}(x)$ is obtained:

$$ \left(\frac{1}{[n]_q!}\left(\frac{1}{q}B_{n,q} - \sum_{s=0}^{n}\binom{n}{s}_q\sum_{l=0}^{s}\binom{l}{s}_q\frac{q-1}{2q^2}G_{n-s,q}B_{l,q}\right)D^n_{q,x} - \cdots - q^n\left(x - \frac{1}{[2]_q\,q} - \frac{1}{q^2} + \frac{q-1}{q^3}\right.\right. $$
$$ \left.\left.-\frac{q-1}{q^3(q+1)}\right)D_{q,x}\right){}_BG_{n,q}(x) - [n]_q\,{}_BG_{n,q}(qx) = 0. \tag{3.34} $$

Example 3.6 Taking $A^{\mathrm{I}}_q(t) = \left(\frac{2}{e_q(t)+1}\right)$ and $A^{\mathrm{II}}_q(t) = \left(\frac{2t}{e_q(t)+1}\right)$ (that is when the 2IqAP $A^{[2]}_{n,q}(x)$ reduce to the qEGP ${}_EG_{n,q}(x)$) in Eqs. (2.1)–(2.3), respectively, so that

$$ t\frac{D_{q,t}\frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_q!}, \tag{3.35} $$
$$ t\frac{D_{q,t}\frac{2t}{e_q(t)+1}}{\frac{2qt}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\beta_n\frac{t^n}{[n]_q!} \tag{3.36} $$

and

$$ \frac{\frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!}, \tag{3.37} $$

respectively.

In view of generating functions (Table 1, II, III), the above equations give

$$ \alpha_n = \frac{1}{2}E_{n-1,q}; \quad \alpha_0 = 0; \quad \alpha_1 = -\frac{1}{2}, \tag{3.38} $$
$$ \beta_n = \frac{1}{2q}G_{n,q}; \quad \beta_0 = \frac{1}{q}; \quad \beta_1 = -\frac{1}{q} \tag{3.39} $$

and

$$ \gamma_n = \frac{q-1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}, \quad n \geq 1; \quad \gamma_0 = \frac{q+1}{2}, \tag{3.40} $$

respectively.

Using Eqs. (3.38)–(3.40) in Eq. (2.6), the following linear homogeneous recurrence relation for the qEGP ${}_EG_{n,q}(x)$ is obtained:

$$ {}_EG_{n,q}(qx) = \frac{1}{[n]_q}\frac{(q+1)q^{n-1}}{2}\,{}_EG_{n,q}(x) + q^n\left(x - \frac{1}{2q} - \frac{q+1}{2q^2} + \frac{q-1}{4q^2}\right){}_EG_{n-1,q}(x) + \frac{1}{[n]_q}\sum_{k=0}^{n-2}\begin{bmatrix} n \\ k \end{bmatrix}_q $$
$$ \times\left(\frac{1}{2}E_{n-k-1,q} + \sum_{s=0}^{n-k}\sum_{l=0}^{s}\begin{bmatrix} n-k \\ s \end{bmatrix}_q\begin{bmatrix} s \\ l \end{bmatrix}_q\frac{q-1}{4q^2}G_{n-k-s,q}\,E_{l,q}\right)q^k\,{}_EG_{k,q}(x), \quad n \geq 1. \tag{3.41} $$

Similarly, substituting the values from Eqs. (3.38)–(3.40) in Eq. (2.18), the following q-difference equation for the qEGP ${}_EG_{n,q}(x)$ is obtained:

$$\left(\frac{1}{[n]_q!}\left(\frac{1}{2}E_{n-1,q}+\sum_{s=0}^{n}\binom{n}{s}_q\sum_{l=0}^{s}\binom{l}{s}_q\frac{q-1}{4q}G_{n-s,q}E_{l,q}\right)D_{q,x}^{n}+\cdots+q^n\left(x-\frac{1}{2\,q}-\frac{q+1}{2q^2}+\frac{q-1}{4q^2}\right)\right.$$
$$\left.+q^n\left(\frac{q+1}{2q}\right)D_{q,x}\right)_E G_{n,q}(x)-[n]_{q\,E}G_{n,q}(qx)=0. \tag{3.42}$$

In the next section, further applications and importance of the 2-iterated and mixed type q-special polynomials are discussed.

4 Further applications

The orthogonal polynomials in general and the classical orthogonal polynomials in particular have been the object of extensive works. They are connected with numerous problems of applied mathematics, theoretical physics, chemistry, approximation theory and several other mathematical branches.

During the last 20 years, there has been a growing interest in multiple orthogonal polynomials. However, it is only recently that the examples of multiple orthogonal polynomials have appeared in the literature. A convenient framework to discuss such examples consists in considering a subclass of multiple orthogonal polynomials known as d-orthogonal polynomials [5,7]. The notion of d-dimensional orthogonality for polynomials [23] is the generalization of ordinary orthogonality for polynomials. The problem of finding all polynomial sequences, which are at the same time q-Appell polynomials and d-orthogonal is considered in[33].

The new investigations and important results related to the 2-iterated q-Appell and mixed type q-special polynomials are derived in [17] and are briefly discussed in Sect. 1, which make these polynomials important from different view points. The results which are derived in Sects. 2 and 3 also acquire special importance.

This paper is a first attempt to establish the recurrence relations and q-difference equations for the mixed type q-special polynomials and can also be taken to solve various problems arising in different areas of science and engineering. These q-recurrence relations and q-difference equations of the 2-iterated q-Appell and mixed type q-special polynomials can be used to study the d-orthogonality property of these polynomials. This is obvious that when these polynomials become orthogonal, these can be useful to other fields such as in wavelet analysis. The series expansions and continuous wavelet transforms can be derived in terms of 2-iterated Appell, 2-iterated q-Appell and mixed type q-Appell polynomials and their particular members. This aspect may be considered in further investigation.

Acknowledgements This work has been done under Post-Doctoral Fellowship (Office Memo No.2/40(38)/2016/R&D-II/1063) awarded to Dr. Mumtaz Riyasat by the National Board of Higher Mathematics, Department of Atomic Energy, Government of India, Mumbai. The authors are thankful to the reviewer(s) for several useful comments and suggestions towards the improvement of this paper.

References

1. Al-Salam, W.A.: q-Appell polynomials. Ann. Mat. Pura Appl. **4**(77), 31–45 (1967)
2. Al-Salam, W.A.: q-Bernoulli numbers and polynomials. Math. Nachr. **17**, 239–260 (1959)
3. Andrews, G.E.; Askey, R.; Roy, R.: Special Functions of Encyclopedia. Mathematics and its Applications. Cambridge University Press, Cambridge (1999)
4. Appell, P.: Sur une classe de polynômes. Ann. Sci. École Norm. Sup. **9**(2), 119–144 (1880)
5. Cheikh, Y.B.; Douak, K.: On the classical d-orthogonal polynomials defined by certain generating functions, I. Bull. Belg. Math. Soc. Simon Stevin **7**, 107–124 (2000)
6. Costabile, F.A.; Longo, E.: A determinantal approach to Appell polynomials. J. Comput. Appl. Math. **234**(5), 1528–1542 (2010)
7. Douak, K.: The relation of the d-orthogonal polynomials to the Appell polynomials. J. Comput. Appl. Math. **70**(2), 279–295 (1996)
8. Ernst, T.: q-Bernoulli and q-Euler polynomials, an umbral approach. Int. J. Differ. Equ. **1**(1), 31–80 (2006). (0973-6069)
9. Ernst, T.: A Comprehensive Treatment of q-Calculus, p. xvi + 491. Birkhäuser/Springer Basel AG, Basel (2012)
10. He, M.X.; Ricci, P.E.: Differential equation of Appell polynomials via the factorization method. J. Comput. Appl. Math. **139**, 231–237 (2002)

11. Keleshteri, M.E.; Mahmudov, N.I.: A q-umbral approach to q-Appell polynomials. arXiv:1505.05067
12. Keleshteri, M.E.; Mahmudov, N.I.: On the class of $2D$ q-Appell polynomials. arXiv:1512.03255v1
13. Khan, S.; Al-Saad, M.W.M.; Khan, R.: Laguerre-based Appell polynomials: properties and applications. Math. Comput. Model. **52**(1–2), 247–259 (2010)
14. Khan, S.; Raza, N.: 2-Iterated Appell polynomials and related numbers. Appl. Math. Comput. **219**(17), 9469–9483 (2013)
15. Khan, S.; Raza, N.: General-Appell polynomials within the context of monomiality principle. Int. J. Anal. **2013** (2013). https://doi.org/10.1155/2013/328032
16. Khan, S.; Riyasat, M.: Differential and integral equations for the 2-iterated Appell polynomials. J. Comput. Appl. Math. **306**, 116–132 (2016)
17. Khan, S.; Riyasat, M.: Determinant approach to the 2-Iterated q-Appell and mixed type q-Appell polynomials. arXiv:1606.04265
18. Khan, S.; Yasmin, G.; Khan, R.; Hassan, N.A.M.: Hermite-based Appell polynomials: properties and applications. J. Math. Anal. Appl. **351**(2), 756–764 (2009)
19. Mahmudov, N.I.: On a class of q-Bernoulli and q-Euler polynomials. Adv. Differ. Equ. **108**, 11 (2013)
20. Mahmudov, N.I.: Difference equations of q-Appell polynomials. Appl. Math. Comput. **245**, 539–543 (2014)
21. Mahmudov, N.I.; Keleshteri, M.E.: On a class of generalized q-Bernoulli and q-Euler polynomials. Adv. Differ. Equ. **115**, 1–10 (2013)
22. Mahmudov, N.I.; Keleshteri, M.E.: q-Extensions for the Apostol type polynomials. J. Appl. Math. **2014** (2014). https://doi.org/10.1155/2014/868167
23. Maroni, P.: L'orthogonalité et les récurrences de polynômes d'ordre supérieur á deux. Ann. Fac. Sci. Toulouse **10**, 105–139 (1989)
24. Özarslan, M.A.: Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials. Adv. Differ. Equ. **116**, 1–13 (2013)
25. Özarslan, M.A.; Yilmaz, B.: A set of finite order differential equations for the Appell polynomials. J. Comput. Appl. Math. **259**, 108–116 (2014)
26. Roman, S.: The theory of the umbral calculus I. J. Math. Anal. Appl. **87**, 58–115 (1982)
27. Roman, S.: The Umbral Calculus. Academic Press, New York (1984)
28. Roman, S.: More on the umbral calculus, with emphasis on the q-umbral calculus. J. Math. Anal. Appl. **107**, 222–254 (1985)
29. Srivastava, H.M.: Some characterizations of Appell and q-Appell polynomials. Ann. Mat. Pura Appl. **4**(130), 321–329 (1982)
30. Srivastava, H.M.: Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Inform. Sci. **5**, 390–444 (2011)
31. Srivastava, H.M.; Choi, J.: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier Science Publishers, Amsterdam (2012)
32. Srivastava, H.M.; Özarslan, M.A.; Yilmaz, B.: Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials. Filomat **28**(4), 695–708 (2014)
33. Zaghouani, A.: On some basic d-orthogonal polynomial sets. Georgian Math. J. **12**, 583–593 (2005)