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Embedding of vector-valued Morrey spaces and separable differential operators

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Abstract
The paper is the first part of a program devoted to the study of the behavior of operator-valued multipliers in Morrey spaces. Embedding theorems and uniform separability properties involving E-valued Morrey spaces are proved. As a consequence, maximal regularity for solutions of infinite systems of anisitropic elliptic partial differential equations are established.

Keywords Differential operators · Maximal regularity · Partial differential equations · Morrey Spaces

1 Introduction

The aim of this note is to study the behavior of some differential operators in Morrey spaces. Useful tools to achieve this goal are embedding properties of these spaces studied in [33–35]. It is worth to mention that weighted spaces are used, in order to introduce weighted variational and quasi-variational inequalities and kinetic equations (see [4–6]).

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Dedicated to Professor Michel Théra in occasion of his 70th birthday with deep esteem and undying friendship.

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The interest of such a general setting raises from the following considerations. Fourier multipliers, in vector-valued function spaces, has been well studied (see e.g. [29,45]) as well as operator-valued Fourier multipliers [7,15,22,25,46]. On the other hand, the study of Morrey spaces has received considerable attention in the last thirty years in different research areas (see e.g. [8–10,16,17,19–21,23,28,31,36,43]). A further motivation comes from the fact that, to our knowledge, nothing is known concerning Morrey estimates for such operator-valued Fourier multipliers and embedding properties of abstract Sobolev–Morrey spaces. Lebesgue multipliers of the Fourier transformation are, in a clear way and in detail, treated in [45], §2.2.1–§2.2.4. We also mention the papers [24,37,47] where boundary value problems (BVPs) for differential-operator equations (DOEs) have been studied.

Our main results are operator-valued multiplier theorems in E-valued Morrey spaces $L^{p,\lambda}(\Omega; E)$. To develop this study, the authors consider the E-valued Sobolev–Morrey type function space $W^{l,p,\lambda}(\Omega; E_0, E) = W^{l,p,\lambda}(\Omega; E) \cap L^{p,\lambda}(\Omega; E_0)$, where Ω is a domain in R^n, E_0 and E are two Banach spaces and E_0 is continuously and densely embedded into E.

Let us introduce the set $E\left(A^{\theta}\right)$ as the space $D\left(A^{\theta}\right)$ equipped with the following norm

$$\|u\|_{E(A^{\theta})} = \left(\|u\|^p + \left\|A^{\theta}u\right\|^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces and θ and p such that $0 < \theta < 1$ and $1 \leq p \leq \infty$. Let us denote by $(E_1, E_2)_{\theta,p}$ the interpolation space obtained from $\{E_1, E_2\}$ by the K-method ([45] §1.3.1), for the above values of p and θ.

In Theorems 4.2 and 4.6 the authors prove that the most regular class of interpolation space E_α, between E_0 and E, is the one such that the mixed differential operators D^α are *bounded* from $W^{l,p,\lambda}(\Omega; E_0, E)$ to $L^{p,\lambda}(\Omega; E_\alpha)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $l = (l_1, l_2, \ldots, l_n)$ are n-tuples of nonnegative integer numbers such that $|\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \leq 1$, and are *compact* from $W^{l,p,\lambda}(\Omega; E_0, E)$ to $L^{p,\lambda}(\Omega; E_\alpha)$ if the last inequality is strict, that is, if $|\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} < 1$.

We point out that these results are sharp because, among the spaces E_α such that the following embedding holds

$$D^\alpha W_p^l(\Omega; E(A), E) \subset L^{p,\lambda}(\Omega; E_\alpha),$$

the space $(E(A), E)_{k,p}$ is the most smooth, i.e. $(E(A), E)_{k,p} \subset E_\alpha$ for all kind of spaces E_α such that the above embedding is valid.

The undertaken study has the purpose to refine and improve the outcomes contained in [3] §9, [42] §1.7 for scalar Sobolev spaces, the upshot contained in [26] for one dimensional vector function spaces, and the achievements obtained in [39–41] for Hilbert-space valued class.

Throughout the paper we refer to the following parameter-dependent differential-operator equation

$$(L+\nu)u = \sum_{|\alpha:l|=1} a_\alpha D^\alpha u + (A+\nu)u + \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u = f, \tag{1.1}$$

where ν is a positive parameter, a_α are complex numbers, A and $A_\alpha(x)$ are linear operators in a Banach space E. We notice that, for $l_1 = l_2 = \cdots = l_n = 2m$, Eq. (1.1) can be written as the following elliptic DOE

$$\sum_{|\alpha|=2m} a_\alpha(x) D_k^{2m} u(x) + Au(x) + \sum_{|\alpha|<2m} A_\alpha(x) D^\alpha u(x) = f(x).$$

We establish that Eq. (1.1) is $L^{p,\lambda}(R^n; E)$-separable, namely, we show that, for all $f \in L^{p,\lambda}(R^n; E)$, there exists a unique solution $u \in W^{l,p,\lambda}(R^n; E(A), E)$ satisfying (1.1) almost everywhere on R^n and there exists a positive constant C independent of f, such that the following coercive estimate:

$$\sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{L^{p,\lambda}(R^n;E)} + \|Au\|_{L^{p,\lambda}(R^n;E)} \le C \|f\|_{L^{p,\lambda}(R^n;E)}$$

is true.

This enables us to state that if $f \in L^{p,\lambda}(R^n; E)$ and u is the solution of (1.1), then all the terms of Eq. (1.1) belong to $L^{p,\lambda}(R^n; E)$ or, equivalently, that all the terms are separable in $L^{p,\lambda}(R^n; E)$.

Moreover, we point out that the above estimate implies that the inverse of the differential operator generated by (1.1) is bounded from $L^{p,\lambda}(R^n; E)$ to $W^{l,p,\lambda}(R^n; E(A), E)$.

The paper is organized as follows. In Sect. 2 we mention the necessary tools from Banach space theory and some background materials. Section 3 is devoted to the proof of multiplier theorems. In Sect. 4 we study continuity and compactness of embedding operators in E-valued Sobolev–Morrey spaces. In Sect. 5 we obtain separability properties and, finally, in Sect. 6 maximal regularity properties of infinite systems of anisotropic

2 Notation and background

Let us introduce the main tools and briefly discuss some consequence of them. Given $\Omega \subset R^n$ a measurable set, E a Banach space and, for $x = (x_1, x_2, \ldots, x_n)$, $\gamma = \gamma(x)$ a positive measurable function on Ω, we set $L_{p,\gamma}(\Omega; E)$ for the Banach space of strongly measurable E-valued functions defined in Ω, endowed with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \gamma(x)\, dx \right)^{\frac{1}{p}} 1 \le p < \infty.$$

We note $L_p = L_p(\Omega; E)$, the space $L_{p,\gamma}(\Omega; E)$ when $\gamma(x) \equiv 1$.

Let us consider $1 < p < \infty$ and $0 \le \lambda < n$. We use the notation $L^{p,\lambda}(R^n; E)$, for the *E-valued Morrey Space* of those functions $f \in L^1_{loc}(R^n; E)$ for which the following quantity is finite

$$||f||^p_{L^{p,\lambda}(R^n;E)} = \sup_{x\in R^n,\, r>0} \frac{1}{r^\lambda} \int_{B_r(x)} \|f(y)\|_E^p\, dy.$$

It is worth emphasize that a Banach space E is a ζ-convex space if there exists a symmetric real-valued function $\zeta(u, v)$, defined in $E \times E$, that is convex with respect to each variable and that satisfies the following properties

$$\zeta(0, 0) > 0, \zeta(u, v) \leq \|u + v\|, \text{ for } \|u\| = \|v\| = 1.$$

We mention that a ζ-convex Banach space E is usually called a UMD space, see for instance [11]. We also recall that E is a UMD space if and only if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in the space $L_p(R; E)$, $\forall p \in (1, \infty)$.
Note that L_p and ℓ_p spaces, as well as Lorentz spaces L_{pq}, $p, q \in (1, \infty)$, belong to the class of UMD spaces. We refer the reader to [11] for further information on the above definitions and comments.

In what follows we need the following definitions.

Definition 2.1 Let γ be a weight function. A Banach space E is called a γ-UMD space if all E-valued martingale difference sequences are unconditional in $L_{p,\gamma}(R^n; E)$, for every $p \in (1, \infty)$, or, equivalently, if there exists a positive constant C_p such that for any martingale $\{f_k, k \in \mathbf{N}_0\}$ (see [14] §5), any choice of signs $\{\varepsilon_k, k \in \mathbf{N}\} \in \{-1, 1\}$ and any $N \in \mathbf{N}$, we have

$$\left\| f_0 + \sum_{k=1}^{N} \varepsilon_k (f_k - f_{k-1}) \right\|_{L_{p,\gamma}(\Omega, E)} \leq C_p \|f_N\|_{L_{p,\gamma}(\Omega, E)}.$$

We assume that a Banach space E has the $h_{p,\gamma}$ property if the Hilbert operator is bounded in $L_{p,\gamma}(R^n; E)$, for all $p \in (1, \infty)$.
Let $\mathbb{C}$ be the set of complex numbers and $0 \leq \varphi < \pi$. We set

$$S_\varphi = \{\xi; \;\; \xi \in \mathbb{C}, \;\; |\arg \xi| \leq \varphi\} \cup \{0\}.$$

A linear operator A is said to be *positive* in a Banach space E and has bound $M > 0$, if its domain $D(A)$ is dense in E and

$$\left\| (A + \xi I)^{-1} \right\|_{B(E)} \leq M (1 + |\xi|)^{-1}, \quad \forall \xi \in S_\varphi, \quad \forall \varphi \in [0 \, , \, \pi),$$

where I is the identity operator in E and $B(E)$ is the space of bounded linear operators on E. The constant M is dependent only on φ but, since we consider φ a fixed angle, we do not need uniformly estimate with respect to φ. Without ambiguity we only write $A + \xi$ instead of $A + \xi I$ and denote it by A_ξ. It is useful to recall ([45] §1.15.1) that there exist fractional powers A^θ of the positive operator A, $-\infty < \theta < \infty$.

We need to introduce the following definition, that hereafter plays an important role.

Denoting by F the Fourier transformation, a function $\Psi \in L^{\infty}\left(R^n; L\left(E_1, E_2\right)\right)$ is called a *multiplier* from $L^{p,\lambda}\left(R^n; E_1\right)$ to $L^{q,\lambda}\left(R^n; E_2\right)$, provided there exists a positive constant C such that

$$\left\| F^{-1}\Psi\left(\xi\right) Fu\right\|_{L^{q,\lambda}\left(R^n; E_2\right)} \leq C\left\|u\right\|_{L^{p,\lambda}\left(R^n; E_1\right)}$$

for all $u \in L^{p,\lambda}\left(R^n; E_1\right)$.

Let us denote by $M_{p,\lambda}^{q,\lambda}\left(E_1, E_2\right)$ the set of all multipliers from $L^{p,\lambda}\left(R^n; E_1\right)$ to $L^{q,\lambda}\left(R^n; E_2\right)$. If $E_1 = E_2 = E$ we simply write $M_{p,\lambda}^{q,\lambda}\left(E\right)$ instead of $M_{p,\lambda}^{q,\lambda}\left(E_1, E_2\right)$.

In the sequel let us consider H a generic set, h a parameter in H and

$$M\left(H\right) = \left\{\Psi_h \in M_{p,\lambda}^{q,\lambda}\left(E_1, E_2\right), h \in H\right\}$$

a collection of multipliers in $M_{p,\lambda}^{q,\lambda}\left(E_1, E_2\right)$.

A family of sets $M\left(H\right) \subset B\left(E_1, E_2\right)$, dependent on $h \in H$, is called a *uniform collection of multipliers* , if there exists a positive constant C, independent of $h \in H$, such that

$$\left\| F^{-1}\Psi_h Fu\right\|_{L^{q,\lambda}\left(R^n; E_2\right)} \leq C\left\|u\right\|_{L^{p,\lambda}\left(R^n; E_1\right)}$$

for all $h \in H$ and $u \in L^{p,\lambda}\left(R^n; E_1\right)$.

A set $K \subset B\left(E_1, E_2\right)$ is said to be R-bounded (see e.g. [15,22,46]), if there exists a positive constant C such that for all $T_1, T_2, \ldots, T_m \in K$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in \mathbf{N}$,

$$\int\limits_0^1 \left\|\sum_{j=1}^m r_j\left(y\right) T_j u_j\right\|_{E_2} dy \leq C\int\limits_0^1 \left\|\sum_{j=1}^m r_j\left(y\right) u_j\right\|_{E_1} dy,$$

where $\left\{r_j\right\}$ is a sequence of independent symmetric $[-1, 1]$-valued random variables on $[0, 1]$. The smallest constant C is called the R-bound of K and is denoted by $R\left(K\right)$.

A family of sets $K\left(h\right) \subset B\left(E_1, E_2\right)$, dependent on the parameter $h \in H$, is called *uniformly R-bounded with respect to* h, if there is a positive constant C such that, for all $T_1, T_2, \ldots, T_m \in K\left(h\right)$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int\limits_0^1 \left\|\sum_{j=1}^m r_j\left(y\right) T_j\left(h\right) u_j\right\|_{E_2} dy \leq C\int\limits_0^1 \left\|\sum_{j=1}^m r_j\left(y\right) u_j\right\|_{E_1} dy,$$

where the constant C is independent of the parameter h, that is

$$\sup_{h\in H} R\left(K\left(h\right)\right) < \infty.$$

In a similar way we can introduce the multipliers in the weighted spaces $L_{p,\gamma}\left(R^n; E\right)$ and define $M_{p,\gamma}^{p,\gamma}\left(E\right)$ as the collection of multipliers in $L_{p,\gamma}\left(R^n; E\right)$.

In view of the next definition we set

$$U_n = \{\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in N \times N \times \cdots \times N, |\beta| \leq n\} \text{ and } \xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2} \cdots \xi_n^{\beta_n}.$$

Definition 2.2 A Banach space E satisfies a *multiplier condition*, with respect to $p \in (1, \infty)$ and a weight function γ, if for every $\Psi \in C^n(R^n \setminus \{0\}; B(E))$ such that

$$\left\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\},$$

is R-bounded, it follows that $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$.

Remark 2.3 It is interesting to observe that the classical multiplier results (see Theorem 1 and 2 in [44]) implies that the space ℓ_p, $p \in (1, \infty)$, satisfies the multiplier condition with respect to p and the weight functions

$$\gamma = |x|^\alpha, \ -1 < \alpha < p-1, \ \gamma = \prod_{k=1}^N \left(1 + \sum_{j=1}^n |x_j|^{\alpha_{jk}}\right)^{\beta_k},$$
$$\alpha_{jk} \geq 0, \quad N \in \mathbf{N}, \quad \beta_k \in R.$$

We recall that a Banach space E satisfies Property (α) (see e.g. [22]) if there exists a constant α such that

$$\left\|\sum_{i,j=1}^N \alpha_{ij}\varepsilon_i\varepsilon'_j x_{ij}\right\|_{L_2\left(\Omega\times\Omega';E\right)} \leq \alpha \left\|\sum_{i,j=1}^N \varepsilon_i\varepsilon'_j x_{ij}\right\|_{L_2(\Omega\times\Omega';E)}$$

for all $N \in \mathbf{N}$, $x_{i,j} \in E$, $\alpha_{ij} \in \{0, 1\}$, $i, j = 1, 2, \ldots, N$, and all choices of independent, symmetric, $\{-1, 1\}$-valued random variables $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, \varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_N$ on probability spaces Ω and Ω'.

For instance, the space $L_p(\Omega)$, $1 \leq p < \infty$, verify Property (α).

A Banach space E is said to have local *unconditional structure* (in short l.u.st.) (see [32]) if there exists a positive constant C with the following property: given any finite dimensional subspace $F \subset E$, there exists a space U, with an unconditional basis $\{u_n\}$, and operators A from F to U and B from U to E such that BA is the identity on F and $\|A\| \cdot \|B\| \cdot \chi_{\{u_n\}} \leq C$.

Let us recall that a function γ is a Muckenhoupt A_p weight (see [30]), i.e. $\gamma \in A_p$, $1 < p < \infty$, if there is a positive constant C such that

$$\left(\frac{1}{|Q|}\int_Q \gamma(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \gamma^{-\frac{1}{p-1}}(x)\,dx\right)^{p-1} \leq C,$$

for all balls $Q \subset R^n$.

The next remark shows a useful property that correlates the above definitions ([38], Theorem 3.7).

Remark 2.4 If E is a UMD space having Property (α), it satisfies the multiplier condition with respect to $\gamma \in A_p$, for $p \in (1, \infty)$.

It is well known (see [25,27]) that any Hilbert space satisfies the multiplier condition. There are, however, Banach spaces which are not Hilbert spaces but satisfy the multiplier condition, for example UMD spaces (see [15,22,46]).

Definition 2.5 We say that a positive operator A is R-positive in the Banach space E, if there exists $\varphi \in [\,0, \pi\,)$ such that the set

$$L_A = \left\{ \xi \left(A + \xi I\right)^{-1} : \xi \in S_\varphi \right\}$$

is R-bounded.

In a Hilbert space, every norm bounded set is R-bounded. As a consequence, in a Hilbert space all positive operators are R-positive.

Let us now consider Ω a domain in R^n and $l = (l_1, l_2, \ldots, l_n)$. We define $W^{l,p,\lambda}(\Omega; E_0, E)$ the space of all functions $u \in L^{p,\lambda}(\Omega; E_0)$ having generalized derivatives $D_k^{l_k} u = \frac{\partial^{l_k}}{\partial x_k^{l_k}} u \in L^{p,\lambda}(\Omega; E)$ and equipped with the norm given by:

$$\|u\|_{W^{l,p,\lambda}(\Omega; E_0, E)} = \|u\|_{L^{p,\lambda}(\Omega; E_0)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{L^{p,\lambda}(\Omega; E)} < \infty.$$

For $E_0 = E$ the space $W^{l,p,\lambda}(\Omega; E_0, E)$ is simply denoted by $W^{l,p,\lambda}(\Omega; E)$.

Let us recall the definition of a *Hardy-Littlewood Maximal function*, a notion which is very important in various areas of analysis including harmonic analysis, PDE's and function theory (see e.g. [18]).

Definition 2.6 Let $f \in L^1_{loc}(R^n; E)$. The Hardy-Littlewood Maximal function of f is defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \|f(y)\|_E \, dy$$

where $B_r(x)$ is a ball centered at $x \in R^n$ with radius $r > 0$.

Let us consider the following anisotropic partial differential equation (PDE)

$$\sum_{|\alpha : l| \le 1} a_\alpha D^\alpha u(x) = f(x),$$

where a_α are complex numbers. It is anisotropic elliptic if, for all $\xi \in R^n$, there exists a positive constant C such that

$$\left| \sum_{|\alpha : l| = 1} a_\alpha \xi^\alpha \right| \ge C \sum_{k=1}^{n} |\xi_k|^{l_k}.$$

The term *anisotropic* means that the principal part could contain generally, different differentiation with respect to different variables.

3 Multiplier theorems

Our aim in this section is to prove a sufficient condition to have multipliers in E-valued Morrey spaces $L^{p,\lambda}(R^n; E)$. In order to obtain this result we make use of the concepts of Hardy-Littlewood maximal function, Muckenhoupt weights A_p and Fourier multipliers theorems in E-valued in L^p spaces. We refer the reader to [2,12,48] for related results.

Theorem 3.1 *Assume that the following conditions are verified:*

(1) *E, E_1 are UMD spaces satisfying Property (α), $\Psi_h \in C^n\left(R^n \setminus \{0\}; B\left(E, E_1\right)\right)$, $h \in H$;*
(2) *$\gamma \in A_p$, $1 < p < \infty$.*

Moreover, if the quantity

$$\sup_{h\in H} R\left(\left\{\xi^\beta D_\xi^\beta \Psi_h\left(\xi\right) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\}\right)$$

is finite, then $\{\Psi_h\}_{h\in H}$ is a uniformly collection of multipliers in $M_{p,\gamma}^{p,\gamma}\left(E, E_1\right)$.
If $n = 1$ then, the result remains true for all the UMD spaces E and E_1.

Proof The theorem is proved, in a similar way as in [1]. □

Remark 3.2 It is easily verifiable that Theorem 3.1 is true if multiplier functions are not dependent on a parameter.

Theorem 3.3 *Let us suppose that all conditions of Theorem 3.1 are true. Then, $\{\Psi_h\}_{h\in H}$ is a uniform collection of multipliers in $L^{p,\lambda}\left(R^n; E\right)$, for every $1 < p < \infty$ and $0 < \lambda < n$.*

Proof We recall that a function $\Psi \in L_\infty\left(R^n; L\left(E\right)\right)$ is a multiplier in the space $L_{p,\gamma}\left(R^n; E\right)$ if there exists a positive constant C such that the operator $u \rightarrow F^{-1}\Psi\left(\xi\right) Fu$ is bounded in $L_{p,\gamma}\left(R^n; E\right)$. This is equivalent to say that the convolution operator $u \rightarrow Ku = \left[F^{-1}\Psi\left(\xi\right)\right] * u$ is bounded in $L_{p,\gamma}\left(R^n; E\right)$ i.e.

$$\|Ku\|_{L_{p,\gamma}(R^n;E)} \leq C\,\|u\|_{L_{p,\gamma}(R^n;E)} \tag{3.1}$$

for all $u \in L_{p,\gamma}\left(R^n; E\right)$.

We get the required result if we prove that estimate (3.1) implies

$$\|Ku\|_{L^{p,\lambda}(R^n;E)} \leq C\,\|u\|_{L^{p,\lambda}(R^n;E)}.$$

Let us fix any $\bar{\gamma} \in]\lambda/n; 1[$. For any $x_0 \in R^n$ and any $r > 0$ we consider $\chi = \chi_{B_r(x_0)}$, the characteristic function of $B_r(x_0)$ and $M\,\chi_{B_r(x_0)}$ the Hardy-Littlewood maximal function of $\chi_{B_r(x_0)}$.

We know that $[(M\chi_{B_r(x_0)})^{\bar{\gamma}}] \in A_1 \subset A_p$ for $0 < \bar{\gamma} < 1$, $1 < p < \infty$ ([13] see also [18]) and from Lemma 8 in [9] we have

$$
\begin{aligned}
\int_{B_r(x_0)} \|K\,u(x)\|_E^p\,dx &= \int_{R^n} \|K\,u(x)\|_E^p \left[\left(\chi_{B_r(x_0)}(x)\right)^{\bar{\gamma}}\right] dx \\
&\leq \int_{R^n} \|K\,u(x)\|_E^p \left[\left(M\chi_{B_r(x_0)}(x)\right)^{\bar{\gamma}}\right] dx \\
&\leq \int_{R^n} \|u(x)\|_E^p \left[\left(M\chi_{B_r(x_0)}(x)\right)^{\bar{\gamma}}\right] dx \\
&= \left\{\int_{B_{2r}} \|u(x)\|_E^p \left(M\chi_{B_r(x_0)}(x)\right)^{\bar{\gamma}} dx \right.\\
&\quad \left. + \sum_{k=2}^{\infty} \int_{B_{2^k r}\setminus B_{2^{k-1}r}} \|u(x)\|_E^p \left(M\chi_{B_r(x_0)}(x)\right)^{\bar{\gamma}} dx \right\} \\
&\leq c\,r^{\lambda}\,\|u\|^p_{L^{p,\lambda}(R^n)} \left\{2^{\lambda} + \sum_{k=2}^{\infty} \frac{2^{\lambda k}}{(2^{(k-1)}-1)^{n\bar{\gamma}}}\right\} \\
&\leq c\,r^{\lambda}\,\|u\|^p_{L^{p,\lambda}(R^n)}.
\end{aligned}
$$

Then, it follows immediately

$$
\|Ku\|_{L^{p,\lambda}(R^n)} \leq C\,\|u\|_{L^{p,\lambda}(R^n)}.
$$

□

4 Embedding theorems in abstract Morrey spaces

In this section, continuity and compactness of embedding operators in E-valued Sobolev–Morrey spaces are derived. Specifically, boundedness and compactness of mixed differential operators in the framework of abstract interpolation of Banach spaces are shown.

Theorem 4.1 *Let $1 < p < \infty$, $\gamma \in A_p$, $0 < \lambda < n$, $l = (l_1, l_2, \ldots, l_n)$ and E be a Banach space. Let us also assume that $\Omega \subset R^n$ is a region such that there exists a bounded linear extension operator from $W^l_{p,\gamma}(\Omega; E)$ to $W^l_{p,\gamma}(R^n; E)$. Then, there exists a bounded linear extension operator from $W^{l,p,\lambda}(\Omega; E)$ to $W^{l,p,\lambda}(R^n; E)$.*

Proof From the assumptions we know that there exists a bounded extension operator P acting from $W^l_{p,\gamma}(\Omega, E)$ to $W^l_{p,\gamma}(R^n, E)$, i.e.

$$
\|Pu\|_{W^l_{p,\gamma}(R^n,E)} \leq C\,\|u\|_{W^l_{p,\gamma}(\Omega,E)}
$$

for all $u \in W^l_{p,\gamma}(\Omega, E)$. Let us fix any $\bar{\gamma} \in]\lambda/n;\,1[$; we know that $[(M\chi_{B_r(x_0)})^{\bar{\gamma}}](x) \in A_1 \subseteq A_p$, for every ball $B_r = B_r(x_0)$ having center $x_0 \in R^n$ and radius $r > 0$. Then, we have

$$
\begin{aligned}
\int_{B_r(x_0)} \|P\, u(x)\|_E^p\, dx &= \int_{R^n} \|P\, u(x)\|_E^p\ (\chi_{B_r(x_0)}(x))^p \left[\left(\chi_{B_r(x_0)}(x)\right)^{\bar{\gamma}}\right] dx \\
&\leq \int_{R^n} \|Pu(x)\|_E^p (\chi_{B_r(x_0)}(x))^p\ [(M\chi_{B_r(x_0)})(x)^{\bar{\gamma}}] dx \\
&\leq \int_{\Omega} \|u(x)\|_E^p (\chi_{B_r(x_0)}(x))^p\ [(M\chi_{B_r(x_0)})(x)^{\bar{\gamma}}] dx \\
&\leq c\, r^{\lambda} \|u\|^p_{L^{p,\lambda}(\Omega,E)}.
\end{aligned}
$$

Repeating the same arguments for the generalized derivatives $D_k^{l_k} Pu$ we obtain the requested inequality

$$
\|Pu\|_{W^{l,p,\lambda}(R^n,E)} \leq C\, \|u\|_{W^{l,p,\lambda}(\Omega,E)} \tag{4.1}
$$

for all $u \in W^{l,p,\lambda}\,(\Omega,\, E)$. □

Theorem 4.2 *Let us suppose that the following assumptions are true:*

(1) *E is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$, A is a R-positive operator in E for $\varphi \in]0, \pi]$;*
(2) *let $0 < \lambda < n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be given and suppose that $l = (l_1, l_2, \ldots, l_n)$ is a n-tuples of nonnegative integer numbers such that*

$$
\kappa = |\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \leq 1 \text{ and } 0 \leq \mu \leq 1 - \kappa;
$$

(3) *$\Omega \subset R^n$ is a region such that there exists a bounded linear extension operator from $W^{l,p,\lambda}\,(\Omega; E\,(A)\,, E)$ to $W^{l,p,\lambda}\,(R^n; E\,(A)\,, E)$.*
Then, the embedding

$$
D^{\alpha} W^{l,p,\lambda}\,(\Omega; E\,(A)\,, E) \subset L^{p,\lambda}\left(\Omega; E\left(A^{1-\kappa-\mu}\right)\right)
$$

is continuous and there exists a positive constant C_μ such that

$$
\begin{aligned}
&\left\|D^{\alpha} u\right\|_{L^{p,\lambda}(\Omega; E(A^{1-\kappa-\mu}))} \\
&\quad \leq C_\mu \left[h^{\mu}\, \|u\|_{W^{l,p,\lambda}(\Omega;E(A),E)} + h^{-(1-\mu)}\, \|u\|_{L^{p,\lambda}(\Omega;E)}\right]
\end{aligned} \tag{4.2}
$$

for all $u \in W^{l,p,\lambda}\,(\Omega; E\,(A)\,, E)$ and every $h > 0$.

Proof We distinguish two cases.
First case: $\Omega = R^n$.

We have that

$$
\left\|D^{\alpha} u\right\|_{L^{p,\lambda}(R^n; E(A^{1-\kappa-\mu}))} \sim \left\|F^{-\prime}\,(i\xi)^{\alpha}\, A^{1-\kappa-\mu} \hat{u}\right\|_{L^{p,\lambda}(R^n;E)}.
$$

Additionally, for every $u \in W^{l,p,\lambda}(R^n; E(A), E)$, we see that

$$\begin{aligned} \|u\|_{W^{l,p,\lambda}(R^n;E(A),E)} &= \|u\|_{L^{p,\lambda}(R^n;E(A))} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{L^{p,\lambda}(R^n;E)} \\ &= \left\| F^{-1}\hat{u} \right\|_{L^{p,\lambda}(R^n;E(A))} + \sum_{k=1}^{n} \left\| F^{-1}\left[(i\xi_k)^{l_k} \hat{u} \right] \right\|_{L^{p,\lambda}(R^n;E)} \\ &\sim \left\| F^{-1}A\hat{u} \right\|_{L^{p,\lambda}(R^n;E)} + \sum_{k=1}^{n} \left\| F^{-1}\left[(i\xi_k)^{l_k} \hat{u} \right] \right\|_{L^{p,\lambda}(R^n;E)}. \end{aligned}$$

Then, prove (4.2) is equivalent to show

$$\begin{aligned} &\left\| F^{-1}(i\xi)^{\alpha} A^{1-\kappa-\mu}\hat{u} \right\|_{L^{p,\lambda}(R^n,E)} \\ &\quad \le C_\mu \left[h^{\mu} \left(\left\| F^{-1}A\hat{u} \right\|_{L^{p,\lambda}(R^n,E)} + \sum_{k=1}^{n} \left\| F^{-1}\left[(i\xi_k)^{l_k} \hat{u} \right] \right\|_{L^{p,\lambda}(R^n,E)} \right) \right. \\ &\qquad \left. + h^{-(1-\mu)} \left\| F^{-1}\hat{u} \right\|_{L^{p,\lambda}(R^n,E)} \right], \end{aligned} \tag{4.3}$$

for a suitable positive constant C_μ. We obtain inequality (4.3), at once, if we prove that $Q_{0h} = \xi^{\alpha} Q_h(\xi)$ and $Q_{kh} = \xi_k^{l_k} Q_h(\xi)$ are uniform collections of multipliers in $L^{p,\lambda}(R^n, E)$, where

$$Q_h(\xi) = h^{\mu} \left(A + \sum_{k=1}^{n} |\xi_k|^{l_k} \right) + h^{-(1-\mu)}, \quad h > 0$$

This fact is proved in a similar way as in [1], Theorem A_2. Really, to achieve this, we prove that the sets

$$\left\{ \xi^{\beta} D^{\beta} \Psi_{i,h}(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n, i = 0, 1, \ldots, n \right\}$$

are R-bounded in E and the R-bounds are independent of h, applying a technique similar to the one used in [40] Lemma 3.1. From [40] Lemma 3.1, we have the existence of a constant $C > 0$ such that

$$\left|\xi^{\beta}\right| \left\| D^{\beta}\Psi_h(\xi) \right\|_{B(E)} \le C, \ \xi \in R^n \setminus \{0\}, \ \beta \in U_n, \tag{4.4}$$

uniformly in h. Using the R-positivity assumption of the operator A and from the above estimate we obtain that the following sets

$$\left\{ A Q_h^{-1}(\xi) : \xi \in R^n \setminus \{0\} \right\}, \left\{ \left(1 + \sum_{k=1}^{n} |\xi_k|^{l_k} + h^{-1} \right) Q_h^{-1}(\xi) : \xi \in R^n \setminus \{0\} \right\}$$

are R-bounded, uniformly respect to h. Furthermore, for $u_{1,}u_2, \ldots, u_m \in E, m \in N$ and $\xi^j = \left(\xi_{1j}, \xi_{2j}, \ldots, \xi_{nj}\right) \in R^n \setminus \{0\}$, we get

$$\left\| \sum_{j=1}^{m} r_j(y) \Psi_h\left(\xi^j\right) u_j \right\|_{L_p(0,1;E)} = \left\| \sum_{j=1}^{m} r_j(y) \xi^\alpha A^{1-\kappa-\mu} Q_h^{-1}\left(\xi^j\right) u_j \right\|_{L_p(0,1;E)}$$
$$= \| \sum_{j=1}^{m} r_j(y) \xi^\alpha \left(1 + \sum_{k=1}^{n} \left|\xi_{kj}\right|^{l_k} + h^{-1}\right)^{-(\kappa+\mu)}$$
$$\cdot \left[\left(1 + \sum_{k=1}^{n} \left|\xi_{kj}\right|^{l_k} + h^{-1}\right) Q_h^{-1}\left(\xi^j\right)\right]^{(\kappa+\mu)} \left[A Q_h^{-1}\left(\xi^j\right)\right]^{1-(\kappa+\mu)} u_j \|_{L_p(0,1;E)},$$

where $\left\{r_j\right\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables in $[0, 1]$. By virtue of Kahane's contraction principle ([15], Lemma 3.5) from the above equality, we obtain

$$\left\| \sum_{j=1}^{m} r_j(y) \; \Psi_h \left(\xi^j\right) \; u_j \right\|_{L_p(0,1;E)}$$
$$\leq M_0 \| \sum_{j=1}^{m} r_j(y) \left[\left(1 + \sum_{k=1}^{n} \left|\xi_{kj}\right|^{l_k} + h^{-1}\right) Q_h^{-1}\left(\xi^j\right)\right]^{(\kappa+\mu)}$$
$$\left[A Q_h^{-1}\left(\xi^j\right)\right]^{1-(\kappa+\mu)} u_j \|_{L_p(0,1;E)}.$$

From (4.4), combining the above estimate and product properties of the collection of R-bounded operators (see e.g. [15], Proposition 3.4), we get that the set $\{\Psi_h(\xi) : \xi \in R^n \setminus \{0\}\}_{h>0}$ is R-bounded, uniformly with respect to h. Analogously, having in mind Kahane's contraction principle and both products and additional properties of the collection of R-bounded operators ([15], Proposition 3.4), we ensure that the sets

$$\left\{\xi^\beta D^\beta \Psi_h(\xi) : \xi \in R^n \setminus \{0\}, \; \beta \in U_n\right\}_{h>0}$$

are R-bounded, uniformly with respect to h. It implies that $\{\Psi_h(\xi) : \xi \in R^n \setminus \{0\}\}_{h>0}$ is a uniform collection of multipliers in $M_{p,\lambda}^{p,\lambda}(E)$ and, therefore, we obtain estimate (4.3).

Second case: Ω is a generic set in R^n.

Let us set B a bounded linear extension operator from $W^{l,p,\lambda}(\Omega; E(A), E)$ to $W^{l,p,\lambda}(R^n; E(A), E)$, and let B_Ω be the restriction operator from R^n to Ω. Then, for any $u \in W^{l,p,\gamma}(\Omega; E(A), E)$, we have

$$\left\|D^\alpha u\right\|_{L^{p,\lambda}(\Omega; E(A^{1-\kappa-\mu}))} = \left\|D^\alpha B_\Omega B u\right\|_{L^{p,\lambda}(\Omega; E(A^{1-\kappa-\mu}))}$$
$$\leq \left\|D^\alpha B u\right\|_{L^{p,\lambda}(R^n; E(A^{1-\kappa-\mu}))}$$

$$\leq C_\mu \left[h^\mu \|Bu\|_{W^{l,p,\lambda}(R^n;E(A),E)} + h^{-(1-\mu)} \|Bu\|_{L^{p,\lambda}(R^n;E)}\right]$$
$$\leq C_\mu \left[h^\mu \|u\|_{W^{l,p,\lambda}(\Omega;E(A)E)} + h^{-(1-\mu)} \|u\|_{L^{p,\gamma}(\Omega;E)}\right],$$

from which follows estimate (4.2). □

Corollary 4.3 *For the isotropic case i.e., $l_1 = l_2 = \cdots = l_n = m$, $\kappa = \frac{|\alpha|}{m} \leq 1$, $1 < p < \infty$, $0 \leq \mu \leq 1 - \kappa$ and $0 < \lambda < n$, the embedding*

$$D^\alpha W^{m,p,\lambda} (\Omega; E (A) , E) \subset L^{p,\lambda} \left(\Omega; E \left(A^{1-\kappa-\mu}\right)\right)$$

is continuous and an estimate of type (4.2) holds.

For $n = 1$, $0 \leq j \leq m - 1$ we get that the embedding

$$D^j W^{m,p,\lambda} (0, 1; E (A) , E) \subset L^{p,\lambda} \left(0, 1; E \left(A^{1-\frac{j}{m}}\right)\right)$$

is continuous.

Theorem 4.4 *Let us suppose that all assumptions of Theorem 4.2 are satisfied.*

Then, for $0 < \mu < 1 - \kappa$ and $0 < \lambda < n$, the embedding

$$D^\alpha W^{l,p,\lambda} (\Omega; E (A), E) \subset L^{p,\lambda} \left(\Omega; (E (A), E)_{\kappa,p}\right)$$

is continuous and there exists a positive constant C_μ such that

$$\left\| D^\alpha u\right\|_{L^{p,\lambda}\left(\Omega;(E(A),E)_{\kappa,p}\right)} \leq C \|u\|_{W^{l,p,\lambda}(\Omega;E(A),E)}$$

for all $u \in W^{l,p,\lambda} (\Omega; E (A), E)$.

Proof Following the line of the proof of Theorem 4.2, it is sufficient to show that an operator function $\Psi (\xi) = \xi^\alpha [A + \sum_{k=1}^n \xi_k^{l_k}]^{-1}$ is a multiplier from $L^{p,\lambda} (R^n; E)$ to $L^{p,\lambda}(R^n; ((E (A) , E)_{\kappa,p}))$. It is proved taking into account R-positivity properties of the operator A and using the definition of the interpolation spaces ([45], §1.14.5). □

Let us now prove the next compactness result, using the s-horn condition (see definition in [3], §7).

Theorem 4.5 *Let E and E_0 be two Banach spaces such that the embedding $E_0 \subset E$ is compact. Let also $\Omega \subset R^n$ be a bounded region satisfying the s-horn condition, $1 < p < n : 1 < p \leq q < p^* = \frac{p\sigma_l}{\sigma_l - p}$, $1 < q_1 \leq q$, $\sigma_l = \sum_{k=1}^n \frac{1}{l_k}$, $0 < \lambda < n$, $\left(\frac{1}{p} - \frac{1}{q}\right) \sum_{k=1}^n \frac{1}{l_k} < 1$ and $\eta \leq n \left(1 - \frac{q_1}{q}\right)$.*

Then, the embedding

$$W^{l,p,\lambda} (\Omega; E_0, E) \subset L^{q_1,\eta} (\Omega; E)$$

is compact.

Proof Using Rellich's Theorem, we have that $W^{l,p}(\Omega; E_0, E)$ is compactly embedded in $L^q(\Omega; E)$, for every $q \in [1, p^*[$. We also have that

$$L^q(\Omega; E) \subset L^{q_1,\eta}(\Omega; E), \quad q_1 \leq q : \frac{n-\eta}{q_1} \geq \frac{n}{q}.$$

According to the fact that $W^{l,p,\lambda}(\Omega; E_0, E) \subset W^{l,p}(\Omega; E_0, E)$, the compactness is established. □

Theorem 4.6 *Suppose that* E *is a Banach space,* $\Omega \subset R^n$ *is a bounded region satisfying the* s*-horn condition and* A^{-1} *is a compact operator in* E*. Let us also assume* $0 < \lambda < n$, $1 < p < n : 1 < p \leq q < p^* = \frac{p\sigma_l}{\sigma_l - p}$, $\sigma_l = \sum_{k=1}^{n} \frac{1}{l_k}$, $1 < q_1 \leq q$ *and* $0 < \lambda \leq n\left(1 - \frac{p}{q}\right)$.

Then, for $0 < \mu \leq 1 - \kappa$, *the embedding*

$$D^\alpha W^{l,p,\lambda}(\Omega; E(A), E) \subset L^{p,\lambda}\left(\Omega; E\left(A^{1-\kappa-\mu}\right)\right)$$

is compact.

Proof Let us consider (4.2) for $h = \|u\|_{L^{p,\lambda}(\Omega;E)} \|u\|^{-1}_{W^{l,p,\lambda}(\Omega;E(A),E)}$.

We obtain, for $0 \leq \mu \leq 1 - \kappa$, the following multiplicative inequality

$$\left\| D^\alpha u \right\|_{L^{p,\lambda}(\Omega; E(A^{1-\kappa-\mu}))} \leq C_\mu \|u\|^{\mu}_{L^{p,\lambda}(\Omega;E)} \|u\|^{1-\mu}_{W^{l,p,\lambda}(\Omega;E(A),E)}. \tag{4.5}$$

Assuming, in Theorem 4.5, $q_1 = p$ and $\lambda = \eta$, we get that the following embedding $W^{l,p,\lambda}(\Omega; E(A), E) \subset L^{p,\lambda}(\Omega; E)$ is compact.

Then, for any bounded sequence $\{u_k\}_{k\in N} \subset W^{l,p,\lambda}(\Omega; E(A), E)$ there exists a subsequence $\left\{u_{k_j}\right\}_{k_j\in N}$ which converges in $L^{p,\lambda}(\Omega; E)$ to an element u. Furthermore, the boundedness of the set $\{u_k\}_{k\in N}$ in $W^{l,p,\lambda}(\Omega; E(A), E)$ and the estimate (4.2) imply the boundedness of the set $\{D^\alpha u_k\}_{k\in N}$ in $L^{p,\lambda}(\Omega; E)$, for $\kappa \leq 1$, i.e. this set is weakly compact in $L^{p,\lambda}(\Omega; E)$. hence, generalized derivatives $D^\alpha u$ of the limit function u exist and verify $D^\alpha u \in L^{p,\lambda}(\Omega; E)$. Moreover, due to closedness of A we get $Au \in L^{p,\lambda}(\Omega; E)$, i.e. $u \in W^{l,p,\lambda}(\Omega; E(A), E)$. Then, from (4.5), for $0 < \kappa \leq 1 - \mu$, we have

$$\begin{aligned}
&\left\| D^\alpha \left(u_{k_j} - u\right)\right\|_{L^{p,\lambda}(\Omega; E(A^{1-\kappa-\mu}))} \\
&\quad \leq C_\mu \left\| \left(u_{k_j} - u\right)\right\|^{\mu}_{L^{p,\lambda}(\Omega;E)} \left\| \left(u_{k_j} - u\right)\right\|^{1-\mu}_{W^{l,p,\lambda}(\Omega;E(A),E)}.
\end{aligned}$$

Due to the boundedness of $\{u_k\}_{k\in N}$ in $W^{l,p,\lambda}(\Omega; E(A), E)$, there exists a positive constant M such that $\|(u_k - u)\|^{1-\mu}_{W^{l,p,\lambda}(\Omega;E(A),E)} \leq M$. Since $\left\|\left(u_{k_j} - u\right)\right\|^{\mu}_{L^{p,\lambda}(\Omega;E)} \to 0$ for $j \to \infty$, the above estimate implies that $\left\| D^\alpha \left(u_{k_j} - u\right)\right\|_{L^{p,\lambda}(\Omega;E(A^{1-\kappa-\mu}))} \to 0$ for $j \to \infty$. Hence, the operator $u \to D^\alpha u$ is compact from $W^{l,p,\lambda}(\Omega; E(A), E)$ to $L^{p,\lambda}\left(\Omega; E\left(A^{1-\kappa-\mu}\right)\right)$ and we reach the conclusion. □

In a similar way we obtain the following result.

Theorem 4.7 *Suppose that all assumptions of Theorem 4.6 are satisfied.*
Then, for $0 < \mu \leq 1 - \kappa$, *the embedding*

$$D^{\alpha} W^{l,p,\lambda} (\Omega; E(A), E) \subset L^{p,\lambda} \left(\Omega; \left((E(A), E)_{\kappa+\mu,p}\right)\right)$$

is compact.

We highlight that for the isotropic case and $n = 1$. From Theorem 4.7, we obtain the following result.

Corollary 4.8 *Let us set* $0 \leq j < m - 1$, $0 < \mu < 1 - \frac{j}{m}$, $1 < p < n$ *and* $0 < \lambda \leq n\left(1 - \frac{p}{q}\right)$.
Then, the embedding

$$D^{j} W^{m,p,\lambda} (0, 1; E(A), E) \subset L^{p,\lambda} \left(0, 1; (E(A), E)_{\frac{j}{m}+\mu}\right)$$

is compact.

Remark 4.9 If $E = H$, $p = q = 2$, $\Omega = (0, T)$, $l_1 = l_2 = \cdots = l_n = m$ and $A = A^{*}$, we obtain a generalization of result Lions–Peetre [26]. Namely, even in the one dimensional case the result of Lions–Peetre has an improvement considering, in general, nonselfadjoint positive operators A.

Corollary 4.10 *If* $E = R$, $A = I$ *and* Ω *is a bounded domain, we obtain an embedding in Sobolev–Morrey spaces* $W^{l,p,\lambda}(\Omega)$. *Precisely, for* $1 < p < n$, $0 < \lambda \leq n\left(1 - \frac{p}{q}\right)$ *and* $\kappa = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \leq 1$, *the embedding* $D^{\alpha} W^{l,p,\lambda}(\Omega) \subset L^{p,\lambda}(\Omega)$ *is compact.*

Example 4.11 For $s \in R^{+}$ let us consider the following space ([45], §1.18.2):

$$l_q^s = \{u; \ \ u = \{u_i\}, i = 1, 2, \ldots, \infty, u_i \in \mathbb{C}, \}$$

equipped with the norm

$$\|u\|_{l_q^s} = \left(\sum_{i=1}^{\infty} 2^{iqs} |u_i|^q\right)^{1/q} < \infty.$$

We point out that $l_p^0 = \ell_p$. Let us also set A an infinite matrix defined in the space ℓ_q such that $D(A) = \ell_q^s$, $A = \left[\delta_{ij} 2^{si}\right]$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, being $i, j = 1, 2, \ldots, \infty$. Since the operator A is R-positive in ℓ_q, from Theorems 4.2 and 4.7, we have:

(1) for $\kappa = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \leq 1$, $1 < p < n$ and $0 < \lambda \leq n\left(1 - \frac{p}{q}\right)$ the embedding $D^{\alpha} W^{l,p,\lambda}\left(\Omega; \ell_q^s, \ell_q\right) \subset L^{p,\lambda}\left(\Omega; \ell_q^{s(1-\kappa)}\right)$ is continuous and there exists a positive constant C_μ such that

$$\left\| D^{\alpha} u \right\|_{L^{p,\lambda}\left(\Omega; l_q^{(1-\kappa-\mu)s}\right)} \leq C_\mu \left[h^{\mu} \|u\|_{W^{l,p,\lambda}\left(\Omega; l_q^s, l_q\right)} + h^{-(1-\mu)} \|u\|_{L^{p,\lambda}(\Omega; l_q)} \right]$$

for all $u \in W^{l,p,\lambda}\left(\Omega; \ell_q^s, \ell_q\right)$ and $h > 0$;

(2) for $\kappa < 1$, $0 < \lambda < n$ and $0 \leq \mu \leq 1 - \kappa$ the following embedding $D^{\alpha} W^{l,p,\lambda}\left(\Omega; \ell_q^s, \ell_q\right) \subset L^{p,\lambda}\left(\Omega; \ell_q^{s(1-\kappa-\mu)}\right)$ is compact.

It should be noted that the above embedding has not been obtained by the authors using a classical method concerning the integral representation of differentiable functions.

5 Separable differential in Morrey spaces

Let us consider the parameter-dependent principal equation

$$Lu \equiv \sum_{|\alpha:l|=1} a_\alpha D^{\alpha} u + (A + \nu)\, u = f, \tag{5.1}$$

where a_α are complex numbers, ν is a complex parameter and A is a linear operator defined in a Banach space E. We want to highlight the fact that A could be an unbounded operator.

By reasoning as in [1] Theorem A_4 we have the following result.

Theorem 5.1 *Let us assume that the following assumptions are true:*

(1) *E is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$ and $0 < \lambda < n$;*

(2) *A is a R-positive operator in E with $\varphi \in [0\ \pi)$, $\nu \in S(\varphi_1)$, $\varphi_1 \in [0\ \pi)$, $\varphi + \varphi_1 < \pi$ and*

$$K(\xi) = \sum_{|\alpha:l|=1} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_n)^{\alpha_n} \in S(\varphi),$$

$$|K(\xi)| \geq C \sum_{k=1}^{n} |\xi_k|^{l_k}, \quad \xi \in R^n.$$

Then, for every $f \in L^{p,\lambda}(R^n; E)$ there is a unique solution u of equation (5.1) that belongs to the space $W^{l,p,\lambda}(R^n; E(A), E)$ and the following coercive uniform estimate holds true:

$$\sum_{|\alpha:l|\leq 1} |\nu|^{1-|\alpha:l|} \left\| D^{\alpha} u \right\|_{L^{p,\lambda}} + \| Au \|_{L^{p,\lambda}} \leq C \, \| f \|_{L^{p,\lambda}}. \tag{5.2}$$

Proof Applying Fourier transform to Eq. (5.1) it follows

$$K(\xi) + (A + \nu)\, \hat{u}(\xi) = \hat{f}(\xi), \tag{5.3}$$

where

$$K(\xi) = \sum_{|\alpha:l|=1} a_{\alpha} (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_n)^{\alpha_n}.$$

Since $K(\xi) \in S(\varphi)$, for every $\xi \in R^n$, we derive that the operator $A + K(\xi)$ is invertible in E. Then, the solution of (5.3) can be expressed as

$$u(x) = \mathcal{F}^{-1} [A + K(\xi) + \nu]^{-1} \hat{f}. \tag{5.4}$$

Thanks to this expression of u, we have

$$\begin{aligned}
\| Au \|_{L^{p,\lambda}} &= \left\| \mathcal{F}^{-1} A [A + K(\xi) + \nu]^{-1} \hat{f} \right\|_{L^{p,\lambda}}, \\
\left\| D^{\alpha} u \right\|_{L^{p,\lambda}} &= \left\| \mathcal{F}^{-1} (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_n)^{\alpha_n} [A + K(\xi) + \nu]^{-1} \hat{f} \right\|_{L^{p,\lambda}}.
\end{aligned}$$

Then, it is enough to prove that

$$\begin{aligned}
\sigma_1(\xi) &= A [A + K(\xi) + \nu]^{-1}, \\
\sigma_2(\xi) &= \sum_{|\alpha:l|\leq 1} (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_n)^{\alpha_n} [A + K(\xi) + \nu]^{-1}
\end{aligned}$$

are multipliers in $L^{p,\lambda}(R^n; E)$. Therefore, we must show that the following collections

$$\begin{aligned}
&\left\{ \xi^{\beta} D^{\beta} \sigma_1(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}, \\
&\left\{ \xi^{\beta} D^{\beta} \sigma_2(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}
\end{aligned}$$

are R-bounded in E, uniformly in $\nu \in S(\varphi_1)$. Thanks to the R-positivity of A, the set

$$\left\{ \sigma_1(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}$$

is R-bounded, uniformly with respect to parameter ν. Similarly to the proof of Theorem 4.2 and having in mind hypothesis (2), we have that the set

$$\left\{ \sigma_2(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}$$

is R-bounded. Furthermore, making use of Kahane's contraction principle, product properties of the collection of R-bounded operators (see e.g. [15], Lemma 3.5, Proposition 3.4) and R-positivity of operator A, we have

$$
\begin{aligned}
&R\left\{\xi^{\beta} D^{\beta} \sigma_1(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\} \leq C, \\
&R\left\{\xi^{\beta} D^{\beta} \sigma_2(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\} \leq C.
\end{aligned} \tag{5.5}
$$

Estimates (5.5) imply that the functions $\sigma_1(\xi)$ and $\sigma_2(\xi)$ are $L^{p,\lambda}(E)$ multipliers. The proof is achieved. □

Let us denote by L_0 the differential operator in $L^{p,\lambda}(R^n; E)$ generated by (5.1) that is

$$
L_0 u \equiv \sum_{|\alpha:l|=1} a_\alpha D^\alpha u + Au.
$$

The domain $D(L_0)$ of L_0 is equal to $W^{l,p,\lambda}(R^n, E(A), E)$.

From Theorem 5.1 we obtain the following consequence.

Corollary 5.2 *Let us assume that all conditions of Theorem 5.1 are satisfied. Then, there exist two positive constants M_1, M_2 such that the solution $u \in W^{l,p,\lambda}(R^n; E(A), E)$ of (5.1) satisfies the following inequalities*

$$
M_1 \|u\|_{W^{l,p,\lambda}(R^n;E(A),E)} \leq \|L_0 u\|_{L^{p,\lambda}(R^n;E)} \leq M_2 \|u\|_{W^{l,p,\lambda}(R^n;E(A),E)}.
$$

Proof The left part of the chain comes from Theorem 5.1.

The right side is obtained from Theorem 4.2. Indeed, according to the last mentioned result, for all $u \in W^{l,p,\lambda}(R^n; E(A), E)$, we have

$$
\begin{aligned}
\|L_0 u\|_{L^{p,\lambda}(R^n;E)} &\leq \sum_{|\alpha:l|=1} |a_\alpha| \left\|D^\alpha u\right\|_{L^{p,\lambda}(R^n;E)} + \|Au\|_{L^{p,\lambda}(R^n;E)} \\
&\leq \max_\alpha |a_\alpha| \sum_{|\alpha:l|=1} \left\|D^\alpha u\right\|_{L^{p,\lambda}(R^n;E)} + \|Au\|_{L^{p,\lambda}(R^n;E)} \\
&\leq M_2 \|u\|_{W^{l,p,\lambda}(R^n;E(A),E)}.
\end{aligned}
$$

□

Corollary 5.3 *Let us suppose that all assumptions of Theorem 5.1 are satisfied. Then, L_0 has a bounded inverse L_0^{-1} from $L^{p,\lambda}(R^n; E)$ into $W^{l,p,\lambda}(R^n; E(A), E)$ and the resolvent operator $(L_0 + \nu)^{-1}$, for $\nu \in S(\varphi_1)$, satisfies the following sharp coercive estimate*

$$
\sum_{|\alpha:l|\leq 1} |\nu|^{1-|\alpha:l|} \left\|(L_0 + \nu)^{-1}\right\|_{B(L^{p,\lambda})} + \left\|A(L_0 + \nu)^{-1}\right\|_{B(L^{p,\lambda})} \leq C,
$$

for a suitable constant $C > 0$.

Proof From Theorem 4.2 we have that, for $\nu \in S(\varphi_1)$, the operator $(L_0+\nu)^{-1}$ is bounded from $L^{p,\lambda}(R^n; E)$ into $W^{l,p,\lambda}(R^n; E(A), E)$ and applying (4.2) the above estimate follows. □

As a natural consequence of Corollary 5.3 we have the following result.

Corollary 5.4 *Let us suppose that all conditions of Theorem 5.1 are true. Then, the operator L_0 is positive in $L^{p,\lambda}(R^n; E)$.*

Let us call L the differential operator in $L^{p,\lambda}(R^n; E)$ generated by (1.1). Namely,

$$Lu = L_0 u + L_1 u, \quad \text{where} \quad L_1 u = \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u,$$

and its domain $D(L)$ is the set $W^{l,p,\lambda}(R^n, E(A), E)$.

Theorem 5.5 *Let us consider that all conditions of Theorem 5.1 hold and let us also suppose that*

$$A_\alpha(x) A^{1-|\alpha:l|-\mu} \in L_\infty\left(R^n; B(E)\right) \quad \textit{for} \quad 0<\mu<1-|\alpha:l|.$$

Then, for all $f \in L^{p,\lambda}(R^n; E)$ and sufficiently large $|\nu|>0$, Eq. (1.1) has a unique solution u that belongs to the space $W^{l,p,\lambda}(R^n; E(A), E)$ and satisfies the following coercive estimate

$$\sum_{|\alpha:l|\le 1} |\nu|^{1-|\alpha:l|} \left\| D^\alpha u \right\|_{L^{p,\lambda}} + \|Au\|_{L^{p,\lambda}} \le C \|f\|_{L^{p,\lambda}}.$$

Proof In view of the above condition on A_α and by virtue of Theorem 4.2 we can state that there exists $h>0$ such that

$$\begin{aligned} \|L_1 u\|_{L^{p,\lambda}} &\le \sum_{|\alpha:l|<1} \left\| A_\alpha(x) D^\alpha u \right\|_{L^{p,\lambda}} \le \sum_{|\alpha:l|<1} \left\| A^{1-|\alpha:l|-\mu} D^\alpha u \right\|_{L^{p,\lambda}} \\ &\le h^\mu \left(\sum_{|\alpha:l|=1} \left\| D^\alpha u \right\|_{L^{p,\lambda}} + \|Au\|_{L^{p,\lambda}} \right) + h^{-(1-\mu)} \|u\|_{L^{p,\lambda}} \end{aligned} \tag{5.6}$$

for $u \in W^{l,p,\lambda}(R^n; E(A), E)$. Then, from estimates (5.2) and (5.6), for $u \in W^{l,p,\lambda}(R^n; E(A), E)$, we obtain

$$\|L_1 u\|_{L^{p,\lambda}} \le C\left[h^\mu \|(L_0+\nu)u\|_{L^{p,\lambda}} + h^{-(1-\mu)} \|u\|_{L^{p,\lambda}}\right]. \tag{5.7}$$

Since $\|u\|_{L^{p,\lambda}} = \frac{1}{\nu} \|(L_0+\nu)u - L_0 u\|_{L^{p,\lambda}}$ for $\nu > 0$, by Corollary 5.4, $\forall u \in W^{l,p,\lambda}(R^n; E(A), E)$, we get

$$\begin{aligned}\|u\|_{L^{p,\lambda}} &\leq \frac{1}{\nu}\,\|(L_0+\nu)\,u\|_{L^{p,\lambda}} + \frac{1}{\nu}\,\|L_0 u\|_{L^{p,\lambda}} \\ &\leq \frac{1}{\nu}\,\|(L_0+\lambda)\,u\|_{L^{p,\lambda}} + \frac{M}{\nu}\left[\sum_{|\alpha:l|=1}\left\|D^{\alpha}u\right\|_{L^{p,\lambda}} + \|Au\|_{L^{p,\lambda}}\right]. \quad (5.8)\end{aligned}$$

For every $u \in W^{l,p,\lambda}(R^n; E(A), E)$, from estimates (5.6) and (5.7), it follows

$$\|L_1 u\|_{L^{p,\lambda}} \leq Ch^{\mu}\,\|(L_0+\nu)\,u\|_{L^{p,\lambda}} + CM\nu^{-1}h^{-(1-\mu)}\,\|(L_0+\nu)\,u\|_{L^{p,\lambda}}. \quad (5.9)$$

Taking suitable h and $\nu : Ch^{\mu} < 1$ and $CMh^{-(1-\mu)} < \nu$, from (5.10) for sufficiently large ν, we have

$$\left\|L_1\,(L_0+\nu)^{-1}\right\|_{B\left(L^{p,\lambda}(R^n;E)\right)} < 1. \quad (5.10)$$

Now, we have the following relations

$$\begin{aligned}(L+\nu) &= L_0+\nu+L_1, \\ (L+\nu)^{-1} &= (L_0+\nu)^{-1}\left[I + L_1\,(L_0+\nu)^{-1}\right]^{-1}.\end{aligned}$$

Hence, using inequality (5.10), Theorem 5.1 and the perturbation theory of linear operators. we obtain that the differential operator $L+\nu$ is invertible from $L^{p,\lambda}(R^n; E)$ into $W^{l,p,\lambda}(R^n; E(A), E)$. This concludes the proof. □

6 Maximal regular infinite systems of anisotropic equations

Let us define the following infinite system of equations

$$\sum_{|\alpha:l|=1} a_{\alpha} D^{\alpha} u_m(x) + d_m(x)\, u_m(x) + \sum_{|\alpha;l|<1}\sum_{j=1}^{\infty} d_{\alpha j}(x)\, D^{\alpha} u_j(x) + \nu u_m(x) = f_m(x), \quad (6.1)$$

for $x \in R^n,\ m = 1, 2, \ldots, \infty, \nu > 0$. Let us also fix

$$\begin{aligned}&D = \{d_m\}, d_m > 0, u = \{u_m\}, Du = \{d_m u_m\}, m = 1, 2, \ldots \infty, \\ &l_q(D) = l_q^s, s > 0\end{aligned}$$

for every $x \in R^n$ and $1 < q < \infty$.

Theorem 6.1 *Let us suppose $p, q \in (1, \infty)$, $0 < \lambda < n$, a_{α}, $d_{\alpha m j} \in L_{\infty}(R^n)$ be such that, for $0 < \mu < 1 - |\alpha : l|$,*

$$\left| \sum_{|\alpha:l|=1} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_n)^{\alpha_n} \right| \geq C \sum_{k=1}^{n} |\xi_k|^{l_k}, \xi \in R^n,$$

$$\sum_{|\alpha:l|<1} \sum_{j,m=1}^{\infty} \left[d_{\alpha j} d_m^{-(1-|\alpha:l|-\mu)} \right]^{q'} < \infty, \text{for a.e. } x \in R^n, \frac{1}{q} + \frac{1}{q'} = 1.$$

Then, for all $f(x) = \{f_m(x)\}_1^\infty \in L^{p,\lambda}\left(R^n; l_q\right)$ and for sufficiently large $|\nu|$, $\nu \in S(\varphi)$, $0 \leq \varphi < \pi$, system (6.1) has a unique solution $u(x) = \{u_m(x)\}_1^\infty$ that belongs to the space $W^{l,p,\lambda}\left(R^n, l_q(D), l_q\right)$ and the uniform coercive estimate

$$\sum_{|\alpha:l|\leq 1} |\nu|^{1-|\alpha:l|} \left\| D^\alpha u \right\|_{L^{p,\lambda}(R^n;l_q)} \leq C \left\| f \right\|_{L^{p,\lambda}(R^n;l_q)}$$

holds.

Proof Let $E = l_q$, A and $A_\alpha(x)$ be infinite matrices, such that

$$A = \left[d_m \delta_{mj} \right], \ A_\alpha(x) = \left[d_{\alpha j}(x) \right], \ j = 1, 2, \ldots \infty.$$

The operator A is obviously positive in ℓ_q. Thus, thanks to Theorem 5.5, the assertion is immediate. □

Remark 6.2 As an application of the above results, considering concrete Banach spaces instead of E, and concrete R-positive differential, pseudo differential operators, or finite, infinite matrices instead of operator A, on the differential-operator equation (1.1), by virtue of Theorem 5.5 we catch different classes of maximal regular partial differential equations or systems of equations.

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