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# Evaluation of various partial sums of Gaussian $q$-binomial sums

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**Abstract** We present three new sets of weighted partial sums of the Gaussian $q$-binomial coefficients. To prove the claimed results, we will use $q$-analysis, Rothe's formula and a $q$-version of the celebrated algorithm of Zeilberger. Finally we give some applications of our results to generalized Fibonomial sums.

**Mathematics Subject Classification** 11B65 · 05A30

**الملخص**

نستعرض ثلاث مجموعات جديدة من المجاميع الجزئية لمجاميع جوس ذات q-حدين. ولإثبات النتائج، سوف نستخدم تحليل q، صيغة روث، ونسخة من نوع q لخوارزمية زيلبرجر المشهورة. وأخيرا نقدم بعض التطبيقات لنتائجنا على مجاميع فيبوناتشي المعممة أحادية الحدود.

## 1 Introduction

The authors of [2] present some results about the partial sums of rows of Pascal's triangle as well as its alternating analogues. As alternating partial sum of Pascal's triangle, from [2] we have

$$\sum_{k\leq m}\binom{r}{k}(-1)^k=(-1)^m\binom{r-1}{m},\qquad \text{integer } m.$$

The authors note that there is no closed form for the partial sum of a row of Pascal's triangle, $S(m,n)=\sum_{k\leq m}\binom{n}{k}$. However, they give a curious example for the partial sum of the row elements multiplied by their distance from the center:

$$\sum_{k\leq m}\binom{r}{k}\left(\frac{r}{2}-k\right)=\frac{m+1}{2}\binom{r}{m+1},\qquad \text{integer } m.$$

Ollerton [11] considered the same partial sum of a row of Pascal's triangle $S(m,n)$ and developed formulae for the sums by use of generating functions.

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We recall a different result from [6]: for any nonnegative integer $t$,

$$\sum_{k\leq t}\binom{2n}{k}\left(1-\frac{k}{n}\right)=\binom{2n-1}{t}.$$

These kinds of partial sums, as well as certain weighted sums (both alternating, and non alternating) are very rare in the current literature. Calkin [1] proved the following curious identity:

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=n\cdot 2^{3n-1}+2^{3n}-3n\binom{2n}{n}2^{n-2}.$$

Hirschhorn [5] established the following two identities on sums of powers of binomial partial sums:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j}=n\cdot 2^{n-1}+2^{n} \quad \text{and} \quad \sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2=n\cdot 2^{2n-1}+2^{2n}-\frac{n}{2}\binom{2n}{n}.$$

Recently, He [4] gave $q$-analogues of the results of Hirschhorn as well as a new version of another result. For example, from [4], we recall that

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\begin{bmatrix}n\\j\end{bmatrix}_q q^{mk+\binom{j}{2}}=\frac{(-q^m;q)_n-q^{m(n+1)}(-1;q)_n}{1-q^m},$$

where the $q$-Pochhammer symbol is $(x;q)_n=(1-x)(1-xq)\ldots(1-xq^{n-1})$ and the Gaussian $q$-binomial coefficients are

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Note that

$$\lim_{q\to 1}\begin{bmatrix}n\\k\end{bmatrix}_q=\binom{n}{k}.$$

Meanwhile, Guo et al. [3] gave some partial sums including $q$-binomial coefficients, for example

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{j=0}^{k}\begin{bmatrix}2n\\j\end{bmatrix}_q\right)^2=\left(\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q\right)\left(\sum_{k=0}^{2n}\begin{bmatrix}2n\\2k\end{bmatrix}_q\right).$$

For later use, we recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n}q^{\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q x^k=\prod_{k=1}^{n}(1+xq^k),$$

and *Rothe's* formula is

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q x^k=(x;q)_n=\prod_{k=0}^{n-1}(1-xq^k).$$

Nowadays, there is increasing interest in deriving $q$-analogues of certain combinatorial statements. This includes, for example, $q$-analogues of certain identities involving harmonic numbers, we refer to [9,10].

Recently, Kılıç and Prodinger [7] computed half Gaussian $q$-binomial sums with certain weight functions. They also gave applications of their results as the generalized Fibonomial sum formulas. For example, they showed that for $n \geq 1$ such that $2n-1 \geq r$

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}(k^2-k(2r+1))} (1+q^k)^{2r+1} = -2^{2r} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q q^{\frac{1}{2}k(k-3)} (1-q^k)^3 = 2 \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix}_q \frac{(1-q)}{q} (1-q^n)(1-q^{2n-1}).$$

More recently, Kılıç and Prodinger [8] computed three types of sums involving products of the Gaussian $q$-binomial coefficients. They are of the following forms: for any real number $a$

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} (a-q^k),$$

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{q^{-k}-a}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k+1}{2}} \frac{a-q^{-k}}{b-q^{-k}}.$$

They also presented interesting applications of their results to generalized Fibonomial and Lucanomial sums.

In this paper, inspired by the partial sums given in [2], we present three sets of weighted partial sums of the Gaussian binomial $q$-binomial coefficients. Before each set, we first give a lemma and then present our main results. To prove the claimed results, we will use Rothe's formula, $q$-analysis and a $q$-version of the celebrated algorithm of Zeilberger. For the sake of simplicity, we will only prove one result from each set.

## 2 The main results

In this section, we will examine each set in separate subsections. We start with the first set of weighted partial sums.

### 2.1 First kind of weighted partial sums

We start with the following lemma to use in proving our first set of results.

**Lemma 2.1** *(i) For odd n,*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-n)} \mathbf{i}^{k(n-k)} (-1)^k = 0.$$

*(ii) For even n,*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-n-1)} \mathbf{i}^{k(n+k)+k} (1-q^k) = 0.$$

*Proof* We only give a proof for the first case (i). The second case (ii) is similarly done. For odd $n$, we write

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}q^{\binom{k}{2}-kn}(-1)^{\binom{k+1}{2}+kn}=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}q^{\binom{k}{2}-kn}(-1)^{kn}\left(\frac{\mathbf{i}+1}{2}\mathbf{i}^k-\frac{\mathbf{i}-1}{2}(-\mathbf{i})^k\right)$$
$$=\frac{\mathbf{i}+1}{2}\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}q^{\binom{k}{2}-kn}(-1)^{kn}\,\mathbf{i}^k-\frac{\mathbf{i}-1}{2}\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}q^{\binom{k}{2}-kn}(-1)^{kn}(-\mathbf{i})^k,$$

which, by Rothe formula, equals

$$\frac{\mathbf{i}+1}{2}(-(-1)^n\mathbf{i}q^{-n};q)_{2n+1}-\frac{\mathbf{i}-1}{2}((-1)^n\mathbf{i}q^{-n};q)_{2n+1}$$
$$=\frac{\mathbf{i}+1}{2}\prod_{k=0}^{2n}(1+(-1)^n\mathbf{i}q^{k-n})-\frac{\mathbf{i}-1}{2}\prod_{k=0}^{2n}(1-(-1)^n\mathbf{i}q^{k-n}).$$

Here we consider two subcases: first, if $n$ is even,

$$\frac{\mathbf{i}+1}{2}\prod_{k=0}^{2n}(1+\mathbf{i}q^{k-n})-\frac{\mathbf{i}-1}{2}\prod_{k=0}^{2n}(1-\mathbf{i}q^{k-n})=2\mathbf{i}\left(\prod_{k=1}^{n}(\mathbf{i}q^{-k}(1+q^{2k}))-\prod_{k=1}^{n}(-\mathbf{i}q^{-k}(1+q^{2k}))\right)$$
$$=2\mathbf{i}^{1-n}q^{-\binom{n+1}{2}}\prod_{k=1}^{n}(1+q^{2k})[(-1)^n-1],$$

which clearly equals 0 for even $n$. The other subcase, $n$ is odd, can be done in a similar way. □

**Theorem 2.2** *For* $n, m \geq 0$

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+2m+4\\ k+m\end{bmatrix}_q(1-q^{2n+m+4-k})(-1)^{\binom{k}{2}+kn}q^{\frac{1}{2}k(k-3)-kn}$$
$$=q^{-n-1}\begin{bmatrix}2n+2m+4\\ m+1\end{bmatrix}_q\begin{cases}(1+q^{n+1})(1-q^{m+n+2}) & \textit{if } n \textit{ is even},\\ -(1-q^{n+1})(1+q^{m+n+2}) & \textit{if } n \textit{ is odd}.\end{cases}$$

*Proof* Consider the LHS of the claim

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+2m+4\\ k+m\end{bmatrix}_q(1-q^{2n+m+4-k})(-1)^{\binom{k}{2}+kn}q^{\frac{1}{2}k(k-3)-kn}$$
$$=(-1)^{mn-\binom{m+1}{2}}q^{\frac{1}{2}m(m+3)+mn}$$
$$\times\sum_{k=m}^{2n+m+1}\begin{bmatrix}2n+2m+4\\ k\end{bmatrix}_q(1-q^{2n+2m+4-k})(-1)^{k(m+n)-\binom{k}{2}}q^{\frac{1}{2}k(k-2(m+n)-3)}.$$

By Lemma 2.1 (ii), we have

$$\sum_{k=0}^{2n+2m+4}\begin{bmatrix}2n+2m+4\\ k\end{bmatrix}_q(1-q^{2n+2m+4-k})(-1)^{k(m+n)-\binom{k}{2}}q^{\frac{1}{2}k(k-2(m+n)-3)}=0.$$

Apart from the constant factor, we consider the LHS of the claim and write

$$-\sum_{k=m}^{2n+m+1}\begin{bmatrix}2n+2m+4\\ k\end{bmatrix}_q(1-q^{2n+2m+4-k})(-1)^{k(m+n)-\binom{k}{2}}q^{\frac{1}{2}k(k-2(m+n)-3)}$$
$$=\sum_{k=0}^{m-1}\begin{bmatrix}2n+2m+4\\ k\end{bmatrix}_q(1-q^{2n+2m+4-k})(-1)^{k(m+n)-\binom{k}{2}}q^{\frac{1}{2}k(k-2(m+n)-3)}$$

$$+\sum_{k=2n+m+2}^{2n+2m+4} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1-q^{2n+2m+4-k}) \times (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)},$$

which, after some rearrangements, equals

$$\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1-q^{2n+2m+4-k})\,(-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)}$$
$$+\sum_{k=0}^{m+2} \begin{bmatrix} 2n+2m+4 \\ m+2-k \end{bmatrix}_q (1-q^{m-k+2}) \times (-1)^{kn+n(m+1)-\binom{k}{2}+\binom{m+1}{2}+1} q^{-\frac{1}{2}(m-k+1)(k+m+2n+2)}$$
$$=\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1-q^{2n+2m+4-k})\,(-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)}$$
$$+\sum_{k=0}^{m+2} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (-1)^{(k+1)(m+n)-\binom{k+1}{2}} q^{-\frac{1}{2}(k-1)(2m-k+2n+4)} \left(1-q^k\right).$$

After further rearrangements, this equals

$$\left(1-q^{2n+2m+4}\right)\left(\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\frac{1}{2}k(k-1)} q^{\frac{1}{2}k(k-2m-2n-3)} + \sum_{k=0}^{m+2} \begin{bmatrix} 2n+2m+3 \\ k-1 \end{bmatrix}_q (-1)^{(k+1)(m+n)-\binom{k+1}{2}} q^{-\frac{1}{2}(k-1)(2m-k+2n+4)}\right)$$
$$=(1-q^{2n+2m+4})\left(\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2m-2n-3)} + \sum_{k=0}^{m+1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}+1} q^{\frac{1}{2}k(k-2m-2n-3)}\right)$$
$$=\left(1-q^{2n+2m+4}\right)\left(\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2m-2n-3)} - \sum_{k=0}^{m+1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2m-2n-3)}\right),$$

which, by telescoping, equals

$$-\left(1-q^{2n+2m+4}\right)\left(\begin{bmatrix} 2n+2m+3 \\ m+1 \end{bmatrix}_q (-1)^{\frac{1}{2}(m+2n)(m+1)} q^{-\frac{1}{2}(m+1)(m+2n+2)} + \begin{bmatrix} 2n+2m+3 \\ m \end{bmatrix}_q (-1)^{\frac{1}{2}m(m+2n+1)} q^{-\frac{1}{2}m(m+2n+3)}\right)$$
$$=-\left(1-q^{2n+2m+4}\right)(-1)^{\frac{1}{2}m(m+2n+1)} q^{-\frac{1}{2}m(m+2n+3)} \times \left(\begin{bmatrix} 2n+2m+3 \\ m+1 \end{bmatrix}_q (-1)^n q^{-(n+1)} + \begin{bmatrix} 2n+2m+3 \\ m \end{bmatrix}_q\right)$$

$$= -(-1)^{\frac{1}{2}m(m+2n+1)}\, q^{-\frac{1}{2}(m+1)(m+2n+2)} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q \times \begin{cases} \left(1+q^{n+1}\right)\left(1-q^{m+n+2}\right) & \text{if } n \text{ is even,} \\ -\left(1-q^{n+1}\right)\left(1+q^{m+n+2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we get

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+2m+4 \\ k+m \end{bmatrix}_q \left(1-q^{2n+m+4-k}\right) (-1)^{\binom{k}{2}+kn}\, q^{\frac{1}{2}k(k-3)-kn}$$
$$= q^{-(n+1)} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q \begin{cases} \left(1+q^{n+1}\right)\left(1-q^{m+n+2}\right) & \text{if } n \text{ is even,} \\ -\left(1-q^{n+1}\right)\left(1+q^{m+n+2}\right) & \text{if } n \text{ is odd,} \end{cases}$$

as claimed. □

We present the following results without proof.

**Theorem 2.3** *For $n \geq 0$ and $m \geq 1$,*

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n+2m-1 \\ k+m \end{bmatrix}_q q^{\binom{k+1}{2}-kn} (-1)^{kn-\binom{k}{2}} = (-1)^n q^n \begin{bmatrix} 2n+2m-1 \\ m-1 \end{bmatrix}_q,$$
$$\sum_{k=0}^{2n} \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{\binom{k}{2}-kn} (-1)^{\binom{k+1}{2}+kn} \left(1-q^{k+m}\right) = (1-q^{2n+2m}) \begin{bmatrix} 2n+2m-1 \\ m-1 \end{bmatrix}_q.$$

**Theorem 2.4** *For $n, m \geq 0$,*

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n+2m+1 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-2n-1)} (-1)^{kn-\binom{k+1}{2}} = \begin{bmatrix} 2n+2m+1 \\ m \end{bmatrix}_q,$$
$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+2m+3 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-2n-3)} (-1)^{\binom{k}{2}+kn}$$
$$= (-1)^n q^{-n-1} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q \begin{cases} (1-q^{n+1})/(1-q^{n+m+2}) & \text{if } n \text{ is odd,} \\ \left(1+q^{n+1}\right)/\left(1+q^{n+m+2}\right) & \text{if } n \text{ is even.} \end{cases}$$

### 2.2 Second kind of weighted partial sums

Now we continue with our second set of sums. Before that, we need the following lemma which could be proven similar to Lemma 2.1.

**Lemma 2.5** *(i) If $n$ is odd,*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-n)} = 0.$$

*(ii) If $n$ is even,*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-n+1)} (-1)^k (1-q^{n-k}) = 0.$$

We start with the following result.

**Theorem 2.6** *For* $n, m \geq 0$,

$$\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-3)} = -q^{-2n-1} \frac{(1-q^{2n+1})(1+q^{2n+m+2})}{(1-q^{m+1})} \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q .$$

*Proof* Consider the LHS of the claim, by taking $k-m$ instead of $k$, we note

$$\begin{aligned} &\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k \, q^{-\frac{1}{2}k(4n-k+3)} \\ &= (-1)^m \, q^{\frac{1}{2}m(m+4n+3)} \sum_{k=m}^{4n+m+1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)} \end{aligned}$$

and by Lemma 2.5 (i),

$$\sum_{k=0}^{4n+2m+3} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)} = 0.$$

Thus, we obtain

$$\begin{aligned} &-\sum_{k=m}^{4n+m+1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)} \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)} \\ &\quad + \sum_{k=4n+m+2}^{4n+2m+3} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)}, \end{aligned}$$

which, by some rearrangements, equals

$$\begin{aligned} &\sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)} \\ &+ \sum_{k=0}^{m+1} \begin{bmatrix} 4n+2m+3 \\ k+4n+m+2 \end{bmatrix}_q (-1)^{k+m} \, q^{-\frac{1}{2}(m-k+1)(k+m+4n+2)} \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)} \\ &\quad - \sum_{k=0}^{m+1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-4n-2m-3)}. \end{aligned}$$

By telescoping, this equals

$$\begin{aligned} &-(-1)^m \, q^{-\frac{1}{2}(m+1)(m+4n+2)} \left( -\begin{bmatrix} 4n+2m+3 \\ m+1 \end{bmatrix}_q + \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q q^{2n+1} \right) \\ &= -(-1)^m \, q^{-\frac{1}{2}(m+1)(m+4n+2)} \frac{(q)_{4n+2m+3}}{(q)_{m+1} \, (q)_{4n+m+3}} \times (-(1-q^{4n+m+3}) + q^{2n+1}(1-q^{m+1})), \end{aligned}$$

which equals

$$
\begin{aligned}
&(-1)^m\, q^{-\frac{1}{2}(m+1)(m+4n+2)}\, \frac{(q)_{4n+2m+3}}{(q)_m\, (q)_{4n+m+3}}\, \frac{(1-q^{2n+1})(1+q^{2n+m+2})}{1-q^{m+1}}\\
&\quad = (-1)^m\, q^{-\frac{1}{2}(m+1)(m+4n+2)} \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q \frac{(1-q^{2n+1})(1+q^{2n+m+2})}{(1-q^{m+1})}.
\end{aligned}
$$

Thus, we derive

$$
\begin{aligned}
&\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k\, q^{\frac{1}{2}k(k-4n-3)}\\
&\quad = -q^{-2n-1}\, \frac{(1-q^{2n+1})(1+q^{2n+m+2})}{(1-q^{m+1})} \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q,
\end{aligned}
$$

as claimed. □

Using similar techniques as in the above proof, we obtain the following identities, for all $n, m \geq 0$,

$$
\begin{aligned}
&\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+4 \\ k+m \end{bmatrix}_q (-1)^k\, q^{\frac{1}{2}k(k-4n-3)}(1-q^{4n+m+4-k})\\
&\quad = -q^{2n+1}(1-q^{4n+2m+4})\, \frac{(1-q^{2n+1})(1+q^{2n+m+2})}{(1-q^{m+1})} \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q,\\
&\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k\, q^{\frac{1}{2}k(k-4n-3)} = \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q, \qquad (1)\\
&\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m-1 \\ k+m \end{bmatrix}_q (-1)^k\, q^{\binom{k+1}{2}-2kn} = q^{2n}\, \frac{(1-q^{2n+1})(1+q^{m+2n})}{1-q^{m-1}} \begin{bmatrix} 4n+2m-1 \\ m-2 \end{bmatrix}_q,\\
&\sum_{k=0}^{4n} \begin{bmatrix} 4n+2m-1 \\ k+m \end{bmatrix}_q q^{\binom{k+1}{2}-2kn}\, (-1)^k = q^{2n} \begin{bmatrix} 4n+2m-1 \\ m-1 \end{bmatrix}_q,\\
&\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2m+4 \\ k+m \end{bmatrix}_q q^{-\frac{1}{2}k(4n-k+3)}\, (-1)^k\, (1-q^{4n+4+m-k}) = (1-q^{4n+4+2m}) \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q,\\
&\sum_{k=0}^{4n} \begin{bmatrix} 4n+2m \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n+1)}\, (-1)^k\, (1-q^{4n+m-k})\\
&\quad = q^{2n}(1-q^{4n+2m}) \begin{bmatrix} 4n+2m-1 \\ m-1 \end{bmatrix}_q.
\end{aligned}
$$

### 2.3 Third kind of weighted partial sums

We start with the following lemma for later use.

**Lemma 2.7** *For $m > 0$ and integer $r$,*

$$
\begin{aligned}
&\sum_{k=0}^{m+r} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q (-1)^k\, q^{\frac{1}{2}k(k-4n-2m-1)}(1+q^{2n+m+1-k})\\
&\quad = (-1)^{m+r} q^{-\frac{1}{2}(m+r)(m+4n-r+1)}(1+q^{1+m+2n}) \begin{bmatrix} 4n+2m+1 \\ m+r \end{bmatrix}_q.
\end{aligned}
$$

*Proof* Denote the left-hand side of this identity by $S[m,r]$. Using the celebrated algorithm of Zeilberger in Mathematica, we obtain

$$S\,[m,r] = (-1)^{m+r} q^{-\frac{1}{2}(m+r)(m+4n-r+1)} \frac{1}{(1-q^{1+m+2n})} \frac{(q)_{4n+2m+2}}{(q)_{4n+m+1-r}\,(q)_{m+r}},$$

which is easily seen to be the same as the right-hand side of the desired identity. □

**Theorem 2.8** *For $n \geq 0$ and $m > 0$,*

$$\sum_{k=0}^{4n+3} \begin{bmatrix} 4n+2m+2 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1-q^k) = -q^{2n+1} \frac{(1-q^{2n+2})(1-q^m)}{(1-q^{m+2n+1})} \begin{bmatrix} 4n+2m+2 \\ m-1 \end{bmatrix}_q .$$

*Proof* Note that

$$q^{\frac{1}{2}m(m+4n+3)} (-1)^m \sum_{k=m}^{4n+m+3} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1-q^{k-m}) = \sum_{k=0}^{4n+3} \begin{bmatrix} 4n+2m+2 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1-q^k)$$

and also

$$\sum_{k=0}^{4n+2m+2} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1-q^{k-m}) = 0.$$

Thus, we write

$$\begin{aligned}
&- \sum_{k=m}^{4n+m+3} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1-q^{k-m}) \\
&= \sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1-q^{k-m}) \\
&\quad + \sum_{k=0}^{m-2} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}(k+1)(k-4n-2m-2)} (1-q^{m+4n+2-k}) \\
&= \begin{bmatrix} 4n+2m+2 \\ m-1 \end{bmatrix}_q q^{-\frac{1}{2}(m-1)(m+4n+4)} (-1)^{m-1} (1-q^{-1}) \\
&\quad + \sum_{k=0}^{m-2} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q (-1)^k \\
&\quad \times \left[ q^{\frac{1}{2}k(k-4n-2m-3)} (1-q^{k-m}) + q^{\frac{1}{2}(k+1)(k-4n-2m-2)} (1-q^{m-k+4n+2}) \right] \\
&= \begin{bmatrix} 4n+2m+2 \\ m-1 \end{bmatrix}_q q^{-\frac{1}{2}(m-1)(m+4n+4)} (-1)^{m-1} (1-q^{-1}) \\
&\quad + q^{-(2n+m+1)} (1-q^{2n+1}) \\
&\quad \times \sum_{k=0}^{m-2} \begin{bmatrix} 4n+2m+2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-1)} (1+q^{2n+m+1-k}),
\end{aligned}$$

which, by Lemma 2.7 with the case $r = -2$, equals

$$\begin{aligned}
&(-1)^{m-1} q^{-\frac{1}{2}(m-1)(m+4n+4)} \\
&\quad \times \left( \begin{bmatrix} 4n+2m+2 \\ m-1 \end{bmatrix}_q (1-q^{-1}) - (1-q^{2n+1})(1+q^{1+m+2n}) \begin{bmatrix} 4n+2m+1 \\ m-2 \end{bmatrix}_q \right) \\
&= (-1)^m q^{-\frac{1}{2}(m-1)(m+4n+4)-1} \\
&\quad \times (1-q^{2n+2})(1+q^{m+2n+1})(1-q^m) \frac{(q)_{4n+2m+1}}{(q)_{m-1}\,(q)_{4n+m+3}}.
\end{aligned}$$

Thus,

$$\sum_{k=0}^{4n+3} \begin{bmatrix} 4n+2m+2 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1-q^k) = -q^{2n+1} \frac{(1-q^{2n+2})(1-q^m)}{(1-q^{m+2n+1})} \begin{bmatrix} 4n+2m+2 \\ m-1 \end{bmatrix}_q,$$

as claimed. □

We present the following results without proof. The proofs use Lemma 2.7 and are similar to that of Theorem 2.8.

**Theorem 2.9** *For $n \geq 0$ and $m \geq 1$,*

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2m+2 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1-q^k) = (1-q^{2n+1})(1+q^{1+m+2n}) \begin{bmatrix} 4n+2m+1 \\ m-1 \end{bmatrix}_q.$$

**Theorem 2.10** *For $n, m \geq 0$,*

$$\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+2 \\ k+m \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-3)} (1-q^k) = -q^{-2n-1}(1+q^{2n+m+1})(1-q^{2n+1}) \begin{bmatrix} 4n+2m+1 \\ m \end{bmatrix}_q,$$

$$\begin{aligned}
&\sum_{k=0}^{4n} \begin{bmatrix} 4n+2m+2 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1-q^k) \\
&\quad = q^{-4n-1} \frac{(1-q^{4n+m+2})(1-q^{2n})}{(1-q^{2n+m+1})} \begin{bmatrix} 4n+2m+2 \\ m+1 \end{bmatrix}_q.
\end{aligned}$$

## 3 Applications

In this section, we will give some applications of our results to the generalized Fibonomial coefficients. Define the non-degenerate second-order linear sequences $\{U_n\}$ and $\{V_n\}$ by, for $n > 1$

$$\begin{aligned}
U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\
V_n &= pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p.
\end{aligned}$$

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = (p \pm \sqrt{p^2+4})/2$.

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U := \frac{U_1 U_2 \ldots U_n}{(U_1 U_2 \ldots U_k) \cdot (U_1 U_2 \ldots U_{n-k})}$$

with $\left\{ {n \atop 0} \right\}_U = \left\{ {n \atop n} \right\}_U = 1$.

When $p = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_F$.

The link between the generalized Fibonomial and Gaussian $q$-binomial coefficients is

$$\left\{ {n \atop k} \right\}_U = \alpha^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

By taking $q = \beta/\alpha$, the Binet formulae are reduced to the following forms:

$$U_n = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n (1+q^n),$$

where $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$.

From each set, we now give one application. From Theorem 2.3, Eq. (1) and Theorem 2.9, we have the following identities by taking $q = \beta/\alpha$, respectively: for $n \geq 0$ and $m \geq 1$,

$$\sum_{k=0}^{2n} \left\{ {2n+2m \atop k+m} \right\}_U (-1)^k U_{k+m} = U_{2n+2m} \left\{ {2n+2m-1 \atop m-1} \right\}_U,$$

$$\sum_{k=0}^{4n+2} \left\{ {4n+2m+3 \atop k+m} \right\}_U (-1)^{\frac{1}{2}k(k-1)} = \left\{ {4n+2m+3 \atop m} \right\}_U,$$

$$\sum_{k=0}^{4n+2} \left\{ {4n+2m+2 \atop k+m} \right\}_U (-1)^{\frac{1}{2}k(k-1)} U_k = U_{2n+1} V_{1+m+2n} \left\{ {4n+2m+1 \atop m-1} \right\}_U.$$

As a showcase we will prove the second identity just above. First we convert the second identity into $q$-form. Thus, it takes the form

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q \alpha^{(k+m)(4n+2m+3-(k+m))} (-1)^{\frac{1}{2}k(k-1)} = \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q \alpha^{m(4n+2m+3-m)}$$

or

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q \alpha^{(k+m)(m-k+4n+3)} (-1)^{\frac{1}{2}k(k-1)} = \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q \alpha^{m(4n+m+3)}$$

or

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k q^{-\frac{1}{2}k(4n-k+3)} = \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q,$$

which was already given as the identity (1).

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