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On a Quillen adjunction between the categories of differential graded and simplicial coalgebras

W. Hermann B. Sore[1]

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Abstract
We prove that the normalization functor of the Dold-Kan correspondence does *not* induce a Quillen equivalence between Goerss' model category of simplicial coalgebras and Getzler–Goerss' model category of differential graded coalgebras.

Keywords Model category · Dold-Kan correspondence · Coalgebras

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Contents

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✉ W. Hermann B. Sore
hermann.sore@gmail.com

1 Departement de Mathematiques et Informatique, UFR/ST, Universite Nazi Boni de Bobo-Dioulasso (ex UPB), 01 BP 1091, Bobo-Dioulasso 01, Burkina Faso

1 Introduction

In [9], the authors consider the monoidal model categories of simplicial modules **SMod** and differential graded modules **DGMod** over a commutative ring A. These categories are Quillen equivalent via the Dold-Kan correspondence and the authors then give sufficient conditions for lifting this Quillen equivalence to the level of the associated categories of monoids, namely, simplicial associative algebras **SAlg** and differential graded algebras **DGAlg** over A. The crucial argument (see [9, Theorem 3.12]) that enables the lifting of this Quillen equivalence is that the functors (Forget: $\mathbf{SAlg} \to \mathbf{SMod}$) and (Forget: $\mathbf{DGAlg} \to \mathbf{DGMod}$) *create* (see [9, Definition 3.2]) the model structures for **SAlg** and **DGAlg** respectively.

In [10, Sect. 4], we consider the monoidal model categories of simplicial vector spaces **SVct** and differential graded vector spaces **DGVct** over a field K. Since these categories are Quillen equivalent via the Dold-Kan correspondence, we investigate whether dual methods from [9] apply to lift this Quillen equivalence to the level of the categories of comonoids, namely, simplicial coassociative coalgebras **ScoAlg** and differential graded coassociative coalgebras **DGcoAlg** over K. However, many problems arise with dualizing the crucial notion of *create*. While the small object argument satisfied in many usual categories, enables the transfer of a model category structure from left to right along a given adjunction, its formal dual the cosmall object argument is harder to achieve. In fact the categories **SVct** and **DGVct** do not have enough cosmall objects to guarantee a right to left transfer of model category structure along the adjunctions (Forget : $\mathbf{ScoAlg} \rightleftarrows \mathbf{SVct}$: Cofree) and (Forget : $\mathbf{DGcoAlg} \rightleftarrows \mathbf{DGVct}$: Cofree) respectively. We refer to [1] and [5] where the authors give conditions for right to left transfer of model category structures. In [10, Sect. 5] we then restrict to the categories of connected simplicial coassociative coalgebras $\mathbf{ScoAlg}_c$ and connected differential graded coassociative coalgebras $\mathbf{DGcoAlg}_c$ over a field K. An object C in $\mathbf{ScoAlg}_c$ (resp. in $\mathbf{DGcoAlg}_c$) is an object in **ScoAlg** (resp. in **DGcoAlg**) with $C_0 = K$. Note that connected coalgebras are conilpotent coalgebras. Moreover the cofree coalgebra on a connected differential graded vector space V (i.e. $V \in \mathbf{DGVct}$ with $V_0 = 0$) is just given by $T'_d(V)$ the tensor coalgebra on V. In contrast, the cofree coalgebra on a connected simplicial vector space W (i.e. $W \in \mathbf{SVct}$ with $W_0 = 0$) is given by a degreewise extension of Sweedler's cofree coalgebra functor on W. Since the functor $T'_d(-)$ has good homological properties, the natural attempt was to check directly whether the connected coalgebra-valued normalization functor induces a Quillen equivalence. But an error (see [11]) occurs with a wrong definition of the cofree coalgebra functor from $\mathbf{SVct}_c$ to $\mathbf{ScoAlg}_c$ and consequently skews the claimed Quillen equivalence between $\mathbf{ScoAlg}_c$ and $\mathbf{DGcoAlg}_c$.

The present paper is then the sequel to [10] and brings a correction to [11]. Our aim remains to check whether the Quillen adjunction ($\widetilde{N}$: $\mathbf{ScoAlg} \leftrightarrows \mathbf{DGcoAlg}$: R) established in [10, Proposition 4.5]can be improved to a Quillen equivalence. For this, we are led to understand the homology vector spaces of the cofree coalgebras $S_d(\mathbb{S}^1) \in \mathbf{DGcoAlg}$ and $S_s\Gamma(\mathbb{S}^1) \in \mathbf{ScoAlg}$. Note that both objects are connected coalgebras. Moreover, both involve Sweedler's cofree coalgebra functor as constructed in [12, Theorem 6.4.1]. Since we do not assume cocommutativity, the definition of this cofree coalgebra functor is very abstract and consequently difficult to use for homological computations. However a more tractable interpretation of this functor is available: pioneered by Peterson-Taft in [8] for 1-dimensional vector spaces, this interpretation has been generalized by Hazewinkel in [4] for higher dimensional vector spaces. By using this generalization by [4], we provide some explicit constructions of multivariable recursive sequences that prove that the normalization functor *cannot* induce a Quillen equivalence between Goerss' model category of simplicial coalgebras and Getzler–Goerss' model category of differential graded coalgebras.

The paper is organized as follows. In Sect. 2, we give a review of Sweedler's cofree coalgebra functor, its interpretation by Peterson-Taft and the generalization by Hazewinkel. In Sect. 3, we recall Getzler–Goerss' model category of differential graded coalgebras and Goerss' model category of simplicial coalgebras. In Sect. 4, we investigate our motivating problem: given the simplicial vector space $\Gamma(\mathbb{S}^1)$ with Sweedler's cofree coalgebra S applied degreewise, we try to evaluate the dimension of the homology vector spaces of its associated Moore complex $\widetilde{N}S_s\Gamma(\mathbb{S}^1)$. As a consequence, we derive the failure of our Quillen adjunction $(\widetilde{N}, R)$ to be a Quillen equivalence.

2 Preliminaries on cofree coalgebras

2.1 The construction of the cofree coalgebra functor

Let K be a fixed field. If A is a K-algebra, it is proven in [12, Proposition 6.0.2] that the vector space

$$A^\circ = \Big\{ f \in A^* \mid \ker f \text{ contains a cofinite ideal } I \text{ of } A \Big\}$$

has a coalgebra structure. Moreover the assignment $A \mapsto A^\circ$ defines a contravariant functor $(-)^\circ : \mathbf{Alg} \to \mathbf{coAlg}$ and by [12, Lemma 6.0.1] if $f : A \to B$ is a morphism of algebras then $f^\circ : B^\circ \to A^\circ$ is the restriction of f^* to B°.

Definition 2.1 Let V be a vector space. A pair (C, π) with $C \in \mathbf{coAlg}$ and $\pi : C \to V$ a map of vector spaces is called a *cofree coalgebra on* V if for any coalgebra D and any map $\nu : D \to V$ of vector spaces there exists a unique coalgebra morphism g such that the diagram

$$\begin{array}{ccc} V & \xleftarrow{\ \nu\ } & D \\ {\scriptstyle \pi}\uparrow & \swarrow {\scriptstyle \exists!\, g} & \\ C & & \end{array}$$

commutes. In other words, the pair (C, π) is couniversal among the pairs (D, ν). Notice that the coalgebra C when it exists is unique up to an isomorphism of coalgebras.

Theorem 2.2 [12, Theorem 6.4.1] *There exists a cofree coalgebra on any* $V \in \mathbf{Vct}$.

If $V \in \mathbf{Vct}$ and $T : \mathbf{Vct} \to \mathbf{Alg}$ denotes the tensor algebra functor, then $((T(V^*)^\circ, \pi)$ with π given by the composition $T(V^*)^\circ \to T(V^*)^* \to V^{**}$ is the cofree coalgebra on the bidual V^{**}. By [12, Lemma 6.4.2] the cofree coalgebra on V is given by (D, ρ) where $D = \sum E$ with the sum taken over all subcoalgebras E such that $\pi(E) \subseteq V$ and ρ is the restriction of π to D.

Corollary 2.3 *The cofree coalgebra construction above defines a functor* $S\colon \mathbf{Vct} \to \mathbf{coAlg}$ *that is right adjoint to the coalgebra forgetful functor* $U\colon \mathbf{coAlg} \to \mathbf{Vct}$.

2.2 The cofree coalgebra on a 1-dimensional vector space

Let V be a 1-dimensional vector space. Then $T(V^*)$ is isomorphic to $K[X]$ and $K[X]^*$ can be identified with $K^{\mathbb{N}}$, the space of all infinite sequences $f = (f_n)_{n\geq 0}$ where $f_n = f(X^n)$. Let $f \in S(V) = K[X]^\circ \subseteq K[X]^*$. Since all non-zero ideals J of $K[X]$ are generated by monic polynomials, they are all cofinite. Let $f(J) = 0$ for such a non-zero J containing a monic polynomial $h(X) = X^r - h_1X^{r-1} - \cdots - h_r$. Then $f(X^m h(X)) = 0$ for $m \geq 0$, *i.e.*,

$$f_n = h_1 f_{n-1} + h_2 f_{n-2} + \cdots + h_r f_{n-r} \quad \text{for} \quad n \geq r.$$

Using this fact [8, Pages 6–7] proves the following important description.

Proposition 2.4 [8, Pages 6–7] *The cofree coalgebra* $S(K)$ *on a one-dimensional vector space can be identified with* $K[X]^\circ$ *which consists of the space of all linearly recursive sequences,* i.e.

sequences $f = (f_n)_{n\geq 0}$ *satisfying for some* $r > 0$ *the relation*

$$f_n = h_1 f_{n-1} + h_2 f_{n-2} + \cdots + h_r f_{n-r} \quad \textit{for} \quad n \geq r. \tag{2.1}$$

2.3 The cofree coalgebra on any finite dimensional vector space

In [4], Hazewinkel gives an appropriate definition of recursiveness in the multivariable case that generalizes the above description by [8]. We give a brief review of his construction. However, we restrict in this section to finite dimensional vector spaces instead of free modules with finite rank over a commutative ring as in Hazewinkel's original generalization paper [4].

Let V be a finite dimensional vector space with $\dim_K V = m \geq 2$ and $\langle X_1, \ldots, X_m \rangle$ be a basis of V. It is well-known that the tensor algebra $T(V)$ can be identified with the non-commutative polynomial algebra $K\langle X_1, \ldots, X_m\rangle$. If we denote by Word $\{1, \ldots, m\}$ the free monoid of all words in the alphabet $\{1, \ldots, m\}$, then a word $\alpha \in$ Word $\{1, \ldots, m\}$ is written $\alpha = \alpha_1 \ldots \alpha_n$ where $n \in \mathbb{N} \setminus \{0\}$ and where the α_i are in $\mathbb{N} \setminus \{0\}$. The empty word is written $\emptyset$. The *length* of α denoted by $\lg(\alpha)$ is then n and $\lg(\emptyset)$ is 0. A basis of $K\langle X_1, \ldots, X_m\rangle$ is given by all monomials $\{X_\alpha\}_\alpha$ where $X_\alpha = X_{\alpha_1} \ldots X_{\alpha_n}$and $X_\emptyset = 1_K$.

With this reminder the tensor algebra $T(V^*)$ can be identified with the non-commutative polynomial algebra $K\left\langle X_1^*, \ldots, X_m^*\right\rangle$. Furthermore the completion of

$T(V)$ given by $\widehat{T(V)} = \prod_{n\geq 0} V^{\otimes n}$ can be identified with the non-commutative power series algebra $K\langle\langle X_1, \ldots, X_m\rangle\rangle$. Since V has a *finite* basis, the completion $\widehat{T(V)}$ can also be identified with $T(V^*)^*$ via

$$\begin{array}{rlcll} \Psi : \widehat{T(V)} & \longrightarrow & (T(V^*))^* & & \\ f & \longmapsto & \Psi(f) & : T(V^*) \longrightarrow & K \\ & & & \varphi \quad \longmapsto & \langle f, \varphi\rangle = \sum \left\langle f^i, \varphi^i\right\rangle \end{array}$$

where $f^i \in V^{\otimes i}, \varphi^i \in (V^*)^{\otimes i}$. Now let f be in $\widehat{T(V)} = K\langle\langle X_1, \ldots, X_m\rangle\rangle$. Then f can be written as $f = \sum_\alpha f_\alpha X_\alpha$ and for a basis element $X^*_\beta \in T(V^*)$, we have $\Psi(f)(X^*_\beta) = \left\langle f, X^*_\beta\right\rangle = f_\beta$.

Definition 2.5 A tensor power series $f \in \widehat{T(V)}$ is *representative* if there exists $k \in \mathbb{N}\setminus\{0\}$ and $(g_i)_{i\in\{1,\ldots,k\}}, (h_i)_{i\in\{1,\ldots,k\}} \in \widehat{T(V)}$ such that

$$f(ab) = \sum_{i=1}^{k} g_i(a)h_i(b) \qquad \forall\, a, b \in T(V^*).$$

It is proven in [4, Theorem 3.14] that TV_{repr}, the vector space of all representative tensor power series over K is the cofree coalgebra over V.

In order to have a more tractable characterization of representative tensor power series, Hazewinkel introduces the following definition (that generalizes linearly recursive sequences).

Definition 2.6 [4, Definition 4.3] Let V be a vector space with basis $\left\{X_j, j \in J\right\}$. A tensor power series $f \in \widehat{T(V)} \cong K\left\langle\left\langle X_j, j \in J\right\rangle\right\rangle$ is

(1) *left recursive* if there is a finite set of monomials $X_{\lambda_i}, i = 1, \ldots, l$ and for some fixed $s > \max\{\lg(\lambda_i), i = 1, \ldots, l\}$, there are coefficients $c_{\gamma,i} \in K$, for each $i = 1, \ldots, l$ and word $\gamma \in \text{Word}(J)$ of length s, such that for $n \geq s$ for each $\alpha \in \text{Word}(J)$ of length n

$$f(\alpha) = \sum_{i=1}^{l} c_{\alpha_{\text{pre}(s)},i}\, f(\lambda_i\alpha_{\text{suf}}) \tag{2.2}$$

where if β, γ are two words over J then $\beta\gamma$ is the concatenation of them and where for a word α of length $\geq s$, $\alpha_{\text{pre}(s)}$ is the prefix of α of length s and α_{suf} is the corresponding suffix so that $\alpha = \alpha_{\text{pre}(s)}\alpha_{\text{suf}}$.

(2) *right recursive* if there is a finite set of monomials $X_{\rho_i}, i = 1, \ldots, r$ and for some fixed $t > \max\{\lg(\rho_i), i = 1, \ldots, r\}$, there are coefficients $d_{\gamma,i} \in K$, for each $i = 1, \ldots, r$ and word $\gamma \in \text{Word}(J)$ of length t, such that for $n \geq t$ for each $\alpha \in \text{Word}(J)$ of length n

$$f(\alpha) = \sum_{i=1}^{l} d_{\alpha_{\text{suf}(t)},i}\, f(\alpha_{\text{pre}}\rho_i) \tag{2.3}$$

where this time $\alpha_{\mathrm{suf}(t)}$ is the suffix of α of length t and α_{pre} is the corresponding prefix so that $\alpha = \alpha_{\mathrm{pre}}\alpha_{\mathrm{suf}(t)}$.

(3) *left (resp. right) recursive with finiteness condition* if it is left (resp. right) recursive and moreover the recursive coefficient matrix $\left(c_{\alpha,i}\right)_{\lg(\alpha)=s,i}$ (resp. $\left(d_{\alpha,i}\right)_{\lg(\alpha)=s,i}$) has only finitely many entries unequal to zero.

(4) *recursive* if it is both left and right recursive.

(5) *recursive with finiteness condition* if it is left recursive with finiteness condition and right recursive with finiteness condition.

In the case of *finitely* many variables, [4, Theorem 5.7] proves that the notions of left recursiveness, right recursiveness, recursiveness and representativeness are all equivalent. Therefore, we obtain the following result as a consequence of [4, Theorems 3.14, 4.6 and 5.7].

Proposition 2.7 [4, Corollary 5.9] *Let $V = K^{\oplus m}$ be an m-dimensional vector space. Then the cofree coalgebra $S(V) = S(K^{\oplus m}) = \left(T((K^{\oplus m})^*)\right)^\circ$ can be identified with $K\left\langle X_1^*, \ldots, X_m^*\right\rangle^\circ$ which consists of the space of all tensor power series in $K\left\langle\left\langle X_1, \ldots, X_m\right\rangle\right\rangle$ that are recursive with finiteness condition.*

Remark 2.8 Note that the Eq. (2.2) of Definition 2.6 (1) means that a power series f is left recursive if there exists some order $s \in \mathbb{N}\setminus\{0\}$ and some coefficients $c_{\beta,\gamma} \in K$ such that each coefficient f_α with α a word of length $\lg(\alpha) \geq s$ may be expressed as

$$f_\alpha = \sum_{\substack{\gamma\in\mathrm{Word}\\ \lg(\gamma)<s}} c_{\alpha_{\mathrm{pre}(s)},\gamma} \quad f_{\gamma\alpha_{\mathrm{suf}}} \tag{2.4}$$

Example 2.9 A recursive power series $f \in K\left\langle\left\langle X_1\right\rangle\right\rangle$ of order $s = 2$ with $\alpha \in \mathrm{Word}\{1\}$ has the following expressions for its coefficients f_α

$$\begin{aligned}
f_{11} &= c_{11,\emptyset} f_\emptyset + c_{11,1} f_1\\
f_{111} &= c_{11,\emptyset} f_1 + c_{11,1} f_{11}\\
f_{1111} &= c_{11,\emptyset} f_{11} + c_{11,1} f_{111}\\
&\vdots\\
f_{1^n} &= \ldots c_{11,\emptyset} f_{1^{n-2}} + c_{11,1} f_{1^{n-1}}
\end{aligned}$$

Hence, by setting $n = \underbrace{1\ldots1}_{n} = 1^n$, $c_{11,\emptyset} = c_0$ and $c_{11,1} = c_1$ one returns to the usual definition of a linearly recursive sequence of order 2.

3 Model category structures on categories of coalgebras

3.1 A model category structure on DGcoAlg

In this section, we consider the category of coassociative, counital differential non-negatively graded coalgebras over a fixed field K, denoted here by **DGcoAlg**. We do *not* assume cocommutativity for **DGcoAlg**. We recall some basic results due to Getzler and Goerss in their unpublished paper [2]. These results and their proofs have been revisited in [10, Sect. 3.1].

There is a categorical equivalence between the category of vector spaces and the category of profinite vector spaces given by the linear dual functor $(-)^*$ and the continuous linear dual functor $(-)'$. This categorical equivalence extends to a categorical equivalence between the category of coalgebras and the category of profinite algebras. This fact is used by Getzler and Goerss to prove the following result.

Proposition 3.1 [2, Proposition 1.10]. *Let V be an object in* **DGVct**. *Then, there is a functor denoted S_d from* **DGVct** *to* **DGcoAlg** *that is right adjoint to the functor U_d that forgets the coalgebra structure:*

(1) *if V is degreewise finite dimensional,*

$$S_d(V) = \left(\widehat{T_d(V^*)}\right)'$$

where $T_d(-)$, $\widehat{(-)}$ and $(-)'$ denote respectively the tensor algebra, the profinite completion and the continuous dual functors.

(2) *for any $V \in$* **DGVct**,

$$S_d(V) = \mathrm{colim}_\alpha(S_d(V_\alpha))$$

with V_α running over finite dimensional subvector spaces of V.

Remark 3.2 For a finite dimensional vector space V, the cofree coagebra $(T(V^*))^\circ$ given in Proposition 2.7 and the cofree coalgebra $\left(\widehat{T(V^*)}\right)'$ given in the previous Proposition 3.1 (here V is viewed as a differential graded vector space with V concentrated in degree 0) are isomorphic since any two-right adjoint of a functor are naturally isomorphic.

Theorem 3.3 [2, Definition 2.3, Theorem 2.8]. *Define $f : C \to D \in$* **DGcoAlg** *to be*

1. *a* weak equivalence *if $H_* f$ is an isomorphism.*
2. *a* cofibration *if f is a degreewise injection of graded vector spaces.*
3. *a* fibration *if f has the right lifting property with respect to acyclic cofibrations.*

With these definitions, **DGcoAlg** *becomes a closed model category.*

3.2 A model category structure on ScoAlg

In this section, we consider the category of coassociative, counital simplicial coalgebras over a fixed field K, denoted here by **ScoAlg**. We *do* not assume cocommutativity for **ScoAlg**. In [3, Sect. 3], Goerss has established a model category structure for *cocommutative* simplicial coalgebras. In [10, Sect. 3.2], we have adapted Goerss' arguments for proving the existence of a model category structure for our category of not necessarily cocommutative simplicial coalgebras **ScoAlg**.

Lemma 3.4 *The forgetful functor U_s from the category of simplicial coalgebras to the category of simplicial vector spaces has a right adjoint S_s.*

Proof The functor S_s is obtained by extending degreewise the cofree coalgebra functor S from the category of vector spaces to the category of not necessarily cocommutative coalgebras as stated in Corollary 2.3. □

Theorem 3.5 [3, Sect. 3]. *Define $f : C \to D \in$* **ScoAlg** *to be*

1. *a* weak equivalence *if $\pi_* f$ is an isomorphism.*
2. *a* cofibration *if f is a levelwise inclusion.*
3. *a* fibration *if f has the right lifting property with respect to acyclic cofibrations.*

With these definitions, **ScoAlg** *becomes a closed model category.*

4 A comparison of the categories ScoAlg and DGcoAlg

4.1 The Dold-Kan correspondence

The Dold-Kan correspondence is a classical result that establishes an equivalence of categories between the category of simplicial objects and the category of differential non-negatively graded objects in every abelian category. In this section, we collect the basics for its construction. We refer to [13, Chapter 8] for more details.

Definition 4.1 Let A be a simplicial object in an abelian category **C**, then

(1) the *differential graded object associated to* A is denoted by CA and is given by $(CA)_n = A_n$ with differential the alternating sum of the face operators:

$$d = \sum_{i=0}^{n} (-1)^i d_i : (CA)_n \to (CA)_{n-1}$$

(2) and the *differential graded object of degenerate simplices associated to* A, denoted by DA is the subcomplex of CA with

$$(DA)_0 = 0 \quad \text{and} \quad (DA)_n = s_0 A_{n-1} + \cdots + s_{n-1} A_{n-1} \quad \text{for} \quad n \geq 1.$$

Definition 4.2 The *normalized differential graded object associated to* A, denoted by NA is the quotient of CA by its subcomplex DA, that is

$$NA = CA/DA.$$

The *normalization functor* N is the functor that associates to a simplicial object its normalized differential graded object.

We also mention the alternative definition of the normalization functor N given by the Moore chain complex

$$(NA)_n = \bigcap_{i=0}^{n-1} \ker(d_i)$$

with differential $(NA)_n \longrightarrow (NA)_{n-1}$ induced from d_n by restriction.

Theorem 4.3 (Dold-Kan correspondence) *Let* $\mathbf{C}$ *be an abelian category. Then, the normalization functor* N *is an equivalence of categories between the category of differential non-negatively graded objects in* $\mathbf{C}$ *and the category of simplicial objects in* $\mathbf{C}$.

We refer to [13, 8.4.4] for an algorithm that describes the inverse Γ of the normalization functor and for a proof of the Dold-Kan correspondence.

Example 4.4 Let us denote by $\mathbb{S}^n$ the n-sphere chain complex. This is the object of **DGVct** which has the field K in degree n and 0 in other degrees. All differentials in $\mathbb{S}^n$ are trivial. We describe the image under the inverse functor Γ of the vector space $\mathbb{S}^n$ for $n \geq 1$ with the help of the algorithm given in Sect. [13, 8.4.4]. If l is an index with $l < n$, then $\left(\Gamma(\mathbb{S}^n)\right)_l$ is the zero vector space 0. For an index $l \geq n$, the vector space $\left(\Gamma(\mathbb{S}^n)\right)_l$ consists of $\binom{l}{n}$ copies of K. Hence restricting to $n = 1$, the diagram of face operators in the simplicial vector space $\Gamma(\mathbb{S}^1)$ is

$$0 \begin{smallmatrix}\longleftarrow\\\longleftarrow\end{smallmatrix} K \begin{smallmatrix}\longleftarrow\\\longleftarrow\\\longleftarrow\end{smallmatrix} K^{\oplus 2} \begin{smallmatrix}\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow\end{smallmatrix} K^{\oplus 3} \begin{smallmatrix}\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow\end{smallmatrix} K^{\oplus 4} \quad \cdots$$

We may represent the face operators involved in $\Gamma(\mathbb{S}^1)$ with matrices where the entries 1 and 0 denote respectively the identity map on K and the trivial map. We give here the face operators involved in the coming sections.

The face operators $d_i : K \longrightarrow 0$ are all trivial.

The face operators $d_i : K^{\oplus 2} \longrightarrow K$ are given by

$$d_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \; d_1 = \begin{pmatrix} 1 & 1 \end{pmatrix},$$
$$d_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The face operators $d_i : K^{\oplus 3} \longrightarrow K^{\oplus 2}$ are given by

$$d_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; d_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; d_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.2 A Quillen adjunction between ScoAlg and DGcoAlg

We refer to [10, Sect. 4.2] for the details of the constructions involved in this paragraph. We would like to compare the model categories of **ScoAlg** and **DGcoAlg** by means of a Quillen adjunction. For this purpose, we have considered the classical Dold-Kan correspondence (N, Γ) between simplicial and differential graded vector spaces as in [13, Chapter 8]. Then, we have viewed the categories of coalgebras as their respective categories of comonoids. Moreover, we have seen in Proposition 3.1 that the forgetful functor U_d from the category of differential graded coalgebras to the category of differential graded vector spaces has a right adjoint S_d. The counterpart result for the categories of simplicial coalgebras and vector spaces is recalled in Lemma 3.4. In this way, the situation to be studied may be illustrated in the diagram

$$\begin{array}{ccc} (\mathbf{SVct}, \widehat{\otimes}, I(K)) & \underset{\Gamma}{\overset{N}{\rightleftarrows}} & (\mathbf{DGVct}, \otimes, K[0]) \\ U_s \uparrow\downarrow S_s & & U_d \uparrow\downarrow S_d \\ \mathbf{ScoAlg} & \xrightarrow{\widetilde{N}} & \mathbf{DGcoAlg}. \end{array}$$

where $\widetilde{N}$ stands for the coalgebra-valued normalization functor. In [10, Sect. 4.2] we prove the following result.

Proposition 4.5 [10, Proposition 4.5] *In the above situation the functor $\widetilde{N}$ has a right adjoint R. Moreover the adjoint pair $(\widetilde{N}, R)$ is a Quillen adjunction.*

4.3 Is the Quillen adjunction $(\widetilde{N}, R)$ a Quillen equivalence?

This section addresses the question whether the Quillen adjunction $(\widetilde{N}, R)$ considered in Proposition 4.5 is a Quillen equivalence.

Definition 4.6 Given two model categories $\mathbf{C}$ and $\mathbf{D}$ and a Quillen adjunction $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ between them, the Quillen pair (F, U) is a *Quillen equivalence* if and only if, for every cofibrant object $C \in \mathbf{C}$ and fibrant object $D \in \mathbf{D}$, a map $f : FC \to D$ is a weak equivalence in $\mathbf{D}$ if and only if its adjoint map $C \to UD$ is a weak equivalence in $\mathbf{C}$.

In Goerss' model category of **ScoAlg** every object is cofibrant. Moreover every object is fibrant in the model structure on the category **DGVct**. Since the functor S_d preserves fibrations, it follows that every cofree coalgebra is fibrant and in particular $S_d(\mathbb{S}^n)$ is fibrant. By considering the Quillen adjunction $(\widetilde{N}, R)$, the fibrant object $S_d(\mathbb{S}^1) \in \mathbf{DGcoAlg}$ and the cofibrant object $RS_d(\mathbb{S}^1) \in \mathbf{ScoAlg}$, we prove that while the identity map $\mathrm{id}\colon RS_d(\mathbb{S}^1) \to RS_d(\mathbb{S}^1)$ is clearly a weak equivalence in **ScoAlg**, its adjoint map $\widetilde{N}RS_d(\mathbb{S}^1) \to S_d(\mathbb{S}^1)$ fails to be a weak equivalence in **DGoAlg**.

Before starting our homological computations, we recall the following definition on ordered alphabets.

Definition 4.7 [7, Sect. 1.2.1] Given finite words x, y over an ordered alphabet, the *radix order* $\preccurlyeq$ is defined by

$$x \preccurlyeq y \iff \begin{cases} \lg(x) < \lg(y) \\ \text{or} \\ \lg(x) = \lg(y) \text{ and } x = uax' \text{ and } y = uby' \end{cases}$$

with a, b letters in the ordered alphabet such that $a \leq b$.

4.3.1 Description of the Moore chain complex associated to $S_s\Gamma(\mathbb{S}^1)$

By [10, Proof of Proposition 4.5] we have $R \circ S_d = S_s \circ \Gamma$. Therefore, we have $\widetilde{N} R S_d(\mathbb{S}^1) = \widetilde{N} S_s \Gamma(\mathbb{S}^1)$ and we are forced to understand the Moore complex associated to $S_s\Gamma(\mathbb{S}^1)$. With the help of the description of $\Gamma(\mathbb{S}^1)$ in Example 4.4 we obtain the simplicial coalgebra $S_s\Gamma(\mathbb{S}^1)$

$$S_s\Gamma(\mathbb{S}^1): \qquad S(0) \leftleftarrows S(K) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; S(K^{\oplus 2}) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; S(K^{\oplus 3}) \ldots$$

by extending degreewise Sweedler's cofree coalgebra functor S given in Corollary 2.3.

Let us consider a face operator

$$d_i : K^{\oplus m} \longrightarrow K^{\oplus (m-1)}$$

with $m \geq 1$ coming from the description of $\Gamma(\mathbb{S}^1)$ in Example 4.4. By applying successively the linear dual functor $(-)^*$ and the tensor algebra functor T to d_i we obtain

$$T\left(d_i^*\right) : T\left[\left(K^{\oplus(m-1)}\right)^*\right] \longrightarrow T\left[\left(K^{\oplus(m)}\right)^*\right].$$

Then, we obtain the following morphism

$$\begin{array}{rccc} T(d_i^*)^* : & \left(T\left[\left(K^{\oplus(m)}\right)^*\right]\right)^* & \longrightarrow & \left(T\left[\left(K^{\oplus(m-1)}\right)^*\right]\right)^* \\ & f & \longmapsto & T(d_i^*)^*(f) \end{array}$$

by applying the linear dual functor to $T\left(d_i^*\right)$. With the help of the introduction of Sect. 2.3, we identify $\left(T\left[\left(K^{\oplus(m)}\right)^*\right]\right)^*$ with the completion $T\left[\widehat{K^{\oplus(m)}}\right]$. Thus, $f \in \left(T\left[\left(K^{\oplus(m)}\right)^*\right]\right)^*$ is viewed as an infinite sequence

$$f = (f_\alpha)_{\alpha \in \text{Word}\{1,\ldots,m\}} = (f_\emptyset, f_1, \ldots, f_m, f_{11}, \ldots, f_\alpha, \ldots)$$

with the radix order on Word $\{1, \ldots, m\}$. Hence, if $K^{\oplus m}$ has a basis $\langle Y_1, \ldots, Y_m \rangle$ and if α is the word $\alpha_1 \ldots \alpha_n$, then

$$f_\alpha = \left\langle f, Y_{\alpha_1}^* \ldots Y_{\alpha_n}^* \right\rangle = f(Y_{\alpha_1}^* \ldots Y_{\alpha_n}^*). \tag{4.1}$$

Furthermore, by considering a basis $\langle X_1, \ldots, X_{m-1}\rangle$ for $K^{\oplus(m-1)}$, the morphism $T(d_i^*)^*(f)$ is defined as the composition

$$\begin{array}{rccccc} T(d_i^*)^*(f): T\left[\left(K^{\oplus(m-1)}\right)^*\right] & \xrightarrow{T(d_i^*)} & T\left[\left(K^{\oplus(m)}\right)^*\right] & \xrightarrow{f} & K \\ (X_k^*)^{\otimes n} & \longmapsto & (d_i^*(X_k^*))^{\otimes n} & \longmapsto & f\left((d_i^*(X_k^*))^{\otimes n}\right) \end{array} \tag{4.2}$$

where by using the definition of d_i, $d_i^*(X_k^*)$ for $k \in \{1, \ldots, m-1\}$ is a linear combination of the basis elements Y_l^* with $l \in \{1, \ldots, m\}$.

It follows that the functor S applied on the face operators d_i gives the maps

$$S(d_i) = T(d_i^*)^\circ \colon \left(T\left[\left(K^{\oplus(m)}\right)^*\right]\right)^\circ \longrightarrow \left(T\left[\left(K^{\oplus(m-1)}\right)^*\right]\right)^\circ$$

that is, by definition the restriction of $T(d_i^*)^*$ to the cofree coalgebra $\left(T(K^{\oplus m})^*\right)^\circ$.

We can now define the normalized or Moore chain complex ([13, Definition 8.3.6]) associated to the simplicial coalgebra $S_s\Gamma(\mathbb{S}^1)$ by

$$\left(\tilde{N}S_s\Gamma(\mathbb{S}^1)\right)_n = \begin{cases} S(0) = K & \text{if } n = 0 \\ \bigcap_{i=0}^{n-1} \ker S(d_i) & \text{if } n \geq 1 \end{cases}$$

with differential $\partial_n \colon \left(\tilde{N}S_s\Gamma(\mathbb{S}^1)\right)_n \longrightarrow \left(\tilde{N}S_s\Gamma(\mathbb{S}^1)\right)_{n-1}$ induced from $S(d_n)$ by restriction.

4.3.2 Computing $H_2\left(\tilde{N}S_s\Gamma(\mathbb{S}^1)\right)$

We apply the process in Sect. 4.3.1 in order to describe the vector spaces and the differentials involved in the computation of $H_2\left(\tilde{N}S_s\Gamma(\mathbb{S}^1)\right)$.

If $K^{\oplus 2} = \langle Y_1, Y_2\rangle$, $K^{\oplus 1} = \langle X_1\rangle$ and $d_0, d_1, d_2 \colon K^{\oplus 2} \longrightarrow K$ are the face operators given in Example 4.4, we obtain the following assignments

$$\begin{aligned} d_0^* &\colon X_1^* \longmapsto Y_1^* \\ d_1^* &\colon X_1^* \longmapsto Y_1^* + Y_2^* \\ d_2^* &\colon X_1^* \longmapsto Y_2^*. \end{aligned}$$

which determine the morphisms $T(d_i^*)^*$ for $i \in \{0, 1, 2\}$. In fact, by using (4.2) we obtain the following composition mappings

$$T(d_0^*)^*: (X_1^*)^{\otimes n} \longmapsto (Y_1^*)^{\otimes n} \longmapsto f\left((Y_1^*)^{\otimes n}\right) = f_{\underbrace{1\ldots1}_{n}}$$

$$T(d_1^*)^*: (X_1^*)^{\otimes n} \longmapsto (Y_1^* + Y_2^*)^{\otimes n} \longmapsto f\left((Y_1^* + Y_2^*)^{\otimes n}\right) = \sum_{\substack{\alpha\in \mathrm{Word}\{1,2\} \\ \lg(\alpha)=n}} f_\alpha$$

$$T(d_2^*)^* : (X_1^*)^{\otimes n} \longmapsto (Y_2^*)^{\otimes n} \longmapsto f\left((Y_2^*)^{\otimes n}\right) = f_{\underbrace{2\ldots2}_{n}} \tag{4.3}$$

respectively.

If $K^{\oplus 3} = \langle Z_1, Z_2, Z_3 \rangle$, $K^{\oplus 2} = \langle Y_1, Y_2 \rangle$ and $d_0, d_1, d_2, d_3 \colon K^{\oplus 3} \longrightarrow K^{\oplus 2}$ are the face operators given in Example 4.4. we obtain the following mappings

$$\begin{aligned} d_0^* &: \begin{cases} Y_1^* \mapsto Z_1^* \\ Y_2^* \mapsto Z_2^* \end{cases} & d_1^* &: \begin{cases} Y_1^* \mapsto \quad Z_1^* \\ Y_2^* \mapsto Z_2^* + Z_3^* \end{cases} \\ d_2^* &: \begin{cases} Y_1^* \mapsto Z_1^* + Z_2^* \\ Y_2^* \mapsto \qquad Z_3^* \end{cases} & d_3^* &: \begin{cases} Y_1^* \mapsto Z_2^* \\ Y_2^* \mapsto Z_3^* \end{cases} \end{aligned} \tag{4.4}$$

which determine the morphisms $T(d_i^*)^*$ for $i \in \{0, 1, 2, 3\}$.

Example 4.8 By using (4.2), the morphism $T(d_1^*)^*$ is given by the following composition assignments

$$\begin{aligned} \left(Y_1^*\right)^{\otimes n} &\longmapsto \quad \left(Z_1^*\right)^{\otimes n} \quad \longmapsto \quad f\left(\left(Z_1^*\right)^{\otimes n}\right) = f_{1^n} \\ \left(Y_2^*\right)^{\otimes n} &\longmapsto \left(Z_2^* + Z_3^*\right)^{\otimes n} \longmapsto f\left(\left(Z_2^* + Z_3^*\right)^{\otimes n}\right) = \sum_{\substack{\alpha \in \mathrm{Word}\{2,3\} \\ \lg(\alpha)=n}} f_\alpha \end{aligned}$$

and by means of (4.1), we display the following example of computation

$$\begin{aligned} (S(d_1)(f_\alpha))_{1211} &= T(d_1^*)^*(f)(Y_1^* Y_2^* Y_1^* Y_1^*) \\ &= f \circ T(d_1^*)(Y_1^* Y_2^* Y_1^* Y_1^*) \\ &= f\left(Z_1^*(Z_2^* + Z_3^*)Z_1^* Z_1^*\right) \\ &= f(Z_1^* Z_2^* Z_1^* Z_1^*) + f(Z_1^* Z_3^* Z_1^* Z_1^*) \\ &= f_{1211} + f_{1311} \end{aligned}$$

Lemma 4.9 *The vector space* $\left(\tilde{N} S_s \Gamma(\mathbb{S}^1)\right)_2 = \ker S(d_0) \cap \ker S(d_1)$ *consists of multirecursive sequences*

$$(f_\alpha)_{\alpha \in \mathrm{Word}\{1,2\}} = (f_\emptyset, f_1, f_2, f_{11}, f_{12}, f_{21}, f_{22}, f_{111}, f_{112}, f_{121}, f_{122}, \ldots)$$

with the radix order on Word $\{1, 2\}$ *such that*

$$\begin{cases} f_{\underbrace{1\ldots1}_{n}} = 0 \\ \displaystyle\sum_{\substack{\alpha \in \mathrm{Word}\{1,2\} \\ \lg(\alpha)=n}} f_\alpha = 0 \end{cases}$$

Proof This result is a clear consequence of the given mappings (4.3). □

Lemma 4.10 *The vector space* $\ker \partial_2$ *consists of multirecursive sequences*

$$(g_\alpha)_{\alpha \in \text{Word}\{1,2\}} = (g_\emptyset, g_1, g_2, g_{11}, g_{12}, g_{21}, g_{22}, g_{111}, g_{112}, g_{121}, g_{122}, \ldots)$$

with the radix order on Word $\{1, 2\}$ *such that*

$$\begin{cases} g_{\underbrace{1\ldots1}_{n}} = 0 = g_{\underbrace{2\ldots2}_{n}} \\ \displaystyle\sum_{\substack{\alpha \in \text{Word}\{1,2\} \\ \text{lg}(\alpha)=n}} g_\alpha = 0 \end{cases}$$

Proof Since $\ker \partial_2 = (\ker S(d_0) \cap \ker S(d_1)) \cap \ker S(d_2)$, the result follows from Lemma 4.9. □

Example 4.11 From Definition 2.6, the prefix of $\alpha \in$ Word $\{1, 2\}$ of length 2 is denoted by $\alpha_{\text{pre}(2)}$. Let us consider the following coefficients for $\alpha_{\text{pre}(2)} \in \{12, 21\}$

$$\begin{cases} c_{\alpha,\gamma} = 0 & \text{if } \alpha_{\text{pre}(2)} \neq \gamma \\ c_{\alpha,\gamma} = 1 & \text{if } \alpha_{\text{pre}(2)} = \gamma. \end{cases} \tag{4.5}$$

Then the sequence $(g_\alpha)_{\alpha \in \text{Word}\{1,2\}}$ defined by

$$g_\alpha = \begin{cases} \begin{cases} 1 & \text{if } \alpha = 12 \\ -1 & \text{if } \alpha = 21 \\ 0 & \text{otherwise} \end{cases} & \text{for } \text{lg}(\alpha) \leq 2 \\ \\ \begin{cases} 1 & \text{if } \alpha_{\text{pre}(2)} = 12 \\ -1 & \text{if } \alpha_{\text{pre}(2)} = 21 \\ 0 & \text{otherwise} \end{cases} & \text{for } \text{lg}(\alpha) \geq 3 \end{cases}$$

is multirecursive of order $s = 3$ since for α with $\text{lg}(\alpha) \geq 3$, it satisfies the relation

$$g_\alpha = \sum_{\substack{\gamma \in \text{Word}\{1,2\} \\ \text{lg}(\gamma)<3}} c_{\alpha_{\text{pre}(3)},\gamma} \quad g_{\gamma\alpha_{\text{suf}}}$$

as observed in Remark 2.8. Moreover this sequence is in $\ker \partial_2$ by the characterization given in the previous Lemma 4.10.

Example 4.12 Let us consider the following coefficients for $\alpha \in \text{Word}\{1,2\}$ and $\alpha_{\text{pre}(2)} \in \{12, 21\}$

$$\begin{cases} d_{\alpha,\gamma} = 0 & \text{if } \alpha_{\text{pre}(2)} \neq \gamma \\ d_{\alpha,\gamma} = -1 & \text{if } \alpha_{\text{pre}(2)} = \gamma. \end{cases} \tag{4.6}$$

Then, the sequence $(h_\alpha)_{\alpha \in \text{Word}\{1,2\}}$ defined by

$$h_\alpha = \begin{cases} -(-1)^{\lg(\alpha)} & \text{if } \alpha_{\text{pre}(2)} = 12 \\ (-1)^{\lg(\alpha)} & \text{if } \alpha_{\text{pre}(2)} = 21 \\ 0 & \text{otherwise} \end{cases}$$

is in $\ker \partial_2$. Moreover it is multirecursive of order $s = 3$ since for α with $\lg(\alpha) \geq 3$, it satisfies the relation

$$h_\alpha = \sum_{\substack{\gamma \in \text{Word}\{1,2\} \\ \lg(\gamma) < 3}} d_{\alpha_{\text{pre}(3)},\gamma}\, h_{\gamma \alpha_{\text{suf}}}.$$

Indeed, if $\alpha_{\text{pre}(2)} \in \{12, 21\}$, then the sum in the previous relation reduces to

$$d_{\alpha_{\text{pre}(3)},12}\, h_{12\alpha_{\text{suf}}} + d_{\alpha_{\text{pre}(3)},21}\, h_{21\alpha_{\text{suf}}}$$

since by (4.6) the coefficients $d_{\alpha,\gamma}$ are zero for $\gamma \in \{\emptyset, 1, 2, 11, 22\}$. Let us consider the case $\alpha_{\text{pre}(2)} = 12$ for instance. We can write $\alpha = 12a\omega$ where $a \in \{1, 2\}$ and $\omega \in \text{Word}\{1,2\}$, in other words ω is the suffix α_{suf} corresponding to $\alpha_{\text{pre}(3)}$. Thus, in the one hand we have

$$h_\alpha = h_{12a\omega} = -(-1)^{\lg(\alpha)} = -(-1)^{\lg(12a\omega)} = (-1)^{\lg(\omega)}.$$

In the other hand we have

$$\begin{aligned} d_{\alpha_{\text{pre}(3)},12} &= d_{12a,12} = -1 \\ h_{12\alpha_{\text{suf}}} &= h_{12\omega} \;\;= -(-1)^{\lg(12\omega)} = -(-1)^{\lg(\omega)} \\ d_{\alpha_{\text{pre}(3)},21} &= d_{12a,21} = 0 \end{aligned}$$

Consequently $h_\alpha = (-1)^{\lg(\omega)} = d_{\alpha_{\text{pre}(3)},12}\, h_{12\alpha_{\text{suf}}}$ satisfying the required recursive relation. The case $\alpha_{\text{pre}(2)} = 21$ is treated similarly and the relation is trivial for $\alpha_{\text{pre}(2)} \notin \{12, 21\}$.

Lemma 4.13 *If* $(f_\alpha)_{\alpha \in \text{Word}\{1,2,3\}} \in \left(\tilde{N} S_s \Gamma(\mathbb{S}^1)\right)_3$, *then* $(\partial_3 f)_{12} = 0 = (\partial_3 f)_{121}$.

Proof By using the mappings given in (4.4) we obtain that

$$\begin{aligned} (S(d_0)(f_\alpha))_{12} &= f_{12} \\ (S(d_1)(f_\alpha))_{12} &= f_{12} + f_{13} \\ (S(d_2)(f_\alpha))_{12} &= f_{13} + f_{23} \end{aligned}$$

Since $(f_\alpha)_\alpha \in \left(\tilde{N} S_s \Gamma(\mathbb{S}^1)\right)_3 = \ker S(d_0) \cap \ker S(d_1) \cap \ker S(d_2)$, it follows that

$$(S(d_3)(f_\alpha))_{12} = f_{23} = 0.$$

A similar argument proves that $(\partial_3 f)_{121} = 0$. □

Proposition 4.14 *The homology vector space $H_2\left(\tilde{N} S_s \Gamma(\mathbb{S}^1)\right)$ is of dimension ≥ 2.*

Proof We exhibit two vectors in $H_2\left(\tilde{N} S_s \Gamma(\mathbb{S}^1)\right)$ that are linearly independent. To this end, we consider the sequences $(g_\alpha)_{\alpha \in \mathrm{Word}\{1,2\}}$ and $(h_\alpha)_{\alpha \in \mathrm{Word}\{1,2\}}$ as given in Examples 4.11 and 4.12 respectively and make the following observations:

(1) Both sequences are in $\ker \partial_2$ by construction.
(2) Let $a, b \in K$ such that $ag_\alpha + bh_\alpha \in \mathrm{im}\partial_3$ for all $\alpha \in \mathrm{Word}\,\{1, 2\}$. Then there exists a sequence $(f_\gamma)_{\gamma \in \mathrm{Word}\{1,2,3\}} \in \left(\tilde{N} S_s \Gamma(\mathbb{S}^1)\right)_3$ such that $ag_\alpha + bh_\alpha = (\partial_3 f)_\alpha$. By using the previous Lemma 4.13 we obtain for $\alpha = 12$ and 121

$$\begin{cases} ag_{12} + bh_{12} &= (\partial_3 f)_{12} \\ ag_{121} + bh_{121} &= (\partial_3 f)_{121} \end{cases} \iff \begin{cases} a - b = 0 \\ a + b = 0 \end{cases}$$

Hence $a = 0 = b$ and the sequences are linearly independent.

These observations suffice to guarantee the required result. □

4.3.3 Homology of $S_d(\mathbb{S}^1)$

In this section we use a Lemma by Goerss and Getzler in [2] to compute the homology vector space $H_*\left(S_d(\mathbb{S}^1)\right)$. As a consequence, we conclude that the Quillen adjunction $(\widetilde{N}, R)$ is *not* a Quillen equivalence.

Lemma 4.15 [2, Lemma 1.12.1] *Let C be a finite dimensional differential graded coalgebra and (V, ∂) be a differential graded vector space which is concentrated in non-negative degrees and finite dimensional in each degree. If $V_0 = 0$ then*

$$\left(C \times S_d(V)\right)^* \cong T_{C^*}\left(C^* \otimes V^* \otimes C^*\right).$$

This Lemma has the following consequence: we can choose the differential graded vector space V to be the n-sphere $\mathbb{S}^n$ with $n \geq 1$ and C to be the 1-dimensional differential graded coalgebra $K[0]$ concentrated in degree zero. Note that $K[0]$ is the terminal object in the category **DGcoAlg** and therefore the product $K[0] \sqcap X$ with any object X in **DGcoAlg** is isomorphic to X. Thus we obtain that

$$\left(S_d(\mathbb{S}^n)\right)^* \cong T_{K[0]^*}\left((\mathbb{S}^n)^*\right) \cong T_{K[0]}\left(\mathbb{S}^{-n}\right) = \mathbb{S}^0 \oplus \mathbb{S}^{-n} \oplus \mathbb{S}^{-2n} \oplus \mathbb{S}^{-3n} \oplus \ldots$$

By applying the continuous dual functor $(-)'$ of [2, Proposition 1.7], we obtain that

$$S_d(\mathbb{S}^1) \cong \left(\left(S_d(\mathbb{S}^1)\right)^*\right)' \cong \mathbb{S}^0 \oplus \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{S}^3 \oplus \ldots$$

and therefore

$$H_*\left(S_d(\mathbb{S}^1)\right) = H_*\left(\mathbb{S}^0 \oplus \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{S}^3 \oplus \dots\right).$$

Proposition 4.16 *The Quillen adjunction $(\widetilde{N}, R)$ is not a Quillen equivalence.*

Proof The map $\tilde{N}RS_d(\mathbb{S}^1) \longrightarrow S_d(\mathbb{S}^1)$ is *not* a weak equivalence since

$$\dim H_2\left(S_d(\mathbb{S}^1)\right) = 1 < \dim H_2\left(\tilde{N}S_s\Gamma(\mathbb{S}^1)\right).$$

However, its adjoint $\mathrm{id}\colon RS_d(\mathbb{S}^1) \longrightarrow RS_d(\mathbb{S}^1)$ is clearly a weak equivalence and therefore the adjunction $(\widetilde{N}, R)$ cannot be a Quillen equivalence. □

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