Journal of Homotopy and Related Structures (2019) 14:293–347
https://doi.org/10.1007/s40062-018-0213-7

Homotopical algebraic context over differential operators

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Received: 19 August 2017 / Accepted: 23 August 2018 / Published online: 18 September 2018

Abstract
Building on our previous work, we show that the category of non-negatively graded chain complexes of $\mathcal{D}_X$-modules – where X is a smooth affine algebraic variety over an algebraically closed field of characteristic zero – fits into a homotopical algebraic context in the sense of Toën and Vezzosi.

Keywords Derived algebraic geometry · Homotopical algebraic context · Model category · Batalin-Vilkovisky complex · Stack over differential operators · Functor of points

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Communicated by Jim Stasheff.

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Introduction

The *classical Batalin-Vilkovisky complex* is, roughly, a kind of resolution of $C^\infty(\Sigma)^{\mathrm{gs}}$. The functions $C^\infty(\Sigma)$ of the shell Σ are obtained by identifying those functions of the infinite jet bundle $J^\infty E$ of the field bundle $E \to X$ that coincide on-shell (quotient). We then get the functions $C^\infty(\Sigma)^{\mathrm{gs}}$ by selecting those on-shell functions that are gauge invariant, i.e., constant along the gauge orbits (intersection). In important particular cases, when working dually, i.e., with spaces instead of function algebras, we first mod out the gauge symmetries, i.e., we consider some space $C := J^\infty E/\mathrm{GS}$, where GS refers to integrated gauge symmetry vector fields gs – thought of as vector fields prolonged to $J^\infty E$. Since, in the function algebra approach, we determine the shell Σ by solving the algebraic infinite jet bundle equation $\mathrm{Alg}(\mathrm{d}\,S) = 0$ that corresponds to the equation $\mathrm{d}\,S = 0$, where S denotes the functional acting on sections of E, it is clear that, in the dual approach, the functional S must be defined on C, i.e., $S \in \mathcal{O}(C)$ and $\mathrm{d}\,S : C \to T^*C$, and that we have then to find those 'points m' in C that satisfy $\mathrm{d}_m\,S = 0$.

When switching to the *context of algebraic geometry*, we start with a quasi-coherent module $\mathcal{E} \in \mathtt{qcMod}(\mathcal{O}_X)$ over the function sheaf $\mathcal{O}_X$ of a *scheme* X. Let now $\mathcal{S}_{\mathcal{O}_X}$ be the corresponding symmetric tensor algebra functor. The quasi-coherent commutative $\mathcal{O}_X$-algebra $\mathcal{S}_{\mathcal{O}_X}\mathcal{E} \in \mathtt{qcCAlg}(\mathcal{O}_X)$ can be viewed as the pushforward $\mathcal{O}_X^E$ of the function sheaf of a vector bundle $E \to X$ (we think about $\mathcal{E}$ as the module of sections of the dual bundle E^*). If X is a *smooth scheme*, the infinite jet functor $\mathcal{J}^\infty$ [4] leads to a sheaf $\mathcal{J}^\infty(\mathcal{O}_X^E) \in \mathtt{qcCAlg}(\mathcal{D}_X)$ of commutative algebras over the sheaf $\mathcal{D}_X$ of rings of differential operators on X, which is quasi-coherent as sheaf of $\mathcal{O}_X$-modules. The spectrum of the latter is the infinite jet bundle $J^\infty E \to X$. This bundle is thus an affine X-$\mathcal{D}_X$-scheme $J^\infty E \in \mathtt{Aff}(\mathcal{D}_X)$.

Since an intersection of schemes may well be not transverse, or, algebraically, since tensor products of commutative rings viewed as certain modules can be badly behaved (tensor product functor only right-exact), these tensor products should be left-derived, i.e., commutative rings or commutative algebras should be replaced by simplicial commutative rings or differential non-negatively graded commutative algebras (category $\mathtt{DGCA}$). Similarly, since quotients of affine schemes can be non-affine (non-trivial automorphism groups), they should be derived, i.e., replaced by groupoids, or, in the case of higher symmetries, by infinity groupoids or simplicial sets (category $\mathtt{SSet}$). For the *functor of points approach* to schemes—schemes are viewed as, say, locally representable sheaves (for the Zariski topology) $G : \mathtt{CA} \to \mathtt{Set}$ from commutative algebras to sets—this means that we pass to functors $F : \mathtt{DGCA} \to \mathtt{SSet}$.

In the following we assume that X is a *smooth affine algebraic variety*, so that we can, roughly speaking, replace sheaves by their global sections. In particular, in the above $\mathcal{D}$-geometric setting, the differential non-negatively graded commutative algebras of the preceding paragraph, i.e., the objects of $\texttt{DGCA}$, become the objects of $\texttt{DG}_+\texttt{qcCAlg}(\mathcal{D}_X)$, i.e., the sheaves of differential non-negatively graded $\mathcal{O}_X$-quasi-coherent commutative $\mathcal{D}_X$-algebras, and, due to the assumption that X is smooth affine, the category $\texttt{DG}_+\texttt{qcCAlg}(\mathcal{D}_X)$ is equivalent [6] to the category $\texttt{DG}\mathcal{D}\texttt{A}$ of differential non-negatively graded commutative algebras over the ring $\mathcal{D} = \mathcal{D}_X(X)$ of global sections of $\mathcal{D}_X$. Let us mention that the latter category is of course the category $\texttt{CMon}(\texttt{DG}\mathcal{D}\texttt{M})$ of commutative monoids in the symmetric monoidal category $\texttt{DG}\mathcal{D}\texttt{M}$ of differential non-negatively graded modules over $\mathcal{D}$ (i.e., the category $\texttt{DG}\mathcal{D}\texttt{M}$ of non-negatively graded chain complexes of $\mathcal{D}$-modules), as well as that, despite the used simplified notation $\texttt{DG}\mathcal{D}\texttt{M}$ and $\texttt{DG}\mathcal{D}\texttt{A}$, the reader should keep in mind the considered non-negative grading and underlying variety X.

It follows that, in $\mathcal{D}$-geometry, the above functors $F : \texttt{DGCA} \to \texttt{SSet}$ become functors

$$F : \texttt{DG}\mathcal{D}\texttt{A} \to \texttt{SSet}.$$

As suggested in the second paragraph, the category $\texttt{DG}\mathcal{D}\texttt{A} \simeq \texttt{DG}_+\texttt{qcCAlg}(\mathcal{D}_X)$ is opposite to the category $\texttt{D\_Aff}(\mathcal{D}_X)$ of *derived affine X-$\mathcal{D}_X$-schemes*. Those functors or presheaves $F : \texttt{DG}\mathcal{D}\texttt{A} \to \texttt{SSet}$ that are actually sheaves (in the sense detailed below) are referred to as *derived X-$\mathcal{D}_X$-stacks* and the model category of presheaves $\texttt{F} := \texttt{Fun}(\texttt{DG}\mathcal{D}\texttt{A}, \texttt{SSet})$ (endowed with its local model structure) models the category of derived X-$\mathcal{D}_X$-stacks [37,38]. The *sheaf condition* is a natural homotopy version of the standard sheaf condition [20]. This homotopical variant is correctly encoded in the fibrant object condition of the *local model structure* of $\texttt{F}$. That structure encrypts both, the model structure of the target and the one of the source [6,7]. More precisely, one starts with the global model structure on $\texttt{F}$, which is the one implemented 'object-wise' by the model structure of the target category $\texttt{SSet}$. The model structure of the source category $\texttt{DG}\mathcal{D}\texttt{A}$ is taken into account via the left Bousfield localization with respect to the weak equivalences of $\texttt{DG}\mathcal{D}\texttt{A}^{\text{op}}$, what leads to a new model category denoted by $\texttt{FO}$. If τ is an appropriate model pre-topology on $\texttt{DG}\mathcal{D}\texttt{A}^{\text{op}}$, it should be possible to define homotopy τ-sheaves of groups, as well as a class H_τ of homotopy τ-hypercovers. The mentioned local model category $\texttt{F}^{\sim,\tau}$ arises now as the left Bousfield localization of $\texttt{FO}$ with respect to H_τ. The local weak equivalences are those natural transformations that induce isomorphisms between all homotopy sheaves. The fibrant object condition in $\texttt{F}^{\sim,\tau}$, which is roughly the descent condition with respect to the homotopy τ-hypercovers, is the searched sheaf or stack condition for derived X-$\mathcal{D}_X$-stacks [37,38]. The notion of derived X-$\mathcal{D}_X$-stack represented by an object in $\texttt{D\_Aff}(\mathcal{D}_X) \simeq \texttt{DG}\mathcal{D}\texttt{A}^{\text{op}}$, i.e., by a derived affine X-$\mathcal{D}_X$-scheme, can easily be defined.

Notice finally that our two assumptions – smooth and affine – on the underlying algebraic variety X are necessary. Exactly the same smoothness condition is indeed used in [4, Remark p. 56], since for an arbitrary singular scheme X, the standard notion of left $\mathcal{D}_X$-module is meaningless and would have to be replaced by its extension defined as quasi-coherent module over the de Rham space X_{dR} of X. On the other hand, the assumption that X is affine is needed to replace the category $\texttt{DG}_+\texttt{qcMod}(\mathcal{D}_X)$ by the category $\mathtt{DG}\mathcal{D}\mathtt{M}$ and to thus avoid the problem of the non-existence of a projective model structure [11]. To be precise, there exists a flat monoidal model structure on chain complexes of sheaves on a well-behaved ringed space [18], but a treatment of the non-affine case based on such a construction is not really expedient for our purpose. However, the confinement to the affine case, does not only allow us to use the artefacts of the model categorical environment, but it may also allow us to extract the fundamental structure of the main actors of the considered problem and to extend these to an arbitrary smooth scheme X [29].

To implement the preceding ideas, one must prove that the triplet ($\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{A}$) is a homotopical algebraic context (HA context) and consider moreover a homotopical algebraic geometric context ($\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{A}$, τ, $\mathbf{P}$) (HAG context). A HA context is a triplet ($\mathtt{C}$, $\mathtt{C}_0$, $\mathtt{A}_0$) made of a symmetric monoidal model category $\mathtt{C}$ and two full subcategories $\mathtt{C}_0 \subset \mathtt{C}$ and $A_0 \subset \mathtt{CMon}(\mathtt{C})$, which satisfy several quite natural but important assumptions that guarantee that essential tools from linear and commutative algebra are still available. Further, $\mathbf{P}$ is a class of morphisms in $\mathtt{DG}\mathcal{D}\mathtt{A}^{\mathrm{op}}$ that is compatible with τ (a priori one may think about τ as being the étale topology and about $\mathbf{P}$ as being a class of smooth morphisms). In this framework, a 1-*geometric derived* X-$\mathcal{D}_X$-*stack* is, roughly, a derived X-$\mathcal{D}_X$-stack, which is obtained as the quotient by a groupoid action – in representable derived X-$\mathcal{D}_X$-stacks – that belongs to $\mathbf{P}$. Hence, $\mathbf{P}$ determines the type of action we consider (e.g., a smooth action, maybe a not really nice action) and determines the type of geometric stack we get.

Let us now come back to the first two paragraphs of this introduction. Since $J^\infty E \in \mathtt{Aff}(\mathcal{D}_X) \subset \mathtt{D\_Aff}(\mathcal{D}_X) \simeq \mathtt{DG}\mathcal{D}\mathtt{A}^{\mathrm{op}}$ is a representable derived X-$\mathcal{D}_X$-stack, it is natural to view $C := J^\infty E/\mathrm{GS}$, or, better, $C := [J^\infty E/\mathrm{GS}]$ as a 1-geometric derived X-$\mathcal{D}_X$-stack (or even an n-geometric one). Further evidence for this standpoint appears in [5,27,28,40].

The full implementation of the above $\mathcal{D}$-geometric [4] extensions of homotopical algebraic geometric ideas [37,38], as well as of the program sketched in the first paragraph *within* this HAG setting over differential operators, is being written down in a separate paper [30]. In the present text, we prove that ($\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{A}$) is indeed a HA context. Let us recall that modules over the non-commutative ring $\mathcal{D}$ of differential operators are rather special. For instance, the category $\mathtt{DG}\mathcal{D}\mathtt{M}$ is closed monoidal, with internal Hom and tensor product taken, not over $\mathcal{D}$, but over $\mathcal{O}$. More precisely, one considers in fact the $\mathcal{O}$-modules given, for $M, N \in \mathtt{DG}\mathcal{D}\mathtt{M}$, by $\mathrm{Hom}_{\mathcal{O}}(M, N)$ and $M \otimes_{\mathcal{O}} N$, and shows that their $\mathcal{O}$-module structures can be extended to $\mathcal{D}$-module structures. This and other specificities must be kept in mind throughout the whole of the paper, and related subtleties have to be carefully checked.

It can be shown that the *new homotopical algebraic* $\mathcal{D}$-*geometric approach* provides in particular a convenient way to encode total derivatives and allows to recover the classical Batalin-Vilkovisky complex as a specific case of the general constructions [30].

1 Monoidal model structure on differential graded $\mathcal{D}$-modules.

In this section, we show that the category $\mathtt{DG}\mathcal{D}\mathtt{M}$ is a symmetric monoidal *model* category. Such a category is the basic ingredient of a Homotopical Algebraic Context.

Definition 1.1 A symmetric monoidal model structure on a category $\mathtt{C}$ is a closed symmetric monoidal structure together with a model structure on $\mathtt{C}$, which satisfy the compatibility axioms:

MMC1. *The monoidal structure* $\otimes : \mathtt{C} \times \mathtt{C} \to \mathtt{C}$ *is a Quillen bifunctor.*

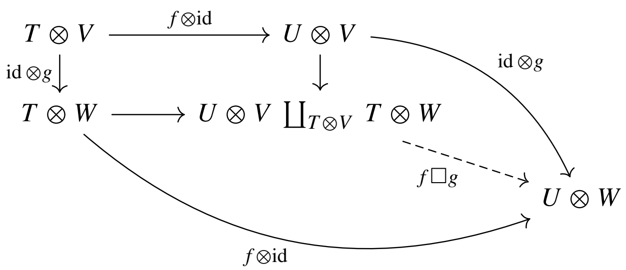
MMC2. *If* $Q\mathbb{I} \xrightarrow{q} \mathbb{I}$ *is the cofibrant replacement of the monoidal unit* $\mathbb{I}$ *(obtained from the functorial 'cofibration - trivial fibration' decomposition of* $\emptyset \to \mathbb{I}$*), then the map*

$$Q\mathbb{I} \otimes C \xrightarrow{q \otimes \mathrm{id}} \mathbb{I} \otimes C$$

is a weak equivalence for every cofibrant $C \in \mathtt{C}$.

We briefly comment on this definition [17].

1. It is known that a morphism of model categories needs not respect the whole model categorical structure – this would be too strong a requirement. The concept of Quillen functor is the appropriate notion of morphism between model categories. In the preceding definition, we ask that $\otimes$ be a Quillen *bi* functor, i.e., that, for any two cofibrations $f : T \to U$ and $g : V \to W$, the universal morphism or *pushout product* $f \Box g$ in the next diagram be a cofibration as well—which is trivial if one of the inducing maps f or g is trivial.



(1.1)

If the model category $\mathtt{C}$ is cofibrantly generated, it suffices to check the pushout axiom MMC1 for generating (trivial) cofibrations.

2. Note that the axiom MMC2 is obviously satisfied if $\mathbb{I}$ is cofibrant.

The category $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$ is an Abelian symmetric monoidal and a finitely generated model category [6] over any smooth affine variety X over an algebraically closed field of characteristic 0.

The monoidal unit is $\mathbb{I} = \mathcal{O} = \mathcal{O}_X(X)$ viewed as concentrated in degree 0 and with zero differential. This complex $(\mathcal{O}, 0)$ is cofibrant if the unique chain map $(\{0\}, 0) \to (\mathcal{O}, 0)$ is a cofibration, i.e., an injective chain map with degree-wise projective cokernel. It is clear that this cokernel is $(\mathcal{O}, 0)$ itself. It is degree-wise projective if and only if $\mathcal{O}$ is a projective $\mathcal{D}$-module. Therefore, the axiom MMC2 is not obvious, if $\mathcal{O}$ is not a flat $\mathcal{D}$-module. The $\mathcal{D}$-module $\mathcal{O}$ is flat if and only if, for any injective $\mathcal{D}$-linear map $M \to N$ between right $\mathcal{D}$-modules, the induced $\mathbb{Z}$-linear map $M \otimes_{\mathcal{D}} \mathcal{O} \to N \otimes_{\mathcal{D}} \mathcal{O}$ is injective as well. Let now $\mathcal{O} = \mathbb{C}[z]$, consider the complex affine line $X = \operatorname{Spec} \mathcal{O}$ and denote by (∂_z) the right ideal of the ring $\mathcal{D} = \mathcal{D}_X(X)$. The right $\mathcal{D}$-linear injection $(\partial_z) \to \mathcal{D}$ induces the morphism

$$(\partial_z) \otimes_{\mathcal{D}} \mathcal{O} \to \mathcal{D} \otimes_{\mathcal{D}} \mathcal{O} \simeq \mathcal{O} \; .$$

Since $\partial_z \simeq \partial_z \otimes_{\mathcal{D}} 1$ is sent to $1 \otimes_{\mathcal{D}} \partial_z 1 \simeq 0$, the kernel of the last morphism does not vanish; hence, in the case of the complex affine line, $\mathcal{O}$ is not $\mathcal{D}$-flat. Eventually, MMC2 is not trivially satisfied.

Before proving that MMC1 and MMC2 hold, we have still to show that the category $\mathtt{DG}\mathcal{D}\mathtt{M}$, which carries a (cofibrantly generated) model structure, is *closed* symmetric monoidal. Let us stress that the equivalent category $\mathtt{DG}_+\mathtt{qcMod}(\mathcal{D}_X)$ is of course equipped with a model structure, but is *a priori* not closed, since the internal Hom of $\mathcal{O}_X$-modules does not necessarily preserve $\mathcal{O}_X$-quasi-coherence (whereas the tensor product of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent). On the other hand, the category $\mathtt{DG}_+\mathtt{Mod}(\mathcal{D}_X)$ is closed symmetric monoidal [34,35], but not endowed with a projective model structure [11] (it has an injective model structure, which, however, is not monoidal [19]). The problem is actually that the category $\mathtt{Mod}(\mathcal{D}_X)$ has not enough projectives. The issue disappears for $\mathtt{qcMod}(\mathcal{D}_X)$, since this category is equivalent to the category $\mathcal{D}\mathtt{M}$ of modules over the ring $\mathcal{D}$.

Let us start with the following observation. Consider a topological space X – in particular a smooth variety – and a sheaf $\mathcal{R}_X$ of unital rings over X, and let $R = \Gamma(X, \mathcal{R}_X)$ be the ring of global sections of $\mathcal{R}_X$. We will also denote the global sections of other sheaves by the Latin letter corresponding to the calligraphic letter used for the considered sheaf. The localization functor $\mathcal{R}_X \otimes_R - : \mathtt{Mod}(R) \to \mathtt{Mod}(\mathcal{R}_X)$ is left adjoint to the global section functor $\Gamma(X, -) : \mathtt{Mod}(\mathcal{R}_X) \to \mathtt{Mod}(R)$:

$$\operatorname{Hom}_{\mathcal{R}_X}(\mathcal{R}_X \otimes_R V, \mathcal{W}) \simeq \operatorname{Hom}_R(V, \operatorname{Hom}_{\mathcal{R}_X}(\mathcal{R}_X, \mathcal{W})) \simeq \operatorname{Hom}_R(V, W), \tag{1.2}$$

for any $V \in \mathtt{Mod}(R)$ and $\mathcal{W} \in \mathtt{Mod}(\mathcal{R}_X)$ [25].

As mentioned above, the category $(\mathtt{Mod}(\mathcal{D}_X), \otimes_{\mathcal{O}_X}, \mathcal{O}_X, \mathcal{H}om_{\mathcal{O}_X})$ is Abelian closed symmetric monoidal. More precisely, for any $\mathcal{N}, \mathcal{P}, \mathcal{Q} \in \mathtt{Mod}(\mathcal{D}_X)$, there is an isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}, \mathcal{Q}) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{Q})) \; . \tag{1.3}$$

Let now $\mathcal{R}_X$ be $\mathcal{O}_X$ or $\mathcal{D}_X$. The preceding Hom functor $\mathcal{H}om_{\mathcal{R}_X}(-,-)$ is the 'internal' Hom of sheaves of $\mathcal{R}_X$-modules, i.e., the functor defined, for any such sheaves $\mathcal{V}, \mathcal{W} \in \mathtt{Mod}(\mathcal{R}_X)$ and for any open $U \subset X$, by

$$\mathcal{H}om_{\mathcal{R}_X}(\mathcal{V}, \mathcal{W})(U) = \operatorname{Hom}_{\mathcal{R}_X|_U}(\mathcal{V}|_U, \mathcal{W}|_U) \; ,$$

where the RHS Hom denotes the morphisms of sheaves of $\mathcal{R}_X|_U$-modules. This set is an Abelian group and an $\mathcal{R}_X(U)$-module, if $\mathcal{R}_X$ is commutative. Hence, by definition, we have

$$\Gamma(X, \mathcal{H}om_{\mathcal{R}_X}(\mathcal{V}, \mathcal{W})) = \operatorname{Hom}_{\mathcal{R}_X}(\mathcal{V}, \mathcal{W}) \; .$$

Recall now that, in the (considered) case of a smooth affine variety X, the global section functor $\Gamma(X,-)$ yields an equivalence

$$\Gamma(X,-):\mathtt{qcMod}(\mathcal{R}_X)\rightarrow\mathtt{Mod}(R):\mathcal{R}_X\otimes_R -$$

of Abelian symmetric monoidal categories. The quasi-inverse $\mathcal{R}_X\otimes_R -$ of $\Gamma(X,-)$ is well-known if $\mathcal{R}_X=\mathcal{O}_X$; for $\mathcal{R}_X=\mathcal{D}_X$, we refer the reader to [16]; the quasi-inverses are both strongly monoidal. If $\mathcal{V}\in\mathtt{qcMod}(\mathcal{R}_X)$ and $\mathcal{W}\in\mathtt{Mod}(\mathcal{R}_X)$, we can thus write $\mathcal{V}\simeq\mathcal{R}_X\otimes_R V$, where $V=\Gamma(X,\mathcal{V})$, and, in view of (1.2), also

$$\mathrm{Hom}_{\mathcal{R}_X}(\mathcal{V},\mathcal{W})=\mathrm{Hom}_{\mathcal{R}_X}(\mathcal{R}_X\otimes_R V,\mathcal{W})\simeq\mathrm{Hom}_R(V,W)\ . \tag{1.4}$$

When applying the global section functor to (1.3), we get

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{P},\mathcal{Q})\simeq\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{N},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\mathcal{Q}))\ ,$$

and, when assuming that $\mathcal{N},\mathcal{P},\mathcal{Q}\in\mathtt{qcMod}(\mathcal{D}_X)$ and using (1.4), we obtain

$$\mathrm{Hom}_{\mathcal{D}}(\Gamma(X,\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{P}),Q)\simeq\mathrm{Hom}_{\mathcal{D}}(N,\Gamma(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\mathcal{Q})))\ ,$$

or, still,

$$\mathrm{Hom}_{\mathcal{D}}(N\otimes_{\mathcal{O}}P,Q)\simeq\mathrm{Hom}_{\mathcal{D}}(N,\mathrm{Hom}_{\mathcal{O}}(P,Q))\ . \tag{1.5}$$

Since any $\mathcal{D}$-module L can be viewed as $\Gamma(X,\mathcal{L})$, where $\mathcal{L}=\mathcal{D}_X\otimes_{\mathcal{D}}L\in\mathtt{qcMod}(\mathcal{D}_X)\subset\mathtt{Mod}(\mathcal{D}_X)$, the Eq. (1.5) proves that $(\mathcal{D}\mathtt{M},\otimes_{\mathcal{O}},\mathcal{O},\mathrm{Hom}_{\mathcal{O}})$ is – just as $(\mathtt{Mod}(\mathcal{D}_X),\otimes_{\mathcal{O}_X},\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X})$ – an Abelian closed symmetric monoidal category. Observe that the internal Hom of $\mathcal{D}\mathtt{M}$ is given by:

$$\mathrm{Hom}_{\mathcal{O}}(-,-)=\Gamma(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_X\otimes_{\mathcal{D}}-,\mathcal{D}_X\otimes_{\mathcal{D}}-))\in\mathcal{D}\mathtt{M}\ . \tag{1.6}$$

Both categories satisfy the AB3 (Abelian category with direct sums) and AB3* (Abelian category with direct products) axioms. It thus follows from [22, Lemma 3.15] that the corresponding categories of chain complexes are Abelian closed symmetric monoidal as well. The tensor product is the usual tensor product $(-\otimes_\bullet-,\delta_\bullet)$ of chain complexes and the internal $(\mathrm{Hom}_\bullet(-,-),d_\bullet)$ is defined, for any complexes $(M_\bullet,d_M)$ and $(N_\bullet,d_N)$ and for any $n\in\mathbb{N}$, by

$$\mathrm{Hom}_n(M_\bullet,N_\bullet)=\begin{cases}\prod_{k\in\mathbb{N}}\mathrm{Hom}_{\mathcal{O}}(M_k,N_{k+n}), & \text{in the case of } \mathtt{DG}\mathcal{D}\mathtt{M}\ ,\\ \prod_{k\in\mathbb{N}}\mathcal{H}om_{\mathcal{O}_X}(M_k,N_{k+n}), & \text{in the case of } \mathtt{DG_+Mod}(\mathcal{D}_X)\ ,\end{cases} \tag{1.7}$$

and, for any $f=(f_k)_{k\in\mathbb{N}}\in\mathrm{Hom}_n(M_\bullet,N_\bullet)$, by

$$(d_nf)_k=d_N\circ f_k-(-1)^n f_{k-1}\circ d_M\ .$$

The closed structure $\mathrm{Hom}_\bullet(-,-)$ of $\mathtt{DG}\mathcal{D}\mathtt{M}$ defines a closed structure $\mathcal{H}om_\bullet(-,-)$ on the equivalent category $\mathtt{DG}_+\mathtt{qcMod}(\mathcal{D}_X)$ via usual transfer

$$\begin{aligned}\mathcal{H}om_\bullet(-,-) &= \mathcal{D}_X \otimes_{\mathcal{D}} (\mathrm{Hom}_\bullet(\Gamma(X,-),\Gamma(X,-))) \\ &= \mathcal{D}_X \otimes_{\mathcal{D}} \left(\prod_k \mathrm{Hom}_{\mathcal{O}}(\Gamma(X,-_k),\Gamma(X,-_{k+\bullet}))\right) \\ &= \mathcal{D}_X \otimes_{\mathcal{D}} \left(\prod_k \Gamma(X,\mathcal{H}om_{\mathcal{O}_X}(-_k,-_{k+\bullet}))\right) ,\end{aligned}$$

where we used (1.6). According to what has been said above, we have the adjunction

$$\mathcal{D}_X \otimes_{\mathcal{D}} - : \mathtt{Mod}(\mathcal{D}) \rightleftarrows \mathtt{Mod}(\mathcal{D}_X) : \Gamma(X,-) ,$$

so that $\Gamma(X,-)$ commutes with limits:

$$\begin{aligned}\mathcal{H}om_\bullet(-,-) &= \mathcal{D}_X \otimes_{\mathcal{D}} \Gamma\left(X, \prod_k \mathcal{H}om_{\mathcal{O}_X}(-_k,-_{k+\bullet})\right) \\ &= \mathcal{D}_X \otimes_{\mathcal{D}} \Gamma\left(X, \mathrm{Hom}_\bullet(-,-)\right) ,\end{aligned}$$

where $\mathrm{Hom}_\bullet(-,-)$ is now the above closed structure of $\mathtt{DG}_+\mathtt{Mod}(\mathcal{D}_X)$. However, since $\mathrm{Hom}_\bullet(-,-)$ is in general not quasi-coherent, the RHS is in the present case not isomorphic to the module $\mathrm{Hom}_\bullet(-,-)$. More precisely, the closed structure on $\mathtt{DG}_+\mathtt{qcMod}(\mathcal{D}_X)$ is given by the coherator of the closed structure on $\mathtt{DG}_+\mathtt{Mod}(\mathcal{D}_X)$. Note also that, since $\mathtt{DG}\mathcal{D}\mathtt{M}$ and $\mathtt{DG}_+\mathtt{qcMod}(\mathcal{D}_X)$ are equivalent symmetric monoidal categories and the internal Hom of the latter is the transfer of the one of the former closed symmetric monoidal category, the second category is closed symmetric monoidal as well (i.e., its monoidal and its closed structures are 'adjoint').

Hence, the

Proposition 1.2 *The category* $(\mathtt{DG}\mathcal{D}\mathtt{M}, \otimes_\bullet, \mathcal{O}, \mathrm{Hom}_\bullet)$ (*resp.*, $(\mathtt{DG}_+\mathtt{qcMod}(\mathcal{D}_X), \otimes_\bullet, \mathcal{O}_X, \mathcal{H}om_\bullet)$) *is Abelian closed symmetric monoidal. The closed structure is obtained by transfer of* (*resp., as the coherator of*) *the closed structure of* $\mathtt{DG}_+\mathtt{Mod}(\mathcal{D}_X)$. *In particular, for any* $N_\bullet, P_\bullet, Q_\bullet \in \mathtt{DG}\mathcal{D}\mathtt{M}$, *there is a* $\mathbb{Z}$*-module isomorphism*

$$\mathrm{Hom}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(N_\bullet \otimes_\bullet P_\bullet, Q_\bullet) \simeq \mathrm{Hom}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(N_\bullet, \mathrm{Hom}_\bullet(P_\bullet, Q_\bullet)) , \tag{1.8}$$

which is natural in $N_\bullet$ *and* $Q_\bullet$.

To examine the axiom MMC1, we need the next proposition. As up till now, we write $\mathcal{D}$ (resp., $\mathcal{O}$) instead of $\Gamma(X,\mathcal{D}_X)$ (resp., $\Gamma(X,\mathcal{O}_X)$).

Proposition 1.3 *If the variety* X *is smooth affine, the module* $\mathcal{D}$ *is projective as* $\mathcal{D}$*- and as* $\mathcal{O}$*-module.*

Proof Projectivity of $\mathcal{D}$ as $\mathcal{D}$-module is obvious, since $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, -) \simeq \mathrm{id}(-)$. Recall now that the sheaf $\mathcal{D}_X$ of differential operators is a filtered sheaf $\mathrm{F}\,\mathcal{D}_X$ of $\mathcal{O}_X$-modules, with filters defined by

$$\mathrm{F}_{-1}\mathcal{D}_X = \{0\} \quad \text{and} \quad \mathrm{F}_i\mathcal{D}_X = \{D \in \mathcal{D}_X : [D, \mathcal{O}_X] \subset \mathrm{F}_{i-1}\mathcal{D}_X\} \ :$$

$\varinjlim_i \mathrm{F}_i\mathcal{D}_X = \mathcal{D}_X$. The graded sheaf $\mathrm{Gr}\,\mathcal{D}_X$ associated to $\mathrm{F}\,\mathcal{D}_X$ is the sheaf, whose terms are defined by

$$\mathrm{Gr}_i\mathcal{D}_X = \mathrm{F}_i\mathcal{D}_X/\mathrm{F}_{i-1}\mathcal{D}_X \ .$$

Consider now, for $i \in \mathbb{N}$, the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathrm{F}_{i-1}\mathcal{D}_X \to \mathrm{F}_i\mathcal{D}_X \to \mathrm{Gr}_i\mathcal{D}_X \to 0 \ .$$

Due to the local freeness of $\mathcal{D}_X$, this is also an exact sequence in $\mathtt{qcMod}(\mathcal{O}_X)$. Since X is affine, we thus get the exact sequence

$$0 \to \Gamma(X, \mathrm{F}_{i-1}\mathcal{D}_X) \to \Gamma(X, \mathrm{F}_i\mathcal{D}_X) \to \Gamma(X, \mathrm{Gr}_i\mathcal{D}_X) \to 0 \tag{1.9}$$

in $\mathtt{Mod}(\mathcal{O})$ – in view of the equivalence of Abelian categories

$$\Gamma(X, -) : \mathtt{qcMod}(\mathcal{O}_X) \rightleftarrows \mathtt{Mod}(\mathcal{O}) \ .$$

However, the functor $\Gamma(X, -)$ transforms a locally free $\mathcal{O}_X$-module of finite rank into a projective finitely generated $\mathcal{O}$-module. We can therefore conclude that $\Gamma(X, \mathrm{Gr}_i\mathcal{D}_X)$ is $\mathcal{O}$-projective, what implies that the sequence (1.9) is split, i.e., that

$$\Gamma(X, \mathrm{F}_i\mathcal{D}_X) = \Gamma(X, \mathrm{Gr}_i\mathcal{D}_X) \oplus \Gamma(X, \mathrm{F}_{i-1}\mathcal{D}_X) \ .$$

An induction and commutation of the left adjoint $\Gamma(X, -)$ with colimits allow to conclude that

$$\mathcal{D} = \Gamma(X, \varinjlim_i \mathrm{F}_i\mathcal{D}_X) = \varinjlim_i \bigoplus_{j=0}^{i} \Gamma(X, \mathrm{Gr}_j\mathcal{D}_X) = \bigoplus_{j=0}^{\infty} \Gamma(X, \mathrm{Gr}_j\mathcal{D}_X) \ .$$

Finally, $\mathcal{D}$ is $\mathcal{O}$-projective as direct sum of $\mathcal{O}$-projective modules.[1] □

Theorem 1.4 *The category* $\mathtt{DG}\mathcal{D}\mathtt{M}$ *is a symmetric monoidal model category.*

Proof of Axiom MMC1. In this proof, we omit the bullets in the notation of complexes. We have to show that the pushout product of two generating cofibrations is a cofibration and that the latter is trivial if one of its factors is a generating trivial cofibration. Recall

[1] We finally observed that the proof of Proposition 1.3 can also be found in [25].

that the generating cofibrations (resp., generating trivial cofibrations) in $\mathtt{DG}\mathcal{D}\mathtt{M}$ are the canonical maps

$$\iota_0 : 0 \to S^0 \quad \text{and} \quad \iota_n : S^{n-1} \to D^n \; (n \geq 1)$$
$$(\text{resp., } \; \zeta_n : 0 \to D^n \; (n \geq 1) \,) \; . \tag{1.10}$$

Here D^n is the n-disc, i.e., the chain complex

$$D^n : \cdots \to 0 \to 0 \to \overset{(n)}{\mathcal{D}} \to \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to \overset{(0)}{0} \quad (n \geq 1) \; , \tag{1.11}$$

whereas S^n is the n-sphere, i.e., the chain complex

$$S^n : \cdots \to 0 \to 0 \to \overset{(n)}{\mathcal{D}} \to 0 \to \cdots \to \overset{(0)}{0} \quad (n \geq 0) \; . \tag{1.12}$$

The map ι_n vanishes, except in degree $n - 1$, where it is the identity map id; the differential in D^n vanishes, except in degree n, where it is the desuspension map s^{-1}.

Step 1. We consider the case of $\iota_m \square \iota_n$ $(m, n \geq 1)$ (the cases m or n is zero and $m = n = 0$ are similar but easier), i.e., we prove that the pushout product in the diagram

$$\begin{array}{ccccc} S^{m-1} \otimes S^{n-1} & \xrightarrow{\iota_m \otimes \mathrm{id}} & D^m \otimes S^{n-1} & \xrightarrow{\mathrm{id} \otimes \iota_n} & \\ \downarrow{\scriptstyle \mathrm{id} \otimes \iota_n} & & \downarrow & & \\ S^{m-1} \otimes D^n & \longrightarrow & \coprod := D^m \otimes S^{n-1} \coprod_{S^{m-1} \otimes S^{n-1}} S^{m-1} \otimes D^n & \overset{\iota_m \square \iota_n}{\dashrightarrow} & D^m \otimes D^n \\ & \searrow{\scriptstyle \iota_m \otimes \mathrm{id}} & & & \nearrow \end{array} \tag{1.13}$$

is a cofibration.

Remark that

$$S^{m-1} \otimes S^{n-1} : \cdots \to 0 \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to 0 \; ,$$
$$D^m \otimes S^{n-1} : \cdots \to 0 \to \overset{(m)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to 0 \; ,$$
$$S^{m-1} \otimes D^n : \cdots \to 0 \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n)}{\mathcal{D}} \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to 0 \; ,$$

and

$$D^m \otimes D^n : \cdots \to 0 \to \overset{(m)}{\mathcal{D}} \otimes \overset{(n)}{\mathcal{D}} \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n)}{\mathcal{D}} \oplus \overset{(m)}{\mathcal{D}}$$
$$\otimes \overset{(n-1)}{\mathcal{D}} \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to 0 \; . \tag{1.14}$$

The non-trivial terms of the differentials are, $s^{-1} \otimes \mathrm{id}$ in $D^m \otimes S^{n-1}$, $\mathrm{id} \otimes s^{-1}$ in $S^{m-1} \otimes D^n$, as well as $s^{-1} \otimes \mathrm{id} + \mathrm{id} \otimes s^{-1}$ and $\mathrm{id} \otimes s^{-1} \oplus s^{-1} \otimes \mathrm{id}$ in $D^m \otimes D^n$.

In an Abelian category pushouts and pullbacks do exist. For instance, the pushout of two morphisms $f : A \to B$ and $g : A \to C$ is the cokernel $(h, k) : B \oplus C \to \mathrm{coker}(f, -g)$ of the morphism $(f, -g) : A \to B \oplus C$. In the Abelian category of chain complexes in an Abelian category, and in particular in $\mathtt{DG}\mathcal{D}\mathtt{M}$, cokernels are taken degree-wise. Hence, in degree $p \in \mathbb{N}$, the pushout of the chain maps $\iota_m \otimes \mathrm{id}$ and $\mathrm{id} \otimes \iota_n$ is the cokernel

$$(h_p, k_p) : (D^m \otimes S^{n-1})_p \oplus (S^{m-1} \otimes D^n)_p \to \mathrm{coker}((\iota_m \otimes \mathrm{id})_p, -(\mathrm{id} \otimes \iota_n)_p) .$$

This cokernel is computed in the category of $\mathcal{D}$-modules and is thus obtained as quotient $\mathcal{D}$-module of the direct sum $(D^m \otimes S^{n-1})_p \oplus (S^{m-1} \otimes D^n)_p$ by the $\mathcal{D}$-submodule generated by

$$\{((\iota_m \otimes \mathrm{id})_p(D \otimes \Delta), -(\mathrm{id} \otimes \iota_n)_p(D \otimes \Delta)) : D \otimes \Delta \in (S^{m-1} \otimes S^{n-1})_p\} .$$

In degree $p \neq m + n - 2$, we divide $\{0\}$ out, and, in degree $p = m + n - 2$, we divide the module

$$\overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \oplus \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}}$$

by the submodule

$$\{(D \otimes \Delta, -D \otimes \Delta) : D \in \overset{(m-1)}{\mathcal{D}}, \Delta \in \overset{(n-1)}{\mathcal{D}}\} .$$

This shows that the considered pushout is

$$\coprod : \cdots \to 0 \to \overset{(m)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \oplus \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n)}{\mathcal{D}} \to \overset{(m-1)}{\mathcal{D}} \otimes \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to 0 . \tag{1.15}$$

The non-trivial term of the pushout differential is direct sum differential $s^{-1} \otimes \mathrm{id} \oplus \mathrm{id} \otimes s^{-1}$ viewed as valued in $\mathcal{D} \otimes \mathcal{D}$.

It is clear that the unique chain map $\iota_m \square \iota_n$, which renders the two triangles commutative, vanishes, except in degrees $m + n - 1$ and $m + n - 2$, where it coincides with the identity. Recall now that the cofibrations of $\mathtt{DG}\mathcal{D}\mathtt{M}$ are the injective chain maps with degree-wise projective cokernel. In view of (1.15) and (1.14), the cokernel of the injective map $\iota_m \square \iota_n$ vanishes in all degrees, except in degree $m + n$, where it is equal to $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$. It thus suffices to show that $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$ is $\mathcal{D}$-projective, i.e., that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}, -) : \mathtt{Mod}(\mathcal{D}) \to \mathtt{Ab}$ is an exact functor valued in Abelian groups. In view of (1.5), we have

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}, -) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, \mathrm{Hom}_{\mathcal{O}}(\mathcal{D}, -)) \simeq \mathrm{Hom}_{\mathcal{O}}(\mathcal{D}, -) : \mathtt{Mod}(\mathcal{D}) \to \mathtt{Ab} .$$

It follows from Proposition 1.3 that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{D}, -)$ is an exact functor $\mathtt{Mod}(\mathcal{O}) \to \mathtt{Mod}(\mathcal{O})$, so also an exact functor $\mathtt{Mod}(\mathcal{D}) \to \mathtt{Ab}$.

Step 2. Take now the pushout product $\zeta_m \Box \iota_n$ ($m, n \geq 1$, see (1.10)) (the other cases are analogous). It is straightforwardly seen that the considered chain map is the map

$$\zeta_m \Box \iota_n : D^m \otimes S^{n-1} \to D^m \otimes D^n \ ,$$

which vanishes in all degrees, except in degrees $m + n - 1$ and $m + n - 2$, where it coincides with the identity. To see that this cofibration is trivial, i.e., induces an isomorphism in homology, we compute the homologies $\mathrm{H}(D^m \otimes S^{n-1})$ and $\mathrm{H}(D^m \otimes D^n)$. Since D^m is acyclic and since $\mathcal{D}$ is $\mathcal{O}$-projective, hence, $\mathcal{O}$-flat (this fact has also been proven independently in [6]), it follows from Künneth's formula [42, Theorem 3.6.3] that $\mathrm{H}(D^m \otimes S^{n-1}) = \mathrm{H}(D^m \otimes D^n) = 0$. Therefore, the map $\mathrm{H}(\zeta_m \Box \iota_n)$ is a $\mathcal{D}$-module isomorphism. □

Proof of Axiom MMC2. Axiom MMC2 holds for $\mathtt{DG}\mathcal{D}\mathtt{M}$, thanks to the following more general result, which will be proven independently in 3.1.

Lemma 1.5 *Let $f : A \to B$ be a weak equivalence in $\mathtt{DG}\mathcal{D}\mathtt{M}$ and let M be a cofibrant object. Then $f \otimes \mathrm{id}_M : A \otimes M \to B \otimes M$ is again a weak equivalence.*

2 Monoidal model structure on modules over differential graded $\mathcal{D}$-algebras

2.1 Modules over commutative monoids

It turned out that $\mathcal{D}$-**geometric Koszul-Tate resolutions** [29] are specific objects of the category

$$\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$$

of commutative monoids in the category $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ of modules in $\mathtt{DG}\mathcal{D}\mathtt{M}$ (see Definition 2.1) over an object $\mathcal{A}$ of the category $\mathtt{CMon}(\mathtt{DG}\mathcal{D}\mathtt{M}) = \mathtt{DG}\mathcal{D}\mathtt{A}$. Moreover, it is known that **model categorical Koszul-Tate resolutions** [7] are cofibrant replacements in the coslice category

$$\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}.$$

The fact that the latter are special $\mathcal{D}$-geometric Koszul-Tate resolutions seems to confirm the natural intuition that there is an isomorphism of categories

$$\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})) \simeq \mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}.$$

Despite the apparent evidence, this equivalence will be proven in detail below (note that, since in this proof the unit elements of the commutative monoids and of the differential graded $\mathcal{D}$-algebras play a crucial role, a similar equivalence for non-unital monoids and non-unital algebras does not hold). Eventually it is clear that an object of the latter under-category is dual to a relative derived affine X-$\mathcal{D}_X$-scheme, where X is the fixed underlying smooth affine algebraic variety (see above). Similar spaces appear [7] in the classical Koszul-Tate resolution, where vector bundles are pulled back over a vector bundle with the same base manifold X.

We first recall the definition of $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$ and explain that this category is closed symmetric monoidal.

Definition 2.1 Let $(\mathtt{C}, \otimes, \mathrm{I}, \underline{\mathrm{Hom}})$ be a closed symmetric monoidal category with all small limits and colimits. Consider an (a commutative) algebra in $\mathtt{C}$, i.e., a commutative monoid $(\mathcal{A}, \mu, \eta)$. The corresponding algebra morphisms are defined naturally and the category of algebras in $\mathtt{C}$ is denoted by $\mathtt{Alg}_{\mathtt{C}}$. A (left) *$\mathcal{A}$-module in* $\mathtt{C}$ is a $\mathtt{C}$-object M together with a $\mathtt{C}$-morphism $\nu : \mathcal{A} \otimes M \to M$, such that the usual associativity and unitality diagrams commute. *Morphisms of $\mathcal{A}$-modules in* $\mathtt{C}$ are also defined in the obvious manner and the category of $\mathcal{A}$-modules in $\mathtt{C}$ is denoted by $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$.

The category of right $\mathcal{A}$-modules in $\mathtt{C}$ is defined analogously. Since $\mathcal{A}$ is commutative, the categories of left and right modules are equivalent (one passes from one type of action to the other by precomposing with the braiding 'com').

The tensor product $\otimes_{\mathcal{A}}$ of two modules $M', M'' \in \mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$ is defined as usual [23, VII.4, Exercise 6] as the coequalizer in $\mathtt{C}$ of the maps

$$\begin{aligned}\psi' := (\nu_{M'} \otimes \mathrm{id}_{M''}) \circ (\mathrm{com} \otimes \mathrm{id}_{M''}),\ \psi'' := \mathrm{id}_{M'} \otimes \nu_{M''} : (M' \otimes \mathcal{A}) \otimes M'' \\ \simeq M' \otimes (\mathcal{A} \otimes M'') \rightrightarrows M' \otimes M''\ .\end{aligned}$$

Since $\mathcal{A} \in \mathtt{Alg}_{\mathtt{C}}$ is commutative, $M' \otimes_{\mathcal{A}} M''$ inherits an $\mathcal{A}$-module structure from those of M' and M'' [38].

Even for an abstract $\mathtt{C}$, one can further define an internal $\underline{\mathrm{Hom}}_{\mathcal{A}}$ in $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$, see Appendix B. Moreover, the expected adjointness property holds,

$$\mathrm{Hom}_{\mathcal{A}}(M \otimes_{\mathcal{A}} M', M'') \simeq \mathrm{Hom}_{\mathcal{A}}(M, \underline{\mathrm{Hom}}_{\mathcal{A}}(M', M''))\ ,$$

and the category of $\mathcal{A}$-modules in $\mathtt{C}$ has all small limits and colimits. We thus get the

Proposition 2.2 [38] *Exactly as the original category* $(\mathtt{C}, \otimes, I, \underline{Hom})$, *the category*

$$(\mathtt{Mod}_{\mathtt{C}}(\mathcal{A}), \otimes_{\mathcal{A}}, \mathcal{A}, \underline{Hom}_{\mathcal{A}})$$

of modules in $\mathtt{C}$ *over* $\mathcal{A} \in \mathtt{Alg}_{\mathtt{C}}$ *is closed symmetric monoidal and contains all small limits and colimits.*

Proposition 2.3 *For any nonzero* $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$, *there exists an isomorphism of categories*

$$\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})) \simeq \mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A},$$

where notation has been introduced above.

Lemma 2.4 *The initial* DG$\mathcal{D}$A $\mathcal{O}$ *can be viewed as a sub-* DG$\mathcal{D}$A *of any nonzero* DG$\mathcal{D}$A $\mathcal{A}$.

Proof It suffices to notice that the (unique) DG$\mathcal{D}$A-morphism $\varphi : \mathcal{O} \to \mathcal{A}$, which is defined by

$$\varphi(f) = \varphi(f \cdot 1_{\mathcal{O}}) = f \cdot \varphi(1_{\mathcal{O}}) = f \cdot 1_{\mathcal{A}} ,$$

is injective, since it is the composition of the injective DG$\mathcal{D}$A-morphism $\mathcal{O} \ni f \mapsto f \otimes 1_{\mathcal{A}} \in \mathcal{O} \otimes_{\mathcal{O}} \mathcal{A}$ and the bijective DG$\mathcal{D}$A-morphism $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{A} \ni f \otimes a \mapsto f \cdot a \in \mathcal{A}$. □

Remark 2.5 In the sequel, $\mathcal{A}$ is assumed to be a nonzero differential graded $\mathcal{D}$-algebra, whenever needed.

Proof of Proposition 2.3 As already said, the category $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$, or, better, $(\mathtt{DG}\mathcal{D}\mathtt{M}, \otimes_\bullet, \mathcal{O}, \mathrm{Hom}_\bullet)$ satisfies all the requirements of Definition 2.1 and the category $(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}), \otimes_{\mathcal{A}}, \mathcal{A}, \underline{\mathrm{Hom}}_{\mathcal{A}})$ has thus exactly the same properties, see Proposition 2.2.

Note also that in the Abelian category DG$\mathcal{D}$M of chain complexes in $\mathcal{D}$M, we get

$$M' \otimes_{\mathcal{A}} M'' = \mathrm{coeq}(\psi', \psi'') = \mathrm{coker}(\psi'' - \psi') ,$$

so that

$$\left(M' \otimes_{\mathcal{A}} M''\right)_n = \mathrm{coker}_n(\psi'' - \psi') = \mathrm{coker}(\psi''_n - \psi'_n) = \left(M' \otimes_\bullet M''\right)_n / \mathrm{im}(\psi''_n - \psi'_n) ,$$

where the $\mathcal{D}$-submodule in the RHS quotient is given by

$$\left\{ \sum_{\text{fin}} \left(m' \otimes (a \triangleleft'' m'') - (-1)^{|a||m'|} (a \triangleleft' m') \otimes m'' \right) : |a| + |m'| + |m''| = n \right\} ,$$

where all sums are finite and where $\triangleleft'$ (resp., $\triangleleft''$) denotes the $\mathcal{A}$-action $\nu_{M'}$ (resp., $\nu_{M''}$). Hence, in all degrees, the tensors $M' \otimes_{\mathcal{A}} M''$ are the tensors $M' \otimes_\bullet M''$ where we identify the tensors $(a \triangleleft' m') \otimes m''$ with the tensors $(-1)^{|a||m'|}\, m' \otimes (a \triangleleft'' m'')$. It is straightforwardly checked that the differential of $M' \otimes_\bullet M''$ stabilizes the submodules, so that the quotient $M' \otimes_{\mathcal{A}} M''$ is again in DG$\mathcal{D}$M. Moreover, a DG$\mathcal{D}$M-morphism $M' \otimes_\bullet M'' \to M$, which vanishes on the submodules, defines a DG$\mathcal{D}$M-morphism $M' \otimes_{\mathcal{A}} M'' \to M$.

Now, an object in $\mathcal{A} \downarrow$ DG$\mathcal{D}$A is a DG$\mathcal{D}$A-morphism $\phi : \mathcal{A} \to M$, i.e., a DG$\mathcal{D}$M-morphism that respects the multiplications and units. The target is an element $M \in$ DG$\mathcal{D}$M and is endowed with two DG$\mathcal{D}$M-morphisms $\mu_M : M \otimes_\bullet M \to M$ and $\eta_M : \mathcal{O} \to M$, which render commutative the usual associativity, unitality and commutativity diagrams.

On the other hand, an object $N \in \mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$ is an $N \in$ DG$\mathcal{D}$M equipped with a DG$\mathcal{D}$M-morphism $\nu : \mathcal{A} \otimes_\bullet N \to N$, for which the associativity and unitality diagrams commute. Moreover, it carries a commutative monoid structure, i.e., there exist $\mathcal{A}$-linear $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphisms $\mu_N : N \otimes_{\mathcal{A}} N \to N$ and $\eta_N : \mathcal{A} \to N$, such that the associativity, unitality and commutativity requirements are fulfilled.

Start from $(\phi : \mathcal{A} \to M) \in \mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$ and set $N = M$ and $\mu_M = -\star-$. Remember that $-\star-$ is $\mathcal{O}$-bilinear associative unital and graded-commutative, and define an $\mathcal{A}$-action on M by

$$a \triangleleft m := \nu(a \otimes m) := \phi(a)\star m \ . \tag{2.1}$$

In view of [6, Proposition 6], the well-defined map ν is a $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism and it can immediately be seen that

$$a' \triangleleft (a'' \triangleleft m) = (a' * a'') \triangleleft m \quad \text{and} \quad 1_{\mathcal{A}} \triangleleft m = m \ ,$$

where $*$ denotes the multiplication in $\mathcal{A}$. Since, we have

$$\begin{aligned} (a \triangleleft m')\star m'' &= \phi(a)\star m'\star m'' = (-1)^{|a||m'|} m'\star\phi(a)\star m'' \\ &= (-1)^{|a||m'|}\, m'\star(a \triangleleft m'') = a \triangleleft (m'\star m'') \ , \end{aligned} \tag{2.2}$$

the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism μ_M is a well-defined $\mathtt{DG}\mathcal{D}\mathtt{M}$ and $\mathcal{A}$-linear morphism μ_N on $M \otimes_{\mathcal{A}} M$. As for η_N, note that η_M is completely defined by $\eta_M(1_{\mathcal{A}}) = \eta_M(1_{\mathcal{O}}) = 1_M$, see Lemma 2.4. Define now an $\mathcal{A}$-linear morphism $\eta_N : \mathcal{A} \to M$ by setting $\eta_N(1_{\mathcal{A}}) = 1_M$. It follows that

$$\eta_N(a) = a \triangleleft \eta_N(1_{\mathcal{A}}) = a \triangleleft 1_M = \phi(a)\star 1_M = \phi(a) \ ,$$

so that η_N is a $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism, which coincides with η_M on $\mathcal{O} \subset \mathcal{A}$:

$$\eta_N(f) = f \cdot 1_M = f \cdot \eta_M(1_{\mathcal{O}}) = \eta_M(f) \ .$$

Conversely, if $N \in \mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$ is given, set $M = N$. The composition of the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $\pi : M \otimes_\bullet M \to M \otimes_{\mathcal{A}} M$ with the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism μ_N is a $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $\mu_M : M \otimes_\bullet M \to M$. The restriction of the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $\eta_N : \mathcal{A} \to M$ to the subcomplex $\mathcal{O} \subset \mathcal{A}$ in $\mathcal{D}\mathtt{M}$ is a $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $\eta_M : \mathcal{O} \to M$. We thus obtain a differential graded $\mathcal{D}$-algebra structure on M with unit $1_M = \eta_M(1_{\mathcal{O}}) = \eta_N(1_{\mathcal{A}})$. Define now a $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $\phi : \mathcal{A} \to M$ by

$$\phi(a) = \nu(a \otimes 1_M) = a \triangleleft 1_M \ . \tag{2.3}$$

This map visibly respects the units and, since $\mu_M = -\star-$ is $\mathcal{A}$-bilinear in the sense of (2.2), it respects also the multiplications.

When starting from a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism ϕ_1 and applying the maps (2.1) and (2.3), we get a $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism

$$\phi_2(a) = a \triangleleft 1_M = \phi_1(a)\star 1_M = \phi_1(a) \ .$$

Conversely, we obtain

$$a \triangleleft_2 m = \phi(a)\star m = (a \triangleleft_1 1_M)\star m = a \triangleleft_1 \mu_N[1_M \otimes m] = a \triangleleft_1 m \ ,$$

with self-explaining notation.

In fact, the two maps we just defined between the objects of the categories $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$, say F and G, are functors and even an isomorphism of categories.

Indeed, if

$$\begin{array}{ccc} & & M' \\ & \overset{\psi'}{\nearrow} & \\ \mathcal{A} & & \Big\downarrow \theta \\ & \underset{\psi''}{\searrow} & \\ & & M'' \ , \end{array}$$

is a morphism Θ in $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$, then $F(\Theta) = \theta$ is a morphism in $\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$ between the modules $F(\psi') = M'$ and $F(\psi'') = M''$, with $\mathcal{A}$-action given by

$$a \triangleleft' m' = \psi'(a)\star' m'$$

and similarly for M''. To prove this claim, it suffices to check that θ is $\mathcal{A}$-linear:

$$\theta(a \triangleleft' m') = \theta(\psi'(a)\star' m') = \psi''(a)\star''\theta(m') = a \triangleleft'' \theta(m') \ .$$

Conversely, if $\pi : N' \to N''$ is a morphism in $\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$ and if $\phi' : \mathcal{A} \to N'$ is the morphism (2.3) in $\mathtt{DG}\mathcal{D}\mathtt{A}$ defined by $a \mapsto a \triangleleft' 1_{N'}$ and similarly for ϕ'', then $G(\pi)$, given by the commutative triangle

$$\begin{array}{ccc} & & N' \\ & \overset{\phi'}{\nearrow} & \\ \mathcal{A} & & \Big\downarrow \pi \\ & \underset{\phi''}{\searrow} & \\ & & N'' \ , \end{array}$$

is a morphism Π in $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$ between $G(N') = \phi'$ and $G(N'') = \phi''$. Eventually, the maps F and G are actually functors, and, as verified above, the composites FG and GF coincide with the corresponding identity functors on objects. It is easily seen that the same holds on morphisms. This is clear for FG, whereas for GF one has to notice that $\phi'(a) = a \triangleleft' 1_{N'} = \psi'(a)\star' 1_{N'} = \psi'(a)$ and analogously for ϕ''. □

2.2 Differential graded $\mathcal{D}$-algebras and modules over them as algebras over a monad

For the purpose of further studying the category $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ of modules in $\mathtt{DG}\mathcal{D}\mathtt{M}$ over an algebra $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$, as well as the category $\mathtt{DG}\mathcal{D}\mathtt{A}$ itself, we rely on results of [36]. To be able to apply the latter, we must view the two preceding categories as categories of algebras over monads.

2.2.1 Differential graded $\mathcal{D}$-algebras

Consider the adjunction

$$\mathcal{S} : \mathtt{DG}\mathcal{D}\mathtt{M} \rightleftarrows \mathtt{DG}\mathcal{D}\mathtt{A} : \mathrm{F},$$

where $\mathcal{S}$ is the graded symmetric tensor product functor and F the forgetful functor (see, for instance, [6]). This Hom-set adjunction can be viewed as a unit-counit adjunction $\langle \mathcal{S}, \mathrm{F}, \eta, \varepsilon \rangle$. It implements a monad $\langle T, \mu, \eta \rangle = \langle \mathrm{F}\,\mathcal{S}, \mathrm{F}\,\varepsilon\mathcal{S}, \eta \rangle$ in $\mathtt{DG}\mathcal{D}\mathtt{M}$.

Proposition 2.6 *The category $\mathtt{DG}\mathcal{D}\mathtt{A}$ of differential graded $\mathcal{D}$-algebras and the Eilenberg-Moore category $\mathtt{DG}\mathcal{D}\mathtt{M}^T$ of T-algebras in $\mathtt{DG}\mathcal{D}\mathtt{M}$ are equivalent.*

Proof The statement is true if the forgetful functor F is monadic. This can be checked using the crude monadicity theorem (see nLab entry 'monadicity theorem'). However, there is a quicker proof. It is known [31] that, if $\mathtt{C}$ is a symmetric monoidal, locally presentable category (see Appendix A), and such that, for any $c \in \mathtt{C}$, the functor $c \otimes \bullet$ respects directed colimits, then the forgetful functor $\mathrm{For} : \mathtt{CMon}(\mathtt{C}) \to \mathtt{C}$ is monadic. Note first that the category $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$ *is* locally presentable. The result can be proven directly, but follows also from [32]. Moreover, this category is Abelian closed symmetric monoidal. In view of closedness, the functor $c \otimes \bullet$ is a left adjoint functor and respects therefore all colimits. Hence, the functor $\mathrm{F} : \mathtt{DG}\mathcal{D}\mathtt{A} \to \mathtt{DG}\mathcal{D}\mathtt{M}$ is monadic.

2.2.2 Modules over a differential graded $\mathcal{D}$-algebra

Let $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$ and consider the adjunction

$$\Sigma : \mathtt{DG}\mathcal{D}\mathtt{M} \rightleftarrows \mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}) : 8\ ,$$

where Σ is the functor $\mathcal{A} \otimes_\bullet -$ and Φ the forgetful functor. Checking that these functors really define an adjunction, so that, for any $M \in \mathtt{DG}\mathcal{D}\mathtt{M}$, the product $\Sigma(M) = \mathcal{A} \otimes_\bullet M$ is the free $\mathcal{A}$-module in $\mathtt{DG}\mathcal{D}\mathtt{M}$, is straightforward. When interpreting this Hom-set adjunction as a unit-counit adjunction $\langle \Sigma, \Phi, \eta, \varepsilon \rangle$, we get an induced monad $\langle U, \mu, \eta \rangle = \langle \Phi\,\Sigma, \Phi\,\varepsilon\,\Sigma, \eta \rangle$ in $\mathtt{DG}\mathcal{D}\mathtt{M}$.

Proposition 2.7 *The category $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ of $\mathcal{A}$-modules in $\mathtt{DG}\mathcal{D}\mathtt{M}$ and the Eilenberg-Moore category $\mathtt{DG}\mathcal{D}\mathtt{M}^U$ of U-algebras in $\mathtt{DG}\mathcal{D}\mathtt{M}$ are equivalent.*

We address the proof of this proposition later on. In view of the requirements of a Homotopical Algebra Context, we will show that the model structure of $\mathtt{DG}\mathcal{D}\mathtt{M}$ can be lifted to $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$:

Theorem 2.8 *The category $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$, $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$, is a cofibrantly generated symmetric monoidal model category that satisfies the monoid axiom (see below). For its monoidal structure we refer to Proposition 2.2. The weak equivalences and fibrations are those $\mathcal{A}$-module morphisms ψ whose underlying $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphisms $\Phi(\psi)$ are weak equivalences or fibrations, respectively. The cofibrations are defined as the morphisms that have the LLP with respect to the trivial fibrations. The set of generating cofibrations (resp., generating trivial cofibrations) is made of the image $\Sigma(I) = \{\mathrm{id}_{\mathcal{A}} \otimes_\bullet \iota_n : \iota_n \in I\}$ (resp., $\Sigma(J) = \{\mathrm{id}_{\mathcal{A}} \otimes_\bullet \zeta_n : \zeta_n \in J\}$) of the set I of generating cofibrations (resp., the set J of generating trivial cofibrations) of $\mathtt{DG}\mathcal{D}\mathtt{M}$.*

The proof will turn out to be a consequence of [36, Theorem 4.1(2)]. For convenience, we recall that this theorem states that, if $\mathtt{C}$, here $\mathtt{DG}\mathcal{D}\mathtt{M}$, is a cofibrantly generated symmetric monoidal model category, which satisfies the monoid axiom and whose objects are small relative to the entire category, then, for any $\mathcal{A} \in \mathtt{CMon}(\mathtt{DG}\mathcal{D}\mathtt{M}) = \mathtt{DG}\mathcal{D}\mathtt{A}$, the category $\mathtt{DG}\mathcal{D}\mathtt{M}^U$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom. The monoidal and model structures are defined as detailed in Theorem 2.8. The model part of this result [36, Proofs of Theorems 4.1(1) and 4.1(2)] is a direct consequence of [36, Lemma 2.3]. This allows in fact to conclude also that the generating sets of cofibrations and trivial cofibrations are the sets $\Sigma(I)$ and $\Sigma(J)$ described in 2.8.

Since any chain complex of $\mathcal{D}$-modules is small relative to all chain maps, any object in $\mathtt{DG}\mathcal{D}\mathtt{M}$ is small relative to all $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphisms. Hence, to finish the proof of Theorem 2.8, it suffices to check that $\mathtt{DG}\mathcal{D}\mathtt{M}$ satisfies the monoid axiom:

Definition 2.9 A monoidal model category $\mathtt{C}$ satisfies the *monoid axiom* [36, Definition 3.3], if any TrivCof $\otimes$ $\mathtt{C}$-cell (a concise definition of cells can be found, for instance, in [6, Appendix 6]), i.e., any cell with respect to the class of the tensor products $\phi \otimes \mathrm{id}_C : C' \otimes C \to C'' \otimes C$ of a trivial cofibration $\phi : C' \to C''$ and the identity of an object $C \in \mathtt{C}$, is a weak equivalence.

If $\mathtt{C}$ is cofibrantly generated and closed symmetric monoidal, the monoid axiom holds if any $J \otimes \mathtt{C}$-cell, where J is a set of generating trivial cofibrations of $\mathtt{C}$, is a weak equivalence [36, Lemma 3.5].

Hence, to prove that $\mathtt{DG}\mathcal{D}\mathtt{M}$ satisfies the monoid axiom, it suffices to show that a $J \otimes_\bullet \mathtt{DG}\mathcal{D}\mathtt{M}$-cell, i.e., a transfinite composition of pushouts of morphisms in $J \otimes_\bullet \mathtt{DG}\mathcal{D}\mathtt{M}$, is a weak equivalence. Since $\mathtt{DG}\mathcal{D}\mathtt{M}$ is a finitely generated model category [6] and the domains and codomains of its generating cofibrations I are finite, i.e., n-small ($n \in \mathbb{N}$), relative to the whole category [17, Lemma 2.3.2], weak equivalences are closed under transfinite compositions [17, Corollary 7.4.2]. Therefore, it is enough to make sure that a pushout of a morphism $\zeta_n \otimes_\bullet \mathrm{id}_M \in J \otimes_\bullet \mathtt{DG}\mathcal{D}\mathtt{M}$ ($n \geq 1$, $M \in \mathtt{DG}\mathcal{D}\mathtt{M}$) is a weak equivalence. Here $\zeta_n : 0 \to D^n$ and

$$D^n : \cdots \to 0 \to 0 \to \overset{(n)}{\mathcal{D}} \to \overset{(n-1)}{\mathcal{D}} \to 0 \to \cdots \to \overset{(0)}{0} \ .$$

Using standard arguments that have already been detailed above, one easily checks that the pushout of $\zeta_n \otimes_\bullet \mathrm{id}_M : 0 \to D^n \otimes_\bullet M$ along a morphism $\phi : 0 \to N$ is given by

$$\begin{array}{ccc} 0 & \xrightarrow{\phi} & N \\ {\scriptstyle \zeta_n \otimes_\bullet \mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle i_2} \\ D^n \otimes_\bullet M & \xrightarrow{i_1} & (D^n \otimes_\bullet M) \oplus N \end{array}$$

Applying Künneth's Theorem to the complexes D^n and M – noticing that both, D^n and $d(D^n)$ (which vanishes, except in degree $n-1$, where it coincides with $\mathcal{D}$), are termwise flat $\mathcal{O}$-modules (see Proposition 1.3; for a direct proof, see [6])—we get, for any m, a short exact sequence

$$0 \to \bigoplus_{p+q=m} H_p(D^n) \otimes H_q(M) \to H_m(D^n \otimes_\bullet M)$$
$$\to \bigoplus_{p+q=m-1} \mathrm{Tor}_1(H_p(D^n), H_q(M)) \to 0\ .$$

Since D^n is acyclic, the central term of this exact sequence vanishes, as the first and the third do. Eventually, the pushout i_2 of $\zeta_n \otimes_\bullet \mathrm{id}_M$ is a weak equivalence, since

$$H(i_2) : H(N) \to H(D^n \otimes_\bullet M) \oplus H(N) \simeq H(N)$$

is obviously an isomorphism.

The category $\mathtt{DG}\mathcal{D}\mathtt{M}$ thus satisfies all the conditions of [36, Theorem 4.1(2)]. It now follows from [36, Proofs of Theorems 4.1(1) and 4.1(2)] that the category $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ is equivalent to the category $\mathtt{DG}\mathcal{D}\mathtt{M}^U$ (the result can also be obtained via the crude monadicity theorem). This completes the proofs of Proposition 2.7 and Theorem 2.8.

Remark 2.10 For any $\mathcal{A} \in \mathtt{CMon}(\mathtt{DG}\mathcal{D}\mathtt{M}) = \mathtt{DG}\mathcal{D}\mathtt{A}$, an $\mathcal{A}$-algebra $A \in \mathtt{Alg}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ is defined in [36] as a monoid $A \in \mathtt{Mon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$. Theorem 4.1(3) in [36] states that $\mathtt{Alg}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ is a cofibrantly generated model category. When choosing $\mathcal{A} = \mathcal{O}$, we find that $\mathtt{Alg}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{O}) = \mathtt{Mon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{O})) = \mathtt{Mon}(\mathtt{DG}\mathcal{D}\mathtt{M})$ is cofibrantly generated. However, the theorem does not treat the case of commutative $\mathcal{O}$-algebras, of commutative monoids in $\mathtt{DG}\mathcal{D}\mathtt{M}$, or, still, of differential graded $\mathcal{D}$-algebras $\mathtt{DG}\mathcal{D}\mathtt{A}$. The fact that $\mathtt{DG}\mathcal{D}\mathtt{A}$ is a cofibrantly generated model category has been proven independently in [6].

Remark 2.11 In the sequel, we write $\mathtt{Mod}(\mathcal{A})$ instead of $\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$, whenever no confusion arises.

2.3 Cofibrant objects in $\mathtt{Mod}(\mathcal{A})$

In this last Subsection, we describe cofibrant $\mathcal{A}$-modules. We need a similar lemma as [7, Lemma 1] that we used to characterize cofibrations in $\mathtt{DG}\mathcal{D}\mathtt{A}$.

Lemma 2.12 *Let* $(\mathcal{A}, d_{\mathcal{A}}) \in \mathtt{DG}\mathcal{D}\mathtt{A}$, $(T, d_T) \in \mathtt{Mod}(\mathcal{A})$, *let* $(g_j)_{j\in J}$ *be a family of symbols of degree* $n_j \in \mathbb{N}$, *and let* $V = \bigoplus_{j\in J} \mathcal{D}\cdot g_j$ *be the free non-negatively graded* $\mathcal{D}$*-module with homogeneous basis* $(g_j)_{j\in J}$.

(i) *To endow the graded* $\mathcal{D}$*-module* $T \oplus \mathcal{A} \otimes_\bullet V$ *– equipped with the natural* $\mathcal{A}$*-module structure induced by the* $\mathcal{A}$*-actions of* T *and* $\mathcal{A} \otimes_\bullet V$ *– with a differential* d *that makes it an* $\mathcal{A}$*-module, it suffices to define*

$$d(g_j) \in T_{n_j-1} \cap d_T^{-1}\{0\} \ , \tag{2.4}$$

to extend d *as* $\mathcal{D}$*-linear map to* V, *and to finally define* d *on* $T \oplus \mathcal{A} \otimes_\bullet V$, *for any* $t \in T_p, a \in \mathcal{A}_k, v \in V_{p-k}$, *by*

$$d(t \oplus a \otimes v) = d_T(t) + d_{\mathcal{A}}(a) \otimes v + (-1)^k a \triangleleft d(v) \ , \tag{2.5}$$

where $\triangleleft$ *is the* $\mathcal{A}$*-action on* T. *The inclusion*

$$(T, d_T) \hookrightarrow (T \oplus \mathcal{A} \otimes_\bullet V, d)$$

is a morphism of $\mathcal{A}$*-modules. Moreover, the differential (2.5) is the unique differential that restricts to* d_T *on* T, *maps* V *into* T *and provides an* $\mathcal{A}$*-module structure on the graded* $\mathcal{D}$*-module* $T \oplus \mathcal{A} \otimes_\bullet V$ *equipped with its natural* $\mathcal{A}$*-action.*

(ii) If $(B, d_B) \in \mathtt{Mod}(\mathcal{A})$ *and* $p \in \mathrm{Hom}_{\mathcal{A}}(T, B)$, *it suffices – to define a morphism* $q \in \mathrm{Hom}_{\mathcal{A}}(T \oplus \mathcal{A} \otimes_\bullet V, B)$ *(where the* $\mathcal{A}$*-module* $(T \oplus \mathcal{A} \otimes_\bullet V, d)$ *is constructed as described in (i)) – to define*

$$q(g_j) \in B_{n_j} \cap d_B^{-1}\{p\, d(g_j)\} \ , \tag{2.6}$$

to extend q *as* $\mathcal{D}$*-linear map to* V, *and to eventually define* q *on* $T \oplus \mathcal{A} \otimes_\bullet V$ *by*

$$q(t \oplus a \otimes v) = p(t) + a \triangleleft q(v) \ , \tag{2.7}$$

where $\triangleleft$ *is the* $\mathcal{A}$*-action on* B. *Moreover, 2.7 is the unique* $\mathcal{A}$*-module morphism* $(T \oplus \mathcal{A} \otimes_\bullet V, d) \to (B, d_B)$ *that restricts to* p *on* T.

Note that Condition (2.4) corresponds to the characterizing lowering condition in relative Sullivan $\mathcal{D}$-algebras [6].

Proof (i) It is straightforward to see that d is a well-defined, degree -1 and $\mathcal{D}$-linear map on $T \oplus \mathcal{A} \otimes_\bullet V$. It squares to zero, since the $\mathcal{A}$-action $- \triangleleft - = \nu$ on T commutes with the differentials on $\mathcal{A} \otimes_\bullet T$ and T,

$$\begin{aligned} d_T(a \triangleleft d(v)) = d_T(\nu(a \otimes d(v))) &= \nu(d_{\mathcal{A}}(a) \otimes d(v) + (-1)^k a \otimes d_T(d(v))) \\ &= d_{\mathcal{A}}(a) \triangleleft d(v) \ , \end{aligned}$$

and thus compensates the other non-vanishing term in $d^2(t + a \otimes v)$. Hence, $T \oplus \mathcal{A} \otimes_\bullet V \in \mathtt{DG}\mathcal{D}\mathtt{M}$. Its natural $\mathcal{A}$-action – also denoted by $- \triangleleft -$ – endows it with an $\mathcal{A}$-module structure, if it commutes with the differentials $d_{\mathcal{A}} \otimes \mathrm{id} + \mathrm{id} \otimes d$ of $\mathcal{A} \otimes_{\bullet} (T \oplus \mathcal{A} \otimes_{\bullet} V)$ and d of $T \oplus \mathcal{A} \otimes_{\bullet} V$. This condition is easily checked, so that $T \oplus \mathcal{A} \otimes_{\bullet} V$ is actually an $\mathcal{A}$-module for the differential d and the $\mathcal{A}$-action

$$a' \triangleleft (t + a'' \otimes v) = a' \triangleleft t + (a'a'') \otimes v \ .$$

It is clear that T is an $\mathcal{A}$-submodule of $T \oplus \mathcal{A} \otimes_{\bullet} V$. Concerning uniqueness, let ∂ be any differential that has the required properties. Then,

$$\begin{aligned} \partial(t + a \otimes v) &= d_T(t) + \partial(a \triangleleft (1_{\mathcal{A}} \otimes v)) \\ &= d_T(t) + d_{\mathcal{A}}(a) \triangleleft (1_{\mathcal{A}} \otimes v) + (-1)^k a \triangleleft \partial(v) \\ &= d_T(t) + d_{\mathcal{A}}(a) \otimes v + (-1)^k a \triangleleft \partial(v) \ , \end{aligned}$$

with

$$\partial(g_j) \in T_{n_j - 1} \cap d_T^{-1}\{0\} \ .$$

(ii) Similar proof. □

We are now prepared to study cofibrant $\mathcal{A}$-modules. This description will be needed later on. Let us recall that the cofibrations in $\mathtt{DG}\mathcal{D}\mathtt{A}$, or, equivalently, in $\mathtt{DG}\mathcal{D}\mathtt{M}^T$ – where T is the composite of the free differential graded $\mathcal{D}$-algebra functor $\mathcal{S}$ (symmetric tensor product functor) and the forgetful functor –, are the retracts of relative Sullivan $\mathcal{D}$-algebras $(\mathcal{B} \otimes \mathcal{S} V, d)$ [6]. We will prove that, similarly, cofibrant objects in $\mathtt{Mod}(\mathcal{A})$, or, equivalently, in $\mathtt{DG}\mathcal{D}\mathtt{M}^U$ – where U is the composite of the free $\mathcal{A}$-module functor $\mathcal{A} \otimes_{\bullet} -$ and the forgetful functor – , are retracts of 'Sullivan $\mathcal{A}$-modules'. If one remembers that the binary coproduct in $\mathtt{Mod}(\mathcal{A})$ (resp., $\mathtt{DG}\mathcal{D}\mathtt{A}$) is the direct sum (resp., tensor product), and that the initial object in $\mathtt{Mod}(\mathcal{A})$ (resp., $\mathtt{DG}\mathcal{D}\mathtt{A}$) is $(\{0\}, 0)$ (resp., $(\mathcal{O}, 0)$), the definition of relative Sullivan $\mathcal{A}$-modules is completely analogous to that of relative Sullivan $\mathcal{D}$-algebras [6]:

Definition 2.13 Let $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$. A *relative Sullivan $\mathcal{A}$-module* (RS$\mathcal{A}$M) is a $\mathtt{Mod}(\mathcal{A})$-morphism

$$(B, d_B) \to (B \oplus \mathcal{A} \otimes_{\bullet} V, d)$$

that sends $b \in B$ to $b + 0 \in B \oplus \mathcal{A} \otimes_{\bullet} V$. Here V is a free non-negatively graded $\mathcal{D}$-module, which admits a homogeneous basis $(m_\alpha)_{\alpha < \lambda}$ that is indexed by a well-ordered set, or, equivalently, by an ordinal λ, and is such that

$$dm_\alpha \in B \oplus \mathcal{A} \otimes_{\bullet} V_{<\alpha} \ , \tag{2.8}$$

for all $\alpha < \lambda$. In the last requirement, we set $V_{<\alpha} := \bigoplus_{\beta<\alpha} \mathcal{D} \cdot m_\beta$. We translate (2.8) by saying that the differential d is *lowering*. A RS$\mathcal{A}$M over $(B, d_B) = (\{0\}, 0)$ is called a *Sullivan $\mathcal{A}$-module* (S$\mathcal{A}$M) $(\mathcal{A} \otimes_{\bullet} V, d)$.

In principle the free $\mathcal{A}$-module functor is applied to $(M, d_M) \in \mathtt{DG}\mathcal{D}\mathtt{M}$ and leads to $(\mathcal{A} \otimes_\bullet M, d_{\mathcal{A}\otimes_\bullet M}) \in \mathtt{Mod}(\mathcal{A})$. In the preceding definition, this functor is taken on $V \in \mathtt{G}\mathcal{D}\mathtt{M}$ and provides a graded $\mathcal{D}$-module with an $\mathcal{A}$-action. The latter is endowed with a lowering differential d such that $(\mathcal{A} \otimes_\bullet V, d) \in \mathtt{Mod}(\mathcal{A})$.

Theorem 2.14 *Let $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$. Any cofibrant object in $\mathtt{Mod}(\mathcal{A})$ is a retract of a Sullivan $\mathcal{A}$-module and vice versa.*

Since we do not use the fact that any retract of a Sullivan module is cofibrant, we will not prove this statement.

Proof By Proposition 2.8, the model category $\mathtt{Mod}(\mathcal{A})$ is cofibrantly generated. Cofibrations are therefore retracts of morphisms in $\Sigma(I)$-cell [17, Proposition 2.1.18 (b)], i.e., they are retracts of transfinite compositions of pushouts of generating cofibrations $\Sigma(I)$.

We start studying the pushout of a generating cofibration

$$\Sigma(\iota_n) := \mathrm{id}_{\mathcal{A}} \otimes_\bullet \iota_n : \mathcal{A} \otimes_\bullet S^{n-1} \to \mathcal{A} \otimes_\bullet D^n$$

along a $\mathtt{Mod}(\mathcal{A})$-morphism $f : \mathcal{A} \otimes_\bullet S^{n-1} \to B$, where $n > 0$ (the case $n = 0$ is simpler). This pushout is given by the square

$$\begin{array}{ccc} (\mathcal{A} \otimes_\bullet S^{n-1}, d_{\mathcal{A}\otimes_\bullet S^{n-1}}) & \xrightarrow{\ f\ } & (B, d_B) \\ {\scriptstyle \Sigma(\iota_n)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ (\mathcal{A} \otimes_\bullet D^n, d_{\mathcal{A}\otimes_\bullet D^n}) & \xrightarrow{\ g\ } & (B \oplus \mathcal{A} \otimes_\bullet S^n, d) \end{array} \quad , \tag{2.9}$$

where the differential d and the $\mathtt{Mod}(\mathcal{A})$-maps g and h are defined as follows.

Observe that $(\mathcal{A} \otimes_\bullet S^{n-1}, d_{\mathcal{A}\otimes_\bullet S^{n-1}})$ meets the requirements of point (i) of Lemma 2.12. Indeed, if 1_{n-1} is the basis of S^{n-1}, the differential δ constructed in Lemma 2.12 satisfies $\delta(1_{n-1}) = 0$ and

$$\delta(a \otimes (\Delta \cdot 1_{n-1})) = d_{\mathcal{A}}(a) \otimes (\Delta \cdot 1_{n-1}) = d_{\mathcal{A}\otimes_\bullet S^{n-1}}(a \otimes (\Delta \cdot 1_{n-1})) ,$$

where $\Delta \cdot 1_{n-1}$ denotes the action of $\Delta \in \mathcal{D}$ on 1_{n-1}. It now follows from point (ii) of Lemma 2.12 that f is completely determined by its value $f(1_{n-1}) \in B_{n-1} \cap d_B^{-1}\{0\}$ (we identify $\Delta \cdot 1_{n-1}$ with $1_{\mathcal{A}} \otimes (\Delta \cdot 1_{n-1})$).

Using again Lemma 2.12.(i), we define d as the unique differential on $B \oplus \mathcal{A} \otimes_\bullet S^n$ satisfying

$$d|_B = d_B \qquad \text{and} \qquad d(1_n) = f(1_{n-1}) .$$

The morphism h is defined as the inclusion of (B, d_B) into $(B \oplus \mathcal{A} \otimes_\bullet S^n, d)$.

As for g, we define it as $h \circ f$ on $\mathcal{A} \otimes_\bullet S^{n-1}$. Then we set $T = \mathcal{A} \otimes_\bullet S^{n-1}$ and $V = S^n$, and observe that the differential ∂ on

$$T \oplus \mathcal{A} \otimes_\bullet V = \mathcal{A} \otimes_\bullet D^n \ ,$$

given by

$$\partial(1_n) = 1_{n-1} \in T_{n-1} \cap d_T^{-1}\{0\} \ ,$$

coincides with the differential $d_{\mathcal{A}\otimes_\bullet D^n}$. In view of this observation, the $\texttt{Mod}(\mathcal{A})$-map g can be defined as the extension of $h \circ f$ to $\mathcal{A} \otimes_\bullet D^n$. Part (ii) of Lemma 2.12 allows to see that g is now fully defined by

$$g(1_n) = 1_n \in (B \oplus \mathcal{A} \otimes_\bullet S^n)_n \cap d^{-1}\{h(f(\partial(1_n)))\} \ .$$

Next, we prove that the diagram (2.9) commutes and is universal among all such diagrams.

A concerns commutativity, note that any element of $\mathcal{A} \otimes_\bullet S^{n-1}$ is a finite sum of elements $a \otimes (D \cdot 1_{n-1}) = a \triangleleft (D \cdot 1_{n-1})$, so that the two $\texttt{Mod}(\mathcal{A})$-maps $h \circ f$ and $g \circ \Sigma(\iota_n)$ coincide if they do on 1_{n-1} – what is a direct consequence of the preceding definitions.

To prove universality, consider any $\mathcal{A}$-module (C, d_C), together with two $\texttt{Mod}(\mathcal{A})$-morphisms

$$p : (\mathcal{A} \otimes_\bullet D^n, d_{\mathcal{A}\otimes_\bullet D^n}) \to (C, d_C)$$

and $q : (B, d_B) \to (C, d_C)$, such that $q \circ f = p \circ \Sigma(\iota_n)$, and show that there is a unique $\texttt{Mod}(\mathcal{A})$-map

$$u : (B \oplus \mathcal{A} \otimes_\bullet S^n, d) \to (C, d_C)$$

that renders commutative the 'two triangles'.

When extending q by means of Lemma 2.12 to a $\texttt{Mod}(\mathcal{A})$-map on $B \oplus \mathcal{A} \otimes_\bullet S^n$, we just have to define $u(1_n) \in C_n \cap d_C^{-1}(p(1_{n-1}))$. Observe that, if u exists, we have necessarily $u|_B = u \circ h = q$ and $u(1_n) = u(g(1_n)) = p(1_n)$. It is easily seen that the latter choice satisfies the preceding conditions and that u is unique. Notice that, obviously, $u \circ h = q$ and that $u \circ g = p$, since this equality holds on 1_n and 1_{n-1}: for 1_{n-1}, we have

$$u(g(1_{n-1})) = u(f(1_{n-1})) = q(f(1_{n-1})) = p(\Sigma(\iota_n)(1_{n-1})) = p(1_{n-1}) \ .$$

Finally (2.9) is indeed the pushout diagram of $\Sigma(\iota_n)$ along f.

The maps in $\Sigma(I)$-cell are the transfinite compositions of such pushout diagrams. A transfinite composition of pushouts is the colimit of a colimit respecting functor

$X : \lambda \to \texttt{Mod}(\mathcal{A})$ (where λ is an ordinal), such that the maps $X_\beta \to X_{\beta+1}$ ($\beta+1 < \lambda$) are pushouts of generating cofibrations $\Sigma(I)$. Let therefore

$$X_0 \to X_1 \to \ldots \to X_\beta \to X_{\beta+1} \to \ldots$$

be such a functor. The successive maps are pushouts in the category $\texttt{Mod}(\mathcal{A})$:

$$X_0 = B, X_1 = B \oplus \mathcal{A} \otimes_\bullet S^{n(1)}, \ldots, X_\beta = B \oplus \mathcal{A} \otimes_\bullet \bigoplus_{\alpha \leq \beta} S^{n(\alpha)}, \ldots,$$
$$X_\omega = B \oplus \mathcal{A} \otimes_\bullet \bigoplus_{\alpha < \omega} S^{n(\alpha)}, X_{\omega+1} = X_\omega \oplus \mathcal{A} \otimes_\bullet S^{n(\omega+1)}, \ldots ,$$

where any $n(\alpha) \in \mathbb{N}$. It follows that the transfinite composition or colimit is

$$\operatorname{colim}_{\alpha<\lambda} X_\alpha = B \oplus \mathcal{A} \otimes_\bullet \bigoplus_{\alpha<\lambda, \alpha \in \mathbf{O}_s} S^{n(\alpha)} ,$$

where $\mathbf{O}_s$ denotes the successor ordinals, or, better, the composition is the $\texttt{Mod}(\mathcal{A})$-map

$$(B, d_B) \to \left(B \oplus \mathcal{A} \otimes_\bullet \bigoplus_{\alpha<\lambda, \alpha \in \mathbf{O}_s} S^{n(\alpha)}, d \right) , \tag{2.10}$$

where d is defined by $d\,|_{X_\alpha} = d_{X_\alpha}$ ($\alpha \in \mathbf{O}_s$) and d_{X_α} is defined inductively by $d_{X_\alpha}|_{X_{\alpha-1}} = d_{X_{\alpha-1}}$ and by $d_{X_\alpha}(1_{n(\alpha)}) = f_\alpha(1_{n(\alpha)-1})$, with self-explaining notation (if $\alpha = \omega + 1$, then $d_{X_{\alpha-1}} = d_{X_\omega}$ is defined by its restrictions to the X_β, $\beta < \omega$). Eventually, any $\Sigma(I)$-cell is a relative Sullivan $\mathcal{A}$-module and any cofibration is a retract of a relative Sullivan $\mathcal{A}$-module.

Let now C be a cofibrant $\mathcal{A}$-module and let QC be its cofibrant replacement, given by the small object argument [17, Theorem 2.1.14]: the $\texttt{Mod}(\mathcal{A})$-map $z' : 0 \to QC$ is in $\Sigma(I)$-cell $\subset$ Cof, hence it is a relative Sullivan $\mathcal{A}$-module. Moreover, in the commutative diagram

$$\begin{array}{ccc} 0 & \xrightarrow{z'} & QC \\ \downarrow z & \overset{\ell}{\nearrow} & \downarrow z'' \\ C & \xrightarrow[\mathrm{id}_C]{} & C \end{array} , \tag{2.11}$$

the right-down arrow z'' is in TrivFib and the left-down z in Cof $=$ LLP(TrivFib), so that the dashed $\texttt{Mod}(\mathcal{A})$-arrow ℓ does exist. The diagram encodes the information $z'' \circ \ell = \mathrm{id}_C$, i.e., the information that the cofibrant $C \in \texttt{Mod}(\mathcal{A})$ is a retract of the Sullivan $\mathcal{A}$-module QC. □

3 Homotopical algebraic Context for $\mathtt{DG}\mathcal{D}\mathtt{M}$

A *Homotopical Algebraic Context* (HAC) is a context that satisfies several minimal requirements for the development of Homotopical Algebra within this setting. Such a context is a triplet $(\mathtt{C}, \mathtt{C}_0, \mathtt{A}_0)$ made of a symmetric monoidal model category $\mathtt{C}$ and two full subcategories $\mathtt{C}_0 \subset \mathtt{C}$ and $A_0 \subset \mathtt{CMon}(\mathtt{C})$, which satisfy assumptions that will be recalled and commented below.

We will show that the triplet

$$(\mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{A})$$

is a HAC. Therefore, some preparation is needed.

3.1 Transfinite filtrations and gradings

We start with a useful lemma. If $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$ and $M \in \mathtt{G}\mathcal{D}\mathtt{M}$, the tensor product $\mathcal{A} \otimes_\bullet M$ can be an object in $\mathtt{Mod}(\mathcal{A})$, in essentially two ways. If M comes with its own differential, i.e., if $M \in \mathtt{DG}\mathcal{D}\mathtt{M}$, the natural choice for the differential on $\mathcal{A} \otimes_\bullet M$ is the standard differential on a tensor product of complexes. If, on the contrary, M has no own differential, the tensor product can be a Sullivan $\mathcal{A}$-module. We will tacitely use the following

Lemma 3.1 *Let $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$, $B \in \mathtt{Mod}(\mathcal{A})$, and $M \in \mathtt{G}\mathcal{D}\mathtt{M}$ such that $\mathcal{A} \otimes_\bullet M \in \mathtt{Mod}(\mathcal{A})$. Then, the $\mathcal{A}$-module $B \otimes_{\mathcal{A}} (\mathcal{A} \otimes_\bullet M)$ and the $\mathcal{A}$-module $B \otimes_\bullet M$ – with canonical $\mathcal{A}$-action and transferred differential coming from the differential of $B \otimes_{\mathcal{A}} (\mathcal{A} \otimes_\bullet M)$ – are isomorphic as $\mathcal{A}$-modules. If $M \in \mathtt{DG}\mathcal{D}\mathtt{M}$ and the differential of $\mathcal{A} \otimes_\bullet M$ is the standard differential, the transferred differential on $B \otimes_\bullet M$ is also the standard differential, so that the isomorphism of $\mathcal{A}$-modules holds with standard differentials.*

Proof We consider first the general case. The $\mathcal{A}$-action on the graded $\mathcal{D}$-module $B \otimes_\bullet M$ is the natural action implemented by the action of B. Set now

$$\imath : B \otimes_{\mathcal{A}} (\mathcal{A} \otimes_\bullet M) \ni b \otimes (a \otimes m) \mapsto (-1)^{|a||b|} a \triangleleft (b \otimes m) \in B \otimes_\bullet M\ . \tag{3.1}$$

It can straightforwardly be checked that i is a well-defined isomorphism of $\mathcal{A}$-modules in $\mathtt{G}\mathcal{D}\mathtt{M}$. Let now d_B (resp., d) be the differential of B (resp., $\mathcal{A} \otimes_\bullet M$). The differential

$$\partial := \imath \circ (d_B \otimes \mathrm{id}_{\otimes_\bullet} + \mathrm{id}_B \otimes d) \circ \imath^{-1} \tag{3.2}$$

makes $B \otimes_\bullet M$ an $\mathcal{A}$-module and $\imath$ an isomorphism of $\mathcal{A}$-modules. The particular case mentioned in the lemma is obvious. □

We also need in the following some results related to λ-filtrations, where $\lambda \in \mathbf{O}$ is an ordinal. Recall first that, if $\mathtt{C}$ is a cocomplete category, the colimit is a functor $\mathrm{colim} : \mathtt{Fun}(\lambda, \mathtt{C}) \to \mathtt{C}$, whose source is the category $\mathtt{Fun}(\lambda, \mathtt{C})$ of diagrams of type λ in $\mathtt{C}$.

Definition 3.2 Let $\lambda \in \mathbf{O}$ be an ordinal and let $\mathtt{C} \in \mathtt{Cat}$ be a category, which is closed under small colimits. An object $C \in \mathtt{C}$ is λ-*filtered*, if it is the colimit $C = \operatorname{colim}_{\beta<\lambda} \mathrm{F}_\beta\, C$ of a λ-sequence of $\mathtt{C}$-monomorphisms, i.e., of a colimit respecting functor $\mathrm{F}\, C : \lambda \to \mathtt{C}$, such that all maps $\mathrm{F}_{\beta,\beta+1}\, C : \mathrm{F}_\beta\, C \to \mathrm{F}_{\beta+1}\, C$, $\beta + 1 < \lambda$, are $\mathtt{C}$-monomorphisms:

$$\mathrm{F}_0\, C \to \mathrm{F}_1\, C \to \ldots \to \mathrm{F}_\gamma\, C \to \ldots$$

The family $(\mathrm{F}_\beta\, C)_{\beta\in\lambda}$ is called a λ-*filtration* of C.

Let $(\mathrm{F}_\beta\, C)_{\beta\in\lambda}$ and $(\mathrm{F}_\beta\, D)_{\beta\in\lambda}$ be λ-filtrations of $C \in \mathtt{C}$ and $D \in \mathtt{C}$, respectively. A $\mathtt{C}$-morphism $f : C \to D$ is *compatible with the* λ-*filtrations*, if it is the colimit $f = \operatorname{colim}_{\beta<\lambda} \varphi_\beta$ of a natural transformation $\varphi : \mathrm{F}\, C \to \mathrm{F}\, D$:

$$\begin{array}{ccccccccc} \mathrm{F}_0\, C & \longrightarrow & \mathrm{F}_1\, C & \longrightarrow & \ldots & \longrightarrow & \mathrm{F}_\gamma\, C & \longrightarrow & \ldots \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_\gamma} & & \\ \mathrm{F}_0\, D & \longrightarrow & \mathrm{F}_1\, D & \longrightarrow & \ldots & \longrightarrow & \mathrm{F}_\gamma\, D & \longrightarrow & \ldots \end{array}$$

In the first two lemmas below, we replace our standard category $\mathtt{DG}\mathcal{D}\mathtt{M}$ by the more general category $\mathtt{DG}\,R\,\mathtt{M}$, where R is, as usual, an arbitrary unital ring. For the model structure on $\mathtt{DG}\,R\,\mathtt{M}$, we refer to [6], as well as to references therein.

Lemma 3.3 *Consider a nonzero ordinal* $\lambda \in \mathbf{O} \setminus \{0\}$, *two* λ-*filtered chain complexes* $C, D \in \mathtt{DG}\,R\,\mathtt{M}$, *with* λ-*filtrations* $(\mathrm{F}_\beta\, C)_{\beta\in\lambda}$ *and* $(\mathrm{F}_\beta\, D)_{\beta\in\lambda}$, *and let* $f : C \to D$ *be a* $\mathtt{DG}\,R\,\mathtt{M}$-*morphism, which is compatible with the filtrations and whose corresponding natural transformation is denoted by* $\varphi : \mathrm{F}\, C \to \mathrm{F}\, D$. *If, for any* $\beta < \lambda$, *the map* $\varphi_\beta : \mathrm{F}_\beta\, C \to \mathrm{F}_\beta\, D$ *is a weak equivalence in* $\mathtt{DG}\,R\,\mathtt{M}$, *then the same holds for* f.

Proof In the following, *we assume temporarily that* $\mathrm{F}_{\beta\gamma} C$ *and* $\mathrm{F}_{\beta\gamma} D$ *are injective, for all* $\beta < \gamma < \lambda$. Note first that, any $\mathtt{DG}\,R\,\mathtt{M}$-map $g : C' \to C''$ induces a $\mathtt{DG}\,R\,\mathtt{M}$-isomorphism $C'/\ker g \simeq \operatorname{im} g$. Hence, for any $\beta \leq \gamma < \lambda$, we get

$$\mathrm{F}_\beta\, C \simeq \operatorname{im}(\mathrm{F}_{\beta\gamma}\, C) \subset \mathrm{F}_\gamma\, C\ .$$

This identification implies that $\mathrm{F}_{\beta\gamma}\, C$ is the canonical injection

$$\mathrm{F}_{\beta\gamma}\, C : \mathrm{F}_\beta\, C \hookrightarrow \mathrm{F}_\beta\, C \subset \mathrm{F}_\gamma\, C$$

and that the differentials ∂_β, ∂_γ of $\mathrm{F}_\beta\, C, \mathrm{F}_\gamma\, C$ satisfy

$$\partial_\gamma|_{\mathrm{F}_\beta\, C} = \partial_\beta\ .$$

The same observation holds for D. For the natural transformation φ, we get

$$\varphi_\gamma|_{\mathrm{F}_\beta\, C} = \varphi_\beta\ .$$

Recall now that a colimit in $\mathtt{DG}\,{}_R\mathtt{M}$, say $C = \mathrm{colim}_{\beta<\lambda} \mathrm{F}_\beta\, C$, is constructed degree-wise in $\mathtt{Mod}(R)$:

$$C_n := \coprod_{\beta<\lambda} \mathrm{F}_{\beta,n}\, C/\sim\ ,$$

where $c_{\beta,n} \sim c_{\gamma,n}$, if there is $\delta \geq \sup(\beta, \gamma), \delta < \lambda$ such that $\mathrm{F}_{\beta\delta}\, C\,(c_{\beta,n}) = \mathrm{F}_{\gamma\delta}\, C\,(c_{\gamma,n})$, i.e., $c_{\beta,n} = c_{\gamma,n}$. It follows that

$$C_n = \bigcup_{\beta<\lambda} \mathrm{F}_{\beta,n}\, C\ . \tag{3.3}$$

The set C_n can be made an object $C_n \in \mathtt{Mod}(R)$ in a way such that the maps $\pi_{\beta,n} : \mathrm{F}_{\beta,n}\, C \to C_n$ become $\mathtt{Mod}(R)$-morphisms and C_n becomes the colimit in $\mathtt{Mod}(R)$ of $\mathrm{F}_n\, C : \lambda \to \mathtt{Mod}(R)$. Due to (3.3), the maps $\pi_{\beta,n}$ are the canonical injections

$$\pi_{\beta,n} : \mathrm{F}_{\beta,n}\, C \hookrightarrow C_n\ .$$

Universality of the colimit allows to conclude that there is a $\mathtt{Mod}(R)$-morphism $\partial_n : C_n \to C_{n-1}$ such that

$$\partial_n|_{\mathrm{F}_{\beta,n}\, C} = \partial_\beta\ . \tag{3.4}$$

We thus get a complex $(C_\bullet, \partial_\bullet) \in \mathtt{DG}\,{}_R\mathtt{M}$, together with $\mathtt{DG}\,{}_R\mathtt{M}$-morphisms

$$\pi_{\beta,\bullet} : \mathrm{F}_{\beta,\bullet}\, C \hookrightarrow C_\bullet\ , \tag{3.5}$$

and this complex is the colimit C in $\mathtt{DG}\,{}_R\mathtt{M}$ of $\mathrm{F}\, C$ [6].

We have still to remove the temporary assumption. Note first the following:

Remark 3.4 If $\lambda \in \mathbf{O}$ and $X \in \mathrm{Fun}(\lambda, \mathtt{C})$ is a λ-diagram in a cocomplete category $\mathtt{C}$, then, for any $\beta < \gamma \leq \lambda$, the map $X_{\beta *}$, which assigns to any $\gamma \setminus \beta$-object α the $\mathtt{C}$-morphism $X_{\beta\alpha} : X_\beta \to X_\alpha$, is a natural transformation between the constant functor X_β and the functor X, both restricted to $\gamma \setminus \beta$. The application of the colimit functor colim : $\mathtt{Fun}(\gamma \setminus \beta, \mathtt{C}) \to \mathtt{C}$ to this natural transformation leads to

$$\mathrm{colim}_{\beta\leq\alpha<\gamma}\, X_{\beta\alpha} : X_\beta \to \mathrm{colim}_{\beta\leq\alpha<\gamma}\, X_\alpha\ . \tag{3.6}$$

Further, a functor $G : \mathtt{D}' \to \mathtt{D}''$ preserves colimits, if, in case (C, ψ) is the colimit of a diagram F in $\mathtt{D}'$, then $(G(C), G(\psi))$ is the colimit of the diagram $G\,F$ in $\mathtt{D}''$. Hence, the functor $X : \lambda \to \mathtt{C}$ preserves colimits means that, for a limit ordinal $\gamma = \mathrm{colim}_{\alpha<\gamma}\, \alpha$ in λ, i.e., for the colimit $(\gamma, \beta < \gamma)$ of the diagram $0 \to 1 \to \cdots \to \alpha \to \alpha + 1 \to \ldots (\gamma)$ in λ, the colimit of the diagram $X_0 \to X_1 \to \cdots \to X_\alpha \to X_{\alpha+1} \to \ldots (\gamma)$ in $\mathtt{C}$ is $(X_\gamma, X_{\beta\gamma})$. In other words,

$$X_\gamma = \mathrm{colim}_{\beta\leq\alpha<\gamma}\, X_\alpha \quad \text{and} \quad X_{\beta\gamma} = \mathrm{colim}_{\beta\leq\alpha<\gamma}\, X_{\beta\alpha}\ . \tag{3.7}$$

We are now prepared to show, by transfinite induction on γ, that the *temporary* hypothesis assuming that $F_{\beta\gamma}C$ (the case of $F_{\beta\gamma}D$ is similar) is injective for $\beta < \gamma < \lambda$, is in fact a consequence of the *actual* assumptions of Lemma 3.3. The induction starts, since $F_{\beta,\beta+1}C$ is injective for $\beta + 1 < \lambda$. The induction assumption is that $F_{\beta\alpha}C$ is injective for $\beta < \alpha < \gamma$. In the case $\gamma \in \mathbf{O}_s$, we have $F_{\beta\gamma}C = F_{\gamma-1,\gamma}C\, F_{\beta,\gamma-1}C$, which is injective, because the first acting map is injective in view of the induction assumption (or the fact that it is identity) and the second map is injective since $(F_\beta C)_{\beta<\lambda}$ is a λ-filtration. If $\gamma \in \mathbf{O}_\ell$, $\gamma = \operatorname{colim}_{\alpha<\gamma}\alpha$, it follows from (3.7), applied to the colimit respecting functor $X = \mathrm{F}\, C$, that

$$F_{\beta\gamma}C : F_\beta C \to F_\gamma C$$

is the map

$$\operatorname{colim}_{\beta\leq\alpha<\gamma} F_{\beta\alpha}C : F_\beta C \to \operatorname{colim}_{\beta\leq\alpha<\gamma} F_\alpha C\ .$$

Moreover, the equation (3.6) shows that the map (3.5) is nothing but $\operatorname{colim}_{\beta\leq\alpha<\lambda} F_{\beta\alpha}C$. When, at the beginning of the proof of Lemma 3.3, the role of λ is played by γ, the temporary assumption is exactly the induction assumption, so that $F_{\beta\gamma} = \operatorname{colim}_{\beta\leq\alpha<\gamma} F_{\beta\alpha}C$ is the natural injection (3.5) for the considered case $\lambda = \gamma$, what eventually removes the temporary assumption.

In the sequel, we omit the subscript $\bullet$, as well as the index n of chain maps and differentials.

When considering both colimits, C and D, we use the above notation, adding a superscript C or D, if confusion has to be avoided. Further, the colimit map $f = \operatorname{colim}_{\beta<\lambda}\varphi_\beta$ is obtained using the universality of the colimit $C = \operatorname{colim}_{\beta<\lambda}\mathrm{F}_\beta\, C$. More precisely, the $\mathtt{DG}\,R\,\mathtt{M}$-morphisms $\varphi_\beta : \mathrm{F}_\beta\, C \to D$ factor through C, i.e.,

$$f|_{\mathrm{F}_\beta\, C} = \varphi_\beta\ . \tag{3.8}$$

We are now prepared to show that the $\mathtt{DG}\,R\,\mathtt{M}$-morphism f induces an isomorphism of graded R-modules in homology.

If the induced degree zero $\mathtt{Mod}(R)$-morphism $H(f)$ is not injective, one of its components $H(f) : H_n(C) \to H_n(D)$, has a non-trivial kernel, i.e., there is a ∂^C-cycle c_n that is not a ∂^C-boundary, such that

$$f\, c_n = \partial^D\, d_{n+1}\ . \tag{3.9}$$

We have

$$c_n = c_{\beta,n} \quad \text{and} \quad d_{n+1} = d_{\gamma,n+1}\ , \tag{3.10}$$

for some $\beta, \gamma < \lambda$. It is clear that $c_{\beta,n}$ is a ∂^C_β-cycle, but not a ∂^C_β-boundary. Moreover,

$$\varphi_\beta\, c_{\beta,n} = f\, c_n = \partial^D_\gamma d_{\gamma,n+1}\ .$$

Depending on whether $\beta \geq \gamma$ or $\beta < \gamma$, this contradicts the fact that $H(\varphi_\beta)$ or that $H(\varphi_\gamma)$ is an isomorphism. Therefore, we finally conclude that $H(f)$ is indeed injective.

As for the surjectivity of $H(f) : H_n(C) \to H_n(D)$, let $v_n \in D_n \cap \ker \partial^D$: $v_n = v_{\beta,n}$ and $v_{\beta,n} \in \mathrm{F}_{\beta,n}\, D \cap \ker \partial^D_\beta$. Since $H(\varphi_\beta) : H(\mathrm{F}_\beta\, C) \to H(\mathrm{F}_\beta\, D)$ is surjective, the homology class $[v_{\beta,n}]_{\mathrm{im}\,\partial^D_\beta}$ is the image by $H(\varphi_\beta)$ of the homology class of some $u_{\beta,n} \in \mathrm{F}_{\beta,n}\, C \cap \ker \partial^C_\beta$. Thus

$$\varphi_\beta u_{\beta,n} = v_{\beta,n} + \partial^D_\beta v_{\beta,n+1} \quad \text{and} \quad f\, u_{\beta,n} = v_n + \partial^D\, v_{\beta,n+1}\ .$$

Since $u_{\beta,n} \in C_n \cap \ker \partial^C$, it follows that $[u_{\beta,n}]_{\mathrm{im}\,\partial^C} \in H_n(C)$ is sent by $H(f)$ to $[v_n]_{\mathrm{im}\,\partial^D} \in H_n(D)$. □

To state and prove the next lemma, we need some preparation.

Consider the setting of Lemma 3.3. The cokernel of any $\mathtt{DG}\,R\,\mathtt{M}$-map $g : C' \to C''$ is computed degree-wise, so that coker $g : C'' \to C''/\operatorname{im} g$, where the RHS differential is induced by the differential of C''.

In our context, we thus get that, for any $\beta + 1 < \lambda$, the cokernel of $\mathrm{F}_{\beta,\beta+1}\, C$ is the $\mathtt{DG}\,R\,\mathtt{M}$-morphism

$$h^C_{\beta+1} : \mathrm{F}_{\beta+1}\, C \ni c_{\beta+1} \mapsto [c_{\beta+1}]_{\mathrm{F}_\beta\, C} \in \mathrm{F}_{\beta+1}\, C/\mathrm{F}_\beta\, C\ .$$

The target complex is denoted by $\mathrm{Gr}_{\beta+1}\, C \in \mathtt{DG}\,R\,\mathtt{M}$ and its differential is the differential $\partial^C_{\beta+1,\sharp}$ induced by $\partial^C_{\beta+1}$. It follows that $\varphi_{\beta+1}$ induces a $\mathtt{DG}\,R\,\mathtt{M}$-map

$$\varphi_{\beta+1,\sharp} : \mathrm{Gr}_{\beta+1}\, C \to \mathrm{Gr}_{\beta+1}\, D\ .$$

It is possible to extend $\mathrm{Gr}\, C$, defined so far on successor ordinals, to a colimit respecting functor $\mathrm{Gr}\, C : \lambda \to \mathtt{DG}\,R\,\mathtt{M}$, which we call *λ-grading associated to the λ-filtration* $\mathrm{F}\, C : \lambda \to \mathtt{DG}\,R\,\mathtt{M}$.

Although we will not need this extension, we will use the precise definition of a colimit respecting functor $F : \mathtt{C} \to \mathtt{D}$. Recall first that, if $J : \mathtt{I} \to \mathtt{C}$ is a $\mathtt{C}$-diagram, its $\mathtt{C}$-colimit, if it exists, is an object $c \in \mathtt{C}$, together with $\mathtt{C}$-morphisms $\eta_i : J_i \to c$, such that $\eta_j J_{ij} = \eta_i$ (i.e., together with a natural transformation η between J and the constant functor c). The functor F is said to be colimit preserving, if the $\mathtt{D}$-colimit of FJ exists and is given by the object $F(c)$, together with the $\mathtt{D}$-morphisms $F(\eta_i) : F(J_i) \to F(c)$ (i.e., the natural transformation is the whiskering of η and F).

Observe now that, since $\mathrm{F}\, C : \lambda \to \mathtt{DG}\,R\,\mathtt{M}$ is colimit respecting by assumption, we have, for $\alpha < \lambda$, $\alpha \in \mathbf{O}_\ell$, $\alpha = \operatorname{colim}_{\beta<\alpha} \beta$,

$$\mathrm{F}_\alpha\, C = \operatorname{colim}_{\beta<\alpha} \mathrm{F}_\beta\, C\ , \quad \text{together with the canonical injections}$$
$$\mathrm{F}_{\beta\alpha}\, C : \mathrm{F}_\beta\, C \hookrightarrow \mathrm{F}_\alpha\, C\ .$$

The same holds for C replaced by D. Since the colimit $\operatorname{colim}_{\beta<\alpha} \varphi_\beta$ is obtained using the universality of $(\mathrm{F}_\alpha\, C, \mathrm{F}_{\bullet\alpha}\, C)$ with respect to the cocone $(F_\alpha D, \mathrm{F}_{\bullet\alpha} D\, \varphi_\bullet)$, this colimit map is the unique $\mathtt{DG}\,R\,\mathtt{M}$-map $m_\alpha : \mathrm{F}_\alpha\, C \to \mathrm{F}_\alpha\, D$, such that $m_\alpha \mathrm{F}_{\beta\alpha}\, C = \mathrm{F}_{\beta\alpha}\, D\, \varphi_\beta$, i.e., such that $m_\alpha|_{\mathrm{F}_\beta\, C} = \varphi_\beta$. Hence, for any limit ordinal $\alpha < \lambda$, we have

$$\varphi_\alpha = \mathrm{colim}_{\beta<\alpha}\, \varphi_\beta \ . \tag{3.11}$$

Lemma 3.5 *Let $\lambda \in \mathbf{O} \setminus \{0\}$, let $C, D \in \mathtt{DG}\,R\,\mathtt{M}$, with λ-filtrations $(\mathrm{F}_\beta\, C)_{\beta\in\lambda}$ and $(\mathrm{F}_\beta\, D)_{\beta\in\lambda}$, and let $f : C \to D$ be a $\mathtt{DG}\,R\,\mathtt{M}$-morphism, which is compatible with the filtrations and whose corresponding natural transformation is denoted by $\varphi : \mathrm{F}\, C \to \mathrm{F}\, D$. Assume that $\varphi_0 : \mathrm{F}_0\, C \to \mathrm{F}_0\, D$ is a weak equivalence and that, for any $\gamma+1 < \lambda$, the induced $\mathtt{DG}\,R\,\mathtt{M}$-map $\varphi_{\gamma+1,\sharp} : \mathrm{Gr}_{\gamma+1}\, C \to \mathrm{Gr}_{\gamma+1}\, D$ is a weak equivalence. Then $\varphi_\beta : \mathrm{F}_\beta\, C \to \mathrm{F}_\beta\, D$ is a weak equivalence, for all $\beta < \lambda$, and therefore f is also a weak equivalence.*

Proof We proceed by transfinite induction. The induction starts, since φ_0 is a weak equivalence by assumption. Let now $\beta < \lambda$ and assume that φ_α is a weak equivalence for all $\alpha < \beta$.

If $\beta \in \mathbf{O}_s$, say $\beta = \gamma + 1$, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{F}_\gamma\, C & \hookrightarrow & \mathrm{F}_{\gamma+1}\, C & \to & \mathrm{Gr}_{\gamma+1}\, C & \to & 0 \\ & & \downarrow\sim & & \downarrow & & \downarrow\sim & & \\ 0 & \to & \mathrm{F}_\gamma\, D & \hookrightarrow & \mathrm{F}_{\gamma+1}\, D & \to & \mathrm{Gr}_{\gamma+1}\, D & \to & 0 \end{array} ,$$

whose rows are exact and whose left (resp., right) vertical arrow φ_γ (resp., $\varphi_{\gamma+1,\sharp}$) is a weak equivalence. The connecting homomorphism theorem now induces in $\mathtt{Mod}(R)$ the diagram

$$\begin{array}{ccccccccccc} \cdots \to & H_{n+1}(\mathrm{Gr}_{\gamma+1}\, C) & \to & H_n(\mathrm{F}_\gamma\, C) & \to & H_n(\mathrm{F}_{\gamma+1}\, C) & \to & H_n(\mathrm{Gr}_{\gamma+1}\, C) & \to & H_{n-1}(\mathrm{F}_\gamma\, C) & \to \cdots \\ & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & \\ \cdots \to & H_{n+1}(\mathrm{Gr}_{\gamma+1}\, D) & \to & H_n(\mathrm{F}_\gamma\, D) & \to & H_n(\mathrm{F}_{\gamma+1}\, D) & \to & H_n(\mathrm{Gr}_{\gamma+1}\, D) & \to & H_{n-1}(\mathrm{F}_\gamma\, D) & \to \cdots \end{array}$$

with exact rows and isomorphisms as non-central vertical arrows. Further the diagram commutes. Indeed, the connecting homomorphism $\Delta^C : H_{n+1}(\mathrm{Gr}_{\gamma+1}\, C) \to H_n(\mathrm{F}_\gamma\, C)$ is defined by

$$\Delta^C [[c_{\gamma+1}]_{\mathrm{F}_\gamma\, C}]_{\mathrm{im}\,\partial^C_{\gamma+1,\sharp}} = [c_\gamma]_{\mathrm{im}\,\partial^C_\gamma}$$

if and only if there exists $c'_{\gamma+1} \in \mathrm{F}_{\gamma+1}\, C$, such that

$$[c_{\gamma+1}]_{\mathrm{F}_\gamma\, C} = [c'_{\gamma+1}]_{\mathrm{F}_\gamma\, C} \quad \text{and} \quad \partial^C_{\gamma+1} c'_{\gamma+1} = c_\gamma \ ,$$

i.e., if and only if

$$\partial^C_{\gamma+1} c_{\gamma+1} = c_\gamma \ .$$

Hence, in the LHS square of the preceding diagram, the top-right composition leads to

$$H(\varphi_\gamma)\, \Delta^C [[c_{\gamma+1}]_{\mathrm{F}_\gamma\, C}]_{\mathrm{im}\,\partial^C_{\gamma+1,\sharp}} = [\varphi_\gamma c_\gamma]_{\mathrm{im}\,\partial^D_\gamma} \ .$$

On the other hand, the left-bottom composition

$$\begin{aligned}\Delta^D\, H(\varphi_{\gamma+1,\sharp})[[c_{\gamma+1}]_{\mathrm{F}_\gamma\, C}]_{\mathrm{im}\,\partial^C_{\gamma+1,\sharp}} &= \Delta^D[\varphi_{\gamma+1,\sharp}[c_{\gamma+1}]_{\mathrm{F}_\gamma\, C}]_{\mathrm{im}\,\partial^D_{\gamma+1,\sharp}} \\ &= \Delta^D[[\varphi_{\gamma+1}c_{\gamma+1}]_{\mathrm{F}_\gamma\, D}]_{\mathrm{im}\,\partial^D_{\gamma+1,\sharp}}\end{aligned}$$

coincides with the value $[\varphi_\gamma c_\gamma]_{\mathrm{im}\,\partial^D_\gamma}$, if and only if

$$\partial^D_{\gamma+1}\varphi_{\gamma+1}c_{\gamma+1} = \varphi_\gamma c_\gamma\ ,$$

what is obviously the case.

It now follows from the Five Lemma that $H_n(\varphi_{\gamma+1})$ is an isomorphism, for all $n \in \mathbb{N}$, i.e., that $\varphi_{\gamma+1} = \varphi_\beta$ is a weak equivalence.

If $\beta \in \mathbf{O}_\ell$, it follows from Lemma 3.3 that φ_β is a weak equivalence. □

The last lemma may be advantageously used to prove Lemma 1.5.

Proof of Lemma 1.5 Recall that our aim is to prove that, if, in $\mathtt{DG}\mathcal{D}\mathtt{M}$, $f : A \to B$ is a weak equivalence and M is a cofibrant object, then $f \otimes \mathrm{id}_M : A \otimes M \to B \otimes M$ is a weak equivalence as well (we omit the subscript $\bullet$ in the tensor product). It follows from the description of the model structure on $\mathtt{DG}\mathcal{D}\mathtt{M}$ [6], that cofibrant objects are exactly those differential graded $\mathcal{D}$-modules that are degree-wise $\mathcal{D}$-projective. In particular, each term of M is $\mathcal{D}$-flat. On the other hand, $\mathcal{D}$ is $\mathcal{O}$-projective and thus $\mathcal{O}$-flat. Therefore, if $0 \to N \to P \to Q \to 0$ is a short exact sequence (SES) in $\mathtt{Mod}(\mathcal{O})$, the free $\mathcal{D}$-module functor $\mathcal{D} \otimes_{\mathcal{O}} \bullet$ on $\mathtt{Mod}(\mathcal{O})$ transforms the considered SES into a new SES in $\mathtt{Mod}(\mathcal{O})$ and even in $\mathtt{Mod}(\mathcal{D})$. Further, left-tensoring the latter sequence over $\mathcal{D}$ by any term M_k, leads to a SES in Abelian groups $\mathtt{Ab}$ and even in $\mathtt{Mod}(\mathcal{O})$. Since $M_k \otimes_{\mathcal{D}} \mathcal{D} \otimes_{\mathcal{O}} \bullet \simeq M_k \otimes_{\mathcal{O}} \bullet$, one deduces that any term M_k of M is also $\mathcal{O}$-flat.

Let now $(M_{\le k}, d_M) \in \mathtt{DG}\mathcal{D}\mathtt{M}$ be the chain complex (M, d_M) truncated at degree $k \in \mathbb{N}$. Then, in the diagram (3.12) below, the top and the bottom rows are ω-filtrations of $A \otimes M$ and $B \otimes M$, respectively. In addition, the product $f \otimes \mathrm{id}_M$ is compatible with these ω-filtrations and is the colimit of the natural transformation $\varphi_\bullet := f \otimes \mathrm{id}_{M_{\le \bullet}}$.

$$\begin{array}{ccccccccc} A\otimes M_{\le 0} & \hookrightarrow & A\otimes M_{\le 1} & \hookrightarrow & \cdots & \hookrightarrow & A\otimes M_{\le n} & \hookrightarrow & \cdots \\ \downarrow{\scriptstyle f\otimes \mathrm{id}_{M_{\le 0}}} & & \downarrow{\scriptstyle f\otimes \mathrm{id}_{M_{\le 1}}} & & & & \downarrow{\scriptstyle f\otimes \mathrm{id}_{M_{\le n}}} & & \\ B\otimes M_{\le 0} & \hookrightarrow & B\otimes M_{\le 1} & \hookrightarrow & \cdots & \hookrightarrow & B\otimes M_{\le n} & \hookrightarrow & \cdots \end{array} \tag{3.12}$$

The morphism $f \otimes \mathrm{id}_M$ is a weak equivalence, if the assumptions of Lemma 3.5 are satisfied. For any $1 \le k+1 < \omega$, the induced map

$$\varphi_{k+1,\sharp} : \mathrm{Gr}_{k+1}(A \otimes M) \to \mathrm{Gr}_{k+1}(B \otimes M) \quad \text{is} \quad f \otimes \mathrm{id}_{M_{k+1}} : A \otimes M_{k+1} \to B \otimes M_{k+1}\ .$$

Moreover, the map

$$\varphi_0 : A \otimes M_{\leq 0} \to B \otimes M_{\leq 0} \quad \text{is} \quad f \otimes \mathrm{id}_{M_0} : A \otimes M_0 \to B \otimes M_0 \ .$$

To show that $f \otimes \mathrm{id}_{M_k}, k \in \mathbb{N}$, is a weak equivalence, we prove the equivalent statement that its mapping cone $\mathrm{Mc}(f \otimes \mathrm{id}_{M_k})$ is acyclic. Notice that

$$\mathrm{Mc}(f \otimes \mathrm{id}_{M_k}) \simeq (\mathrm{Mc}(f))[-k] \otimes M_k \ , \tag{3.13}$$

as $\mathtt{DG}\mathcal{D}\mathtt{M}$, since M_k has zero differential. To find that the RHS is acyclic, it suffices to consider the involved complexes in $\mathtt{DG}\mathcal{O}\mathtt{M}$, to recall that M_k is $\mathcal{O}$-flat and that, since f is weak equivalence, $H((\mathrm{Mc}(f))[-k]) = 0$. The looked for acyclicity then follows from Künneth's formula. □

3.2 HAC condition 1: properness

The first of the HAC assumptions mentioned at the beginning of this section is the condition HAC1 [38, Assumption 1.1.0.1].

HAC1. The underlying model category $\mathtt{C}$ is proper, pointed, and, for any $c', c'' \in \mathtt{C}$, the morphisms

$$Qc' \coprod Qc'' \to c' \coprod c'' \to Rc' \prod Rc'' \ , \tag{3.14}$$

where Q (resp., R) denotes the cofibrant (resp., the fibrant) replacement functor, are weak equivalences. Moreover, the homotopy category $\mathtt{Ho}(\mathtt{C})$ of $\mathtt{C}$ is additive.

Assumption **HAC1** implies that $\mathrm{Hom}_{\mathtt{C}}(c', c'')$ is an Abelian group. This fact and the homotopy part of the assumption allow to understand that the idea is to require that $\mathtt{C}$ be a kind of 'weak' additive or Abelian category.

Let us briefly explain the different parts of condition HAC1. Properness is defined as follows [14, Def. 13.1.1]:

Definition 3.6 A model category $\mathtt{C}$ is said to be:

(1) left proper, *if every pushout of a weak equivalence along a cofibration is a weak equivalence,*
(2) right proper, *if every pullback of a weak equivalence along a fibration is a weak equivalence,*
(3) proper, *if it is both, left proper and right proper.*

Pointed means that the category has a zero object 0. The first morphism in (3.14) comes from the composition of the weak equivalences $Qc' \to c'$ and $Qc'' \to c''$ with the canonical maps $c' \to c' \coprod c''$ and $c'' \to c' \coprod c''$, respectively. As for the second, note that, in addition to the weak equivalence $c' \to Rc'$, we have also the map $c' \to 0 \to Rc''$, hence, finally, a map $c' \to Rc' \prod Rc''$. Similarly, there is a map $c'' \to Rc' \prod Rc''$, so, due to universality, there exists a map $c' \coprod c'' \to Rc' \prod Rc''$.

We now check HAC1 for the basic model category $\mathtt{DG}\mathcal{D}\mathtt{M}$ of the present paper.

Properness of $\mathtt{DG}\mathcal{D}\mathtt{M}$ will be dealt with in Theorem 3.7 below. Since $\mathtt{DG}\mathcal{D}\mathtt{M}$ is Abelian, hence, additive, it has a zero object – in the present situation $(\{0\}, 0)$ – . As for the arrows in (3.14), note that the coproduct and the product of chain complexes of modules are computed degree-wise and that finite coproducts and products of modules coincide and are just direct sums. Since the direct sum of two quasi-isomorphisms is a quasi-isomorphism, the first canonical arrow is a quasi-isomorphism. Recall moreover, that, in $\mathtt{DG}\mathcal{D}\mathtt{M}$, every object is fibrant, so that we can choose the identity as fibrant replacement functor R. Therefore, the second canonical arrow is the identity map and is thus a quasi-isomorphism. Further, the homotopy category $\mathtt{Ho}(\mathtt{DG}\mathcal{D}\mathtt{M})$ is equivalent to the derived category $\mathbb{D}^+(\mathtt{Mod}(\mathcal{D}))$, which is triangulated and thus additive. This additive structure on the derived category can be transferred to the homotopy category (so that the functor that implements the equivalence becomes additive).

We are now left with verifying properness of $\mathtt{DG}\mathcal{D}\mathtt{M}$. By [14, Corollary 13.1.3], a model category all of whose objects are fibrant is right proper: it is easily seen that this is the case, not only for $\mathtt{DG}\mathcal{D}\mathtt{M}$, but also for $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$. We will check that these three categories are left proper as well, so that

Theorem 3.7 *The model categories* $\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{A}$ *and* $\mathtt{Mod}(\mathcal{A})$ *are proper.*

Proof Since $\mathcal{O} \in \mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{O}) = \mathtt{DG}\mathcal{D}\mathtt{M}$, it suffices to prove the statement for $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$, where $\mathcal{A}$ is any object of $\mathtt{DG}\mathcal{D}\mathtt{A}$. Below, the letter $\mathtt{C}$ denotes systematically any of the latter categories, $\mathtt{DG}\mathcal{D}\mathtt{A}$ or $\mathtt{Mod}(\mathcal{A})$.

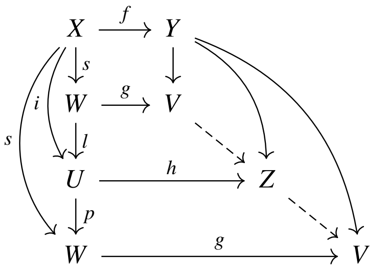
We already mentioned that $\mathtt{C}$ is cofibrantly generated, so that any cofibration is a retract of a map in I-cell, where I is the set of generating cofibrations [17, Proposition 2.1.18(b)]. More precisely, the small object argument allows to factor any $\mathtt{C}$-morphism $s : X \to W$ as $s = p\,i$, with $i \in I\text{-cell} \subset \text{Cof}$ and $p \in I\text{-inj} = \text{TrivFib}$. If $s \in \text{Cof}$, it has the LLP with respect to p. Hence, the commutative diagram

$$\begin{array}{ccccc} X & = & X & = & X \\ \downarrow s & & \downarrow i & & \downarrow s \\ W & \xrightarrow[l]{} & U & \overset{\sim}{\underset{p}{\twoheadrightarrow}} & W \end{array} \quad , \tag{3.15}$$

where l is the lift and where $p\,l = 1$.

We must show that the pushout $g : W \to V$ of a weak equivalence $f : X \to Y$ along the cofibration $s : X \to W$ is a weak equivalence.

Note first that, if $h : U \to Z$ is the pushout of f along i, then g is a retract of h. To see this, consider the following commutative diagram, where the dashed arrows come from the universality of a pushout:



. (3.16)

Due to the uniqueness property encrypted in any universal construction, the composite of the two dashed arrows is the identity of V. Hence, g is indeed a retract of h. As weak equivalences are closed under retracts, it thus suffices to show that the pushout h is a weak equivalence.

We will actually prove that the pushout of a weak equivalence along any map in I-cell, i.e., along any transfinite composition of pushouts of maps in I, is a weak equivalence.

Step 1. In this step, we explain – separately in each of the two categories $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$ – why the pushout of a weak equivalence along a pushout of a map in I, i.e., along a pushout of a generating cofibration, is again a weak equivalence.

In $\mathtt{DG}\mathcal{D}\mathtt{A}$, see [7, Example 1], any pushout of a generating cofibration is a (minimal) relative Sullivan $\mathcal{D}$-algebra $(T, d_T) \hookrightarrow (T \otimes \mathcal{S}S^n, d)$, where d is defined as described in [7, Lemma 1]. Similarly, in $\mathtt{Mod}(\mathcal{A})$, see Proof of Theorem 2.14, any pushout of a generating cofibration is a relative Sullivan $\mathcal{A}$-module $(T, d_T) \hookrightarrow (T \oplus \mathcal{A} \otimes S^n, d)$, where d is defined as detailed in Lemma 2.12.

We first examine the $\mathtt{DG}\mathcal{D}\mathtt{A}$-case. Here the pushout

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X \otimes \mathcal{S}S^n \\ \downarrow f & & \downarrow f\otimes \mathrm{id}_{\mathcal{S}S^n} \\ Y & \xrightarrow{i_Y} & Y \otimes \mathcal{S}S^n \end{array} .$$

of a weak equivalence $f : (X, \partial) \xrightarrow{\sim} (Y, \delta)$ along a relative Sullivan $\mathcal{D}$-algebra $(X, \partial) \hookrightarrow (X \otimes \mathcal{S}S^n, \partial^{(1)})$ is made of

– the relative Sullivan $\mathcal{D}$-algebra $(Y, \delta) \hookrightarrow (Y \otimes \mathcal{S}S^n, \delta^{(1)})$, whose differential $\delta^{(1)}$ is given by

$$\delta^{(1)}(1_n) := f(\partial^{(1)}(1_n)) \in Y_{n-1} \cap \delta^{-1}\{0\} , \tag{3.17}$$

where 1_n is the basis of S^n, and

– the $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism $f \otimes \mathrm{id}_{\mathcal{S}S^n} : (X \otimes \mathcal{S}S^n, \partial^{(1)}) \to (Y \otimes \mathcal{S}S^n, \delta^{(1)})$.

Reference [7, Lemma 1(i)] allows to see that (3.17) defines a relative Sullivan $\mathcal{D}$-algebra. Since the RHS arrow in the above diagram is necessarily an extension ε of $i_Y f : X \to Y \otimes \mathcal{S}S^n$, we apply [7, Lemma 1(ii)] to the morphism $i_Y f$. Therefore we note that the relative Sullivan $\mathcal{D}$-algebra $X \otimes \mathcal{S}S^n$ is actually constructed according to [7, Lemma 1(i)]. Indeed, in view of the first paragraph below [7, Lemma 1], since the differential $\partial^{(1)}$ restricts to ∂ on X and satisfies $\partial^{(1)}(1_n) \in X_{n-1} \cap \partial^{-1}\{0\}$, it is necessarily given by Eq. (9) in [7, Lemma 1]. Hence, the reference [7, Lemma 1(ii)] can be used and the extension ε is fully defined by

$$\varepsilon(1_n) := 1_Y \otimes 1_n \in (Y \otimes \mathcal{S}S^n)_n \cap (\delta^{(1)})^{-1}\{f\,\partial^{(1)}(1_n)\}\ .$$

The extending $\texttt{DG}\mathcal{D}\texttt{A}$-morphism ε is then given, for any $x \in X$ and any $\sigma \in \mathcal{S}S^n$, by $\varepsilon(x \otimes \sigma) = f(x) \otimes \sigma$, so that $\varepsilon = f \otimes \mathrm{id}_{\mathcal{S}S^n}$. As concerns universality, let $h : Y \to E$ and $k : X \otimes \mathcal{S}S^n \to E$ be $\texttt{DG}\mathcal{D}\texttt{A}$-maps, such that $k\, i_X = h\, f$, and define the 'universality map' $\mu : Y \otimes \mathcal{S}S^n \to E$ as extension of h (using the same method as for ε), by setting

$$\mu(1_n) := k(1_X \otimes 1_n) \in E_n \cap d_E^{-1}\{h\,\delta^{(1)}(1_n)\}\ .$$

To check the latter condition on d_E, it suffices to note that, on 1_n, we have

$$d_E\, k = k\,\partial^{(1)} = k\, i_X\,\partial^{(1)} = h\, f\,\partial^{(1)} = h\,\delta^{(1)}\ ,$$

due to (3.17). Further, the condition $\mu\, i_Y = h$ is satisfied by construction, and to see that $\mu\,\varepsilon = k$, we observe that

$$\begin{aligned}\mu(\varepsilon(x \otimes \sigma)) = h(f(x))\star_E \mu(\sigma) \quad \text{and} \quad & k(x \otimes \sigma) \\ & = k(x \otimes 1_{\mathcal{O}})\star_E k(1_X \otimes \sigma) = h(f(x))\star_E \mu(1_Y \otimes \sigma)\ ,\end{aligned}$$

where $k(1_X \otimes \sigma)$ coincides with $\mu(1_Y \otimes \sigma)$, since both maps are $\texttt{DG}\mathcal{D}\texttt{A}$-maps and $k(1_X \otimes 1_n)$ coincides with $\mu(1_Y \otimes 1_n)$, by definition. Eventually, uniqueness of the 'universality map' is easily checked.

As $f \otimes \mathrm{id}_{\mathcal{S}S^n}$ is a weak equivalence in $\texttt{DG}\mathcal{D}\texttt{A}$ if it is a weak equivalence in $\texttt{DG}\mathcal{D}\texttt{M}$, we continue working in the latter category. Notice first that, if Z denotes X or Y and if $d^{(1)}$ denotes $\partial^{(1)}$ or $\delta^{(1)}$, the differential $d^{(1)}$ stabilizes the graded $\mathcal{D}$-submodule $Z_k = Z \otimes \mathcal{S}^{\le k} S^n$ ($k \in \mathbb{N} = \omega$) of $Z \otimes \mathcal{S}S^n$ [7, Lemma 1(i)]. Hence, the restriction

$$\varphi_k := f \otimes \mathrm{id}_{\mathcal{S}^{\le k} S^n} : X_k \to Y_k$$

of the $\texttt{DG}\mathcal{D}\texttt{M}$-map $f \otimes \mathrm{id}_{\mathcal{S}S^n}$ is itself a $\texttt{DG}\mathcal{D}\texttt{M}$-map. Moreover, the injections $Z_{k\ell} : Z_k \to Z_\ell$ ($k \le \ell$) are canonical $\texttt{DG}\mathcal{D}\texttt{M}$-maps, so that we have a functor $Z_* : \omega \to \texttt{DG}\mathcal{D}\texttt{M}$, with obvious colimit $Z \otimes \mathcal{S}S^n$. Since $\varphi_* : X_* \to Y_*$ is a natural transformation between the ω-filtrations of $X \otimes \mathcal{S}S^n$ and $Y \otimes \mathcal{S}S^n$, with colimit $f \otimes \mathrm{id}_{\mathcal{S}S^n}$, it remains to prove that the diagram

$$\begin{array}{ccccccccc} X & \hookrightarrow & X_1 & \hookrightarrow & \cdots & \hookrightarrow & X_k & \hookrightarrow & \cdots \\ \sim\downarrow f & & \downarrow \varphi_1 & & & & \downarrow \varphi_k & & \\ Y & \hookrightarrow & Y_1 & \hookrightarrow & \cdots & \hookrightarrow & Y_k & \hookrightarrow & \cdots \end{array} \tag{3.18}$$

satisfies the hypotheses of Lemma 3.5, i.e., that for any k, the $\mathtt{DG}\mathcal{D}\mathtt{M}$-map $\varphi_{k,\sharp}$ induced by φ_k between the k-terms of the ω-gradings associated to the two ω-filtrations, is a weak equivalence. This will be done independently in Lemma 3.8, what then concludes our argument in the $\mathtt{DG}\mathcal{D}\mathtt{A}$-case.

In the category $\mathtt{Mod}(\mathcal{A})$, the pushout

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X \oplus \mathcal{A} \otimes S^n \\ \sim\downarrow f & & \downarrow f\otimes \mathrm{id}_{\mathcal{A}\otimes S^n} \\ Y & \xrightarrow{i_Y} & Y \oplus \mathcal{A} \otimes S^n \end{array}$$

of a weak equivalence $f : (X, \partial) \xrightarrow{\sim} (Y, \delta)$ along a relative Sullivan $\mathcal{A}$-module $(X, \partial) \hookrightarrow (X \oplus \mathcal{A} \otimes S^n, \partial^{(1)})$ is made of

- the relative Sullivan $\mathcal{A}$-module $(Y, \delta) \hookrightarrow (Y \oplus \mathcal{A} \otimes S^n, \delta^{(1)})$, whose differential $\delta^{(1)}$ is determined by

$$\delta^{(1)}(1_n) = f(\partial^{(1)}(1_n)) \in Y_{n-1} \cap \delta^{-1}\{0\} \;,$$

  and
- the $\mathtt{Mod}(\mathcal{A})$-morphism $f \oplus \mathrm{id}_{\mathcal{A}\otimes S^n} : (X \oplus \mathcal{A} \otimes S^n, \partial^{(1)}) \to (Y \oplus \mathcal{A} \otimes S^n, \delta^{(1)})$.

This statement can be understood similarly to (but more easily than) its counterpart in the $\mathtt{DG}\mathcal{D}\mathtt{A}$-case (replace Sullivan algebras and [7, Lemma 1] by Sullivan modules and Lemma 2.12).

As above, since $f \oplus \mathrm{id}_{\mathcal{A}\otimes S^n}$ is a weak equivalence in $\mathtt{Mod}(\mathcal{A})$ if it is a weak equivalence in $\mathtt{DG}\mathcal{D}\mathtt{M}$, we continue working in the latter category. Notice also that the rows of the preceding diagram are 2-filtrations of $X \oplus \mathcal{A} \otimes S^n$ and $Y \oplus \mathcal{A} \otimes S^n$, respectively, that $f \oplus \mathrm{id}_{\mathcal{A}\otimes S^n}$ is compatible with these filtrations and that the corresponding natural transformation φ_* is defined by the vertical arrows of the diagram. Since f is a weak equivalence, and the induced $\mathtt{DG}\mathcal{D}\mathtt{M}$-map $\varphi_{1,\sharp}$ between the 1-terms of the associated 2-gradings is, as $\mathtt{DG}\mathcal{D}\mathtt{M}$-map, the identity $\mathrm{id}_{\mathcal{A}\otimes S^n}$, the map $f \oplus \mathrm{id}_{\mathcal{A}\otimes S^n}$ is a weak equivalence, thanks to Lemma 3.5.

From here to the end of this proof, we consider the two cases, $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$, again simultaneously and denote both categories by $\mathtt{C}$. We have just shown that the pushout of any weak equivalence along the pushout of any generating cofibration is itself a weak equivalence. In the sequel, we denote the pushout of a generating cofibration, or, better, the corresponding relative Sullivan $\mathcal{D}$-algebra $(X, \partial) \hookrightarrow (X \otimes \mathcal{S}S^n, \partial^{(1)})$ or relative Sullivan $\mathcal{A}$-module $(X, \partial) \hookrightarrow (X \oplus \mathcal{A} \otimes S^n, \partial^{(1)})$, by

$$X^{(0,1)} : (X^{(0)}, \partial^{(0)}) \hookrightarrow (X^{(1)}, \partial^{(1)}) \quad \text{or even} \quad X^{(\beta,\beta+1)} : (X^{(\beta)}, \partial^{(\beta)}) \hookrightarrow (X^{(\beta+1)}, \partial^{(\beta+1)}) \;, \tag{3.19}$$

where β is an ordinal.

Step 2. In this second step, we finally show that the pushout of a weak equivalence $\phi^{(0)} : X^{(0)} \to Y^{(0)}$ in $\mathtt{C}$ along a $\mathtt{C}$-map in I-cell, i.e., along a transfinite composition of pushouts of maps in I, is again a weak equivalence. More precisely, such a composition is the colimit

$$\mathrm{colim}_{\beta<\lambda}\, X^{(0,\beta)} : X^{(0)} \to \mathrm{colim}_{\beta<\lambda}\, X^{(\beta)}$$

of a colimit respecting functor $X^{(*)} : \lambda \to \mathtt{C}$ ($\lambda \in \mathbf{O}$), such that any map $X^{(\beta,\beta+1)} : X^{(\beta)} \hookrightarrow X^{(\beta+1)}$ ($\beta + 1 < \lambda$) is the pushout of a map in I, i.e., is a Sullivan 'object' of the type (3.19).

It might be helpful to notice that the considered transfinite composition is given, in the $\mathtt{Mod}(\mathcal{A})$-case, by (2.10), and in the $\mathtt{DG}\mathcal{D}\mathtt{A}$-case, by

$$X^{(0)} \to X^{(0)} \otimes \mathcal{S}\left(\bigoplus_{\beta<\lambda,\beta\in\mathbf{O}_s} S^{n(\beta)}\right),$$

see [7, Proof of Theorem 4(i)].

Step 2.a. The idea is to first construct the following commutative diagram:

$$\begin{array}{ccccccccccc} X^{(0)} & \xrightarrow{X^{(0,1)}} & X^{(1)} & \xrightarrow{X^{(1,2)}} & \ldots \longrightarrow & X^{(\beta)} & \xrightarrow{X^{(\beta,\beta+1)}} & X^{(\beta+1)} & \longrightarrow \ldots & X^{(\gamma)} & \ldots \\ \sim\downarrow\phi^{(0)} & & \downarrow\phi^{(1)} & & & \downarrow\phi^{(\beta)} & & \downarrow\phi^{(\beta+1)} & & \downarrow\phi^{(\gamma)} & \\ Y^{(0)} & \xrightarrow{Y^{(0,1)}} & Y^{(1)} & \xrightarrow{Y^{(1,2)}} & \ldots \longrightarrow & Y^{(\beta)} & \xrightarrow{Y^{(\beta,\beta+1)}} & Y^{(\beta+1)} & \longrightarrow \ldots & Y^{(\gamma)} & \ldots \end{array} \tag{3.20}$$

Figure: Pushout along an I-cell

More precisely, for $\gamma < \lambda$, we will build, by transfinite induction,

- a colimit respecting functor $Y_\gamma^{(*)} : \gamma + 1 \to \mathtt{C}$ with injective elementary maps $Y^{(\beta,\beta+1)}$ ($\beta + 1 < \gamma + 1$) and
- a natural transformation $\phi_\gamma^{(*)}$ between $X_\gamma^{(*)}$ and $Y_\gamma^{(*)}$,

such that $\phi^{(\gamma)}$ is a weak equivalence and

$$Y^{(0)} \xrightarrow{Y^{(0,\gamma)}} Y^{(\gamma)} \xleftarrow{\phi^{(\gamma)}} X^{(\gamma)}$$

is the pushout of

$$Y^{(0)} \xleftarrow{\phi^{(0)}} X^{(0)} \xrightarrow{X^{(0,\gamma)}} X^{(\gamma)}.$$

This construction is based on the assumption that $Y_\alpha^{(*)}$ and $\phi_\alpha^{(*)}$ have been constructed with the mentioned properties, for any $\alpha < \gamma$.

The induction starts since the requirements concerning $Y_0^{(*)}$ and $\phi_0^{(*)}$ are obviously fulfilled and $\phi^{(0)}$ is a weak equivalence.

We first examine the case $\gamma \in \mathbf{O}_s$. We can begin with the functor $Y^{(*)}_{\gamma-1}$, the natural transformation $\phi^{(*)}_{\gamma-1}$ and the square of the pushout $\phi^{(\gamma-1)}$. Then we build the pushout

$$Y^{(\gamma-1)} \overset{Y^{(\gamma-1,\gamma)}}{\longrightarrow} Y^{(\gamma)} \overset{\phi^{(\gamma)}}{\longleftarrow} X^{(\gamma)}$$

of

$$Y^{(\gamma-1)} \overset{\phi^{(\gamma-1)}}{\longleftarrow} X^{(\gamma-1)} \overset{X^{(\gamma-1,\gamma)}}{\longrightarrow} X^{(\gamma)}$$

as in Step 1. It follows from the induction assumption and the description in Step 1 that there is a canonical functor $Y^{(*)}_\gamma$ that has the required properties, as well as a canonical natural transformation $\phi^{(*)}_\gamma$. Moreover, the map $\phi^{(\gamma)}$ is a weak equivalence, and, since the outer square of two pushout squares is a pushout square, the map $\phi^{(\gamma)}$ has the requested pushout property.

If $\gamma = \mathrm{colim}_{\beta<\gamma}\, \beta \in \mathbf{O}_\ell$, note that, since colimits commute, the searched pushout

$$\mathrm{colim}(Y^{(0)} \longleftarrow X^{(0)} \longrightarrow X^{(\gamma)})$$

of $\phi^{(0)}$ along $X^{(0,\gamma)} : X^{(0)} \to X^{(\gamma)}$, i.e., along $\mathrm{colim}_{\beta<\gamma}\, X^{(0,\beta)} : X^{(0)} \to \mathrm{colim}_{\beta<\gamma}\, X^{(\beta)}$, is equal to

$$\begin{aligned}&\mathrm{colim}_{\beta<\gamma}\,\mathrm{colim}(Y^{(0)} \overset{\phi^{(0)}}{\longleftarrow} X^{(0)} \overset{X^{(0,\beta)}}{\longrightarrow} X^{(\beta)})\\ &= \mathrm{colim}_{\beta<\gamma}\,(Y^{(0)} \overset{Y^{(0,\beta)}}{\longrightarrow} Y^{(\beta)} \overset{\phi^{(\beta)}}{\longleftarrow} X^{(\beta)})\ .\end{aligned} \tag{3.21}$$

Of course, the functors $Y^{(*)}_\alpha$ (resp., the natural transformations $\phi^{(*)}_\alpha$), $\alpha < \gamma$, define a functor $Y^{(*)} : \gamma \to \mathtt{C}$ with the same properties (resp., a natural transformation $\phi^{(*)} : X^{(*)} \to Y^{(*)}$). The functor $Y^{(*)}$ can be extended by

$$Y^{(\gamma)} := \mathrm{colim}_{\beta<\gamma}\, Y^{(\beta)} \quad \text{and} \quad Y^{(\alpha,\gamma)} := \mathrm{colim}_{\alpha\le\beta<\gamma}\, Y^{(\alpha,\beta)}\ ,$$

as colimit respecting functor $Y^{(*)}_\gamma$ with injective elementary maps. Similarly, the natural transformation $\phi^{(*)}$ can be extended, via the application of the colimit functor,

$$\phi^{(\gamma)} := \mathrm{colim}_{\beta<\gamma}\, \phi^{(\beta)} : X^{(\gamma)} \to Y^{(\gamma)}\ ,$$

to a natural transformation $\phi^{(*)}_\gamma$. Hence, the colimit (3.21) is given by

$$Y^{(0)} \overset{Y^{(0,\gamma)}}{\longrightarrow} Y^{(\gamma)} \overset{\phi^{(\gamma)}}{\longleftarrow} X^{(\gamma)}\ .$$

It now suffices to check that $\phi^{(\gamma)}$ is a weak equivalence in $\mathtt{DG}\mathcal{D}\mathtt{M}$. Since, as easily seen, $X^{(*)}$ (resp., $Y^{(*)}$) is a γ-filtration of $X^{(\gamma)}$ (resp., $Y^{(\gamma)}$), since $\phi^{(\gamma)}$ is filtration-compatible with associated natural transformation $\phi^{(*)}$, and since $\phi^{(\alpha)}$, $\alpha < \gamma$, is a weak equivalence, it follows from Lemma 3.3 that $\phi^{(\gamma)}$ is a weak equivalence as well.

Step 2.b. The pushout of $\phi^{(0)}$ along

$$\mathrm{colim}_{\gamma<\lambda}\, X^{(0,\gamma)} : X^{(0)} \to \mathrm{colim}_{\gamma<\lambda}\, X^{(\gamma)}$$

is given by Eq. (3.21) with γ replaced by λ (and β by γ). It is straightforwardly checked that $Y^{(*)}$ (resp., $\phi^{(*)}$) is a functor defined on λ (resp., a natural transformation between such functors). Hence, the colimit (3.21) is given by

$$Y^{(0)} \overset{\mathrm{colim}_{\gamma<\lambda} Y^{(0,\gamma)}}{\longrightarrow} \mathrm{colim}_{\gamma<\lambda}\, Y^{(\gamma)} \overset{\mathrm{colim}_{\gamma<\lambda} \phi^{(\gamma)}}{\longrightarrow} \mathrm{colim}_{\gamma<\lambda}\, X^{(\gamma)} .$$

Since $X^{(*)}$, $Y^{(*)}$ are λ-filtrations of $\mathrm{colim}_{\gamma<\lambda}\, X^{(\gamma)}$ and $\mathrm{colim}_{\gamma<\lambda}\, Y^{(\gamma)}$, respectively, and the considered pushout $\mathrm{colim}_{\gamma<\lambda}\, \phi^{(\gamma)}$ of $\phi^{(0)}$ is filtration-compatible, it follows from Lemma 3.3 that this pushout is a weak equivalence.

To complete the proof of Theorem 3.7, it remains to show that the following lemma, which we state separately for future reference, holds. □

Lemma 3.8 *Diagram* (3.18) *satisfies the assumptions of Lemma 3.5.*

Proof For $X \otimes \mathcal{S}S^n$, the term $X^{k+1} := X_{k+1}/X_k$ $(k+1 < \omega)$ of the ω-grading, associated to the ω-filtration with filters $X_\ell = X \otimes \mathcal{S}^{\leq \ell} S^n$ $(\ell < \omega)$, is isomorphic as $\mathtt{DG}\mathcal{D}\mathtt{M}$-object to

$$X^{k+1} \simeq X \otimes \mathcal{S}^{k+1} S^n ,$$

where the RHS is endowed with the usual tensor product differential. A similar statement holds for $Y \otimes \mathcal{S}S^n$. Moreover, when read through the preceding isomorphisms, say $\mathcal{I}_X$ and $\mathcal{I}_Y$, the $\mathtt{DG}\mathcal{D}\mathtt{M}$-map $\varphi_{k+1,\sharp} : X^{k+1} \to Y^{k+1}$ induced by $\varphi_{k+1} = f \otimes \mathrm{id}_{\mathcal{S}^{\leq k+1} S^n}$, is the $\mathtt{DG}\mathcal{D}\mathtt{M}$-map

$$\mathcal{I}_Y\, \varphi_{k+1,\sharp}\, \mathcal{I}_X^{-1} = f \otimes \mathrm{id}_{\mathcal{S}^{k+1} S^n} .$$

Since $\varphi_0 = f$ is a weak equivalence, it remains to show that $f \otimes \mathrm{id}_{\mathcal{S}^{k+1} S^n}$ is a weak equivalence, for all $k+1 < \omega$, or, still, that $f \otimes \mathrm{id}_{\mathcal{S}^k S^n}$ is a weak equivalence, for all $1 \leq k < \omega$.

Just as in Eq. (3.13), we have here

$$\mathrm{Mc}(f \otimes \mathrm{id}_{\mathcal{S}^k S^n}) \simeq \mathrm{Mc}(f)[-kn] \otimes \mathcal{S}^k S^n ,$$

as $\mathtt{DG}\mathcal{D}\mathtt{M}$-object, since $\mathcal{S}^k S^n$ has zero differential. We now proceed as in [6, Sections 7.5 and 8.7]: The symmetrisation map σ induces a short exact sequence

$$0 \to \ker^k \sigma \overset{i}{\to} \bigotimes^k S^n \overset{\sigma}{\to} \mathcal{S}^k S^n \to 0$$

in the Abelian category $\mathtt{DG}\mathcal{O}\mathtt{M}$. Since this sequence canonically splits, we get the $\mathtt{G}\mathcal{O}\mathtt{M}$-isomorphism

$$H(\mathrm{Mc}(f)[-kn] \otimes \bigotimes^{k} S^n) \simeq H(\mathrm{Mc}(f)[-kn] \otimes \ker^k \sigma) \;\oplus\; H(\mathrm{Mc}(f)[-kn] \otimes \mathcal{S}^k S^n)\ .$$

To prove the weak equivalence condition, it suffices to show that the LHS-homology vanishes. Assume that the claim is proven for $0 \leq k-1 < \omega$. The induction starts since f is a weak equivalence, i.e., a quasi-isomorphism. The fact that

$$H(\mathrm{Mc}(f)[-kn] \otimes \bigotimes^{k-1} S^n \otimes S^n) = 0$$

is then a consequence of the Künneth formula for complexes and the previously mentioned fact that $\mathcal{D}$ is $\mathcal{O}$-flat. □

3.3 HAC condition 2: combinatoriality

All the requirements of the second axiom HAC2 [38, Assumption 1.1.0.2] of a Homotopical Algebraic Context have been established above, except the combinatoriality condition for the model structure of $\mathtt{Mod}(\mathcal{A})$. For future reference, we will also prove the combinatoriality of the model structures of $\mathtt{DG}\mathcal{D}\mathtt{M}$ and of $\mathtt{DG}\mathcal{D}\mathtt{A}$. A reader, who is interested in set-theoretical size issues and **universes**, finds all relevant information in Appendix C.

Roughly, a combinatorial model category is a well manageable type of model category, in the sense that it is generated from small ingredients: it is a category

– in which any object is the colimit of small objects from a given set of generators, and
– which carries a cofibrantly generated model structure, i.e., a model structure whose cofibrations (resp., trivial cofibrations) are generated by sets I (resp., J) of generating morphisms whose sources are small.

More precisely,

Definition 3.9 A **combinatorial model category**, is a locally presentable category that is endowed with a cofibrantly generated model structure.

For **locally presentable categories**, i.e., categories that are locally κ-presentable for some regular cardinal κ, we refer to Appendix A. Aspects of the foundational background of and further details on combinatorial model categories are available in [1,8]. Eventually, a category that satisfies all the conditions of a locally presentable category, except that it is not necessarily cocomplete, is referred to as an **accessible category**.

Our categories of interest, $\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{A}$, and $\mathtt{Mod}(\mathcal{A})$, are cofibrantly generated model categories. In particular, they are (complete and) cocomplete, so that, to prove their combinatoriality, it suffices to prove their accessibility. As for $\mathtt{DG}\mathcal{D}\mathtt{M}$, we mentioned in the proof of Proposition 2.6 that it is locally presentable, hence, accessible. Regarding the accessibility of $\mathtt{DG}\mathcal{D}\mathtt{A} \simeq \mathtt{DG}\mathcal{D}\mathtt{M}^T$ (see Proposition 2.6) and $\mathtt{Mod}(\mathcal{A}) \simeq \mathtt{DG}\mathcal{D}\mathtt{M}^U$ (see Proposition 2.7), we recall [1, 2.78] that a category of algebras over a monad is accessible, if the monad is. Furthermore [1, 2.16], a monad (V, μ, η) over a category $\mathtt{C}$ is accessible, if its endofunctor $V : \mathtt{C} \to \mathtt{C}$ is accessible. Finally, a functor $G : \mathtt{C}' \to \mathtt{C}''$ is called accessible, if it is accessible for some regular cardinal κ, i.e., if $\mathtt{C}'$ and $\mathtt{C}''$ are κ-accessible categories, and if G preserves κ-directed colimits. Summarizing, to prove that $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$ are accessible, we only need to show that both, $T = F\mathcal{S}$ and $U = \Phi\,\Sigma$, preserve κ-directed colimits. In fact, since the left adjoints $\mathcal{S}$ and Σ respect all colimits, it suffices to reassess the right adjoints $F : \mathtt{DG}\mathcal{D}\mathtt{A} \to \mathtt{DG}\mathcal{D}\mathtt{M}$ and $\Phi : \mathtt{Mod}(\mathcal{A}) \to \mathtt{DG}\mathcal{D}\mathtt{M}$. However, in [6], we showed that F commutes with directed colimits (and κ-directed ones), and the proof for Φ is similar. Hence,

Proposition 3.10 *The (proper) model categories* $\mathtt{DG}\mathcal{D}\mathtt{M}$*,* $\mathtt{DG}\mathcal{D}\mathtt{A}$*, and* $\mathtt{Mod}(\mathcal{A})$ *are combinatorial model categories.*

3.4 HAC condition 3: cofibrancy and equivalence-invariance

As above, we choose $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$. The condition HAC3 [38, Assumption 1.1.0.3] asks that, for any cofibrant $M \in \mathtt{Mod}(\mathcal{A})$, the functor

$$- \otimes_{\mathcal{A}} M : \mathtt{Mod}(\mathcal{A}) \to \mathtt{Mod}(\mathcal{A})$$

respect weak equivalences. The requirement is not really surprising. Indeed, to avoid ‘equivalence-invariance breaking’ in the model category $\mathtt{Mod}(\mathcal{A})$ via tensoring by M, this operation should preserve weak equivalences – at least for ‘good’ objects M, i.e., for cofibrant ones. This is similar to tensoring, in the category $\mathtt{Mod}(R)$ of modules over a ring R, by an R-module M, what is an operation that respects injections for ‘good’ objects M, i.e., for flat R-modules.

Proposition 3.11 *Let* $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$*. For every cofibrant* $M \in \mathtt{Mod}(\mathcal{A})$*, the functor*

$$- \otimes_{\mathcal{A}} M : \mathtt{Mod}(\mathcal{A}) \to \mathtt{Mod}(\mathcal{A})$$

preserves weak equivalences.

Proof We have to prove that, if $f : P \to Q$ is a weak equivalence in $\mathtt{Mod}(\mathcal{A})$, then $f \otimes_{\mathcal{A}} \mathrm{id}_M$ is a weak equivalence as well.

By Lemma 2.14, the module M is a retract of a Sullivan $\mathcal{A}$-module, i.e., there exist a Sullivan $\mathcal{A}$-module $\mathcal{A} \otimes V$ and $\mathtt{Mod}(\mathcal{A})$-morphisms $i : M \to \mathcal{A} \otimes V$ and $j : \mathcal{A} \otimes V \to M$ such that $j \circ i = \mathrm{id}_M$. Since the diagram

$$\begin{array}{ccccc} P \otimes_{\mathcal{A}} M & \xrightarrow{\mathrm{id}_P \otimes_{\mathcal{A}} i} & P \otimes_{\mathcal{A}} (\mathcal{A} \otimes V) & \xrightarrow{\mathrm{id}_P \otimes_{\mathcal{A}} j} & P \otimes_{\mathcal{A}} M \\ \downarrow{\scriptstyle f \otimes_{\mathcal{A}} \mathrm{id}_M} & & \downarrow{\scriptstyle f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A} \otimes V}} & & \downarrow{\scriptstyle f \otimes_{\mathcal{A}} \mathrm{id}_M} \\ Q \otimes_{\mathcal{A}} M & \xrightarrow{\mathrm{id}_Q \otimes_{\mathcal{A}} i} & Q \otimes_{\mathcal{A}} (\mathcal{A} \otimes V) & \xrightarrow{\mathrm{id}_Q \otimes_{\mathcal{A}} j} & Q \otimes_{\mathcal{A}} M \end{array} \tag{3.22}$$

is a retract diagram in $\texttt{Mod}(\mathcal{A})$, it suffices to show that $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}\otimes V}$ is a weak equivalence.

The proof of Lemma 2.14 shows that $\mathcal{A} \otimes V$ is the colimit of the λ-sequence $X : \lambda \to \texttt{Mod}(\mathcal{A})$ defined by

$$X_\beta = \mathcal{A} \otimes \bigoplus_{\alpha \leq \beta, \alpha \in \mathbf{O}_s} S^{n(\alpha)} ,$$

where $S^{n(\alpha)}$ is the sphere $\mathcal{D} \cdot 1_{n(\alpha)}$, $n(\alpha) \in \mathbb{N}$, see Eq. 2.10. If we shift the index α of the generators $1_{n(\alpha)}$ by -1, we get

$$X_\beta = \mathcal{A} \otimes V_{<\beta} , \quad \text{where } V_{<\beta} = \bigoplus_{\alpha-1<\beta} \mathcal{D} \cdot 1_{n(\alpha-1)} .$$

As the morphisms $X_{\beta,\beta+1}$ are the canonical injections, the $\mathcal{A}$-module $\mathcal{A} \otimes V$ is equipped with the λ-filtration $\mathrm{F}_\beta(\mathcal{A} \otimes V) = X_\beta$, i.e., with the λ-filtration

$$0 \hookrightarrow \mathcal{A} \otimes V_{<1} \hookrightarrow \cdots \hookrightarrow \mathcal{A} \otimes V_{<\beta} \hookrightarrow \cdots$$

Since $\texttt{Mod}(\mathcal{A})$ is a closed monoidal category, the tensor product $P \otimes_{\mathcal{A}} -$ is a left adjoint functor and thus preserves colimits. Hence,

$$0 \hookrightarrow P \otimes_{\mathcal{A}} (\mathcal{A} \otimes V_{<1}) \hookrightarrow \cdots \hookrightarrow P \otimes_{\mathcal{A}} (\mathcal{A} \otimes V_{<\beta}) \hookrightarrow \cdots$$

is a λ-filtration of the $\mathcal{A}$-module $P \otimes_{\mathcal{A}} (\mathcal{A} \otimes V)$, or, in view of Lemma 3.1,

$$0 \hookrightarrow P \otimes V_{<1} \hookrightarrow \cdots \hookrightarrow P \otimes V_{<\beta} \hookrightarrow \cdots$$

is a λ-filtration of the $\mathcal{A}$-module $P \otimes V$. If we tensor by Q instead of P, we get an analogous λ-filtration for $Q \otimes V$. Moreover, one easily checks (see Eq. 3.1) that

$$\iota \circ (f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}\otimes V}) \circ \iota^{-1} = f \otimes \mathrm{id}_V ,$$

so that the used identifications imply that the $\texttt{Mod}(\mathcal{A})$-morphism $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}\otimes V}$ coincides with the $\texttt{Mod}(\mathcal{A})$-morphism $f \otimes \mathrm{id}_V$. Since the weak equivalences in $\texttt{Mod}(\mathcal{A})$ are those $\texttt{Mod}(\mathcal{A})$-morphisms that are weak equivalences in $\texttt{DG}\mathcal{D}\texttt{M}$, it actually suffices to show that $f \otimes \mathrm{id}_V : P \otimes V \to Q \otimes V$ is a weak equivalence in $\texttt{DG}\mathcal{D}\texttt{M}$.

We already mentioned (see paragraph above Proposition 3.10) that the forgetful functor $\Phi : \texttt{Mod}(\mathcal{A}) \to \texttt{DG}\mathcal{D}\texttt{M}$ respects directed colimits (alternatively we may argue that Φ preserves filtered colimits as right adjoint between two accessible categories). Thus, the λ-filtrations of $P \otimes V$ and $Q \otimes V$ in $\texttt{Mod}(\mathcal{A})$ are also λ-filtrations in $\texttt{DG}\mathcal{D}\texttt{M}$. Let now φ be the natural transformation between the $\texttt{DG}\mathcal{D}\texttt{M}$-filtration functors $\mathrm{F}_\beta(P \otimes V) = P \otimes V_{<\beta}$ and $\mathrm{F}_\beta(Q \otimes V) = Q \otimes V_{<\beta}$, defined by $\varphi_\beta = f \otimes \mathrm{id}_{V_{<\beta}}$:

$$\begin{array}{ccccccccc} 0 & \hookrightarrow & P \otimes V_{<1} & \hookrightarrow & \cdots & \hookrightarrow & P \otimes V_{<\beta}, & \hookrightarrow & \cdots \\ \downarrow{\scriptstyle \varphi_0=0} & & \downarrow{\scriptstyle \varphi_1 = f\otimes \mathrm{id}_{V_{<1}}} & & & & \downarrow{\scriptstyle \varphi_\beta = f\otimes \mathrm{id}_{V_{<\beta}}} & & \\ 0 & \hookrightarrow & Q \otimes V_{<1} & \hookrightarrow & \cdots & \hookrightarrow & Q \otimes V_{<\beta} & \hookrightarrow & \cdots \end{array} . \tag{3.23}$$

The colimit $\mathrm{colim}_{\beta<\lambda}\,\varphi_\beta$ is given by $f \otimes \mathrm{id}_V$, so that the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $f \otimes \mathrm{id}_V$ is compatible with the considered $\mathtt{DG}\mathcal{D}\mathtt{M}$-filtrations.

In order to apply Lemma 3.5, note that φ_0 is a weak equivalence and look at

$$\varphi_{\beta+1,\sharp} : \mathrm{Gr}_{\beta+1}(P \otimes V) \to \mathrm{Gr}_{\beta+1}(Q \otimes V)\ ,$$

where

$$\mathrm{Gr}_{\beta+1}(P \otimes V) = P \otimes V_{<\beta+1} / P \otimes V_{<\beta}\ .$$

Observe first that, we have a $\mathtt{G}\mathcal{D}\mathtt{M}$-isomorphism $\jmath : \mathrm{Gr}_{\beta+1}(P \otimes V) \to P \otimes S^{n(\beta+1)}$ and denote by δ the pushforward

$$\delta = \jmath \circ \partial^{P\otimes V}_{\beta+1,\sharp} \circ \jmath^{-1}$$

of the differential of $\mathrm{Gr}_{\beta+1}(P \otimes V)$. As, in view of Equation (3.2), the differential $\partial^{P\otimes V}_{\beta+1}$ of $\mathrm{F}_{\beta+1}(P \otimes V) = P \otimes V_{<\beta+1}$ is the pushforward

$$\imath \circ (d_P \otimes \mathrm{id}_\otimes + \mathrm{id}_P \otimes d) \circ \imath^{-1}$$

of the differential of $P \otimes_{\mathcal{A}} (\mathcal{A} \otimes V_{<\beta+1})$, and as d is the lowering differential of the Sullivan $\mathcal{A}$-module $\mathcal{A} \otimes V$ (i.e., $d\, 1_{n(\beta)} \in \mathcal{A} \otimes V_{<\beta}$), we get, for any argument in $P \otimes S^{n(\beta+1)}$,

$$\begin{aligned} \delta(p \otimes \Delta \cdot 1_{n(\beta)}) &= \jmath[\partial^{P\otimes V}_{\beta+1}(p \otimes \Delta \cdot 1_{n(\beta)})] = \jmath[d_P\, p \otimes \Delta \cdot 1_{n(\beta)} \\ &\quad +(-1)^{|p|}\imath(p \otimes \Delta \cdot d\, 1_{n(\beta)})] \\ &= d_P\, p \otimes \Delta \cdot 1_{n(\beta)} = (d_P \otimes \mathrm{id}_{S^{n(\beta+1)}})(p \otimes \Delta \cdot 1_{n(\beta)}) \\ &=: d_\otimes(p \otimes \Delta \cdot 1_{n(\beta)})\ , \end{aligned}$$

where $d_\otimes$ the natural differential on $P \otimes S^{n(\beta+1)}$. It follows that

$$d_\otimes = \jmath \circ \partial^{P\otimes V}_{\beta+1,\sharp} \circ \jmath^{-1}\ ,$$

so that $\jmath$ is a $\mathtt{DG}\mathcal{D}\mathtt{M}$-isomorphism and that we can identify $(\mathrm{Gr}_{\beta+1}(P \otimes V), \partial^{P\otimes V}_{\beta+1,\sharp})$ and $(P \otimes S^{n(\beta+1)}, d_\otimes)$ as differential graded $\mathcal{D}$-modules (and similarly for P replaced by Q). It is now easily checked that, when read through these isomorphisms, the morphism $\varphi_{\beta+1,\sharp}$ is the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism

$$f \otimes \mathrm{id}_{S^{n(\beta+1)}} : P \otimes S^{n(\beta+1)} \to Q \otimes S^{n(\beta+1)}\ .$$

In view of Lemma 3.5, it finally suffices to prove that $f \otimes \mathrm{id}_{S^{n(\beta+1)}}$ is a weak equivalence.

Via the by now standard argument, we get

$$\mathrm{Mc}(f \otimes \mathrm{id}_{S^{n(\beta+1)}}) \simeq (\mathrm{Mc}\, f)[-n(\beta+1)] \otimes S^{n(\beta+1)} \ .$$

Since f is a weak equivalence by assumption, Künneth's formula gives

$$\begin{aligned} H_\bullet(\mathrm{Mc}(f \otimes \mathrm{id}_{S^{n(\beta+1)}})) &\simeq H_\bullet((\mathrm{Mc}\, f)[-n(\beta+1)] \otimes S^{n(\beta+1)}) \\ &\simeq H_{\bullet-n(\beta+1)}(\mathrm{Mc}\, f) \otimes S^{n(\beta+1)} = 0 \ , \end{aligned}$$

so that $f \otimes \mathrm{id}_{S^{n(\beta+1)}}$ is a weak equivalence. This completes the proof. □

3.5 HAC condition 4: base change and equivalence-invariance

In this section, we investigate the condition HAC4 [38, Assumption 1.1.0.4].

It actually deals with the categories $\mathtt{CMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$ and $\mathtt{NuCMon}(\mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A}))$ of unital and non-unital commutative monoids in $\mathtt{Mod}(\mathcal{A})$. In [38], the HAC conditions are formulated over an underlying category, which is not necessarily $\mathtt{DG}\mathcal{D}\mathtt{M}$, but any combinatorial symmetric monoidal model category $\mathtt{C}$. The category of non-unital monoids appears in Assumption 1.1.0.4, only since $\mathtt{C}$ is not necessarily additive [38, Remark 1.1.0.5]. In our present case, the category $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$ is Abelian and thus additive, so that the condition on non-unital algebras is redundant here.

Just as there is an adjunction $\mathcal{S} : \mathtt{DG}\mathcal{D}\mathtt{M} \rightleftarrows \mathtt{CMon}(\mathtt{DG}\mathcal{D}\mathtt{M}) : F$, see Subsection 2.2.1, we have an adjunction

$$\mathcal{S}_{\mathcal{A}} : \mathtt{Mod}(\mathcal{A}) \rightleftarrows \mathtt{CMon}(\mathtt{Mod}(\mathcal{A})) : F_{\mathcal{A}} \ ,$$

which is defined exactly as $\mathcal{S} \dashv F$, except that the tensor product is not over $\mathcal{O}$ but over $\mathcal{A}$. Hence, it is natural to define weak equivalences (resp., fibrations) in $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$, as those morphisms that are weak equivalences (resp., fibrations) in $\mathtt{Mod}(\mathcal{A})$, or, equivalently, in $\mathtt{DG}\mathcal{D}\mathtt{M}$. Assumption 1.1.0.4 now reads
HAC 4.

- The preceding classes of weak equivalences and fibrations endow $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ with a combinatorial proper model structure.
- For any cofibrant $\mathcal{B} \in \mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$, the functor

$$\mathcal{B} \otimes_{\mathcal{A}} - : \mathtt{Mod}(\mathcal{A}) \to \mathtt{Mod}(\mathcal{B})$$

respects weak equivalences.

The axiom is easily understood. Recall first that the category $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ is isomorphic to the category $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$, see Proposition 2.3. Moreover, in [7] and [29], we emphasized the importance of a base change, i.e., of the replacement of $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$ by $\mathcal{B} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$ (we actually passed from $\mathcal{A} = \mathcal{O}$ to $\mathcal{B} = \mathcal{J}$, where $\mathcal{J}$ was interpreted as the function algebra of an infinite jet bundle). This suggests to reflect upon a functor from $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ to $\mathtt{CMon}(\mathtt{Mod}(\mathcal{B}))$, or, simply, from $\mathtt{Mod}(\mathcal{A})$ to $\mathtt{Mod}(\mathcal{B})$. The *natural* transition functor is $\mathcal{B} \otimes_{\mathcal{A}} -$, provided $\mathcal{B}$ is not only an object $\mathcal{B} \in \mathtt{DG}\mathcal{D}\mathtt{A}$ but an object $\mathcal{B} \in \mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$. Just as the functor $- \otimes_{\mathcal{A}} M$ in HAC3, the functor $\mathcal{B} \otimes_{\mathcal{A}} -$ is required to preserve weak equivalences, at least for cofibrant objects $\mathcal{B} \in \mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$. HAC4 further asks that the above-defined weak equivalences and fibrations implement a model structure on $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ and that cofibrancy be with respect to this structure. Finally, exactly as the so far considered model categories $\mathtt{DG}\mathcal{D}\mathtt{M}$, $\mathtt{DG}\mathcal{D}\mathtt{A}$, and $\mathtt{Mod}(\mathcal{A})$, the model category $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ must be combinatorial and proper.

Note that there are important examples that do not satisfy this axiom. For instance, it does not hold if the underlying category is the category $\mathtt{C} = \mathtt{DG}R\,\mathtt{M}$ of non-negatively graded chain complexes of modules over a commutative ring R with nonzero characteristic. Our task is to show that it *is* valid, if R is replaced by the non-commutative ring $\mathcal{D}$ of characteristic 0. On the other hand, the assumption HAC4 is essential in proving, for instance, the existence of an analog of the module $\Omega_{B/A}$ of relative differential 1-forms. The existence of this cotangent complex '$\Omega_{B/A}$' is on its part the main ingredient in the definition of smooth and étale morphisms.

Proposition 3.12 *The category* $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$*, whose morphisms are weak equivalences (resp., fibrations) if they are weak equivalences (resp., fibrations) in* $\mathtt{DG}\mathcal{D}\mathtt{M}$*, and whose morphisms are cofibrations if they have in* $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ *the left lifting property with respect to trivial fibrations, is a combinatorial proper model category.*

Proof The categorical isomorphism

$$\mathtt{CMon}(\mathtt{Mod}(\mathcal{A})) \simeq \mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$$

of Proposition 2.3 allows endowing $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ with the model structure of $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$: a $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$-morphism is a weak equivalence (resp., a fibration, a cofibration), if it is a weak equivalence (resp., a fibration, a cofibration) in $\mathtt{DG}\mathcal{D}\mathtt{A}$ [15]. Hence, a $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$-morphism is a weak equivalence (resp., a fibration), if it is a weak equivalence (resp., a fibration) in $\mathtt{DG}\mathcal{D}\mathtt{M}$ [6]. We know that these definitions provide $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ with a model structure, so that a $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$-morphism is a cofibration if and only if it has in $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ the left lifting property with respect to trivial fibrations. It follows that the distinguished classes of Proposition 3.12 equip $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ with a model structure. In addition, in view of [15, Theorem 2.8], this model category is proper and cofibrantly generated. Its generating cofibrations (resp., trivial cofibrations) are obtained from the cofibrations I (resp., trivial cofibrations J) of $\mathtt{DG}\mathcal{D}\mathtt{A}$ by means of the left adjoint functor

$$L_{\otimes} : \mathtt{DG}\mathcal{D}\mathtt{A} \ni \mathcal{B} \mapsto (\mathcal{A} \to \mathcal{A} \otimes \mathcal{B}) \in \mathtt{CMon}(\mathtt{Mod}(\mathcal{A})) : \mathrm{For}\ . \tag{3.24}$$

It remains to show that the category $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ is accessible. Remark that the monad For $L_{\otimes}$ coincides with the coproduct functor $\mathcal{A} \otimes - : \mathtt{DG}\mathcal{D}\mathtt{A} \ni \mathcal{B} \mapsto \mathcal{A} \otimes \mathcal{B} \in \mathtt{DG}\mathcal{D}\mathtt{A}$, and that, if For is monadic, we have the equivalence of categories

$\mathtt{DG}\mathcal{D}\mathtt{A}^{\mathcal{A}\otimes -} \simeq \mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$. To prove the accessibility of $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$, it thus suffices to show that $\mathcal{A}\otimes -$ respects directed colimits *and* that For satisfies the requirements of the monadicity theorem. However, the coproduct functor $\mathcal{A} \otimes -$ of $\mathtt{DG}\mathcal{D}\mathtt{A}$ commutes with colimits and in particular with directed ones. The first condition of the monadicity theorem asks that For reflect isomorphisms, what is easily checked. The second condition asks that $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ admit coequalizers of reflexive pairs and that For preserve them. Since $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ is a model category, it has *all* coequalizers. Finally, when applying For to the coequalizer of two parallel morphisms in $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$, we get the coequalizer in $\mathtt{DG}\mathcal{D}\mathtt{A}$ of the images by For of the considered parallel morphisms. Indeed, as for the universality of this coequalizer-candidate in $\mathtt{DG}\mathcal{D}\mathtt{A}$, any second coequalizer-candidate can be canonically lifted to $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$, and universality in $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$ provides a unique factorization-morphism in $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$, whose projection via For is a factorization-morphism in $\mathtt{DG}\mathcal{D}\mathtt{A}$. It can further be seen that the latter is unique, what completes the proof of the accessibility of $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$. □

The next proposition ensures that also Part 2 of HAC4 is satisfied.

Proposition 3.13 *Let $\mathcal{A}$ be an object in $\mathtt{DG}\mathcal{D}\mathtt{A}$ and let $\mathcal{B}$ be a cofibrant object in $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$. The functor*

$$\mathcal{B} \otimes_{\mathcal{A}} - : \mathtt{Mod}(\mathcal{A}) \to \mathtt{Mod}(\mathcal{B})$$

preserves weak equivalences.

Proof By assumption the morphism $\phi_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ is a cofibration in $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A})) \simeq \mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$, i.e., a cofibration in $\mathtt{DG}\mathcal{D}\mathtt{A}$. Consider now, in $\mathtt{DG}\mathcal{D}\mathtt{A}$, the cofibration - trivial fibration factorization of $\phi_{\mathcal{B}}$ constructed in [7, Theorem 5]:

$$\begin{array}{ccc} (\mathcal{A}, d_{\mathcal{A}}) & \rightarrowtail & (\mathcal{A}\otimes \mathcal{S}V, d_2) \\ \downarrow & \nearrow & \downarrow\sim \\ (\mathcal{B}, d_{\mathcal{B}}) & \xrightarrow[\mathrm{id}_{\mathcal{B}}]{} & (\mathcal{B}, d_{\mathcal{B}}) \end{array} . \tag{3.25}$$

The dashed arrow in this diagram exists in view of the left lifting property of cofibrations with respect to trivial fibrations. The diagram

$$\begin{array}{ccccc} & & (\mathcal{A}, d_{\mathcal{A}}) & & \\ & \swarrow & \downarrow & \searrow & \\ (\mathcal{B}, d_{\mathcal{B}}) & \dashrightarrow & (\mathcal{A}\otimes \mathcal{S}V, d_2) & \xrightarrow{\sim} & (\mathcal{B}, d_{\mathcal{B}}) \end{array} \tag{3.26}$$

now shows that $\mathcal{A} \to \mathcal{B}$ is a retract of $\mathcal{A} \to \mathcal{A}\otimes \mathcal{S}V$ in $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A}$, or, equivalently, that $\mathcal{B}$ is a retract of $\mathcal{A}\otimes \mathcal{S}V$ in $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ and so in $\mathtt{Mod}(\mathcal{A})$.

Let now $f : P \to Q$ be a weak equivalence in $\mathtt{Mod}(\mathcal{A})$. Since, as easily checked, $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{B}}$ is a retract of $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}\otimes\mathcal{S}V}$ in $\mathtt{Mod}(\mathcal{A})$, it suffices to show that the latter morphism is a weak equivalence in $\mathtt{Mod}(\mathcal{A})$. Indeed, in this case $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{B}}$ is also a weak equivalence in $\mathtt{Mod}(\mathcal{A})$, so a weak equivalence in $\mathtt{DG}\mathcal{D}\mathtt{M}$; it follows that the $\mathtt{Mod}(\mathcal{B})$-morphism $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{B}}$ is a weak equivalence in $\mathtt{Mod}(\mathcal{B})$, what then completes the proof.

If we use the identification detailed in Lemma 3.1, the morphism

$$f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}\otimes\mathcal{S}V} : P \otimes_{\mathcal{A}} (\mathcal{A} \otimes \mathcal{S}V) \to Q \otimes_{\mathcal{A}} (\mathcal{A} \otimes \mathcal{S}V)$$

becomes

$$f \otimes \mathrm{id}_{\mathcal{S}V} : P \otimes \mathcal{S}V \to Q \otimes \mathcal{S}V$$

and it suffices to prove that $f \otimes \mathrm{id}_{\mathcal{S}V}$ is a weak equivalence in $\mathtt{Mod}(\mathcal{A})$, i.e., in $\mathtt{DG}\mathcal{D}\mathtt{M}$.

It is known from [7, Theorem 5 and Section 6.2] that $(\mathcal{A} \otimes \mathcal{S}V, d_2)$ is the colimit in $\mathtt{DG}\mathcal{D}\mathtt{A}$ of a λ-sequence of injections

$$\begin{aligned}(\mathcal{A}, d_{\mathcal{A}}) \hookrightarrow (\mathcal{A} \otimes \mathcal{S}V_{<1}, d_{2,<1}) \hookrightarrow (\mathcal{A} \otimes \mathcal{S}V_{<2}, d_{2,<2}) \hookrightarrow \cdots \\ \cdots (\mathcal{A} \otimes \mathcal{S}V_{<\omega}, d_{2,<\omega}) \hookrightarrow (\mathcal{A} \otimes \mathcal{S}V_{<\omega+1}, d_{2,<\omega+1}) \hookrightarrow \cdots\end{aligned} \tag{3.27}$$

where $\mathcal{A} \otimes \mathcal{S}V_{<\beta}$ has the usual meaning (see above) and where $d_{2,<\beta}$ is the restriction of d_2 . Since colimits in an undercategory are computed as colimits in the underlying category, the commutative $\mathtt{Mod}(\mathcal{A})$-monoid $(\mathcal{A}\otimes\mathcal{S}V, d_2)$ is also the colimit of the preceding λ-sequence in $\mathcal{A} \downarrow \mathtt{DG}\mathcal{D}\mathtt{A} \simeq \mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$. Moreover, as a coslice category of an accessible category is accessible, the categories $\mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$ and $\mathtt{Mod}(\mathcal{A})$ are both accessible. It follows that the forgetful functor $F_{\mathcal{A}} : \mathtt{CMon}(\mathtt{Mod}(\mathcal{A})) \to \mathtt{Mod}(\mathcal{A})$ commutes with filtered colimits as right adjoint $\mathcal{S}_{\mathcal{A}} \dashv F_{\mathcal{A}}$ between accessible categories. Hence, the sequence (3.27) is a λ-filtration of $\mathcal{A} \otimes \mathcal{S}V$ in $\mathtt{Mod}(\mathcal{A})$. We can now argue as in the proof of Proposition 3.11: the sequence

$$\begin{aligned}(P, d_P) \hookrightarrow (P \otimes \mathcal{S}V_{<1}, \delta_{P,<1}) \hookrightarrow (P \otimes \mathcal{S}V_{<2}, \delta_{P,<2}) \hookrightarrow \cdots \\ \cdots (P \otimes \mathcal{S}V_{<\omega}, \delta_{P,<\omega}) \hookrightarrow (P \otimes \mathcal{S}V_{<\omega+1}, \delta_{P,<\omega+1}) \hookrightarrow \cdots\end{aligned}$$

is a λ-filtration of $(P \otimes \mathcal{S}V, \delta_P)$ – in $\mathtt{Mod}(\mathcal{A})$, as well as in $\mathtt{DG}\mathcal{D}\mathtt{M}$. Here

$$\delta_P = \iota \circ (d_P \otimes \mathrm{id}_{\otimes} + \mathrm{id}_P \otimes d_2) \circ \iota^{-1}$$

is the differential $d_P \otimes \mathrm{id}_{\otimes} + \mathrm{id}_P \otimes d_2$ pushed forward from $P \otimes_{\mathcal{A}} (\mathcal{A} \otimes \mathcal{S}V)$ to $P \otimes \mathcal{S}V$. Let now φ be, as in the proof of Proposition 3.11, the natural transformation between the $\mathtt{DG}\mathcal{D}\mathtt{M}$-filtration functors $F_\beta(P \otimes \mathcal{S}V) = P \otimes \mathcal{S}V_{<\beta}$ and $F_\beta(Q \otimes \mathcal{S}V) = Q \otimes \mathcal{S}V_{<\beta}$, defined by $\varphi_\beta = f \otimes \mathrm{id}_{\mathcal{S}V_{<\beta}}$:

$$\begin{array}{ccccccccc}
(P, d_P) & \hookrightarrow & (P \otimes \mathcal{S}V_{<1}, \delta_{P,<1}) & \hookrightarrow & \cdots & \hookrightarrow & (P \otimes \mathcal{S}V_{<\beta}, \delta_{P,<\beta}) & \hookrightarrow & \cdots \\
\downarrow{\scriptstyle \varphi_0 = f} & & \downarrow{\scriptstyle \varphi_1 = f\otimes \mathrm{id}_{\mathcal{S}V_{<1}}} & & & & \downarrow{\scriptstyle \varphi_\beta = f\otimes \mathrm{id}_{\mathcal{S}V_{<\beta}}} & & \\
(Q, d_Q) & \hookrightarrow & (Q \otimes \mathcal{S}V_{<1}, \delta_{Q,<1}) & \hookrightarrow & \cdots & \hookrightarrow & (Q \otimes \mathcal{S}V_{<\beta}, \delta_{Q,<\beta}) & \hookrightarrow & \cdots
\end{array} . \tag{3.28}$$

It follows again that the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $f \otimes \mathrm{id}_{\mathcal{S}V}$ is compatible with the $\mathtt{DG}\mathcal{D}\mathtt{M}$-filtrations. To show that $f \otimes \mathrm{id}_{\mathcal{S}V}$ is a weak equivalence in $\mathtt{DG}\mathcal{D}\mathtt{M}$, it suffices to prove that the $\varphi_\beta = f \otimes \mathrm{id}_{\mathcal{S}V_{<\beta}}$, $\beta < \lambda$, are weak equivalences, see Lemma 3.3. This proof will be a transfinite induction on $\beta < \lambda$.

The induction starts, since $\varphi_0 = f$ is a weak equivalence by assumption. We thus must show that φ_β, $\beta < \lambda$, is a weak equivalence, assuming that the φ_α, $\alpha < \beta$, are weak equivalences.

The limit ordinal case $\beta \in \mathbf{O}_\ell$ is a direct consequence of Lemma 3.3.

Let now $\beta \in \mathbf{O}_s$ be the successor of an ordinal γ .

To simplify notation, we denote the differential graded $\mathcal{D}$-module

$$F_\gamma(P \otimes \mathcal{S}V) = (P \otimes \mathcal{S}V_{<\gamma}, \delta_{P,<\gamma}) \quad (\text{resp., } F_\gamma(Q \otimes \mathcal{S}V) = (Q \otimes \mathcal{S}V_{<\gamma}, \delta_{Q,<\gamma}))$$

by $(P', d_{P'})$ (resp., $(Q', d_{Q'})$) and we denote the morphism $\varphi_\gamma = f \otimes \mathrm{id}_{\mathcal{S}V_{<\gamma}}$ by f'. The isomorphism

$$P \otimes \mathcal{S}V_{<\beta} \simeq P \otimes \mathcal{S}V_{<\gamma} \otimes \mathcal{S}(\mathcal{D} \cdot 1_{n(\gamma)}) = P' \otimes \mathcal{S}S^{n(\gamma)} \tag{3.29}$$

of graded $\mathcal{D}$-modules (it just replaces $\odot$ by $\otimes$ and vice versa, so that we will use it tacitly) allows to push the differential

$$\delta_{P,<\beta} = \iota \circ (d_P \otimes \mathrm{id}_\otimes + \mathrm{id}_P \otimes d_2|_{\mathcal{A}\otimes\mathcal{S}V_{<\beta}}) \circ \iota^{-1}$$

of $P \otimes \mathcal{S}V_{<\beta}$ forward to a differential $\partial_{P,<\beta}$ of $P' \otimes \mathcal{S}S^{n(\gamma)}$ and to thus obtain an isomorphic differential graded $\mathcal{D}$-module structure on $P' \otimes \mathcal{S}S^{n(\gamma)}$. The lowering property of d_2 induces a kind of lowering property for $\partial_{P,<\beta}$:

$$(\partial_{P,<\beta} - d_{P'} \otimes \mathrm{id}_{\mathcal{S}S^{n(\gamma)}})(P' \otimes \mathcal{S}^{k+1}S^{n(\gamma)}) \subset P' \otimes \mathcal{S}^k S^{n(\gamma)}. \tag{3.30}$$

Indeed, let $p \otimes \odot_i v_{\alpha_i} \otimes \odot_j s_j$ be an element in $P' \otimes \mathcal{S}^{k+1}S^{n(\gamma)}$ (notation is self-explaining, in particular $\alpha_i < \gamma$ and $s_j = D_j \cdot 1_{n(\gamma)}$). We have

$$\begin{aligned}
\partial_{P,<\beta}(p \otimes \odot_i v_{\alpha_i} \otimes \odot_j s_j) &= \delta_{P,<\beta}(p \otimes \odot_i v_{\alpha_i} \odot \odot_j s_j) \\
&= \iota(d_P \otimes \mathrm{id}_\otimes + \mathrm{id}_P \otimes d_2|_{\mathcal{A}\otimes\mathcal{S}V_{<\beta}}) \\
&\quad (p \otimes (1_\mathcal{A} \otimes (\odot_i v_{\alpha_i} \odot \odot_j s_j))) .
\end{aligned} \tag{3.31}$$

When noticing that

$$1_\mathcal{A} \otimes (\odot_i v_{\alpha_i} \odot \odot_j s_j) = (1_\mathcal{A} \otimes \odot_i v_{\alpha_i}) \Diamond \Diamond_j (1_\mathcal{A} \otimes s_j) ,$$

where $\Diamond$ is the multiplication in $\mathcal{A} \otimes \mathcal{S}V_{<\beta}$, and when remembering that d_2 is a derivation of $\Diamond$, we see that the expression in Eq. (3.31) reads

$$\begin{aligned}
&\iota \big(d_P\, p \otimes (1_{\mathcal{A}} \otimes (\odot_i v_{\alpha_i} \odot \odot_j s_j)) \\
&\quad + (-1)^{|p|} p \otimes d_2(1_{\mathcal{A}} \otimes \odot_i v_{\alpha_i}) \odot \odot_j s_j\big) \\
&\quad \pm \iota \left((-1)^{|p|} p \otimes (1_{\mathcal{A}} \otimes \odot_i v_{\alpha_i}) \Diamond \sum_j \pm (1_{\mathcal{A}} \otimes s_1) \Diamond \ldots \Diamond d_2(1_{\mathcal{A}} \otimes s_j) \Diamond \ldots \Diamond (1_{\mathcal{A}} \otimes s_{k+1}) \right) .
\end{aligned} \tag{3.32}$$

The first term (two first rows) is equal to

$$\begin{aligned}
&\iota \left(d_P\, p \otimes (1_{\mathcal{A}} \otimes \odot_i v_{\alpha_i}) \otimes \odot_j s_j + (-1)^{|p|} p \otimes d_2(1_{\mathcal{A}} \otimes \odot_i v_{\alpha_i}) \otimes \odot_j s_j\right) \\
&\quad = \iota \left((d_P \otimes \mathrm{id}_{\otimes} + \mathrm{id}_P \otimes d_2|_{\mathcal{A}\otimes \mathcal{S}V_{<\gamma}})(p \otimes (1_{\mathcal{A}} \otimes \odot_i v_{\alpha_i}))\right) \otimes \odot_j s_j \\
&\quad = (\delta_{P,<\gamma} \otimes \mathrm{id}_{\mathcal{S}S^{n(\gamma)}})(p \otimes \odot_i v_{\alpha_i} \otimes \odot_j s_j) \\
&\quad = (d_{P'} \otimes \mathrm{id}_{\mathcal{S}S^{n(\gamma)}})(p \otimes \odot_i v_{\alpha_i} \otimes \odot_j s_j) .
\end{aligned}$$

Since $d_2(1_{\mathcal{A}} \otimes s_j) \in \mathcal{A} \otimes \mathcal{S}V_{<\gamma}$, the remaining term in Eq. (3.32) is an element of $P' \otimes \mathcal{S}^k S^{n(\gamma)}$, so that the claim (3.30) holds true.

The $\mathtt{DG}\mathcal{D}\mathtt{M}$-isomorphism (3.29) and the lowering property (3.30) are of course also valid for $Q \otimes \mathcal{S}V_{<\beta} \simeq Q' \otimes \mathcal{S}S^{n(\gamma)}$. Recall now that it remains to prove that

$$\varphi_\beta = f \otimes \mathrm{id}_{\mathcal{S}V_{<\beta}} : (P \otimes \mathcal{S}V_{<\beta}, \delta_{P,<\beta}) \to (Q \otimes \mathcal{S}V_{<\beta}, \delta_{Q,<\beta})$$

is a weak equivalence in $\mathtt{DG}\mathcal{D}\mathtt{M}$, i.e., that

$$f' \otimes \mathrm{id}_{\mathcal{S}S^{n(\gamma)}} : (P' \otimes \mathcal{S}S^{n(\gamma)}, \partial_{P,<\beta}) \to (Q' \otimes \mathcal{S}S^{n(\gamma)}, \partial_{Q,<\beta})$$

is a weak equivalence. In view of the afore-detailed lowering property (3.30), it suffices to replicate the proof of the $\mathtt{DG}\mathcal{D}\mathtt{A}$-case in Step 1 of the proof of Theorem 3.7.

Acknowledgements The work of G. Di Brino was partially supported by the National Research Fund, Luxembourg, and the Marie Curie Actions Program of the European Commission (FP7-COFUND).

Appendices

In the following appendices, notation is the same as in the main part of the text.

Appendix A: Locally presentable categories

Recall that an infinite cardinal κ is *regular*, if no set of cardinality κ is the union of less than κ sets of cardinality less than κ. For instance, if $\kappa = \aleph_0 = \omega$, no countable set is a finite union of finite sets, so that $\aleph_0 = \omega$ is regular.

Let $(I, \leq)$ be a *directed poset*, i.e., a partially ordered set in which every pair of elements has an upper bound, i.e., for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. We view this poset as a category $\mathtt{I}$ whose morphisms $i \to j$ correspond to the inequalities $i \leq j$. A diagram of type $\mathtt{I}$ in a category $\mathtt{C}$ is a direct system and its limit is a *direct limit* or *directed colimit*. More generally, for a regular cardinal κ, a κ*-directed poset* $(J, \leq)$ is a poset in which every subset of cardinality less than κ has an upper bound. Then a colimit over a diagram of type $\mathtt{J}$ in a category $\mathtt{C}$ is called a κ*-directed colimit*. For $\kappa = \aleph_0$, we recover the preceding notion of directed colimit.

A finitely presented (left) module over a ring R is an R-module that is generated by a finite number of its elements, which satisfy a finite number of relations. The categorical substitute for this idea is a category all of whose elements are directed colimits $\varinjlim_i c_i = \bigsqcup_i c_i / \sim$ of some generating objects c_i. This leads to the concept of locally κ-presentable category: such a category is, roughly, a category that comes equipped with a (small) subset S of κ-small objects that generate all objects under κ-directed colimits.

Remember first that the idea of smallness of an object $c \in \mathtt{C}$ is that the covariant Hom functor $\mathrm{Hom}_{\mathtt{C}}(c, \bullet)$ commutes with a certain type of colimits. This actually means (see, for instance, [6]) that any morphism $c \to \operatorname{colim}_i c_i$ out of the small c into a certain type of colimit $\operatorname{colim}_i c_i$ factors through one of the maps $c_j \to \operatorname{colim}_i c_i$. If κ is a regular cardinal, a κ*-small*, κ*-compact, or* κ*-presentable* object $c \in \mathtt{C}$ is an object, such that $\mathrm{Hom}_{\mathtt{C}}(c, \bullet)$ commutes with κ-directed colimits. An object is called *small*, if it is κ-small, for some regular κ.

Combining the two last paragraphs, we get the

Definition 4.1 For a regular cardinal κ, a *locally* κ*-presentable category* $\mathtt{C}$ is

1. a locally small category
2. that has all small colimits
3. and admits a *set* $S \subset \mathrm{Ob}(\mathtt{C})$ of κ-small objects, such that any object in $\mathtt{C}$ is the κ-directed colimit of objects in S.

A category is termed a *locally presentable category*, if it is locally κ-presentable, for some regular κ.

Appendix B: Internal Hom in modules over a commutative monoid

Let $(\mathtt{C}, \otimes, \mathrm{I}, \underline{\mathrm{Hom}})$ be a closed symmetric monoidal category with all small limits and colimits.

In the main part of the present text, we recalled that the category $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$ of modules in $\mathtt{C}$ over a commutative monoid $\mathcal{A}$ in $\mathtt{C}$ is also a closed symmetric monoidal category with all small limits and colimits. However, we did not define the internal $\underline{\mathrm{Hom}}_{\mathcal{A}}$ of this category of modules, at least not in the considered abstract setting. Remember first that $\mathtt{C}$ and $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$ are endowed with bifunctors Hom and $\mathrm{Hom}_{\mathcal{A}}$, and that $\mathtt{C}$ is in addition equipped with an internal $\underline{\mathrm{Hom}}$. Before considering the internal $\underline{\mathrm{Hom}}_{\mathcal{A}}$, recall still that a closed monoidal category $\mathtt{C}$ can be *equivalently defined* as a monoidal category together with, for any two objects C' and C'', a $\mathtt{C}$-

object $\underline{\mathrm{Hom}}(C', C'')$ and a $\mathtt{C}$-morphism

$$\mathrm{ev}_{C',C''} : \underline{\mathrm{Hom}}(C', C'') \otimes C' \to C'' ,$$

which are universal in the sense that, for every $\mathtt{C}$-object X and $\mathtt{C}$-morphism $f : X \otimes C' \to C''$, there exists a unique $\mathtt{C}$-morphism $h : X \to \underline{\mathrm{Hom}}(C', C'')$ such that $f = \mathrm{ev}_{C',C''} \circ (h \otimes \mathrm{id}_{C'})$. Indeed, if we start for instance from the usual definition, the existence of $\mathrm{ev}_{C',C''}$ comes from

$$\mathrm{Hom}(X, \underline{\mathrm{Hom}}(C', C'')) \simeq \mathrm{Hom}(X \otimes C', C'') ,$$

when choosing $X = \underline{\mathrm{Hom}}(C', C'')$. Moreover, if $h : X \to \underline{\mathrm{Hom}}(C', C'')$, we get a $\mathtt{C}$-map

$$f := \mathrm{ev}_{C',C''} \circ (h \otimes \mathrm{id}_{C'}) : X \otimes C' \to C'' .$$

Conversely, if $f \in \mathrm{Hom}(X \otimes C', C'')$, there exists a unique $h \in \mathrm{Hom}(X, \underline{\mathrm{Hom}}(C', C''))$, such that $\mathrm{ev}_{C',C''} \circ (h \otimes \mathrm{id}_{C'}) = f$. Now, if $M', M'' \in \mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$, the $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$-object $\underline{\mathrm{Hom}}_{\mathcal{A}}(M', M'')$ should be the kernel of the ‘ map

$$\underline{\mathrm{Hom}}(M', M'') \ni f \mapsto f \circ \mu_{M'} - \mu_{M''} \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \in \underline{\mathrm{Hom}}(\mathcal{A} \otimes M', M'') \text{ ' } .$$

To put this idea right, we consider the isomorphism

$$\mathrm{Hom}(\underline{\mathrm{Hom}}(M', M''), \underline{\mathrm{Hom}}(\mathcal{A} \otimes M', M'')) \simeq \mathrm{Hom}(\underline{\mathrm{Hom}}(M', M'') \otimes \mathcal{A} \otimes M', M'') ,$$

and define the preceding kernel as the equalizer of the pair of parallel $\mathtt{C}$-arrows

$$\begin{aligned} &\mathrm{ev}_{M',M''} \circ (\mathrm{id} \otimes \mu_{M'}), \mu_{M''} \circ (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{ev}_{M',M''}) \circ (\mathrm{com} \otimes \mathrm{id}_{M'}) \\ &\quad \in \mathrm{Hom}(\underline{\mathrm{Hom}}(M', M'') \otimes \mathcal{A} \otimes M', M'') . \end{aligned}$$

This $\mathtt{C}$-object inherits an $\mathcal{A}$-module structure.

Appendix C: Universes

It is well-known that the set of all sets is not a set but a proper class. In the following, we consider a pyramid of types of set. Start with some type of set on top of the pyramid and call it the 0-sets. Then the set of all 0-sets is not a 0-set, but a set of a next, more general, type, say a 1-set. Similarly, the set of all 1-sets is not a 1-set but a 2-set, and so on. Finally, the union of all types of set is the proper class of all sets.

The adequate formalization of the idea of set of all sets of a certain type is the notion of Grothendieck universe (∗). A Grothendieck universe a (very large) set U, whose elements are sets and which is closed under all standard set-theoretical operations. More precisely [2],

Definition 4.2 A **universe** is a set U that satisfies the axioms:

(1) if $x \in U$ and $y \in x$, then $y \in U$,
(2) if $x, y \in U$, then the set $\{x, y\}$ is an element of U,
(3) if $x \in U$, then the set $\mathcal{P}(x)$ of all subsets of x is an element of U,
(4) if $I \in U$ and $x_i \in U$, for all $i \in I$, then $\bigcup_{i \in I} x_i \in U$,
(5) $\mathbb{N} \in U$.

The preceding axioms allow to prove many additional closure properties, but it is not impossible to leave a universe. The elements of U are termed U-**sets**. In particular, U is the set of all U-sets (see $(*)$). As suggested above $U \notin U$, but there exists a pyramid of universes $U \in V \in W \in \ldots$, so that any element of U is also an element of V and of W, and so on. It is therefore natural to think about the union of all Grothendieck universes as the proper class of all sets. Moreover, this interpretation implies that any set belongs to some universe (Grothendieck's axiom).

We continue with a number of basic definitions.

A set S is U-**small**, if S is isomorphic to a U-set (not all authors distinguish between U-set and U-small set). The category $U-\mathtt{Set}$ is the category with objects all the U-sets and with morphisms all the maps between two U-sets. Both, the collection $\mathrm{Ob}(U-\mathtt{Set})$ of objects and the collection $\mathrm{Mor}(U-\mathtt{Set})$ of morphisms are sets, although no U-sets, but we can speak about the category $U-\mathtt{Set}$ without having to pass to proper classes.

Moreover, a U-**category** $\mathtt{C}$, or, better, a **locally** U-**small category** $\mathtt{C}$, is a category such that, for any $c', c'' \in \mathtt{C}$, the set $\mathrm{Hom}_{\mathtt{C}}(c', c'')$ is U-small. In [2], a category $\mathtt{C}$ is viewed as the set $\mathrm{Mor}(\mathtt{C})$ of its arrows (containing the subset of identity arrows, i.e., the subset $\mathrm{Ob}(\mathtt{C})$ of objects). Hence, $\mathtt{C} \in U$ and $\mathtt{C}$ is U-small can be given the usual meanings. More precisely, if $\mathtt{C} \simeq \mathrm{Mor}(\mathtt{C}) \in U$, then $\mathrm{Ob}(\mathtt{C}) \in \mathcal{P}(\mathtt{C}) \in U$: for $\mathtt{C} \in U$, we have $\mathrm{Ob}(\mathtt{C}) \in U$ and $\mathrm{Mor}(\mathtt{C}) \in U$, i.e., objects and morphisms are U-sets. Similarly, if a category $\mathtt{C} \simeq \mathrm{Mor}(\mathtt{C})$ is U-small, it is easily seen that $\mathrm{Ob}(\mathtt{C})$ and $\mathrm{Mor}(\mathtt{C})$ are U-small sets. Let us stress that:

Remark 4.3 Contrarily to a U-set S, which is just a set $S \in U$, a U-category $\mathtt{C}$ is not a category $\mathtt{C} \in U$: A U-category is a locally U-small category in the above sense, whereas a category $\mathtt{C} \in U$ is a category such that $\mathrm{Ob}(\mathtt{C}), \mathrm{Mor}(\mathtt{C}) \in U$. Note that, in view of what has been said above, any category $\mathtt{C}$ belongs to U, is U-small, and is locally U-small, for some universe U.

The necessity to change from a universe V to a larger universe $W \ni V$ appears in particular when speaking about generalized spaces. If $\mathtt{C}$ denotes some category of spaces, its Yoneda dual category

$$\mathtt{C}^{\vee} := \mathtt{Fun}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set}) ,$$

i.e., the category of contravariant $\mathtt{Set}$-valued functors defined on $\mathtt{C}$, or, still, the category of presheaves defined on $\mathtt{C}$, may be viewed as a category of generalized spaces. In our work, the category

$$\mathtt{SC}^{\vee}_{V} := \mathtt{Fun}(\mathtt{C}^{\mathrm{op}}, V-\mathtt{SSet})$$

of simplicial presheaves on $\mathtt{C}$ with respect to V will play an important role. We start recalling some fundamental results [2]:

Proposition 4.4 *Consider a universe V, two categories $\mathtt{C}, \mathtt{D}$, as well as the category $\mathtt{Fun}(\mathtt{C}, \mathtt{D})$ of functors from $\mathtt{C}$ to $\mathtt{D}$.*

(1) *If $\mathtt{C}, \mathtt{D} \in V$ (resp., are V-small), the category $\mathtt{Fun}(\mathtt{C}, \mathtt{D})$ is an element of V (resp., is V-small).*
(2) *If $\mathtt{C}$ is V-small and $\mathtt{D}$ is a V-category, the category $\mathtt{Fun}(\mathtt{C}, \mathtt{D})$ is a V-category.*
(3) *If $\mathtt{C}$ is V-small, the category $\check{\mathtt{C}}_V$ is a V-category.*
(4) *If $\mathtt{C}$ is a V-category, the category $\check{\mathtt{C}}_V$ is not necessarily a V-category.*

Remark 4.5 Usually authors do not specify the universe in which they work, assuming implicitly that their constructions and results hold in *any universe V*. However, sometimes set-theoretical size issues force them to pass to a higher universe $W \ni V$. In this case, their theory is (considered as) valid in *any universes $V \in W$*. One says that the theory has been *universally quantified* over 1,2, or several universes, and one speaks about the **universal polymorphism approach**. If the passage to higher universes is also implicit, one speaks about **typical ambiguity**. However, this ambiguity, although often used and even sometimes recommended, can be dangerous [33, Remarks 1.3.2 and 2.5.12].

In our paper, we start with the category $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$, which is locally U-small for some universe U (it is clear that the categories $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$ are also locally U-small). However, Ob($\mathtt{DG}\mathcal{D}\mathtt{M}$) and Mor($\mathtt{DG}\mathcal{D}\mathtt{M}$) can be sets that belong only to a higher universe $V \ni U$, so that $\mathtt{DG}\mathcal{D}\mathtt{M}$ is then V-small (and the same holds for $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{Mod}(\mathcal{A})$). Since $\mathtt{DG}\mathcal{D}\mathtt{M}$ is V-small, the category

$$\check{\mathtt{DG}\mathcal{D}\mathtt{M}}_V = \mathtt{Fun}(\mathtt{DG}\mathcal{D}\mathtt{M}^{\mathrm{op}}, V{-}\mathtt{Set})$$

is locally V-small (4.4) and thus it is W-small for some higher universe $W \ni V$. When considering the V-small category $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{A}^{\mathrm{op}}$, we conclude that

$$\mathtt{S}\,\check{\mathtt{DG}\mathcal{D}\mathtt{A}^{\mathrm{op}}}_V = \mathtt{Fun}(\mathtt{DG}\mathcal{D}\mathtt{A}, V{-}\mathtt{SSet})$$

is locally V-small and W-small [37, Appendix A.1].

The preceding paragraph explains the idea behind the introduction of the three universes $U \in V \in W$ in [38]. In the present paper, we work implicitly in an arbitrary universe U that we need a priori not mention. However, since typical ambiguity can lead to problems, we mention explicitly the change of universe each time it is required. In fact, this is not necessary until we pass to simplicial presheaves.

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