Arab. J. Math. (2018) 7:113–145
https://doi.org/10.1007/s40065-017-0190-4

Arabian Journal of Mathematics

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# On wave equation: review and recent results

Received: 10 October 2017 / Accepted: 1 November 2017 / Published online: 27 November 2017


**Abstract** The aim of this paper is to give an overview of results related to nonlinear wave equations during the last half century. In this regard, we present results concerning existence, decay and blow up for classical nonlinear equations. After that, we discuss briefly some important results of the variable-exponent Lebesgue and Sobolev spaces. Results related to nonexistence and blow up for wave equations with non-standard nonlinearities (nonlinearities involving variable exponents) are given in more detail. Finally, we present some recent decay and blow up results together with their proofs.

**Mathematics Subject Classification** 35L05 · 35L70 · 35B44 · 35B35

**الملخص**

الهدف من هذه الورقة هو إعطاء لمحة عامة عن النتائج المتعلقة بمعادلة الأمواج غير الخطية خلال نصف القرن الماضي. وبهذا الصدد، نعرض نتائج وجود، اضمحلال، وانفجار لمعادلات غير خطية كلاسيكية. بعد ذلك، نناقش بإيجاز بعض النتائج الهامة لفضاءات لوبيغ وصوبوليف ذات أسس متغيرة. بعض النتائج المتعلقة بعدم الوجود والانفجار لمعادلات موجية بحدود غير خطية تنطوي على أسس متغيرة تعطى بشيء من التفصيل. وأخيرا نقدم بعض نتائج الاضمحلال والانفجار مع براهينها.

**Contents**



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## 1 Literature review

A considerable and great effort has been devoted to the study of linear and nonlinear wave equations in the case of constant and variable-exponent nonlinearities. Our aim here is to give an overview of the existing results and introduce some other ones.

### 1.1 Decay in the case of constant exponents

There is an extensive literature on the stabilization of the wave equation by internal or boundary feedbacks. Zuazua [95] proved the exponential stability of the energy for the wave equation by a locally distributed internal feedback depending linearly on the velocity. Komornik [37] and Nakoa [72] extended the result of Zuazua [95] by considering the case of a nonlinear damping term with a polynomial growth near the origin. Martinez [53,54] studied a damped wave equation and used the piecewise multiplier technique combined with some nonlinear integral inequalities to establish explicit decay rate estimates. These decay estimates are not optimal for some cases including the case of the polynomial growth. The following initial-boundary value problem of the Kirchhoff equation with a general dissipation of the form

$$\begin{cases} u_{tt} - \phi\Big(\int_\Omega |\nabla u|^2\Big)\Delta u + \sigma(t)g(u_t) = 0, & \text{in } \Omega \times [0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$ and $\phi$, $\sigma$ and $g$ are given functions and the functions $(u_0, u_1)$ are the given initial data, was considered by many authors in the literature. For instance, in the case when $g = \sigma = 0$, the one-dimensional equation of (1.1) was first introduced by Kirchhoff [35] in 1876, and then was called the Kirchhoff string after his name. When $\sigma = 1$, $\phi(r) = r^\alpha$ $(\alpha \geq 1)$ and $g(x) = \tau x$ $(\tau > 0)$, problem (1.1) was treated by Nishihara and Yamada [73]. They proved the existence and uniqueness of a global solution and the polynomial decay for small data $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with $u_0 \neq 0$ . In [76], Ono extended the work of [73] to the case where $\phi(r) = r$ and $\sigma(t) \equiv (1+t)^{-\delta}$, $\delta < \frac{1}{3}$ using the decay lemma of Nakao [69]. Benaissa and Guesmia [18] extended the results obtained by Ono [76] and proved an existence and uniqueness theorem of a global solution in Sobolev spaces to the problem (1.1) when $\phi(r) = r$, $g(v) = v$ and general functions $\sigma$. Also, they obtained an explicit and general decay rate, depending on $\sigma$, $g$ and $\phi$, for the energy of solutions of (1.1) without any growth assumption on $g$ and $\phi$ at the origin, and on $\sigma$ at infinity. Also, the following problem

$$\begin{cases} u_{tt} - \Delta u + g(u_t) + f(u) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega$ is a bounded region in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, was considered by many authors. For instance, in the case when $f(u) = |u|^{p-2}u$, $g(u_t) = |u_t|^{m-2}u_t$, $m, p > 2$, Nakao [70] showed that (1.2) has a unique global weak solution if $0 \leq p-2 \leq 2/(n-2)$, $n \geq 3$ and a global unique strong solution if $p-2 > 2/(n-2)$, $n \geq 3$. In addition to global existence, the issue of the decay rate was also addressed. In both cases it has been shown that the energy of the solution decays algebraically if $m > 2$ and decays exponentially if $m = 2$. This improved an earlier result in [68], where Nakao studied the problem in an abstract setting and established a theorem concerning decay of the solution energy only for the case $m - 2 \leq 2/(n-2)$, $n \geq 3$. Also in a joint work, Nakao and Ono [71] extended this result to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \lambda^2(x)u + \rho(u_t) + f(u) = 0, & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $\rho(u_t)$ behaves like $|u_t|^\beta u_t$ and $f(u)$ behaves like $-bu|u|^\alpha$. In this case the authors required that the initial data be small enough in the $H^1 \times L^2$ norm and with compact supports. In [56] Messaoudi considered problem (1.2) in the case $f(u) = bu|u|^{p-2}$, $g(u_t) = a(1 + |u_t|^{m-2})u_t$, $a, b > 0$, $p, m > 2$, and showed that, for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem has a unique global solution with energy decaying exponentially. Benaissa and Messaoudi [16] studied (1.2), for $f(u) = -bu|u|^{p-2}$, and $g(u_t) = a(1 + |u_t|^{m-2})u_t$, and showed that, for suitably chosen initial data, the problem possesses a global weak solution which decays exponentially even if $m \geq 2$. In [28], Guesmia looked into the following problem

$$\begin{cases} u_{tt} - \Delta u + h(\nabla u) + g(u_t) + f(u) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $\Omega$ is a bounded open domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$ and $f, g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuous nonlinear functions satisfying some general properties. He obtained uniform decay of strong and weak solutions under weak growth assumptions on the feedback function and without any control of the sign of the derivative of the energy related to the above equation. Guesmia and Messaoudi [29], considered (1.4) with $h(\nabla u) = -\nabla\phi \cdot \nabla u$, where $\phi \in W^{1,\infty}(\Omega)$, and proved local and global existence results and showed that weak solutions decay either algebraically or exponentially depending on the rate of growth of $g$. Pucci and Serrin [78] discussed the stability of the following problem

$$\begin{cases} u_{tt} - \Delta u + Q(x,t,u,u_t) + f(x,u) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.5}$$

and proved that the energy of the solution is a Lyapunov function. Although they did not discuss the issue of the decay rate, they did show that in general the energy goes to zero as $t$ approaches infinity. They also considered an important special case of (1.5), when $Q(x,t,u,u_t) = a(t)t^\alpha u_t$ and $f(x,u) = V(x)u$, and showed that the behavior of the solution depends crucially on the parameter $\alpha$. Precisely, they showed if $|\alpha| \leq 1$ then the rest field is asymptotically stable. On the other hand, when $|\alpha| > 1$ there are solutions that do not approach zero or approach nonzero function $\phi(x)$ as $t \to \infty$. In [27], Guesmia studied the following elasticity system

$$\begin{cases} \partial_{tt}u_i - \sigma_{ij,j} + \ell_i(x, \partial_t u_i) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty) \\ u_i(0) = u_i^0, \partial_t u_i(0) = u_i^1, & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $\ell_i(x, \partial_t u_i) = b_i(x)g_i(\partial_t u_i)$, $b_i$'s $\in L^\infty(\Omega)$, are bounded nonnegative functions and $g_i$'s are non-decreasing continuous real-valued functions satisfying certain conditions. He proved precise decay estimates of the energy for the system (1.6) with some localized dissipations. Zuazua [96] considered the following damped semilinear wave equation

$$u_{tt} - \Delta u + \alpha u + f(u) + a(x)u_t = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

with $\alpha > 0$. He proved the exponential decay of the energy of the solution under suitable conditions on the functions $f$ and $a$. In [17], Benaissa and Mokeddem looked into the following equation

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \sigma(t)\operatorname{div}\left(|\nabla u_t|^{m-2}\nabla u_t\right) = 0,$$

where $\sigma$ is a positive function, $p, m \geq 2$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a regular boundary. They gave an energy-decay estimate for the solutions and extended the results of Yang [92] and Messaoudi [59]. Cavalcanti and Guesmia [19] looked into the following problem

$$\begin{cases} u_{tt} - \Delta u + F(x,t,u,\nabla u) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Gamma_0 \times (0, +\infty) \\ u + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)\mathrm{d}s = 0, & \text{on } \partial\Gamma_1 \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.7}$$

where $\Omega$ is a bounded region in $\mathbb{R}^n$ whose boundary is partitioned into disjoint sets $\Gamma_0, \Gamma_1$, $g$ is the relaxation function which satisfies some assumptions. They proved that the dissipation given by the memory term is strong enough to assure exponential (or polynomial) decay provided the relaxation function also decays exponentially (or polynomially). In both cases the solution decays with the same rate of the relaxation function. This result was later generalized by Messaoudi and Soufyane [60], where relaxation functions of general decay type were considered. Alabau-Boussouira [1] used some weighted integral inequalities and convexity arguments and proved a semi-explicit formula which leads to decay rates of the energy in terms of the behavior of the nonlinear feedback near the origin, for which the optimal exponential and polynomial decay rate estimates are only special cases. The following problem has been widely studied in the literature:

$$\begin{cases} u_{tt} - \Delta u + \alpha(t)g(u_t) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{1.8}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $g, \alpha$ are specific functions. For instance, when $\alpha \equiv 1$ and $g$ satisfies

$$c_1 \min\{|s|, |s|^q\} \leq |g(s)| \leq c_2 \max\{|s|, |s|^{1/q}\},$$

where $c_1, c_2 > 0$ are constants and $q > 1$, it was proved that

$$E(t) \leq C\big(E(0)\big)t^{-2/(q-1)}, \quad \forall t > 0,$$

and for $q = 1$, the decay rate is exponential (see [36,43]). In the presence of a weak frictional damping, Benaissa and Messaoudi [15] treated problem (1.8) for $g$ having a polynomial growth near the origin, and established energy decay results depending on $\alpha$ and $h$. Decay results for arbitrary growth of the damping term have been considered for the first time in the work of Lasiecka and Tataru [44]. They showed that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. Mustafa and Messaoudi [65] considered (1.8) and established an explicit and general decay rate result, using some properties of convex functions. Their result was obtained without imposing any restrictive growth assumption on the frictional damping term. Wu and Xue [91] studied the following quasilinear hyperbolic equation

$$u_{tt} - \psi(t)\,\mathrm{div}(|\nabla u_t|^{p-2}\nabla u_t) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + \mu|u_t|^{\alpha}u_t = 0,$$

where $\mu, \alpha \geq 0$, and $p \geq 2$ are constants, the functions $\sigma_i$ $(i = 1, 2, \ldots, n)$ and $\psi$ are nonlinear, the domain $\Omega$ is bounded in $\mathbb{R}^n$ $(n \geq 1)$, with a regular boundary. They investigated, using the multiplier methods, the stability of weak solutions and obtained an explicit estimation for the rate of the decay. In 2015, Mokeddem and Mansour [64] revisited the problem considered in [17] with some modifications. Precisely they treated the equation

$$u_{tt} - \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \sigma(t)(u_t - \mathrm{div}(|\nabla u_t|^{m-2}\nabla u_t)) = 0,$$

and gave the same decay result. Recently, Cavalcanti et al. [20] treated the following damped wave problem

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t - \mathrm{div}(b(x)\nabla u_t) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.9}$$

where $\Omega$ is a bounded open domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$ and $a, b : \Omega \to \mathbb{R}^+$ are nonnegative functions satisfying specific conditions. Under appropriate assumptions on the coefficients and the initial data $(u_0, u_1)$, they proved stabilization results for problem (1.9). Taniguchi [83] studied the following problem with nonlinear boundary condition:

$$\begin{cases} u_{tt}(t) - \rho(t)\Delta u(t) + b(x)u_t(t) = f(u(t)), & \text{on } \Omega \times (0, T), \\ u(t) = 0, & \text{on } \Gamma_0 \times (0, T), \\ \frac{\partial u(t)}{\partial \nu} + \gamma(u_t(t)) = 0, & \text{on } \Gamma_1 \times (0, T) \\ u(0) = u_0, u_t(0) = u_1, & \text{in } \Omega, \end{cases} \tag{1.10}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \phi$. Under some conditions on $\|u_0\|$ and $E(0)$, the global existence and exponential decay of the energy $E(t)$ of weak solution to (1.10) were established.

### 1.2 Blowup in the case of constant exponents

The first study of finite-time blowup of solutions of hyperbolic PDEs of the form

$$u_{tt} - \Delta u = f(u)$$

goes back to early 70s in the work of Levine [46] and Ball [10]. The interaction between the damping and the source terms was considered by Levine for an equation of the form

$$u_{tt} - \Delta u + a u_t = f(u).$$

He introduced the concavity method and showed that solutions with negative initial energy blow up at finite time. This method was later improved by Kalantarov and Ladyzhenskaya [34] to accommodate more general cases. After 20 years, Georgiev and Todorova [26] extended Levine's result to the nonlinearly damped equation

$$u_{tt} - \Delta u + a|u_t|^m u_t = b|u|^p u, \quad \text{in } \Omega \times (0, \infty),$$

for $a, b, m, p > 0$. In their work, Georgiev and Todorova introduced a different method and determined appropriate relations between the nonlinearities in the damping and the source, for which there is either global existence or alternatively finite-time blowup. Precisely, they showed that solutions with negative energy exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and initial energy is 'sufficiently' negative. This result was later generalized to an abstract setting and to unbounded domains by Levine et al. [47], Levine and Serrin [48], Levine and Park [49], and Messaoudi [55,57]. In all these papers, the authors showed that no solution with negative or sufficiently negative energy can be extended on $[0, \infty)$, if the nonlinearity dominates the damping effect $(p > m)$. Vitillaro [87] combined the arguments in [45] and [26] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy. For more results concerning blowup and nonexistence, we mention here the work of Vitillaro [88], Todorova [84], Todorova and Vitillaro [85], Wang [89], Liu [51], Wu [90], and the very recent book of Al'shin et al. [2]. For the nonlinear Kirchhoff-type problem of the form

$$\begin{cases} u_{tt} - (\int_\Omega |D^m u|^2 \mathrm{d}x)^q \Delta u + u_t|u_t|^r = |u|^p u, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.11}$$

where $p, q, r \geq 0$, $\Omega$ is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$ and a unit outer normal $\nu$, several results concerning global existence and blowup have been established; see in this regard [11–13,74,75], and the references therein. Messaoudi and Said-Houari [58] considered the nonlinear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}\big(|\nabla u|^{\alpha-2}\nabla u\big) - \operatorname{div}\big(|\nabla u_t|^{\beta-2}\nabla u_t\big) + a|u_t|^{m-2}u_t = b|u|^{p-2}u,$$

where $a, b > 0$, $\alpha, \beta, m, p > 2$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a regular boundary. They proved, under appropriate conditions on $\alpha, \beta, p, m > 2$, a global nonexistence result for solutions associated with negative initial energy. Chen et al. [21] considered the following nonlinear $p$-Laplacian-wave equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t + q(x,u) = f(x)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $2 \leq p < n$ and $f, q$ are given functions. They established global existence and uniqueness under appropriate conditions on the initial data and the functions $f, q$. They also discussed the long-time behavior of the solution. Ibrahim and Lyaghfouri [32] considered the following equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p_*-2}u$$

in $\mathbb{R}^n$, $n \geq 3$, and $2 < p < n$, $p_* = \frac{pn}{n-p}$ is the critical Sobolev exponent. Under appropriate assumptions on the initial data, they proved the finite-time blow up of solutions and, hence, extended a result by Galaktionov and Pohozaev [23]. Ye [93] investigated the blowup property of solutions of a quasilinear hyperbolic system of equations and proved that certain solutions with positive initial energy explode in finite time and he also gave estimation for the solution lifespan. Recently, Kafini and Messaoudi [33] studied a nonlinear wave equation with damping and delay terms and showed, under suitable hypotheses on the initial data, that the solution energy explodes in a finite time. For more results, we refer the reader to [14,24,31,81].

### 1.3 Blowup in the case of variable-exponent nonlinearities

In recent years, a great deal of attention has been given to the investigation of nonlinear models of hyperbolic, parabolic and elliptic equations with variable exponents of nonlinearity. For instance, some models from physical phenomena like flows of electro-rheological fluids or fluids with temperature-dependent viscosity, filtration processes in a porous media, nonlinear viscoelasticity, and image processing, give rise to such problems. More details on the subject can be found in [3] and [4]. Regarding hyperbolic problems with nonlinearities of variable-exponent type, only few works have appeared. For instance, Antontsev [6] considered the equation

$$u_{tt} - \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u) - \alpha\Delta u_t = b(x,t)u|u|^{\sigma(x,t)-2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $\alpha > 0$ is a constant and $a, b, p, \sigma$ are given functions. For specific conditions on $a, b, p, \sigma$, he proved some blowup results, for certain solutions with non-positive initial energy. He also discussed the case when $\alpha = 0$ and established a blowup result. Subsequently, Antontsev [5] discussed the same equation and proved a local and a global existence of some weak solutions under certain hypotheses on the functions $a, b, p, \sigma$. He also established some blowup results for certain solutions having non-positive initial energy. Guo and Gao [30] looked into the same problem of [6] and established several blowup results for certain solutions associated with negative initial energy. Precisely, they took $\sigma(x,t) = \sigma > 2$, a constant, and established a result of blowup in finite time. For the case $\sigma(x,t) = \sigma(x)$, they claimed the same blowup result but no proof has been given. This work is considered to be an improvement for that of [6]. In [82], Sun et al. looked into the following equation

$$u_{tt} - \operatorname{div}(a(x,t)\nabla u) + c(x,t)u_t|u_t|^{q(x,t)-1} = b(x,t)u|u|^{p(x,t)-1}$$

in a bounded domain, with Dirichlet-boundary conditions, and established a blowup result for solutions with positive initial energy. They also gave lower and upper bounds for the blowup time and provided a numerical illustrations for their result. Recently, Messaoudi and Talahmeh [61] studied

$$u_{tt} - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) + \mu u_t = |u|^{p(x)-2}u, \tag{1.12}$$

with Dirichlet-boundary conditions and for $\mu \geq 0$. They proved a blowup result for certain solutions with arbitrary positive initial energy. This result generalized that of Korpusov [39] established for (1.12), with $m$ and $p$ constants. This latter result was later extended by the same authors in [62] to an equation of the form

$$u_{tt} - \operatorname{div}(|\nabla u|^{r(\cdot)-2}\nabla u) + a|u_t|^{m(\cdot)-2}u_t = b|u|^{p(\cdot)-2}u,$$

where $a, b > 0$ are constants and the exponents of nonlinearity $m$, $p$ and $r$ are given functions. They proved a finite-time blowup result for the solutions with negative initial energy and for certain solutions with positive energy. Very recently, Messaoudi et al. [63] studied

$$u_{tt} - \Delta u + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, \tag{1.13}$$

where $a$, $b$ are positive constants. They established the existence of a unique weak solution using the Faedo–Galerkin method under suitable assumptions on the variable exponents $m$ and $p$ and they also proved the finite-time blow up of solutions and gave a two-dimension numerical example to illustrate the blowup result. Yunzhu Gao and Wenjie Gao [25] studied a nonlinear viscoelastic equation with variable exponents and proved the existence of weak solutions using the Faedo–Galerkin method under suitable assumptions. Autuori et al. [9] looked into a nonlinear Kirchhoff system in the presence of the $\overrightarrow{p}(x,t)$-Laplace operator, a nonlinear force $f(t,x,u)$ and a nonlinear damping term $Q = Q(t,x,u,u_t)$. They established a global nonexistence result under suitable conditions on $f, Q, p$. For more results concerning the blowup of hyperbolic problems, we refer the reader to Antontsev and Ferreira [7] and the book by Antontsev and Shmarev [8].

## 2 Preliminaries

### 2.1 History of variable-exponent Lebesgue spaces

Variable Lebesgue spaces were first introduced by Orlicz in 1931 in his article [77]. He started by looking for necessary and sufficient conditions on a real sequence $(y_i)$ under which $\sum x_i y_i$ converges, for any other real sequence $(x_i)$ such that $\sum x_i^{p_i}$ converges, where $(p_i)$ is a sequence of real numbers with $p_i > 1$. Furthermore, he also considered the variable-exponent function space $L^{p(\cdot)}$ on the real line. Orlicz later concentrated much on the theory of the function spaces that were named after him. In the theory of Orlicz spaces, the space $L^{\varphi}$ is defined as follows:

$$L^{\varphi} := \left\{ u : \Omega \to \mathbb{R} \text{ such that } \varrho(\lambda u) = \int_{\Omega} \varphi(\lambda |u(x)|)\mathrm{d}x < +\infty \right\},$$

for some $\lambda > 0$, where $\varphi$ is a real-valued function which may depend on $x$ and satisfies some additional conditions. Putting certain properties of $\varrho$ in an abstract setting, a more general class of function spaces, called modular spaces, was first studied by Nakano [66,67]. Following the work of Nakano, modular spaces were investigated by several people, most importantly by groups at Sapporo (Japan), Voronezh (U.S.S.R.), and Leiden (the Netherlands). An explicit version of modular function spaces was investigated by Polish Mathematicians, like Hudzik and Kaminska.

The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the Orlicz space $L^{\varphi_{p(\cdot)}}(\Omega)$, where

$$\varphi_{p(\cdot)}(t) = t^{p(\cdot)} \quad \text{or} \quad \varphi_{p(\cdot)}(t) = \frac{t^{p(\cdot)}}{p(\cdot)},$$

with $p(\cdot) : \Omega \to [1, \infty]$ as a measurable function.

Variable-exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. Their results originated in 1961 in a paper by Tsenov [86]. The Luxemburg norm was introduced by Sharapudinov for the Lebesgue space. He showed that this space is Banach if the exponent $p(\cdot)$ satisfies $1 < \operatorname{essinf} p \leq \operatorname{esssup} p < +\infty$. In the mid-80s, Zhikov [94] started a new line of investigation of variable-exponent spaces, by considering variational integrals with non-standard growth conditions. The next major step in the study of variable-exponent spaces was by Kovacik and Rakosnik [40] in the early 90s. In their paper, they established many of the basic properties of Lebesgue and Sobolev spaces in $\mathbb{R}^n$.

### 2.2 Variable-exponent Lebesgue Spaces

In this section, we present some preliminary facts about the Lebesgue spaces with variable exponents.

**Definition 2.1** Let $X$ be a $\mathbb{K}-$vector space. A function $\varrho : X \longrightarrow [0, \infty]$ is said to be *left-continuous* if the mapping $\lambda \longmapsto \varrho(\lambda x)$ is left-continuous on $[0, \infty)$, for every $x \in X$; that is,

$$\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x), \quad \forall\ x \in X.$$

**Definition 2.2** Let $X$ be a $\mathbb{K}-$vector space. A function $\varrho : X \longrightarrow [0, \infty]$ is called a *semimodular* on $X$ if the following properties hold:

(a) $\varrho(0) = 0$.
(b) $\varrho(\lambda x) = \varrho(x)$, for all $x \in X$ and $\lambda \in \mathbb{K}$, with $|\lambda| = 1$.
(c) $\varrho$ is convex.
(d) $\varrho$ is left-continuous.
(e) $\varrho(\lambda x) = 0$, for all $\lambda > 0$ implies $x = 0$.

A semimodular is called *modular* if

(f) $\varrho(x) = 0$ implies $x = 0$

A semimodular is called *continuous* if

(g) the mapping $\lambda \longmapsto \varrho(\lambda x)$ is continuous on $[0, \infty)$ for all $x \in X$

*Examples 2.3* Let $L^0(\Omega)$ be the set of all Lebesgue-measurable functions defined on $\Omega$. If $1 \leq p < +\infty$, then

$$\varrho_p(f) := \int_\Omega |f(x)|^p \mathrm{d}x$$

defines a continuous modular on $L^0(\Omega)$.

**Theorem 2.4** *Let $\varrho$ be a semimodular on $X$. Then, the mapping $\lambda \to \varrho(\lambda x)$ is non-decreasing on $[0, \infty)$ for every $x \in X$. Moreover,*

$$\begin{aligned} \varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda| \varrho(x) \quad \text{for all } |\lambda| \leq 1, \\ \varrho(\lambda x) = \varrho(|\lambda| x) \geq |\lambda| \varrho(x) \quad \text{for all } |\lambda| \geq 1. \end{aligned} \tag{2.1}$$

*Proof* • Assume that $0 \leq \lambda < \mu$, then $0 \leq \frac{\lambda}{\mu} < 1$. So for a fixed $x \in X$ we have, by convexity and non-negativeness of $\varrho$ and the fact that $\varrho(0) = 0$,

$$\varrho(\lambda x) = \varrho\left(\frac{\lambda}{\mu}(\mu x) + \left(1 - \frac{\lambda}{\mu}\right) \cdot 0\right) \leq \frac{\lambda}{\mu}\varrho(\mu x) + \left(1 - \frac{\lambda}{\mu}\right)\varrho(0) = \frac{\lambda}{\mu}\varrho(\mu x) \leq \varrho(\mu x).$$

Hence, for any fixed $x \in X$, we have

$$\varrho(\lambda x) \leq \varrho(\mu x), \qquad \text{for} \qquad 0 \leq \lambda < \mu.$$

- For $\lambda \neq 0$ we have

$$\varrho(\lambda x) = \varrho\left(\frac{\lambda}{|\lambda|}|\lambda| x\right) = \varrho(|\lambda| x) \quad \left(\text{since } \left|\frac{\lambda}{|\lambda|}\right| = 1\right).$$

- For $|\lambda| \leq 1$, we have

$$\varrho(|\lambda| x) = \varrho(|\lambda| x + (1 - |\lambda|)0) \leq |\lambda| \varrho(x) + (1 - |\lambda|)\varrho(0) = |\lambda| \varrho(x).$$

Therefore,

$$\varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda| \varrho(x), \quad \forall x \in X.$$

- For $|\lambda| \geq 1$, we have

$$\varrho(x) = \varrho\left(\frac{1}{|\lambda|}|\lambda| x\right) \leq \frac{1}{|\lambda|}\varrho(|\lambda| x) + \left(1 - \frac{1}{|\lambda|}\right)\varrho(0) = \frac{1}{|\lambda|}\varrho(|\lambda| x).$$

Thus,

$$\varrho(\lambda x) = \varrho(|\lambda x|) \geq |\lambda| \varrho(x), \qquad \forall x \in X.$$

□

**Definition 2.5** Let $(A, \Sigma, \mu)$ be a $\sigma$-finite, complete measure space. We define $\mathcal{P}(A, \mu)$ to be the set of all $\mu$-measurable functions $p : \Omega \to [1, \infty]$. The functions $p \in \mathcal{P}(A, \mu)$ are called variable exponents on $A$. We set

$$p_1 := \operatorname{essinf}_{y \in A} p(y) \text{ and } p_2 := \operatorname{esssup}_{y \in A} p(y).$$

If $p_2 < +\infty$, then we call $p$ a bounded variable exponent. If $p \in \mathcal{P}(A, \mu)$, then we define $p' \in \mathcal{P}(A, \mu)$ by

$$\frac{1}{p(y)} + \frac{1}{p'(y)} = 1, \quad \text{where } \frac{1}{\infty} := 0.$$

The function $p'$ is called the dual variable exponent of $p$. In the special case when $\mu$ is the $n$-dimensional Lebesgue measure and $\Omega$ is an open subset of $\mathbb{R}^n$, we abbreviate $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$.

**Definition 2.6** We define the Lebesgue space with a variable-exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R};\ \text{measurable in } \Omega :\ \varrho_{p(\cdot)}(\lambda u) < \infty,\ \text{for some } \lambda > 0\},$$

where

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x$$

is a modular. We equip $L^{p(\cdot)}(\Omega)$ with the following Luxembourg-type norm

$$\|u\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}.$$

*Examples 2.7* Let $p(x) = x$ on $\Omega = (1, 2)$. Then $\|1\|_{p(\cdot)} = 1$. Indeed,

$$\varrho_{p(\cdot)}(1/\lambda) = \int_1^2 \lambda^{-x} \mathrm{d}x = \frac{\lambda - 1}{\lambda^2 \ln \lambda}.$$

Since $\varrho_{p(\cdot)}(1) = 1$, then, by definition of $\|1\|_{p(\cdot)}$, we have $\|1\|_{p(\cdot)} \le 1$. On the other hand, it is easy to check that $\varrho_{p(\cdot)}(1/\lambda) > 1$, for $0 < \lambda < 1$. This gives $\|1\|_{p(\cdot)} \ge 1$. Hence, we conclude that $\|1\|_{p(\cdot)} = 1$.

**Lemma 2.8** *If $p(x) \equiv p$, where $p$ is constant. Then,*

$$\|u\|_{p(\cdot)} = \lambda_0 = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}. \tag{2.2}$$

*Proof* Since $\varrho_{p(\cdot)}(u/\lambda_0) = 1$, then

$$\|u\|_{p(\cdot)} \le \lambda_0. \tag{2.3}$$

Next, using property of Inf, there exists a sequence $\{\lambda_k\}_{k=1}^\infty$ such that $\lambda_k \ge \|u\|_{p(\cdot)}$, with

$$\varrho_{p(\cdot)}(u/\lambda_k) \le 1 \ \text{ and } \ \lambda_k \to \|u\|_{p(\cdot)}.$$

Since, $\varrho_{p(\cdot)}(u/\lambda_k) = \frac{1}{(\lambda_k)^p} \int_\Omega |u|^p \le 1$, then we have

$$\lambda_0 \le \|u\|_{p(\cdot)}. \tag{2.4}$$

Combining (2.3) and (2.4) gives (2.2). □

**Definition 2.9** We say that a function $q : \Omega \to \mathbb{R}$ is log-Hölder continuous on $\Omega$, if there exist $A > 0$ and $0 < \delta < 1$ such that

$$|q(x) - q(y)| \le -\frac{A}{\log|x - y|},\ \text{ for all } x, y \in \Omega,\quad \text{with } |x - y| < \delta. \tag{2.5}$$

*Examples 2.10* Let $p(x) = x^2 + 1$ be defined on $\Omega = B(0, 1)$. Then $p : \Omega \to \mathbb{R}$ is log-Hölder continuous on $\Omega$. Indeed, Let $(x, y), (x_0, y_0) \in \Omega$, with $|(x, y) - (x_0, y_0)| < \delta$ and $0 < \delta < 1$. Then,

$$\begin{aligned} |p(x, y) - p(x_0, y_0)| &= |x^2 - x_0^2| \\ &= |x - x_0||x + x_0| \\ &\le \frac{4 \log \delta}{\log \delta} \\ &\le -\frac{A}{\log|(x, y) - (x_0, y_0)|}, \end{aligned}$$

where $A = 4 \log(1/\delta)$. Hence, $p$ is log-Hölder continuous.

**Lemma 2.11** (Unit ball property) *Let $p \in \mathcal{P}(A, \mu)$ and $f \in L^{p(\cdot)}(A, \mu)$. Then*

(i) $\|f\|_{p(\cdot)} \le 1$ *if and only if* $\varrho_{p(\cdot)}(f) \le 1$

(ii) *If* $\|f\|_{p(\cdot)} \leq 1$, *then* $\varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$
(iii) *If* $\|f\|_{p(\cdot)} \geq 1$, *then* $\|f\|_{p(\cdot)} \leq \varrho_{p(\cdot)}(f)$
(iv) $\|f\|_{p(\cdot)} \leq 1 + \varrho_{p(\cdot)}(f)$

*Proof* (i) If $\varrho_{p(\cdot)}(f) \leq 1$, then $\|f\|_{p(\cdot)} \leq 1$ by definition of $\|\cdot\|_{p(\cdot)}$. On the other hand, if $\|f\|_{p(\cdot)} \leq 1$, then $\varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1$ for all $\lambda > 1$. Since $\varrho_{p(\cdot)}$ is left-continuous it follows that $\varrho_{p(\cdot)}(f) \leq 1$.
(ii) The claim is obvious for $f = 0$, so assume that $0 < \|f\|_{p(\cdot)} \leq 1$. By (i) and $\left\| f/\|f\|_{p(\cdot)} \right\| = 1$, it follows that $\varrho_{p(\cdot)}\big(f/\|f\|_{p(\cdot)}\big) \leq 1$. Since $\|f\|_{p(\cdot)} \leq 1$, it follows from (2.1) that $\varrho_{p(\cdot)}(f)/\|f\|_{p(\cdot)} \leq 1$. This implies $\varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$.
(iii) Assume that $\|f\|_{p(\cdot)} > 1$. Then $\varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) > 1$ for $1 < \lambda < \|f\|_{p(\cdot)}$ and by (2.1) it follows that $1 < \frac{\varrho_{p(\cdot)}(f)}{\lambda}$. which implies $\lambda < \varrho_{p(\cdot)}(f)$, for $1 < \lambda < \|f\|_{p(\cdot)}$. Since $\lambda$ is arbitrary, we have $\|f\|_{p(\cdot)} \leq \varrho_{p(\cdot)}(f)$.
(iv) This follows immediately from (ii) and (iii).

□

We also state, without proof, some useful results from [41].

**Lemma 2.12** *If* $1 < p_1 \leq p(x) \leq p_2 < +\infty$ *holds, then*

$$\min\{\|u\|^{p_1}_{p(\cdot)}, \|u\|^{p_2}_{p(\cdot)}\} \leq \varrho_{p(\cdot)}(u) \leq \max\{\|u\|^{p_1}_{p(\cdot)}, \|u\|^{p_2}_{p(\cdot)}\},$$

*for any* $u \in L^{p(\cdot)}(\Omega)$.

**Theorem 2.13** *If* $p \in \mathcal{P}(A, \mu)$, *then* $L^{p(\cdot)}(A, \mu)$ *is a Banach space.*

**Lemma 2.14** *If* $p : \Omega \to [1, \infty)$ *is a measurable function with* $p_2 < +\infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

**Lemma 2.15** (Young's inequality) *Let* $p, q, s \in \mathcal{P}(\Omega)$ *such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \textit{for a.e} \quad y \in \Omega.$$

*Then for all* $a, b \geq 0$,

$$\frac{(ab)^{s(\cdot)}}{s(\cdot)} \leq \frac{(a)^{p(\cdot)}}{p(\cdot)} + \frac{(b)^{q(\cdot)}}{q(\cdot)}. \tag{2.6}$$

*By taking* $s = 1$, *and* $1 < p, q < +\infty$, *then we have for any* $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad \forall a, b \geq 0,$$

*where* $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. *For* $p = q = 2$, *we have*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

**Lemma 2.16** (Hölder's Inequality) *Let* $p, q, s \in \mathcal{P}(\Omega)$ *such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \textit{for a.e} \quad y \in \Omega.$$

*If* $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{q(\cdot)}(\Omega)$, *then* $fg \in L^{s(\cdot)}(\Omega)$ *and*

$$\|fg\|_{s(\cdot)} \leq 2\, \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

*By taking* $p = q = 2$, *we have the Cauchy–Schwarz inequality.*

## 2.3 Variable-exponent Sobolev spaces

In this section we study some functional analysis-type properties of Sobolev spaces with variable exponents. We start by recalling the definition of weak derivative.

**Definition 2.17** (*Weak derivative*) Let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial^{|\alpha|}\psi}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \psi g \, \mathrm{d}x,$$

for all $\psi \in C_0^\infty(\Omega)$, then $g$ is called a *weak partial derivative* of $u$ of order $\alpha$. The function $g$ is denoted by $\partial_\alpha u$ or $\frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}$.

**Definition 2.18** Let $k \in \mathbb{N}$. We define the space $W^{k,p(\cdot)}(\Omega)$ by

$$W^{k,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) \text{ such that } \partial_\alpha u \in L^{p(\cdot)}(\Omega),\ \forall\, |\alpha| \le k\}.$$

We define a semimodular on $W^{k,p(\cdot)}(\Omega)$ by

$$\varrho_{W^{k,p(\cdot)}(\Omega)}(u) = \sum_{0 \le |\alpha| \le k} \varrho_{L^{p(\cdot)}(\Omega)}(\partial_\alpha u).$$

This induces a norm given by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \varrho_{W^{k,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \le 1\right\} := \sum_{0 \le |\alpha| \le k} \|\partial_\alpha u\|_{p(\cdot)}.$$

For $k \in \mathbb{N}$, the space $W^{k,p(\cdot)}(\Omega)$ is called Sobolev space and its elements are called Sobolev functions. Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$ and

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \ \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

**Theorem 2.19** *Let $p \in \mathcal{P}(\Omega)$. The space $W^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if $p$ is bounded, and reflexive if $1 < p_1 \le p_2 < +\infty$.*

**Definition 2.20** Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The Sobolev space $W_0^{k,p(\cdot)}(\Omega)$ "with zero boundary trace" is the closure in $W^{k,p(\cdot)}(\Omega)$ of the set of $W^{k,p(\cdot)}(\Omega)-$functions with compact support, i.e.,

$$W_0^{k,p(\cdot)}(\Omega) = \overline{\{u \in W^{k,p(\cdot)}(\Omega) : u = u\chi_K \text{ for a compact } K \subset \Omega\}}.$$

*Remark 2.21* [41] Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. Then

(i) The space $H_0^{k,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.
(ii) $H_0^{k,p(\cdot)}(\Omega) \subset W_0^{k,p(\cdot)}(\Omega)$.
(iii) If $p$ is log-Hölder continuous on $\Omega$, then $W_0^{k,p(\cdot)}(\Omega) = H_0^{k,p(\cdot)}(\Omega)$.
(iv) The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W^{-1,p'(\cdot)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

**Theorem 2.22** *Let $p \in \mathcal{P}(\Omega)$. The space $W_0^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if $p$ is bounded, and reflexive if $1 < p_1 \le p_2 < +\infty$.*

**Theorem 2.23** (Poincaré's inequality) *Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies the Log-Hölder continuity property, then*

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}, \quad \textit{for all} \;\; u \in W_0^{1,p(\cdot)}(\Omega),$$

*where the positive constant $C$ depends on $p(\cdot)$ and $\Omega$ only. In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$.*

*If $p = 2$, then we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.*

*Remark 2.24* Contrary to the constant-exponent case, there is no Poincaré inequality version for modular. The following example shows that the Poincaré inequality does not, in general, hold in a modular form.

*Examples 2.25* [41] Let $p : (-2, 2) \longrightarrow [2, 3]$ be a Lipschitz continuous exponent defined by

$$p(x) = \begin{cases} 3, & \text{if } x \in (-2,-1) \cup (1,2) \\ 2, & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -2x+1, & \text{if } x \in \left[-1, -\frac{1}{2}\right] \\ 2x+1, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let $u_\mu$ be a Lipschitz function defined by

$$u_\mu(x) = \begin{cases} \mu x + 2\mu, & \text{if } x \in (-2,-1] \\ \mu, & \text{if } x \in (-1,1) \\ -\mu x + 2\mu, & \text{if } x \in [1,2). \end{cases}$$

Then

$$\frac{\varrho(u_\mu)}{\varrho(u'_\mu)} = \frac{\int_{-2}^{2} |u_\mu|^{p(x)}\,\mathrm{d}x}{\int_{-2}^{2} |u'_\mu|^{p(x)}\,\mathrm{d}x} \ge \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \mu^2\,\mathrm{d}x}{2\int_{-2}^{-1} \mu^3\,\mathrm{d}x} = \frac{1}{2\mu} \to \infty$$

as $\mu \to 0^+$.

We end this section with some essential embedding results. See [41].

**Lemma 2.26** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p : \Omega \to (1, \infty)$ is a measurable function such that*

$$1 < p_1 \le p(x) \le p_2 < +\infty, \quad \textit{for a.e. } x \in \Omega.$$

*If $p(x), q(x) \in C(\overline{\Omega})$ and $q(x) < p^*(x)$ in $\overline{\Omega}$ with* $p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \textit{if } p_2 < n \\ \infty, & \textit{if } p_2 \ge n. \end{cases}$

*Then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

As a special case we have

**Corollary 2.27** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p : \overline{\Omega} \to (1, \infty)$ is a continuous function such that*

$$2 \le p_1 \le p(x) \le p_2 < \frac{2n}{n-2}, \quad n \ge 3. \tag{2.7}$$

*Then the embedding $H^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.*

## 3 Existence

In this section we discuss a wave problem in the presence of a nonlinear damping, where the nonlinearity is of variable-exponent type. We establish the well posedness, using the Galerkin method and adopting the steps of the book [50] used for standard linearities.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m(\cdot)-2}u_t = 0, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{P}$$

where $m(\cdot)$ is a given continuous function on $\Omega$ satisfying

$$2 \le m_1 \le m(x) \le m_2 < \frac{2n}{n-2}, \quad n \ge 3, \tag{3.1}$$

with

$$m_1 := \text{ess inf}_{x\in\Omega} m(x), \quad m_2 := \text{ess sup}_{x\in\Omega} m(x),$$

and the log-Hölder continuity condition:

$$|m(x) - m(y)| \le -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta, \tag{3.2}$$

with $A > 0, \quad 0 < \delta < 1$.

**Theorem 3.1** *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that the exponent $m(\cdot)$ satisfies conditions (3.1) and (3.2). Then problem (P) has a unique weak solution such that*

$$u \in L^\infty((0,T), H_0^1(\Omega)), \quad u_t \in L^\infty((0,T), L^2(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)),$$
$$u_{tt} \in L^2((0,T), H^{-1}(\Omega)), \ \forall\, T > 0.$$

*Proof* **Uniqueness**
Suppose that ($P$) has two solutions $u$ and $v$. Then, $w = u - v$ satisfies

$$\begin{cases} w_{tt} - \Delta w + u_t|u_t|^{m(\cdot)-2} - v_t|v_t|^{m(\cdot)-2} = 0, & \text{in } \Omega \times (0,T) \\ w(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = w_t(x,0) = 0, & \text{in } \Omega. \end{cases}$$

Multiply the equation by $w_t$ and integrate over $\Omega$, to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega w_t^2 + \int_\Omega |\nabla w|^2\right] + \int_\Omega (u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2})(u_t - v_t)\mathrm{d}x = 0.$$

Integrate over $(0,t)$, to get

$$\int_\Omega (w_t^2 + |\nabla w|^2) + 2\int_0^t \int_\Omega (u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2})(u_t - v_t)\mathrm{d}x\mathrm{d}s = 0.$$

Using the inequality

$$(|\mathrm{a}|^{m(x)-2}\mathrm{a} - |\mathrm{b}|^{m(x)-2}\mathrm{b})(\mathrm{a} - \mathrm{b}) \ge 0,$$

for all a,b $\in \mathbb{R}$ and a.e. $x \in \Omega$, we have

$$\int_\Omega \left(w_t^2 + |\nabla w|^2\right) = 0;$$

which implies that $w = C = 0$, since $w = 0$ on $\partial\Omega$. Hence, the uniqueness.

**Existence**

Let $\{v_j\}_{j=1}^{\infty}$ be an orthonormal basis of $H_0^1(\Omega)$, with

$$-\Delta v_j = \lambda_j v_j, \quad \text{in} \quad \Omega, \quad v_j = 0, \quad \text{on} \quad \partial\Omega.$$

We define the finite-dimensional subspace $V_k = \text{span}\{v_1, \dots, v_k\}$. By normalization, we have $\|v_j\|_2 = 1$. We look for functions

$$u^k(x,t) = \sum_{j=1}^{k} a_j(t) v_j$$

which satisfy the following approximate problems

$$\begin{cases} \int_\Omega u_{tt}^k(x,t)v_j(x)\mathrm{d}x + \int_\Omega \nabla u^k(x,t)\nabla v_j(x)\mathrm{d}x + \int_\Omega |u_t^k(x,t)|^{m(x)-2}u_t^k(x,t)v_j(x)\mathrm{d}x = 0, \\ u^k(x,0) = u_0^k, \ u_t^k(x,0) = u_1^k, \quad \forall j = 1,2,\dots,k, \end{cases} \tag{3.3}$$

where $u_0^k = \sum_{i=1}^k (u_0, v_i)v_i$ and $u_1^k = \sum_{i=1}^k (u_1, v_i)v_i$ are two sequences in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively, such that

$$u_0^k \to u_0 \ \text{ in } \ H_0^1(\Omega) \ \text{ and } u_1^k \to u_1 \ \text{ in } \ L^2(\Omega).$$

This generates the system of $k$ ordinary differential equations

$$\begin{cases} a_j''(t) + \lambda_j a_j(t) = G_j(a_1'(t), \dots, a_k'(t)), \\ a_j(0) = (u_0, v_j), a_j'(0) = (u_1, v_j), \quad \forall j = 1,2,\dots,k, \end{cases} \tag{3.4}$$

where

$$G_j(a_1'(t), \dots, a_k'(t)) = -\int_\Omega \Big|\sum_{i=1}^k a_i'(t)v_i(x)\Big|^{m(x)-2} \sum_{i=1}^k a_i'(t)v_i(x)v_j(x)\mathrm{d}x.$$

This system can be solved by standard ODE theory. Hence, we obtain functions

$$a_j : [0, t_k) \to \mathbb{R}, \ 0 < t_k < T.$$

Next, we have to show that $t_k = T, \ \forall k \geq 1$. We multiply the equation in (3.3) by $a_j'(t)$ and sum over $j$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega (|u_t^k(x,t)|^2\mathrm{d}x + |\nabla u^k(x,t)|^2)\mathrm{d}x\right] + \int_\Omega |u_t^k(x,t)|^{m(x)}\mathrm{d}x = 0.$$

Integration over $(0, t)$ gives

$$\begin{aligned} &\frac{1}{2}\int_\Omega (|u_t^k(x,t)|^2\mathrm{d}x + |\nabla u^k(x,t)|^2)\mathrm{d}x + \int_0^t \int_\Omega |u_t^k(x,s)|^{m(x)}\mathrm{d}x\mathrm{d}s \\ &= \frac{1}{2}\int_\Omega (|u_1^k|^2 + |\nabla u_0^k|^2)\mathrm{d}x \\ &\leq \frac{1}{2}\int_\Omega (u_1^2 + |\nabla u_0|^2)\mathrm{d}x = C. \end{aligned} \tag{3.5}$$

So, we have

$$\sup_{(0,t_k)} \int_\Omega |u_t^k(x,t)|^2\mathrm{d}x + \sup_{(0,t_k)} \int_\Omega |\nabla u^k(x,t)|^2\mathrm{d}x + \int_0^{t_k} \int_\Omega |u_t^k(x,s)|^{m(x)}\mathrm{d}x\mathrm{d}s \leq \widetilde{C}. \tag{3.6}$$

Thus, the solution can be extended to $[0, T)$ and, in addition, we have

$$\begin{aligned} &(u^k) \text{ is a bounded sequence in } L^\infty((0,T), H_0^1(\Omega)) \\ &(u_t^k) \text{ is a bounded sequence in } L^\infty((0,T), L^2(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)). \end{aligned}$$

Therefore, we can extract a subsequence $(u^\ell)$ such that

$$\begin{aligned} u^\ell &\to u \text{ weakly} * \text{ in } L^\infty((0,T), H_0^1(\Omega)) \\ u_t^\ell &\to u_t \text{ weakly} * \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ and weakly in } L^{m(\cdot)}(\Omega \times (0,T)). \end{aligned}$$

We can conclude by Lion's Lemma [50] that $u \in C\big([0,T], L^2(\Omega)\big)$ so that $u(x,0)$ has a meaning. Since $(u_t^\ell)$ is bounded in $L^{m(\cdot)}(\Omega \times (0,T))$ then $(|u_t^\ell|^{m(x)-2}u_t^\ell)$ is bounded in $L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega \times (0,T))$; hence, up to a subsequence,

$$|u_t^\ell|^{m(\cdot)-2}u_t^\ell \to \psi \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}\big(\Omega \times (0,T)\big).$$

We have to show that $\psi = |u_t|^{m(\cdot)-2}u_t$. In (3.3), we use $u^\ell$ instead of $u^k$ and integrate over $(0,t)$ to obtain

$$\int_\Omega u_t^\ell v_j - \int_\Omega u_1^\ell v_j + \int_0^t \int_\Omega \nabla u^\ell \cdot \nabla v_j + \int_0^t \int_\Omega |u_t^\ell|^{m(x)-2}u_t^\ell v_j = 0, \quad \forall j < \ell.$$

As $\ell$ goes to $+\infty$, we easily check that

$$\int_\Omega u_t v_j - \int_\Omega u_1 v_j + \int_0^t \int_\Omega \nabla u \cdot \nabla v_j + \int_0^t \int_\Omega \psi v_j = 0, \quad \forall j \geq 1.$$

Consequently,

$$\int_\Omega u_t v - \int_\Omega u_1 v + \int_0^t \int_\Omega \nabla u \cdot \nabla v + \int_0^t \int_\Omega \psi v = 0, \quad \forall v \in H_0^1(\Omega).$$

All terms define absolute continuous functions; so we get, for a.e $t \in [0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t v + \int_\Omega \big(\nabla u \cdot \nabla v + \psi v\big) = 0, \quad \forall v \in H_0^1(\Omega). \tag{3.7}$$

This implies that

$$u_{tt} - \Delta u + \psi = 0, \text{ in } D'\big(\Omega \times (0,T)\big). \tag{3.8}$$

For simplicity, let $A(v) = |v|^{m(x)-2}v$ and define

$$X^\ell = \int_0^T \int_\Omega (A(u_t^\ell) - A(v))(u_t^\ell - v)\mathrm{d}t \geq 0, \quad \forall v \in L^{m(\cdot)}((0,T)H_0^1(\Omega)).$$

So, using (3.5) and replacing $u^k$ by $u^\ell$, we get

$$\begin{aligned} X^\ell = &\frac{1}{2}\int_\Omega (|u_1^\ell|^2 + |\nabla u_0^\ell|^2) - \frac{1}{2}\int_\Omega |u_t^\ell(x,T)|^2 \\ &- \frac{1}{2}\int_\Omega |\nabla u^\ell(x,T)|^2 - \int_0^T \int_\Omega A(u_t^\ell)v - \int_0^T \int_\Omega A(v)(u_t^\ell - v). \end{aligned} \tag{3.9}$$

Taking $\ell \to \infty$, we obtain

$$\begin{aligned} 0 \leq \limsup_\ell X^\ell \leq &\frac{1}{2}\int_\Omega \big(u_1^2 + |\nabla u_0|^2\big) - \frac{1}{2}\int_\Omega |u_t(x,T)|^2 \\ &- \frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 - \int_0^T \int_\Omega \psi v - \int_0^T \int_\Omega A(v)(u_t - v). \end{aligned} \tag{3.10}$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, replacing $v$ by $u_t$ in (3.7) and integrating over $(0,T)$, we arrive at

$$\frac{1}{2}\int_\Omega |u_t(x,T)|^2 - \frac{1}{2}\int_\Omega u_1^2 + \frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 - \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \int_0^T \int_\Omega \psi u_t = 0. \tag{3.11}$$

Addition of (3.10) and (3.11) yields

$$0 \leq \limsup_{\ell} X^{\ell} \leq \int_0^T \int_{\Omega} \psi u_t - \int_0^T \int_{\Omega} \psi v - \int_0^T \int_{\Omega} A(v)(u_t - v).$$

That is,

$$\int_0^T \int_{\Omega} (\psi - A(v))(u_t - v)\mathrm{d}t \geq 0, \quad \forall v \in L^{m(\cdot)}((0,T) H_0^1(\Omega)).$$

Hence,

$$\int_0^T \int_{\Omega} \big(\psi - A(v)\big)(u_t - v)\mathrm{d}t \geq 0, \quad \forall v \in L^{m(\cdot)}\big(\Omega \times (0,T)\big),$$

by density of $H_0^1(\Omega)$ in $L^{m(\cdot)}(\Omega)$ (Lemma 2.14).

Let $v = \lambda w + u_t$, for $w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big)$. So, we get

$$-\lambda \int_0^T \int_{\Omega} \big(\psi - A(\lambda w + u_t)\big) w \geq 0, \quad \forall \lambda \neq 0, \ \forall w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big).$$

For $\lambda > 0$, we have

$$\int_0^T \int_{\Omega} (\psi - A(\lambda w + u_t)) w \leq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0,T)).$$

As $\lambda \to 0$ and using the continuity of $A$ with respect to $\lambda$, we get

$$\int_0^T \int_{\Omega} (\psi - A(u_t)) w \leq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0,T)).$$

Similarly, for $\lambda < 0$, we get

$$\int_0^T \int_{\Omega} (\psi - A(u_t)) w \geq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0,T)).$$

This implies that $\psi = A(u_t)$. Hence, (3.7) becomes

$$\int_{\Omega} (u_{tt} v + \nabla u \cdot \nabla v + |u_t|^{m(x)-2} u_t v) = 0, \ \forall v \in L^{m(\cdot)}((0,T) \times H_0^1(\Omega)),$$

which gives

$$u_{tt} - \Delta u + |u_t|^{m(x)-2} u_t = 0, \quad \text{in } D'(\Omega \times (0,T)).$$

To handle the initial conditions, we note that

$$\begin{aligned} u^l &\rightharpoonup u \text{ weakly} * \text{ in } L^{\infty}((0,T), H_0^1(\Omega)) \\ u_t^l &\rightharpoonup u_t \text{ weakly} * \text{ in } L^{\infty}((0,T), L^2(\Omega)). \end{aligned} \tag{3.12}$$

Thus, using Lions' Lemma [50], we obtain, up to a subsequence,

$$u^l \to u \quad \text{in } C([0,T], L^2(\Omega)). \tag{3.13}$$

Therefore, $u^l(x,0)$ makes sense and $u^l(x,0) \to u(x,0)$ in $L^2(\Omega)$. Also we have that

$$u^l(x,0) = u_0^l(x) \to u_0(x) \quad \text{in } H_0^1(\Omega).$$

Hence,

$$u(x,0) = u_0(x). \tag{3.14}$$

As in [52], let $\phi \in C_0^\infty(0,T)$ and replacing $(u^k)$ by $(u^l)$, we obtain, from (3.3) and for any $j \le l$,

$$\begin{aligned} &-\int_0^T \int_\Omega u_t^l(x,t)v_j(x)\phi'(t)\mathrm{d}x\mathrm{d}t \\ &\quad = -\int_0^T \int_\Omega \nabla u^\ell(x,t)\nabla v_j(x)\phi(t)\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega |u_t^\ell(x,t)|^{m(x)-2}u_t^\ell(x,t)v_j(x)\phi(t)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.15}$$

As $l \to +\infty$, we obtain that

$$\begin{aligned} &-\int_0^T \int_\Omega u_t(x,t)v_j(x)\phi'(t)\mathrm{d}x\mathrm{d}t \\ &\quad = -\int_0^T \int_\Omega \nabla u(x,t)\nabla v_j(x)\phi(t)\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega |u_t(x,t)|^{m(x)-2}u_t(x,t)v_j(x)\phi(t)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.16}$$

for all $j \ge 1$. This implies

$$\begin{aligned} &-\int_0^T \int_\Omega u_t(x,t)v(x)\phi'(t)\mathrm{d}x\mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \big[\Delta u - |u_t(x,t)|^{m(x)-2}u_t(x,t)\big]v(x)\phi(t)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.17}$$

for all $v \in H_0^1(\Omega)$. This means $u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T),H^{-1}(\Omega))$ and $u$ solves the equation

$$u_{tt} - \Delta u + |u_t|^{m(\cdot)-2}u_t = 0. \tag{3.18}$$

Thus, $u_t \in L^\infty([0,T),L^2(\Omega)),\quad u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T),H^{-1}(\Omega))$. Consequently,

$$u_t \in C([0,T),H^{-1}(\Omega)). \tag{3.19}$$

So, $u_t^l(x,0)$ makes sense (see [52, p.116]). It follows that

$$u_t^l(x,0) \to u_t(x,0) \quad \text{in } H^{-1}(\Omega).$$

But

$$u_t^l(x,0) = u_1^l(x) \to u_1(x) \quad \text{in } L^2(\Omega).$$

Hence,

$$u_t(x,0) = u_1(x). \tag{3.20}$$

This ends the proof of Theorem (3.1). □

## 4 Decay

For the best of our knowledge, there are not many stability results for hyperbolic problem involving non-standard nonlinearities. The only works, we are aware of, are that by Ferreira and Messaoudi [22], where they studied a nonlinear viscoelastic plate equation with a lower order perturbation of a $\overrightarrow{p}(x,t)-$Laplacian operator of the form

$$u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)}u + \int_0^t h(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t + f(u) = 0,$$

where

$$\Delta_{\overrightarrow{p}(x,t)}u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\right), \quad \overrightarrow{p} = (p_1,p_2,\ldots,p_n)^T$$

and $h \geq 0$ is a memory kernel that decays at a general rate and $f$ is a nonlinear function. They proved a general decay result under appropriate assumptions on $h$, $f$, and the variable-exponent $\overrightarrow{p}(x,t)-$Laplacian operator. Also, the work of Yunzhu Gao and Wenjie Gao [25], where they considered the following nonlinear viscoelastic hyperbolic problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $m(x),\ p(x)$ are continuous functions in $\Omega$ such that

$$1 < \inf_{x\in\Omega} m(x) \leq m(x) \leq \sup_{x\in\Omega} m(x) < +\infty, \quad 1 < \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) < +\infty$$

and

$$\forall z, \xi \in \Omega,\ |z-\xi| < 1,\ |m(z) - m(\xi)| + |p(z) - p(\xi)| \leq \omega\big(|z-\xi|\big),$$

where

$$\limsup_{t\to 0^+} \omega(\tau) \ln\left(\frac{1}{\tau}\right) = C < +\infty.$$

They also assumed that

(i) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1$ function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)\mathrm{d}s = \ell > 0;$$

(ii) There exists $\eta > 0$ such that

$$g'(t) \leq -\eta g(t), \quad t \geq 0$$

and proved the existence of the weak solution to problem (4.1).

To establish our decay result, we need the following well-known lemma.

**Lemma 4.1** [38] *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function. Assume that there exist $\sigma > 0,\ \omega > 0$ such that*

$$\int_s^{\infty} E^{1+\sigma}(t)dt \leq \frac{1}{\omega} E^{\sigma}(0)E(s) = \widetilde{c}E(s),\ \forall s > 0.$$

*Then, for all $t \geq 0$,*

$$E(t) \leq \begin{cases} \frac{cE(0)}{(1+t)^{1/\sigma}}, & \text{if } \sigma > 0. \\ cE(0)e^{-\omega t}, & \text{if } \sigma = 0. \end{cases}$$

We define the energy of the solution by

$$E(t) = \frac{1}{2}\int_{\Omega} (u_t^2 + |\nabla u|^2)\mathrm{d}x,$$

which satisfies

$$E'(t) = -\int_{\Omega} |u_t|^{m(x)}\mathrm{d}x \leq 0.$$

This means that $E(t)$ is a nonincreasing function.

**Theorem 4.2** *Suppose the conditions of Theorem 3.1 hold. Then there exist two constants* $c,\ \alpha > 0$ *such that the energy satisfies,* $\forall t \geq 0$,

$$E(t) \leq \begin{cases} \frac{cE(0)}{(1+t)^{2/(m_2-2)}}, & \text{if } m_2 > 2. \\ \\ ce^{-\alpha t}, & \text{if } m(x) = 2, \end{cases}$$

where

$$m_2 := \text{ess sup}_{x\in\Omega} m(x).$$

*Proof* Multiplying (P) by $uE^q(t)$, for $q > 0$ to be specified later, and integrating over $\Omega \times (s, T),\ T > s$, we obtain that

$$\int_s^T E^q(t) \int_\Omega (uu_{tt} - u\Delta u + |u_t|^{m(x)-2} u_t u) = 0,$$

which implies that

$$\int_s^T E^q(t) \int_\Omega \left[(u_t u)_t - u_t^2 + |\nabla u|^2 + |u_t|^{m(x)-2} u_t u\right] = 0.$$

It follows that

$$\begin{aligned} &\int_s^T E^q(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega uu_t \mathrm{d}x + \int_s^T E^q(t) \int_\Omega \left(u_t^2 + |\nabla u|^2\right) \mathrm{d}x - 2\int_s^T E^q(t) \int_\Omega u_t^2 \\ &+ \int_s^T E^q(t) \int_\Omega |u_t|^{m(x)-2} u_t u = 0. \end{aligned} \tag{4.2}$$

Using the definition of $E(t)$ and the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E^q(t) \int_\Omega uu_t \mathrm{d}x\right) = qE^{q-1}(t)E'(t) \int_\Omega uu_t \mathrm{d}x + E^q(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega uu_t \mathrm{d}x,$$

Equation (4.2) becomes

$$\begin{aligned} 2\int_s^T E^{q+1}(t)\mathrm{d}t &= -\int_s^T \frac{\mathrm{d}}{\mathrm{d}t}\left(E^q(t) \int_\Omega uu_t \mathrm{d}x\right) + q\int_s^T E^{q-1}(t)E'(t) \int_\Omega uu_t \mathrm{d}x \\ &+2\int_s^T E^q(t) \int_\Omega u_t^2 - \int_s^T E^q(t) \int_\Omega |u_t|^{m(x)-2} u_t u. \end{aligned} \tag{4.3}$$

**Estimates**

$$\begin{aligned} \cdot\ \left|-\int_s^T \frac{\mathrm{d}}{\mathrm{d}t}\left(E^q(t) \int_\Omega uu_t \mathrm{d}x\right)\right| &= \left|E^q(s) \int_\Omega uu_t(x,s)\mathrm{d}x - E^q(T) \int_\Omega uu_t(x,T)\mathrm{d}x\right| \\ &\leq E^q(s)\left[\frac{1}{2}\int_\Omega u^2(x,s)\mathrm{d}x + \frac{1}{2}\int_\Omega u_t^2(x,s)\mathrm{d}x\right] \\ &\quad + E^q(T)\left[\frac{1}{2}\int_\Omega u^2(x,T)\mathrm{d}x + \frac{1}{2}\int_\Omega u_t^2(x,T)\mathrm{d}x\right] \\ &\leq E^q(s)\left[\frac{1}{2}c_* \int_\Omega |\nabla u|^2(x,s)\mathrm{d}x + \frac{1}{2}\int_\Omega u_t^2(x,s)\mathrm{d}x\right] \\ &\quad + E^q(T)\left[\frac{1}{2}c_* \int_\Omega |\nabla u|^2(x,T)\mathrm{d}x + \frac{1}{2}\int_\Omega u_t^2(x,T)\mathrm{d}x\right], \end{aligned}$$

where $c_*$ is the embedding constant. Since $E(t)$ is nonincreasing, then we have

$$\begin{aligned}\left|-\int_s^T \frac{\mathrm{d}}{\mathrm{d}t}\left(E^q(t)\int_\Omega uu_t\mathrm{d}x\right)\right| &\le cE^{q+1}(s)+cE^{q+1}(T)\\ &\le cE^{q+1}(s)\\ &\le cE^q(0)E(s)=\widetilde{c}E(s).\end{aligned} \tag{4.4}$$

$$\begin{aligned}\cdot\ \left|q\int_s^T E^{q-1}(t)E'(t)\int_\Omega uu_t\mathrm{d}x\right| &\le c\int_s^T E^{q-1}(t)\big(-E'(t)\big)E(t)\mathrm{d}t\\ &= cE^{q+1}(s)-cE^{q+1}(T)\\ &\le cE^{q+1}(s)\le \widetilde{c}E(s).\end{aligned} \tag{4.5}$$

For the third term of the right-hand side of (4.3), we set, as in [38],

$$\Omega_+=\{x\in\Omega\ /\ |u_t(x,t)|\ge 1\}\quad \text{and}\quad \Omega_-=\{x\in\Omega\ /\ |u_t(x,t)|<1\}$$

and exploit Hölder's and Young's inequalities and (3.1) as follows

$$\begin{aligned}\left|2\int_s^T E^q(t)\int_\Omega u_t^2\right| &= \left|2\int_s^T E^q(t)\left[\int_{\Omega_+}u_t^2+\int_{\Omega_-}u_t^2\right]\right|\\ &\le c\int_s^T E^q(t)\left[\left(\int_{\Omega_+}|u_t|^{m_1}\right)^{2/m_1}+\left(\int_{\Omega_-}|u_t|^{m_2}\right)^{2/m_2}\right]\\ &\le c\int_s^T E^q(t)\left[\left(\int_{\Omega}|u_t|^{m(x)}\right)^{2/m_1}+\left(\int_{\Omega}|u_t|^{m(x)}\right)^{2/m_2}\right]\\ &\le c\int_s^T E^q(t)\left[\left(-E'(t)\right)^{2/m_1}+\left(-E'(t)\right)^{2/m_2}\right]\\ &\le c\int_s^T E^q(t)\left(-E'(t)\right)^{2/m_2}\mathrm{d}t+c\int_s^T E^q(t)\left(-E'(t)\right)^{2/m_1}\mathrm{d}t\\ &\le c\varepsilon\int_s^T E^{q+1}(t)\mathrm{d}t+c_\varepsilon\int_s^T(-E'(t))^{2(q+1)/m_2}\mathrm{d}t\\ &\quad +c\varepsilon\int_s^T(E(t))^{qm_1/(m_1-2)}\mathrm{d}t+c_\varepsilon\int_s^T(-E'(t))\mathrm{d}t.\end{aligned}$$

We choose $q$ such that $q=\frac{m_2}{2}-1$ and notice that $\frac{qm_1}{m_1-2}=q+1+\frac{m_2-m_1}{m_1-2}$, if $m_1>2$. Hence, we arrive at

$$\begin{aligned}\left|2\int_s^T E^q(t)\int_\Omega u_t^2\right| &\le c\varepsilon\int_s^T E^{q+1}(t)\mathrm{d}t+c\varepsilon(E(0))^{(m_2-m_1)/(m_1-2)}\int_s^T E^{q+1}(t)\mathrm{d}t+c_\varepsilon E(s)\\ &\le \widetilde{c}\varepsilon\int_s^T E^{q+1}(t)\mathrm{d}t+c_\varepsilon E(s).\end{aligned} \tag{4.6}$$

The case $m_1=2$ is similar.

For the last term of (4.3), we use Young's inequality with $p(x)=\frac{m(x)}{m(x)-1}$, $p'(x)=m(x)$; so for a.e. $x\in\Omega$, we have

$$|u_t|^{m(x)-1}|u|\le\varepsilon|u|^{m(x)}+c_\varepsilon(x)|u_t|^{m(x)},$$

where

$$c_\varepsilon(x)=\varepsilon^{1-m(x)}(m(x))^{-m(x)}(m(x)-1)^{m(x)-1}.$$

Therefore,

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_t u\right| &\le \varepsilon \int_s^T E^q(t)\int_\Omega |u|^{m(x)}\mathrm{d}x + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\mathrm{d}x\\ &\le \varepsilon \int_s^T E^q(t)\left[\int_\Omega |u|^{m_1}\mathrm{d}x + \int_\Omega |u|^{m_2}\mathrm{d}x\right]\\ &\quad + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\mathrm{d}x\\ &\le \varepsilon \int_s^T E^q(t)\left[c_1\|\nabla u\|_2^{m_1} + c_2\|\nabla u\|_2^{m_2}\right]\\ &\quad + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\mathrm{d}x.\end{aligned}$$

Using the definition of $E(t)$, we get

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_t u\right| &\le \varepsilon c_1' \int_s^T E^{q+1}(t)(E(t))^{\frac{m_1}{2}-1} + \varepsilon c_2' \int_s^T E^{q+1}(t)(E(t))^{\frac{m_2}{2}-1}\\ &\quad + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\mathrm{d}x\\ &\le c'\varepsilon((E(0))^{\frac{m_1}{2}-1} + (E(0))^{\frac{m_2}{2}-1})\int_s^T E^{q+1}(t)\mathrm{d}t\\ &\quad + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\mathrm{d}x. \end{aligned}\tag{4.7}$$

Therefore, a combination of (4.3)–(4.7) leads to

$$\begin{aligned}2\int_s^T E^{q+1}(t)\mathrm{d}t &\le c\varepsilon(1 + \left((E(0))^{\frac{m_1}{2}-1} + (E(0))^{\frac{m_2}{2}-1}\right)\int_s^T E^{q+1}(t)\mathrm{d}t\\ &+ c_\varepsilon E(s) + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\mathrm{d}x.\end{aligned}\tag{4.8}$$

We then pick $\varepsilon > 0$ so small that

$$c\varepsilon\Big(1 + ((E(0))^{\frac{m_1}{2}-1} + (E(0))^{\frac{m_2}{2}-1}\Big) < 1.$$

Once $\varepsilon$ is fixed, then $c_\varepsilon(x) \le M$, where $M$ is constant since $m(x)$ is bounded.
Thus, we arrive at

$$\begin{aligned}\int_s^T E^{q+1}(t)\mathrm{d}t &\le cE(s) + M\int_s^T E^q(t)\int_\Omega |u_t|^{m(x)}\mathrm{d}x\\ &\le cE(s) - M\int_s^T E^q(t)E'(t)\mathrm{d}t\\ &\le cE(s) + \frac{M}{q+1}\Big(E^{q+1}(s) - E^{q+1}(T)\Big)\\ &\le c\big(E(s) + E^{q+1}(s)\big)\\ &\le c\big(1 + E^q(0)\big)E(s) = \widetilde{c}E(s), \quad \forall\, T > s > 0.\end{aligned}$$

By taking $T \to \infty$, we get

$$\int_s^\infty E^{\frac{m_2}{2}}(t)\mathrm{d}t \le \widetilde{c}E(s).$$

Hence, Komornik's Lemma (with $\sigma = \frac{m_2}{2} - 1$) implies the desired result. □

## 5 Numerical study

In this section, we present two numerical applications to illustrate the decay results in Theorem 4.2. We consider the initial-boundary value problem $(P)$ on the domain $B(0,1)\times(0,T)$, where $B(0,1)$ is the unit disk in $\mathbb{R}^2$. We take the initial data $u_0(x_1,x_2)=1-x_1^2-x_2^2$ and $u_1(x_1,x_2)=0$. We consider the following applications:

1. *Polynomial decay* Here we take the variable-exponent $m(x_1,x_2)=\lfloor x_1\rfloor^2+4$, where $m_2=5$, and numerically verify that
$$E(t)\le c\,E(0)(1+t)^{-2/3},\quad \forall t\ge 0,$$
for some $c>0$.
2. *Exponential decay* We take $m(x_1,x_2)=2$ and we numerically verify that
$$E(t)\le ce^{-\alpha t},\quad \forall t\ge 0,$$
for some $c>0$ and $\alpha>0$.

In the above applications, the exponent function $m(\cdot)$ satisfies conditions (3.1) and (3.2). Below we introduce a numerical scheme of $(P)$ using finite-element and finite-difference methods for the space and time discretization, respectively. Extensive details about these methods can be found in [42,79,80].

### 5.1 Numerical method

We discretize the wave equation
$$u_{tt}-\Delta u+|u_t|^{m(\cdot)-2}u_t=0,\ \text{ in }\ B(0,1)\times(0,T),\tag{5.1}$$
using finite differences for the time variable and a finite-element method for the space variable $x=(x_1,x_2)\in B(0,1)$.

We first discretize the wave equation in time. The time interval $[0,T]$ is divided into $N$ equal subintervals $[t_j,t_{j+1}]$, $j=0,1,\dots,N-1$, where $t_j=j\tau$ and $\tau=T/N$ is the time step. Denoting
$$U^{(j)}:=u(x,y,t_j),$$
we approximate the time derivatives of $u(x,t)$ at $t=t_j$, for $j=1,2,\dots,N-1$, using the following finite-difference formulas:
$$U_t^{(j)}=\frac{U^{(j)}-U^{(j-1)}}{\tau},$$
$$U_{tt}^{(j)}=\frac{U^{(j+1)}-2U^{(j)}+U^{(j-1)}}{\tau^2}.$$
Then, a semi-discrete formulation of (5.1) at $t=t_{j+1}$ reads
$$U_{tt}^{(j+1)}-\Delta U^{(j+1)}+|U_t^{(j)}|^{m-2}U_t^{(j+1)}=0,\tag{5.2}$$
for $j=1,2,...,N-1$. Note that we have used the history data $U^{(j)}$ and $U_t^{(j)}$ to make the formulation linear in $U^{(j+1)}$, which will be essential for the space discretization. The above formulation is defined for $j=0$ by taking $U^{(0)}=u_0(0)$ and $U_t^{(0)}=u_1(0)$.

Now, we discretize the space variable. Let $B_h$ be a triangulation of $B(0,1)$, with a maximal diameter size $h$. Let $\mathcal{P}_h:=P_1(B_h)\cap H_0(B_h)$, where $H_0(B_h)$ denotes the linear Lagrangian finite-dimensional space.

Multiplying (5.2) by a test function $w_h\in\mathcal{P}_h$, integrating by parts and using the boundary condition $U_h^{(j+1)}=0$ on $\partial B_h$, we obtain the full discrete, weak formulation problem: given initial data $U_h^{(0)}:=u_h(0)$ and $U_{h,t}^{(0)}:=u_{1,h}(0)$, find $U_h^{(j+1)}\in\mathcal{P}_h$ that satisfies
$$(U_{h,tt}^{(j+1)}+|U_{h,t}^{(j)}|^{m-2}U_{h,t}^{(j+1)},w_h)+(\nabla U_h^{(j+1)},\nabla w_h)=0,\quad j=0,1,\dots,N-1,\tag{5.3}$$
for all $w_h\in\mathcal{P}_h$. Here, $(\cdot,\cdot)$ denotes the $L^2$ inner product.

The above numerical problem is advanced, and at this stage we assume that the discrete solution of (5.3) is unique and converges to the exact solution of $(P)$, in the $H^1$-norm, as $(h,\tau)\to(0,0)$.

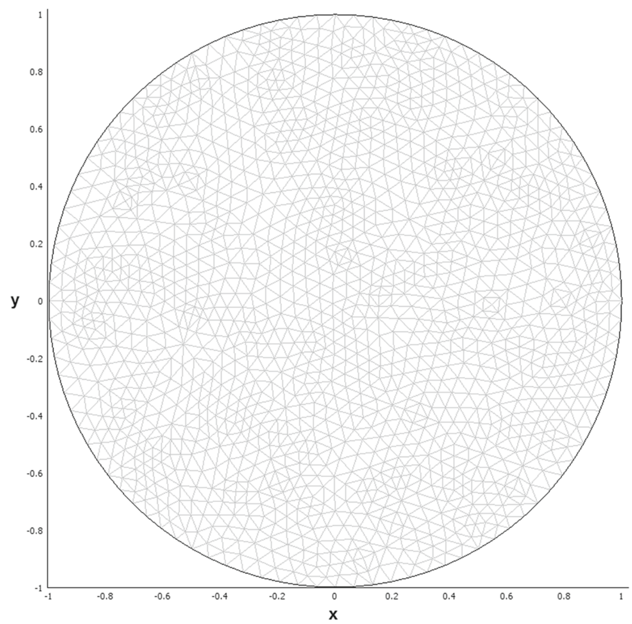



**Fig. 1** $\Omega_h$

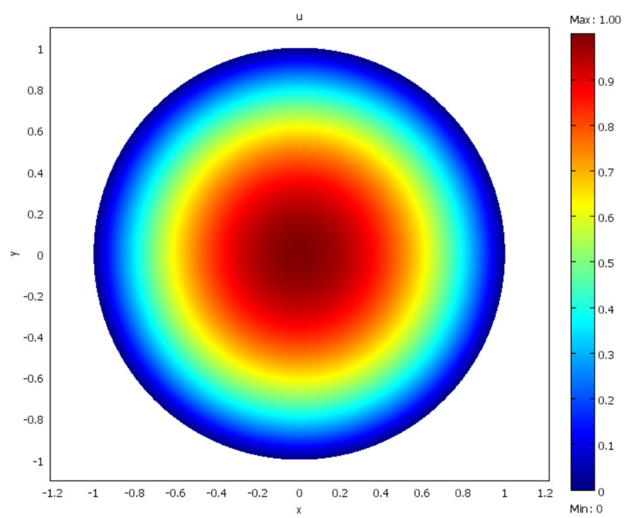



**Fig. 2** Initial data: $u_h^{(0)}$

### 5.2 Numerical results

In this section, we present the decay results of the full discrete model (5.3). The numerical results are obtained using Matlab.

Figure 1 shows the triangulation $\Omega_h$, which consists of 7808 degrees of freedom and 3024 elements. We take a small enough time step $\tau = 0.01$, large enough time span $T = 100$ and $\alpha = 0.1$.

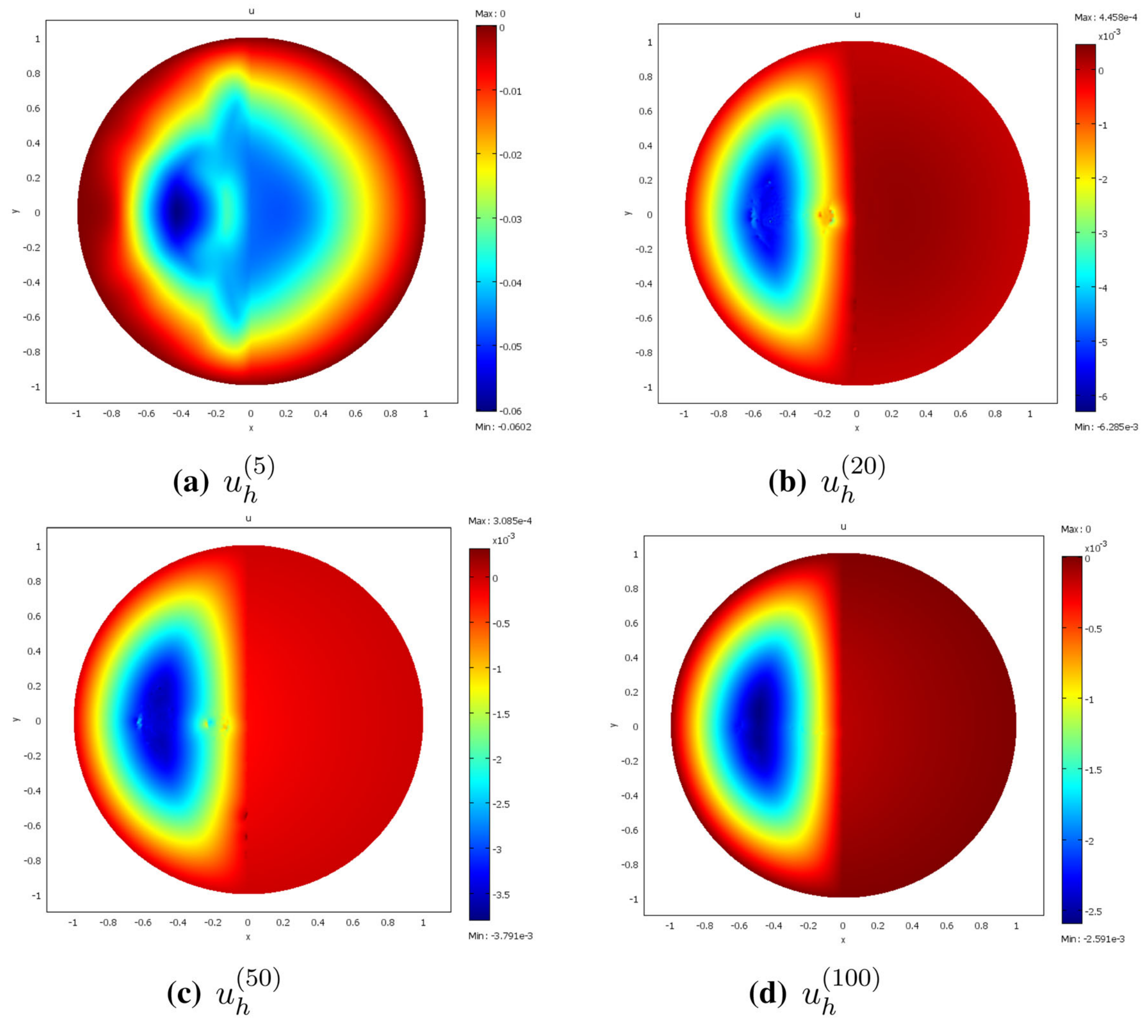
Figure 2 shows the initial data $u_h^{(0)} = 1 - x_1^2 - x_2^2$ projected in the finite-element space $\mathcal{P}_h$.

Now, we present the numerical results corresponding to the polynomial and exponential decay applications.

1. *Polynomial decay* Figure 3 lists the solutions $u_h^{(n)}$ corresponding to $t_n = 5, 20, 50, 100$.
   Figure 4 shows that energy function satisfies

$$E(t_n)(1 + t_n)^{2/3} \leq 4 \;\text{ for }\; t_n \in \{0, 1, \ldots, 100\}.$$

   This implies that the numerical solution $u_h^{(n)}$ has a polynomial decay, which agrees with Theorem 4.2.




**(a)** $u_h^{(5)}$ **(b)** $u_h^{(20)}$

**(c)** $u_h^{(50)}$ **(d)** $u_h^{(100)}$

**Fig. 3** Numerical solutions $u_h^{(n)}$ for the polynomial decay application. **a** $u_h^{(5)}$. **b** $u_h^{(20)}$. **c** $u_h^{(50)}$. **d** $u_h^{(100)}$

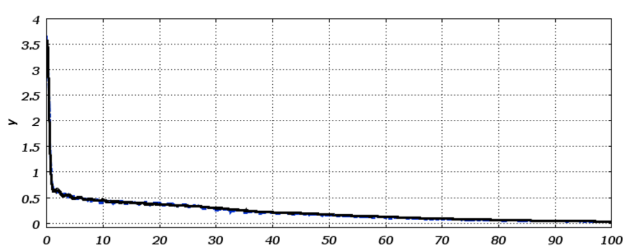



**Fig. 4** Polynomial decay: $y = E(t)(1+t)^{2/3}$

2. *Exponential decay* In this case, the numerical solutions $u_h^{(n)}$ make the energy function $E(t_n)$ satisfy

$$E(t_n)e^{0.1t} \leq 4 \text{ for } t_n \in \{0, 1, \ldots, 100\}.$$

This implies that the numerical solution $u_h^{(n)}$ has an exponential decay, which verifies Theorem 4.2.

In Figs. 4 and 5, the corresponding curves are plotted using $\tau = 0.1$ (the dashed curve) and using $\tau = 0.01$ (the solid curve). In both applications the two curves are coinciding, which indicates that the corresponding numerical solutions are promising.

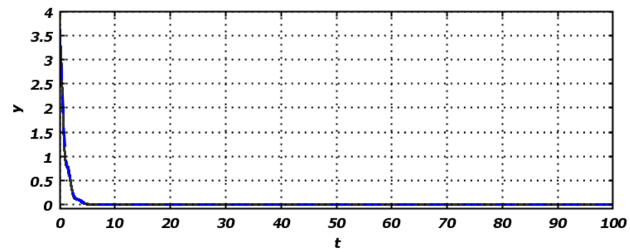



**Fig. 5** Exponential decay: $y = E(t)e^{0.1t}$

In conclusion, the numerical findings of the above applications verify the conclusion of the polynomial and exponential decay results of Theorem 4.2.

## 6 Blowup

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u = a|u|^{p(\cdot)-2}u, & \text{in } \Omega \times (0, T) \\ u(x, t) = 0, & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{Q}$$

where $a > 0$ and $p$ is a given continuous function on $\Omega$ satisfying

$$2 \le p_1 \le p(x) \le p_2 < 2\frac{n-1}{n-2}, \quad n \ge 3, \tag{6.1}$$

with

$$p_1 := \text{ess inf}_{x\in\Omega}\, p(x), \quad p_2 := \text{ess sup}_{x\in\Omega}\, p(x),$$

and the log-Hölder continuity condition:

$$|p(x) - p(y)| \le -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \quad \text{with } |x-y| < \delta, \tag{6.2}$$

$A > 0, \quad 0 < \delta < 1.$

Using a semigroup method, we let $\upsilon = u_t$ and denote by

$$\Phi = (u, \upsilon)^T, \quad \Phi(0) = \Phi_0 = (u_0, u_1)^T \text{ and } J(\Phi) = \left(0, a|u|^{p(\cdot)-2}u\right)^T.$$

Therefore, (Q) can be rewritten as an initial-value problem:

$$\begin{cases} \Phi_t + \mathcal{A}\Phi = J(\Phi) \\ \Phi(0) = \Phi_0, \end{cases} \tag{Q1}$$

where the linear operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = (-\upsilon, -\Delta u)^T.$$

The state space of $\Phi$ is the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega),$$

equipped with the inner product

$$\langle \Phi, \widetilde{\Phi} \rangle_{\mathcal{H}} = \int_\Omega (\nabla u \cdot \nabla \widetilde{u} + \upsilon\widetilde{\upsilon})\mathrm{d}x,$$

for all $\Phi = (u, \upsilon)^T$ and $\widetilde{\Phi} = (\widetilde{u}, \widetilde{\upsilon})^T$ in $\mathcal{H}$. The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \{\Phi \in \mathcal{H} : u \in H^2(\Omega) \cap H_0^1(\Omega), \ \upsilon \in H_0^1(\Omega)\}.$$

Then, we have the following local existence result.

**Theorem 6.1** *Assume that the exponent $p$ satisfies conditions (6.1) and (6.2). Then for any $\Phi_0 \in \mathcal{H}$, problem (Q) has a unique local weak solution $\Phi \in C(\Omega; \mathcal{H})$. That is,*

$$u \in L^\infty((0,T), H_0^1(\Omega)), \quad u_t \in L^\infty((0,T), L^2(\Omega)), \quad u_{tt} \in L^2((0,T), H^{-1}(\Omega)). \tag{6.3}$$

*Proof* First, for all $\Phi \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = -\int_\Omega \nabla \upsilon \cdot \nabla u \, \mathrm{d}x - \int_\Omega \upsilon \Delta u = -\int_\Omega \nabla \upsilon \cdot \nabla u \, \mathrm{d}x + \int_\Omega \nabla \upsilon \cdot \nabla u \, \mathrm{d}x = 0. \tag{6.4}$$

Therefore, $\mathcal{A}$ is a monotone operator.

To show that $\mathcal{A}$ is maximal, we prove that for each

$$F = (f, g)^T \in \mathcal{H},$$

there exists $V = (u, \upsilon)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})V = F$. That is,

$$\begin{cases} u - \upsilon = f \\ \upsilon - \Delta u = g. \end{cases} \tag{6.5}$$

We deduce from (6.5) that

$$u - \Delta u = G, \tag{6.6}$$

where

$$G = f + g \in L^2(\Omega). \tag{6.7}$$

Now we define, over $H_0^1(\Omega)$, the bilinear and linear forms

$$B(u, w) = \int_\Omega uw + \int_\Omega \nabla u \cdot \nabla w, \quad L(w) = \int_\Omega Gw.$$

It is easy to verify that $B$ is continuous and coercive and $L$ is continuous on $H_0^1(\Omega)$. Then, Lax–Milgram theorem implies that the equation

$$B(u, w) = L(w), \quad \forall w \in H_0^1(\Omega), \tag{6.8}$$

has a unique solution $u \in H_0^1(\Omega)$. Hence, $\upsilon = u - f \in H_0^1(\Omega)$. Thus, $V \in \mathcal{H}$.

Using (6.6), we get

$$\int_\Omega uw + \int_\Omega \nabla u \cdot \nabla w = \int_\Omega Gw, \quad \forall\, w \in H_0^1(\Omega).$$

The elliptic regularity theory, then, implies that $u \in H^2(\Omega)$. Therefore,

$$V = (u, \upsilon)^T \in D(\mathcal{A}).$$

Consequently, $I + \mathcal{A}$ is surjective and then $\mathcal{A}$ is maximal.

Finally, we show that $J : \mathcal{H} \to \mathcal{H}$ is locally Lipchitz. So, if we set

$$f(s) = a|s|^{p(x)-2}s \quad \text{then} \quad |f'(s)| = a|p(x) - 1|\, |s|^{p(x)-2}.$$

Hence,

$$\begin{aligned} \|J(\Phi) - J(\widetilde{\Phi})\|_{\mathcal{H}}^2 &= \|(0, a|u|^{p(\cdot)-2}u - a|\widetilde{u}|^{p(\cdot)-2}\widetilde{u})\|_{\mathcal{H}}^2 \\ &= \|a|u|^{p(\cdot)-2}u - a|\widetilde{u}|^{p(\cdot)-2}\widetilde{u})\|_{L^2}^2 \\ &= \|f(u) - f(\widetilde{u})\|_{L^2}^2. \end{aligned}$$

As a consequence of the mean value theorem, we have, for $0 \le \theta \le 1$,

$$\begin{aligned}\|J(\Phi) - J(\widetilde{\Phi})\|_{\mathcal{H}}^2 &= \|f'(\theta u + (1-\theta)\widetilde{u})(u - \widetilde{u})\|_{L^2}^2 \\ &= a\||p(x)-1||\theta u + (1-\theta)\widetilde{u}|^{p(x)-2}(u-\widetilde{u})\|_{L^2}^2 \\ &\le C\||\theta u + (1-\theta)\widetilde{u}|^{p(x)-2}(u-\widetilde{u})\|_{L^2}^2 \\ &\le C\int_\Omega |\theta u + (1-\theta)\widetilde{u}|^{2(p(x)-2)}|u-\widetilde{u}|^2 \mathrm{d}x.\end{aligned}$$

As $u, \widetilde{u} \in H_0^1(\Omega)$, we then use the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, Hölder's inequality and (6.1), to obtain

$$\begin{aligned}\int_\Omega |\theta u + (1-\theta)\widetilde{u}|^{2(p(x)-2)}|u-\widetilde{u}|^2 \mathrm{d}x &\le \left(\int_\Omega |u-\widetilde{u}|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{n}} \left[\left(\int_\Omega |\theta u + (1-\theta)\widetilde{u}|^{n(p_2-2)}\mathrm{d}x\right)^{\frac{2}{n}}\right. \\ &\quad \left. + \left(\int_\Omega |\theta u + (1-\theta)\widetilde{u}|^{n(p_1-2)}\mathrm{d}x\right)^{\frac{2}{n}}\right] \\ &\le \|u-\widetilde{u}\|_{L^{\frac{2n}{n-2}}}^2 \left[\|\theta u + (1-\theta)\widetilde{u}\|_{L^{n(p_2-2)}}^{2(p_2-2)}\right. \\ &\quad \left. + \|\theta u + (1-\theta)\widetilde{u}\|_{L^{n(p_1-2)}}^{2(p_1-2)}\right] \\ &\le C\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2 \left[\left(\|u\|_{L^{n(p_2-2)}} + \|\widetilde{u}\|_{L^{n(p_2-2)}}\right)^{2(p_2-2)}\right. \\ &\quad \left. + \left(\|u\|_{L^{n(p_1-2)}} + \|\widetilde{u}\|_{L^{n(p_1-2)}}\right)^{2(p_1-2)}\right] \\ &\le C\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2 \left[\left(\|u\|_{H_0^1(\Omega)} + \|\widetilde{u}\|_{H_0^1(\Omega)}\right)^{2(p_2-2)}\right. \\ &\quad \left. + \left(\|u\|_{H_0^1(\Omega)} + \|\widetilde{u}\|_{H_0^1(\Omega)}\right)^{2(p_1-2)}\right].\end{aligned}$$

Thus,

$$\left\|J(\Phi) - J(\widetilde{\Phi})\right\|_{\mathcal{H}}^2 \le C(u, \widetilde{u})\left\|\Phi - \widetilde{\Phi}\right\|_{\mathcal{H}}^2.$$

Therefore, $J$ is locally Lipchitz. The proof of Theorem 6.1 is completed. See [38]. □

Next, we show that the solution (6.3) blows up in finite time if (6.1) holds and $E(0) < 0$, where

$$E(t) := \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2\right]\mathrm{d}x - a\int_\Omega \frac{|u|^{p(x)}}{p(x)}\mathrm{d}x. \tag{6.9}$$

We also denote by $\varrho(u) = \int_\Omega |u(x)|^{p(x)}\mathrm{d}x$. First, we establish several lemmas needed for the proof of our blowup result.

**Lemma 6.2** *Suppose the conditions of Corollary 2.27 hold. Then there exists a positive $C > 1$, depending on $\Omega$ only, such that*

$$\varrho^{\frac{s}{p_1}}(u) \le C\left(\|\nabla u\|_2^2 + \varrho(u)\right), \tag{6.10}$$

*for any $u \in H_0^1(\Omega)$ and $2 \le s \le p_1$.*

*Proof* If $\varrho(u) > 1$, then $\varrho^{\frac{s}{p_1}}(u) \leq \varrho(u) \leq C\big(\|\nabla u\|_2^2 + \varrho(u)\big)$, where $C > 1$.

If $\varrho(u) \leq 1$ then, by Lemma 2.11, $\|u\|_{p(\cdot)} \leq 1$. Then, Corollary 2.27 and Lemma 2.12 imply

$$\varrho^{\frac{s}{p_1}}(u) \leq \varrho^{\frac{2}{p_1}}(u) \leq [\max\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\}]^{\frac{2}{p_1}} = \|u\|_{p(\cdot)}^2 \leq C\|\nabla u\|_2^2.$$

Therefore, (6.10) follows. □

As a special case, we have

**Corollary 6.3** *Let the assumptions of Lemma 6.2 hold. Then we have*

$$\|u\|_{p_1}^s \leq C\big(\|\nabla u\|_2^2 + \|u\|_{p_1}^{p_1}\big), \tag{6.11}$$

*for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p_1$.*

We set

$$H(t) := -E(t)$$

and use $C$ to denote a generic positive constant depending on $\Omega$ only. As a result of (6.9) and (6.10), we have

**Corollary 6.4** *Let the assumptions of Lemma 6.2 hold. Then we have*

$$\varrho^{\frac{s}{p_1}}(u) \leq C\big(|H(t)| + \|u_t\|_2^2 + \varrho(u)\big), \tag{6.12}$$

*for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p_1$.*

As a special case, we have

**Corollary 6.5** *Let the assumptions of Lemma 6.2 hold. Then we have*

$$\|u\|_{p_1}^s \leq C\big(|H(t)| + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\big), \tag{6.13}$$

*for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p_1$.*

**Lemma 6.6** *Let the assumptions of Lemma 6.2 hold and let $u$ be the solution of problem (Q), with $E(0) < 0$. Then,*

$$\varrho(u) \geq C\|u\|_{p_1}^{p_1}. \tag{6.14}$$

*Proof* We have

$$\varrho(u) = \int_\Omega |u|^{p(x)}\mathrm{d}x = \int_{\Omega_+} |u|^{p(x)}\mathrm{d}x + \int_{\Omega_-} |u|^{p(x)}\mathrm{d}x,$$

where

$$\Omega_+ = \{x \in \Omega \ / \ |u(x,t)| \geq 1\} \ \text{ and } \ \Omega_- = \{x \in \Omega \ / \ |u(x,t)| < 1\}.$$

so we get

$$\varrho(u) \geq \int_{\Omega+} |u|^{p_1} + \int_{\Omega_-} |u|^{p_2} \geq \int_{\Omega+} |u|^{p_1} + c_1\left(\int_{\Omega-} |u|^{p_1}\right)^{\frac{p_2}{p_1}}.$$

This implies that

$$c_2\big(\varrho(u)\big)^{\frac{p_1}{p_2}} \geq \int_{\Omega-} |u|^{p_1} \ \text{ and } \ \varrho(u) \geq \int_{\Omega+} |u|^{p_1}.$$

This yields

$$c_2\big(\varrho(u)\big)^{\frac{p_1}{p_2}} + \varrho(u) \geq \|u\|_{p_1}^{p_1}. \tag{6.15}$$

Since

$$0 < H(0) \le H(t) \le \frac{a}{p_1}\varrho(u),$$

then (6.15) leads to

$$\varrho(u)\left[1 + c_2\left(\frac{p_1}{a}H(0)\right)^{\frac{p_1}{p_2}-1}\right] \ge \|u\|_{p_1}^{p_1}.$$

Hence, (6.14) follows. □

Our main blowup result reads as follows:

**Theorem 6.7** *Let the conditions of Theorem 6.1 be fulfilled. Assume further that (6.1) holds and*

$$E(0) < 0. \tag{6.16}$$

*Then the solution (6.3) blows up in finite time.*

*Proof* We multiply Eq. (Q) by $u_t$ and integrate over $\Omega$ to get

$$E'(t) = 0, \tag{6.17}$$

hence $H'(t) = 0$ and

$$0 < H(0) = H(t) \le \frac{a}{p_1}\varrho(u), \tag{6.18}$$

for every $t$ in $[0, T)$, by virtue of (6.16). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)\mathrm{d}x, \tag{6.19}$$

for $\varepsilon$ small to be chosen later and

$$0 < \alpha \le \min\left\{1, \frac{p_1 - 2}{2p_1}\right\}. \tag{6.20}$$

By taking the derivative of (6.19) and using equation (Q), we obtain

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega \left[u_t^2 - |\nabla u|^2\right] + \varepsilon a\int_\Omega |u|^{p(x)}. \tag{6.21}$$

Add and subtract $\varepsilon(1-\eta)p_1 H(t)$, for $0 < \eta < 1$, from the right-hand side of (6.21), to arrive at

$$\begin{aligned} L'(t) \ge\ & \varepsilon(1-\eta)p_1 H(t) + \varepsilon a\eta\int_\Omega |u|^{p(x)} + \varepsilon\Big(\frac{(1-\eta)p_1}{2} + 1\Big)\|u_t\|_2^2 \\ & + \varepsilon\Big(\frac{(1-\eta)p_1}{2} - 1\Big)\|\nabla u\|_2^2. \end{aligned} \tag{6.22}$$

For $\eta$ small enough, we see that

$$L'(t) \ge \varepsilon\beta\big[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \varrho(u)\big], \tag{6.23}$$

where

$$\beta = \min\left\{(1-\eta)p_1, a\eta, \frac{(1-\eta)p_1}{2} + 1, \frac{(1-\eta)p_1}{2} - 1\right\} > 0.$$

Therefore, using (6.14), we arrive at

$$L'(t) \ge \gamma\varepsilon\big[H(t) + \|u_t\|_2^2 + \varrho(u)\big] \ge \gamma\varepsilon\big[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\big], \tag{6.24}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad \text{for all} \quad t \geq 0.$$

Next, we would like to show that

$$L'(t) \geq \Gamma L^{\frac{1}{1-\alpha}}(t), \quad \text{for all} \quad t \geq 0, \tag{6.25}$$

where $\Gamma$ is a positive constant depending on $\varepsilon\gamma$ and $C$ (the constant of Corollary 6.3). Once (6.25) is established, we obtain in a standard way the finite-time blowup of $L(t)$. To prove (6.25), we first note that

$$\left| \int_\Omega u u_t(x,t) \mathrm{d}x \right| \leq \|u\|_2 \|u_t\|_2 \leq C \|u\|_{p_1} \|u_t\|_2,$$

which implies

$$\left| \int_\Omega u u_t(x,t) \mathrm{d}x \right|^{\frac{1}{1-\alpha}} \leq C \|u\|_{p_1}^{\frac{1}{1-\alpha}} \|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Again Young's inequality gives

$$\left| \int_\Omega u u_t(x,t) \mathrm{d}x \right|^{1/(1-\alpha)} \leq C[\|u\|_{p_1}^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}], \tag{6.26}$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p_1$ by (6.20).

Therefore, (6.26) becomes

$$\left| \int_\Omega u u_t(x,t) \mathrm{d}x \right|^{1/(1-\alpha)} \leq C[\|u\|_{p_1}^{s} + \|u_t\|_2^2],$$

where $s = 2/(1-2\alpha) \leq p_1$. Using Corollary 6.5, we obtain

$$\left| \int_\Omega u u_t(x,t) \mathrm{d}x \right|^{1/(1-\alpha)} \leq C[H(t) + \|u\|_{p_1}^{p_1} + \|u_t\|_2^2], \quad \text{for all} \quad t \geq 0. \tag{6.27}$$

Finally, by noting that

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left[ H^{(1-\alpha)}(t) + \varepsilon \int_\Omega u u_t(x,t) \mathrm{d}x \right]^{1/(1-\alpha)} \\ &\leq 2^{1/(1-\alpha)} \left[ H(t) + \left| \int_\Omega u u_t \right|^{1/(1-\alpha)} \right] \end{aligned}$$

and combining it with (6.24) and (6.27), the inequality (6.25) is established. A simple integration of (6.25) over $(0,t)$ then yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t \alpha/(1-\alpha)}. \tag{6.28}$$

Therefore, (6.28) shows that $L(t)$ blows up in finite time

$$T^* \leq \frac{1-\alpha}{\Gamma \alpha [L(0)]^{\alpha/(1-\alpha)}}, \tag{6.29}$$

where $\Gamma$ and $\alpha$ are positive constant with $\alpha < 1$ and $L$ is given by (6.19) above. This completes the proof. □

*Remark 6.8* The estimate (6.29) shows that the larger $L(0)$ is, the quicker the blow up takes place.

**Acknowledgements** The authors thank King Fahd University of Petroleum and Minerals for its support. This work is supported by KFUPM under Project no. FT 161004.

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