

Classification and properties of the π -submaximal subgroups in minimal nonsolvable groups

Wenbin Guo¹ · Danila O. Revin^{1,2,3} 

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Abstract Let π be a set of primes. According to H. Wielandt, a subgroup H of a finite group X is called a π -submaximal subgroup if there is a monomorphism $\phi : X \rightarrow Y$ into a finite group Y such that X^ϕ is subnormal in Y and $H^\phi = K \cap X^\phi$ for a π -maximal subgroup K of Y . In his talk at the celebrated conference on finite groups in Santa-Cruz (USA) in 1979, Wielandt posed a series of open questions and among them the following problem: to describe the π -submaximal subgroup of the minimal nonsolvable groups and to study properties of such subgroups: the pronormality, the intravariancy, the conjugacy in the automorphism group etc. In the article, for every set π of primes, we obtain a description of the π -submaximal subgroup in minimal nonsolvable groups and investigate their properties, so we give a solution of Wielandt's problem.

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✉ Danila O. Revin
revin@math.nsc.ru

Wenbin Guo
wbguo@ustc.edu.cn

¹ Department of Mathematics, University of Science and Technology of China, Hefei 230026, People's Republic of China

² Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia

³ Novosibirsk State University, Novosibirsk, Russia 630090

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1 Introduction

In the paper, all groups are finite and G always denotes a finite group. Moreover, π denotes some given set of primes and π' is the set of primes p such that $p \notin \pi$. For a natural number n , let $\pi(n)$ be the set of prime divisors of n and let n_π be the π -share of n , that is, the greatest divisor m of n such that $\pi(m) \subseteq \pi$. Clearly, $n = n_\pi n_{\pi'}$ and $(n_\pi, n_{\pi'}) = 1$. For a group G , let $\pi(G) = \pi(|G|)$. A group G is called a π -group if $\pi(G) \subseteq \pi$.

A subgroup H of G is said to be π -maximal if H is maximal with respect to inclusion in the set of π -subgroups of G . We denote by $m_\pi(G)$ the set of π -maximal subgroups of G .

A group G is called a \mathcal{D}_π -group (or write $G \in \mathcal{D}_\pi$) if any two subgroups in $m_\pi(G)$ are conjugate. It is known that $G \in \mathcal{D}_\pi$ if and only if the complete analog of the Sylow Theorem holds for π -subgroups of G , that is,

- (1) G possesses a π -Hall subgroup (that is, a subgroup of order $|G|_\pi$),
- (2) every two π -Hall subgroups are conjugate in G ,
- (3) every π -subgroup of G is contained in some π -Hall subgroup.

In particular, in a \mathcal{D}_π -group, the π -maximal subgroups are exactly the π -Hall subgroups. The Sylow Theorem states that $G \in \mathcal{D}_p$ for any group G and any prime p . The Hall–Chunikhin Theorem [3, 13, 14] states that a group G is solvable if and only if $G \in \mathcal{D}_\pi$ for every set π of primes. But in general, for a given set π , the class of \mathcal{D}_π -groups is wider than the class of solvable groups since every π -group is a \mathcal{D}_π -group.

We denote by $\text{Hall}_\pi(G)$ the set of π -Hall subgroups of a group G . The \mathcal{D}_π -groups and the groups containing π -Hall subgroups are well investigated (see a survey [31] and next works [7, 10, 25, 34]). In particular, it is proved that $G \in \mathcal{D}_\pi$ if and only if every composition factor of G is a \mathcal{D}_π -group (see [22, Theorem 7.7], [31, Theorem 6.6], [6, Chapter 2, Theorem 6.15]) and the simple \mathcal{D}_π -groups are described in [20, Theorem 3] (see also [31, Theorem 6.9]). These results are based on the description of Hall subgroups in the finite simple groups and the following nice properties of π -Hall subgroups: if N is normal and H is π -Hall subgroups in G , then $HN/N \in \text{Hall}_\pi(G/N)$ and $H \cap N \in \text{Hall}_\pi(N)$.

The Hall–Chunikhin Theorem shows that, if G is nonsolvable, then $G \notin \mathcal{D}_\pi$ for some π . It is clear that $m_\pi(G) \neq \emptyset$ and $\text{Hall}_\pi(G) \subseteq m_\pi(G)$ for any group G but, in general, $m_\pi(G) \not\subseteq \text{Hall}_\pi(G)$ and it may happen that $\text{Hall}_\pi(G) = \emptyset$. It is natural to try to find a description for π -maximal subgroups similar to ones for π -Hall subgroups. A hardness is that the π -maximal subgroups have no properties similar to the mentioned above properties of π -Hall subgroups.

For example [38, (4.2)], if X and Y are groups and $X \notin \mathcal{D}_\pi$ for some π , then, for any π -subgroup (not only for π -maximal) K of Y , there is a π -maximal subgroup H of the regular wreath product $X \wr Y$ such that the image of H under the natural epimorphism $X \wr Y \rightarrow Y$ coincides with K .

In general, the intersection of a π -maximal subgroup H with a normal subgroup N of G is not a π -maximal subgroup of N . For example, it is easy to show that a Sylow 2-subgroup H of $G = PGL_2(7)$ is $\{2, 3\}$ -maximal in G but $H \cap N \notin m_{\{2,3\}}(N)$ for $N = PSL_2(7)$.

But the situation with intersections of π -maximal and normal subgroups of a finite group is not so dramatic in comparing with the situation with images under homomorphisms since the following statement holds.

Proposition 1.1 (Wielandt–Hartley Theorem) *Let G be a finite group, let N be a subnormal subgroup of G and $H \in m_\pi(G)$. Then $H \cap N = 1$ if and only if N is a π' -group.*

For the case where N normal in G , Wielandt’s proof of this statement can be found in [39, 13.2], and Hartly’s one in [11, Lemmas 2 and 3]. The proof of the above proposition in the general case see [26, Theorem 7].

In view of Proposition 1.1, it is natural to consider the following concept. According to H. Wielandt, a subgroup H of a group X is called a π -submaximal subgroup if there is a monomorphism $\phi : X \rightarrow Y$ into a group Y such that X^ϕ is subnormal in Y and $H^\phi = K \cap X^\phi$ for a π -maximal subgroup K of Y . We denote by $sm_\pi(X)$ the set of π -submaximal subgroups of X .

Evidently, $m_\pi(G) \subseteq sm_\pi(G)$ for any group G . The inverse inclusion does not hold in general as one can see in the above example: any Sylow 2-subgroup of $PSL_2(7)$ is $\{2, 3\}$ -submaximal but is not $\{2, 3\}$ -maximal.

In view of the Wielandt–Hartley Theorem, $1 \in sm_\pi(G)$ if and only if G is a π' -group.¹ Moreover, in contrast with π -maximal subgroups, the following property of π -submaximal subgroups holds:

$$\begin{aligned} \text{If } H \in sm_\pi(G) \text{ and } N \text{ is a (sub)normal subgroup of } G, & \quad (*) \\ \text{then } H \cap N \in sm_\pi(N). & \end{aligned}$$

This property shows that, in some sense, the behavior of π -submaximal subgroups is similar to ones of π -Hall subgroups under taking of intersections with (sub)normal subgroups. Note that properties of π -submaximal subgroups are investigated in [8, 9, 37, 39] (see also [12, 38]).

By using the closeness of the class of \mathcal{D}_π -group under taking extensions [22, Theorem 7.7], (see also [31, Theorem 6.6] and [6, Chapter 2, Theorem 6.15]) one can show that $G \in \mathcal{D}_\pi$ if and only if every two π -submaximal subgroup of G are conjugate. In particular, if $G \in \mathcal{D}_\pi$, then $sm_\pi(G) = m_\pi(G) = Hall_\pi(G)$.

In view of the Hall–Chunikhin Theorem, it is natural to consider some “critical” situation where G is non-solvable (and $G \notin \mathcal{D}_\pi$ for some π) but all subgroups of G are solvable (and so they are \mathcal{D}_π -groups). In this situation, G possesses more than one conjugacy class of π -submaximal subgroups. In the paper, we consider the following

¹ Note that, in Wielandt’s talk [38, (5.4)(a)], one can find a more strong version of this statement: if $H \in sm_\pi(G)$ then $N_G(H)/H$ is a π' -group. For the case where G is simple see Lemma 2.9.

problem which was posed by H. Wielandt in his talk² at the well-known conference on finite groups in Santa-Cruz in 1979 [38, Frage (g)]:

Problem 1 (H. Wielandt 1979) To describe the π -submaximal subgroups of the minimal nonsolvable groups. To study properties of such subgroups: conjugacy classes, the pronormality, the intravariancy,³ the conjugacy in the automorphism group etc.

A group G is *minimal nonsolvable* if G is nonsolvable but every proper subgroup of G is solvable. As known, G is a minimal nonsolvable group if and only if $G/\Phi(G)$ is a finite minimal simple group (that is, a non-abelian finite simple group S such that S is minimal nonsolvable), where $\Phi(G)$ is the Frattini subgroup of G that is the intersection of the maximal subgroups of G . In 1968, J. Thompson [29, Corollary 1] proved that S is a minimal simple group if and only if S is isomorphic to a group in the following list \mathcal{T} (we will call it as the Thompson list):

- (1) $L_2(2^p)$ where p is a prime;
- (2) $L_2(3^p)$ where p is an odd prime;
- (3) where p is a prime such that $p > 3$ and $p^2 + 1 \equiv 0 \pmod{5}$;
- (4) $Sz(2^p)$ where p is an odd prime;
- (5) $L_3(3)$.

Actually, the properties emphasized by Wielandt in Problem 1 play an important role in the study of π -maximal and, in particular, π -Hall subgroups. For a π -Hall subgroup H of G , the intravariancy means that H can be lifted to a π -Hall subgroup K of $\text{Aut}(G)$ such that $K \cap \text{Inn}(G)$ coincides with the image of H in $\text{Inn}(G)$ under the natural epimorphism $G \rightarrow \text{Inn}(G)$ [31, Proposition 4.8]. One of important results concerning with π -Hall subgroups of the finite simple groups is the statement that if a finite simple group S contains a π -Hall subgroup, then the number of the conjugacy classes of π -Hall subgroups in S is a bounded π -number [23, Theorem 1.1], [31, Theorem 3.4]. This statement has many consequences for arbitrary finite groups: criteria of the existence [24] and the conjugacy [30], closeness of the class of \mathcal{D}_π -groups under taking extensions, some analog of the Frattini argument for Hall subgroups [25] etc. The strongest results on Hall subgroups turned out to be formulated in term of the pronormality [7, 32–34]: the pronormality of Hall subgroups in the finite simple groups, the existence of a pronormal π -subgroup in any group containing π -Hall subgroups etc.

In the paper, for every set π of primes and for every minimal nonsolvable group G , we solve Problem 1. More precisely, in the first, we reduce this problem to the case where G is a minimal simple group by proving the following statement:

Proposition 1.2 *Let π be a set of primes. Let G be a finite group and let $N = F(G)$ be the Fitting subgroup of G (that is, the greatest normal nilpotent subgroup of G). Suppose $H \in \text{sm}_\pi(G)$. Then the following statements hold:*

² This problem is one of ten open problems in this talk [38]. Another problem [38, Frage (i)] is the following conjecture: a subgroup A of a group G is subnormal if $N_A(H \cap A)/(H \cap A)$ is π' -group for every set π and every $H \in \text{m}_\pi(G)$. This is a converse statement to the strong version of the Wielandt–Hartley Theorem (see previous remark) and it was proved by P. Kleidman in his famous work [17].

³ In Wielandt's terminology, a subgroup H of a group G is said to be *invariant* if the conjugacy class of H in G is $\text{Aut}(G)$ -invariant, that is, for any $\alpha \in \text{Aut}(G)$, there is $g \in G$ such that $H^\alpha = H^g$. Recall, a subgroup H of a group G is said to be *pronormal* if H and H^g are conjugate in $\langle H, H^g \rangle$ for any $g \in G$.

- (1) $m_\pi(G/N) = \{KN/N \mid K \in m_\pi(G)\}$.
- (2) $H \cap N$ coincides with the π -Hall subgroup $O_\pi(N)$ of N .
- (3) $HN/N \in sm_\pi(G/N)$.
- (4) HN/N is pronormal in G/N if and only if H is pronormal in G .
- (5) H is intravariant in G if and only if the conjugacy class of HN/N in G/N is invariant under the image $\overline{\text{Aut}}(\tilde{G})$ of the map $\text{Aut}(G) \rightarrow \text{Aut}(G/N)$ given by the rule $\phi \mapsto \bar{\phi}$ where $\bar{\phi} : Ng \mapsto Ng^\phi$ for $\phi \in \text{Aut}(G)$.

If G is a minimal nonsolvable group, then $F(G) = \Phi(G)$. Thus, for every π -submaximal subgroups H of a minimal nonsolvable group G , the image of H in the corresponding minimal simple group $\tilde{G} = G/\Phi(G)$ is a π -submaximal subgroup of \tilde{G} . In order to solve Problem 1, we only need to describe the π -submaximal subgroup of the minimal simple groups, that is, in the groups of the Thompson list. It is known [5, Corollary 1.7.10] that $\pi(G) = \pi(\tilde{G})$, and it is clear that, if $\pi \cap \pi(\tilde{G}) = \emptyset$, then $sm_\pi(G) = m_\pi(G) = \{1\}$; if $\pi \cap \pi(\tilde{G}) = \{p\}$, then $sm_\pi(G) = m_\pi(G) = \text{Syl}_p(G)$; and if $\pi(\tilde{G}) \subseteq \pi$, then $sm_\pi(G) = m_\pi(G) = \{G\}$. A description of the π -submaximal subgroup in the minimal simple groups for the remaining cases is given in the following Theorem.

Theorem 1.1 *Let π be a set of primes and S a minimal simple group. Suppose that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$. Then representatives of the conjugacy classes of π -submaximal subgroups of S , the information of their structure, π -maximality, pronormality, intravariancy, and the action of $\text{Aut}(S)$ on the set of conjugacy classes of π -submaximal subgroups can be specify in the corresponding Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 below.*

Corollary 1.1 *For every set π of primes, the π -submaximal subgroups of any minimal nonsolvable group are pronormal.*

Note that our results depend on Thompson’s classification of the minimal simple groups and do not depend on the classification of finite simple groups.

1.0 Notation in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11

According to [1,2,18], we use the following notation.

- ε denotes either +1 or -1 and the sing of this number.
- C_n denotes the cyclic group of order n .
- E_q denotes the elementary abelian group of order q where q is a power of a prime.
- D_{2n} denotes the dihedral group of order $2n$, i. e. $D_{2n} = \langle x, y \mid x^n = y^2 = 1, x^y = zx^{-1} \rangle$. Note that $D_4 \cong E_4$.
- SD_{2^n} denotes the semi-dihedral group of order 2^n , i. e. $SD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, x^y = x^{2^{n-2}+1} \rangle$.
- S_n means a symmetric group of degree n .
- A_n denotes the alternating group of degree n .
- $GL_n(q)$ denotes the general linear group of degree n over the field of order q .

$PGL_n(q)$ denotes the projective general linear group of degree n over the field of order q .

$L_n(q) = PSL_n(q)$ denotes the projective special linear group of degree n over the field of order q .

r_+^{1+2n} denotes the extra special group of order r^{1+2n} and of exponent r where r is an odd prime.

$A : B$ means a split extension of a group A by a group B (A is normal).

A^n denotes the direct product of n copies of A .

A^{m+n} means $A^m : A^n$.

The conditions in the column ‘‘Cond.’’ are necessary and sufficient for the existence and the π -submaximality of corresponding H . If a cell in this column is empty, then it means that the corresponding π -submaximal subgroup always exists.

In the column ‘‘ H ’’ the structure of corresponding H is given.

The conditions in the column ‘‘is not π -max. if’’ are necessary and sufficient for corresponding H to be not π -maximal in S . If either this column is skipped or a cell in this column is empty, then the corresponding subgroup is π -maximal.

A number n in the column ‘‘NCC’’ is equal to the number of conjugacy classes of π -submaximal subgroups of S isomorphic to corresponding subgroup H and, if $n > 1$, then in the same column the action of $\text{Aut}(S)$ on these classes is described. The symbol ‘‘ \checkmark ’’ in the column ‘‘Pro.’’ means that the corresponding subgroup H is pronormal in S .

The symbol ‘‘ \checkmark ’’ in the column ‘‘Intra.’’ means that the corresponding subgroup H is intravariant in S . If a cell in this column is empty, then H is not intravariant.

1.1 The π -submaximal subgroups in $S = L_2(q)$, where $q = 2^p$, p is prime, for π such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$

$$|S| = q(q - 1)(q + 1), \quad \pi(S) = \{2\} \cup \pi(q - 1) \cup \pi(q + 1).$$

See Tables 1 and 2.

Table 1 The π -submaximal subgroups of $S = L_2(q)$, where $q = 2^p$, p is a prime. Case: $2 \notin \pi$. Notation: $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon)$, $\varepsilon \in \{+, -\}$. In any cases H is π -maximal

	Cond.	H	NCC	Pro.	Intra.
1	$\pi_+ \neq \emptyset$	$C_{(q-1)\pi}$	1	\checkmark	\checkmark
2	$\pi_- \neq \emptyset$	$C_{(q+1)\pi}$	1	\checkmark	\checkmark

Table 2 The π -submaximal subgroups of $S = L_2(q)$, where $q = 2^p$, p is a prime. Case: $2 \in \pi$. Notation: $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon)$, $\varepsilon \in \{+, -\}$. In any cases H is π -maximal

	Cond.	H	NCC	Pro.	Intra.
1		$E_q : C_{(q-1)\pi}$	1	\checkmark	\checkmark
2	$\pi_+ \neq \emptyset$	$D_{2(q-1)\pi}$	1	\checkmark	\checkmark
3	$\pi_- \neq \emptyset$	$D_{2(q+1)\pi}$	1	\checkmark	\checkmark

1.2 The π -submaximal subgroups in $S = L_2(q)$, where $q = 3^p$, p is odd prime, for π such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$

$$|S| = \frac{1}{2}q(q-1)(q+1), \quad \pi(S) = \{3\} \cup \pi(q-1) \cup \pi(q+1).$$

See Tables 3 and 4.

1.3 The π -submaximal subgroups in $S = L_2(q)$, where q is a prime, $q^2 \equiv -1 \pmod{5}$, for π such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$

$$|S| = \frac{1}{2}q(q-1)(q+1), \quad \pi(S) = \{q\} \cup \pi(q-1) \cup \pi(q+1).$$

See Tables 5 and 6.

Table 3 The π -submaximal subgroups of $S = L_2(q)$, where $q = 3^p$, p is an odd prime. Case: $2 \notin \pi$. Notation: $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon)$, $\varepsilon \in \{+, -\}$. In any cases H is π -maximal

	Cond.	H	NCC	Pro.	Intra.
1	$3 \in \pi$	$E_q : C_{(q-1)\pi}$	1	✓	✓
2	$3 \notin \pi$ and $\pi_+ \neq \emptyset$	$C_{(q-1)\pi}$	1	✓	✓
3	$\pi_- \neq \emptyset$	$C_{(q+1)\pi}$	1	✓	✓

Table 4 The π -submaximal subgroups of $S = L_2(q)$, where $q = 3^p$, p is an odd prime. Case: $2 \in \pi$. Notation: $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon)$, $\varepsilon \in \{+, -\}$, $\phi \in \text{Aut}(S) \setminus \text{Inn}(S)$, $|\phi| = p$. In any cases H is π -maximal

	Cond.	H	NCC	Pro.	Intra.
1	$3 \in \pi$	$E_q : C_{\frac{1}{2}(q-1)\pi}$	1	✓	✓
2	$\pi_+ \neq \{2\}$	$D_{(q-1)\pi}$	1	✓	✓
3		$D_{(q+1)\pi}$	1	✓	✓
4	$3 \in \pi$	A_4	1	✓	✓

Table 5 The π -submaximal subgroups of $S = L_2(q)$, where $q > 3$ is a prime, $q^2 \equiv -1 \pmod{5}$. Case: $2 \notin \pi$. Notation: $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon)$, $\varepsilon \in \{+, -\}$. In any cases H is π -maximal

	Cond.	H	NCC	Pro.	Intra.
1	$q \in \pi$	$C_q : C_{(q-1)\pi}$	1	✓	✓
2	$q \notin \pi$ and $\pi_+ \neq \emptyset$	$C_{(q-1)\pi}$	1	✓	✓
3	$\pi_- \neq \emptyset$	$C_{(q+1)\pi}$	1	✓	✓

Table 6 The π -submaximal subgroups of $S = L_2(q)$, where $q > 3$ is a prime, $q^2 \equiv -1 \pmod{5}$, in the case $2 \in \pi$. Notation: $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon)$, $\varepsilon \in \{+, -\}$, $\delta \in \text{Aut}(S) \setminus \text{Inn}(S)$, $|\delta| = 2$, $\text{Aut}(S) = \langle \text{Inn}(S), \delta \rangle \cong S : \langle \delta \rangle \cong PGL_2(q)$

Cond.	H	NCC	H is not π -max. if	Pro.	Intra.
1 $q \in \pi$	$C_q : C_{\frac{1}{2}(q-1)_\pi}$	1		✓	✓
2 either $\pi_+ \neq \{2\}$, or $3 \notin \pi$ or $q \equiv 1 \pmod{8}$	$D_{(q-1)_\pi}$	1	either $\pi_+ = \{2\}$, $3 \in \pi$, and $q \equiv 41 \pmod{48}$ * or $\pi_+ = \{2, 3\}$ and $q \equiv 7, 31$ $\pmod{72}$ **	✓	✓
3 either $\pi_- \neq \{2\}$, or $3 \notin \pi$, or $q \equiv -1$ $\pmod{8}$	$D_{(q+1)_\pi}$	1	either $\pi_- = \{2\}$, $3 \in \pi$, and $q \equiv 7 \pmod{48}$ * or $\pi_- = \{2, 3\}$ and $q \equiv 41, 65$ $\pmod{72}$ **	✓	✓
4 $3 \in \pi$ and $q \equiv \pm 3$ $\pmod{8}$	A_4	1		✓	✓
5 $3 \in \pi$ and $q \equiv \pm 1$ $\pmod{8}$	S_4	2 permuted by δ		✓	

* $H \cong D_8$ and H is contained in a π -maximal subgroup isomorphic to S_4

** $H \cong D_6$ and H is contained in a π -maximal subgroup isomorphic to S_4

1.4 The submaximal π -subgroups in $S = Sz(q)$, where $q = 2^p$, p is odd prime, for π such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$

$$|S| = q^2(q - 1)(q^2 + 1) = q^2(q - 1)(q - r + 1)(q + r + 1),$$

$$\text{where } r = \sqrt{2q} = 2^{(p+1)/2},$$

$$\pi(S) = \{2\} \cup \pi(q - 1) \cup \pi(q - r + 1) \cup \pi(q + r + 1),$$

See Tables 7 and 8.

Table 7 The π -submaximal subgroups of $S = Sz(q)$, where $q = 2^p$, p is an odd prime. Case: $2 \notin \pi$. Notation: $r = \sqrt{2q} = 2^{(p+1)/2}$, $\pi_0 = \pi \cap \pi(q - 1)$, $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon r + 1)$, $\varepsilon \in \{+, -\}$. In any cases H is π -maximal

Cond.	H	NCC	Pro.	Intra.
1 $\pi_0 \neq \emptyset$	$C_{(q-1)_\pi}$	1	✓	✓
2 $\pi_+ \neq \emptyset$	$C_{(q-r+1)_\pi}$	1	✓	✓
3 $\pi_- \neq \emptyset$	$C_{(q+r+1)_\pi}$	1	✓	✓

Table 8 The π -submaximal subgroups of $S = Sz(q)$, where $q = 2^p$, p is an odd prime. Case: $2 \in \pi$. Notation: $r = \sqrt{2q} = 2^{(p+1)/2}$, $\pi_0 = \pi \cap \pi(q - 1)$, $\pi_\varepsilon = \pi \cap \pi(q - \varepsilon r + 1)$, $\varepsilon \in \{+, -\}$. In any cases H is π -maximal

Cond.	H	NCC	Pro.	Intra.
1	$E_q^{1+1} : C_{(q-1)_\pi}$	1	✓	✓
2 $\pi_0 \neq \emptyset$	$C_{(q-1)_\pi} : C_4$	1	✓	✓
3 $\pi_+ \neq \emptyset$	$D_{2(q-r+1)_\pi}$	1	✓	✓
4 $\pi_- \neq \emptyset$	$D_{2(q+r+1)_\pi}$	1	✓	✓

1.5 The π -submaximal subgroups of $S = L_3(3)$, for π such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$

$$|S| = 2^4 \cdot 3^3 \cdot 13, \quad \pi(S) = \{2, 3, 13\},$$

See Tables 9, 10, and 11.

2 Preliminaries

We write $M \triangleleft G$ if M is a maximal subgroup of G , that is, $M < G$ and $M \leq H \leq G$ implies that either $H = M$ or $H = G$. Moreover, we write $H \trianglelefteq G$ and $H \trianglelefteq\trianglelefteq G$ if H is a normal or subnormal subgroup of G , respectively.

Lemma 2.1 *Let S be a minimal simple group. Then representatives of the conjugacy classes of maximal subgroups of S , the information on their structure, conjugacy classes, and the action of $\text{Aut}(S)$ on the set of the conjugacy classes of maximal subgroups can be specify in the corresponding Tables 12, 13, 14, 15 and 16.*

Proof See [1, Theorem 2.1.1, Tables 8.1–8.4 and 8.16], [16, Theorem II.8.27], [2], [27, Theorem 9]. □

Table 9 The π -submaximal subgroups of $S = L_3(3)$. Case: $\pi \cap \pi(S) = \{3, 13\}$. In any cases H is π -maximal

	H	NCC	Pro.	Intra.
1	$C_{13} : C_3$	1	✓	✓
2	3_+^{2+1}	1	✓	✓

Table 10 The π -submaximal subgroups of $S = L_3(3)$. Case: $\pi \cap \pi(S) = \{2, 13\}$. In any cases H is π -maximal

	H	NCC	Pro.	Intra.
1	C_{13}	1	✓	✓
2	SD_{16}	1	✓	✓

Table 11 The π -submaximal subgroups of $S = L_3(3)$. Case: $\pi \cap \pi(S) = \{2, 3\}$ $\gamma \in \text{Aut}(S) \setminus \text{Inn}(S)$, $|\gamma| = 2$, $\text{Aut}(S) = \langle \text{Inn}(S), \gamma \rangle$

	H	NCC	H is not π -max. if	Pro.	Intra.
1	$E_{3^2} : GL_2(3)$	2 permuted by γ		✓	
2	$3_+^{1+2} : C_2^2$	1	always; $H \leq E_{3^2} : GL_2(3)$	✓	✓
3	$GL_2(3)$	1	always; $H \leq E_{3^2} : GL_2(3)$	✓	✓
4	S_4	1		✓	✓

Table 12 Maximal subgroups of $S = L_2(q)$ where $q = 2^p$, p is prime

M	NCC
$E_q : C_{(q-1)}$	1
$D_{2(q-1)}$	1
$D_{2(q+1)}$	1

Table 13 Maximal subgroups of $S = L_2(q)$ where $q = 3^p$, p is an odd prime

M	NCC
$C_q : C_{\frac{1}{2}(q-1)}$	1
D_{q-1}	1
D_{q+1}	1
A_4	1

Table 14 Maximal subgroups of $S = L_2(q)$ where $q > 3$ is prime, $q^2 \equiv -1 \pmod{5}$

M	NCC	Conditions
$C_q : C_{\frac{1}{2}(q-1)}$	1	
$D_{(q-1)}$	1	$q \neq 7$
$D_{(q+1)}$	1	$q \neq 7$
A_4	1	$q \equiv \pm 3 \pmod{8}$
S_4	2 permuted by δ	$q \equiv \pm 1 \pmod{8}$

Here $\delta \in \text{Aut}(S) \setminus \text{Inn}(S)$, $|\delta| = 2$, $\text{Aut}(S) = \langle \text{Inn}(S), \delta \rangle \cong S : \langle \delta \rangle \cong PGL_2(q)$

Table 15 Maximal subgroups of $S = Sz(q)$ where $q = 2^p$, p is an odd prime

M	NCC
$E_q^{1+1} : C_{(q-1)}$	1
$C_{(q-1)} : C_4$	1
$D_{2(q-r+1)}$	1
$D_{2(q+r+1)}$	1

Table 16 Maximal subgroups of $S = L_3(3)$

M	NCC
$E_{32} : GL_2(3)$	2 permuted by γ
$C_{13} : C_3$	1
S_4	1

Here $\gamma \in \text{Aut}(S) \setminus \text{Inn}(S)$, $|\gamma| = 2$, $\text{Aut}(S) = \langle \text{Inn}(S), \gamma \rangle$

Lemma 2.2 *Let S be a minimal simple group, $M \triangleleft G$ for some G such that $\text{Inn}(S) \leq G \leq \text{Aut}(S)$, and $\text{Inn}(S) \not\leq M$. Then $G = MS$ and one of the following statement holds.*

- (1) $M \cap \text{Inn}(S) \triangleleft \text{Inn}(S)$;
- (2) $S \cong L_2(7)$, $G = \text{Aut}(S)$, $M \cong D_{12}$ and $M \cap \text{Inn}(S) \cong D_6$;
- (3) $S \cong L_2(7)$, $G = \text{Aut}(S)$, $M \cong D_{16}$ and $M \cap \text{Inn}(S) \cong D_8$;
- (4) $S \cong L_3(3)$, $G = \text{Aut}(S)$, $M \cong GL_2(3) : C_2$ and $M \cap \text{Inn}(S) \cong GL_2(3)$;

(5) $S \cong L_3(3)$, $G = \text{Aut}(S)$, $M \cong 3_+^{1+2} : D_8$ and $M \cap \text{Inn}(S) \cong 3_+^{1+2} : C_2^2$.

Proof See [1, Theorem 2.1.1, Tables 8.1–8.4 and 8.16] and [2]. □

Lemma 2.3 (see [35, Statements 7 and 9]) *Let H_1, \dots, H_n be subnormal subgroups of G . Then $K = \langle H_1, \dots, H_n \rangle$ is subnormal in G and every composition factor of K is isomorphic to a composition factor of one of H_1, \dots, H_n .*

Lemma 2.4 (see [15, Lemma 1] and [5, Lemma 1.7.5]) *Let A be a normal subgroup and H be a π -Hall subgroup of G . Then $H \cap A \in \text{Hall}_\pi(A)$ and $HA/A \in \text{Hall}_\pi(G/A)$.*

Lemma 2.5 (see [36] and [5, Theorem 1.10.1]) *If G has a nilpotent π -Hall subgroup, then $G \in \mathcal{D}_\pi$.*

Recall, a group G is said to be π -separable if G has a (sub)normal series

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_n = 1$$

such that every section G_{i-1}/G_i is either a π -group or a π' -group.

Lemma 2.6 *If G is π -separable, then $G \in \mathcal{D}_\pi$ and $\text{sm}_\pi(G) = \text{m}_\pi(G) = \text{Hall}_\pi(G)$.*

Proof The statement $G \in \mathcal{D}_\pi$ is proved in [28, Chapter V, Theorem 3.7]. Thus, in view of

$$\text{Hall}_\pi(G) \subseteq \text{m}_\pi(G) \subseteq \text{sm}_\pi(G),$$

it is sufficient to prove $\text{sm}_\pi(G) \subseteq \text{Hall}_\pi(G)$. Assume that $H \in \text{sm}_\pi(G)$. Without loss of generality, we may assume that $G \trianglelefteq X$ and $H = K \cap G$ for some $K \in \text{m}_\pi(X)$. Let $Y = \langle G^X \rangle$ be the normal closure of G in X . Then Y is π -separable by Lemma 2.3. Moreover, KY is π -separable, so $KY \in \mathcal{D}_\pi$. Hence $K \in \text{m}_\pi(HY) = \text{Hall}_\pi(HY)$. Clearly, $G \trianglelefteq Y$. Hence $H = K \cap G \in \text{Hall}_\pi(G)$ by Lemma 2.4. □

Lemma 2.7 (see [11, Lemma 2]) *Let A be a normal subgroup of G let $H \in \text{m}_\pi(G)$. Then $N_A(H \cap A)/(H \cap A)$ is a π' -group.*

The following lemma was stated without a proof in [38, 5.3]. Here we give a proof for it.

Lemma 2.8 *Let π be a set of primes. Then, for a subgroup H of a non-abelian simple group S , the following conditions are equivalent.*

- (1) $H \in \text{sm}_\pi(S)$.
- (2) *There exists a group G such that S is the socle of G , G/S is a π -group, and $H = S \cap K$ for some $K \in \text{m}_\pi(G)$.*
- (3) $H = S \cap K$ for some $K \in \text{m}_\pi(\text{Aut}(S))$ where S is identified with $\text{Inn}(S)$.

Proof (3) \Rightarrow (2) : It is sufficient to take $G = SK$ where K as in (3).

(2) \Rightarrow (1) : It follows from the definition.

(1) \Rightarrow (3). Take a group X of the smallest order among all groups G such that S can be embedded into G as a subnormal subgroup and $H = S \cap K$ for some $K \in m_\pi(G)$. Let $A = \langle S^X \rangle$. Then the simplicity of S implies that A is a minimal normal subgroup in X . Since X has the smallest possible order, we obtain $X = KA$.

Now we show that X normalizes S . As a consequence, $A = \langle S^X \rangle = S$. Clearly, $H = K \cap S \leq N_K(S)$. Hence $H \leq S \cap N_K(S) \leq S \cap K = H$ and $H = S \cap N_K(S)$. We claim that $N = N_K(S)$ is a π -maximal subgroup of $X_0 = SN$, so by minimality of the order of X we obtain $X = X_0 \leq N_X(S)$. Assume that $N \leq U \leq X_0$ for a π -group U . Since $SN \leq SU \leq X_0 = SN$, we have $SU = SN$. Note that

$$U/(U \cap S) \cong US/S = NS/S \cong N/(N \cap S).$$

In order to prove that $N = U$, we only need to show that $U \cap S = N \cap S$. Let $U_0 = U \cap S$ and let g_1, \dots, g_m be a right transversal of N in K . Then the subgroups $S_i = S^{g_i}, i = 1, \dots, m$, are pairwise different and

$$A = \langle S_1, \dots, S_m \rangle \cong S_1 \times \dots \times S_m.$$

Put $V = \langle U_0^{g_1}, \dots, U_0^{g_m} \rangle$. Then K normalizes V . Indeed, let $x \in K$. Since K acts by the right multiplication on $\{Ng_1, \dots, Ng_m\}$, there are a permutation σ of $\{1, \dots, m\}$ and some elements $t_1, \dots, t_m \in N$ such that

$$g_i x = t_i g_{i\sigma}, \quad i = 1, \dots, m.$$

Since $t_i \in N \leq U$ for all i , we have that t_i normalizes $U_0 = U \cap S$. Hence

$$\begin{aligned} V^x &= \langle U_0^{g_1 x}, \dots, U_0^{g_m x} \rangle = \langle U_0^{t_1 g_{1\sigma}}, \dots, U_0^{t_m g_{m\sigma}} \rangle \\ &= \langle U_0^{g_{1\sigma}}, \dots, U_0^{g_{m\sigma}} \rangle = \langle U_0^{g_1}, \dots, U_0^{g_m} \rangle = V. \end{aligned}$$

It follows that K normalizes V . Since $K \in m_\pi(X)$, we have that $V \leq K$. Consequently,

$$U \cap S = U_0 \leq V \cap S \leq K \cap S = H = N \cap S \leq U \cap S.$$

Thus, $U \cap S = N \cap S$ and $U = N$.

Now we have that $X = KS$, so $C_K(S) \trianglelefteq X$ and $C_K(S)$ is a π -group. Let

$$\bar{} : X \rightarrow X/C_K(S)$$

be the natural epimorphism. Note that its restriction to S is an embedding of S into the almost simple group \bar{X} with the socle $\bar{S} \cong S$. It is easy to see that $\bar{K} \in m_\pi(\bar{X})$ and $\bar{H} = \bar{K} \cap \bar{S}$. By the choice of X , we obtain $C_K(S) = 1$, so X is almost simple.

Thus, we can consider that $X \leq \text{Aut}(S)$. Let $M \in m_\pi(\text{Aut}(S))$ such that $K \leq M$. Then $M \cap S \leq S \leq X$ and K normalizes $M \cap S$. This implies that $M \cap S \leq K$ and $M \cap S = K \cap S = H$. \square

Lemma 2.9 *Let S be a finite simple group and let $H \in \text{sm}_\pi(S)$. Then $N_S(H)/H$ is a π' -group.*

Proof We identify S with $\text{Inn}(S)$. By Lemma 2.8, $H = K \cap S$, where $K \in m_\pi(\text{Aut}(S))$. Since $S \trianglelefteq \text{Aut}(S)$, Lemma 2.7 implies that $N_S(H)/H$ is a π' -group. \square

Lemma 2.10 *Let $S = L_2(q)$ where q is a power of some prime p . Suppose that π is a set of primes such that $2 \notin \pi$ while $p \in \pi$. Then the following statements are equivalent.*

- (1) $S \in \mathcal{D}_\pi$.
- (2) $\text{Aut}(S) \in \mathcal{D}_\pi$.
- (3) $\pi \cap \pi(S) \subseteq \{p\} \cup \pi(q - 1)$.

Proof See [21, Theorems A, 2.5, and, 3.3]. \square

3 Proof of Proposition 1.2

In this section, $N = F(G)$ is the Fitting subgroup of G (i. e. the largest normal nilpotent subgroup of G). Suppose and $H \in \text{sm}_\pi(G)$.

- (1) $m_\pi(G/N) = \{KN/N \mid K \in m_\pi(G)\}$.

Proof Take $K \in m_\pi(G)$. Suppose, $KN/N \leq L/N$ where L/N is a π -group. Then $L \in \mathcal{D}_\pi$ by Lemma 2.6. Since $K \in m_\pi(L)$, we have $K \in \text{Hall}_\pi(L)$ and $KN/N \in \text{Hall}_\pi(L/N) = \{L/N\}$ by Lemma 2.4. Hence, $KN/N \in m_\pi(G/N)$ and

$$\{KN/N \mid K \in m_\pi(G)\} \subseteq m_\pi(G/N).$$

Conversely, assume that $L/N \in m_\pi(G/N)$. By Lemma 2.6, $L \in \mathcal{D}_\pi$ and $L/N = KN/N$ for a fixed $K \in \text{Hall}_\pi(L)$. We show that $K \in m_\pi(G)$. In fact, assume that $K \leq V$ for a π -subgroup V of G . Then $L \leq VN$ and $L = VN$ since $L/N \in m_\pi(G/N)$. In view of $K \in \text{Hall}_\pi(L)$ and $L \in \mathcal{D}_\pi$, we have $K = V$ and $K \in m_\pi(G)$. \square

- (2) $H \cap N$ coincides with the π -Hall subgroup $O_\pi(N)$ of N .

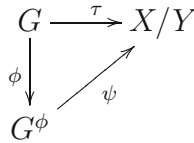
Proof Since N is nilpotent, N has a unique π -Hall subgroup. Hence we only need to show that $H \cap N \in \text{Hall}_\pi(N)$. By property (*) of π -submaximal subgroups, $H \cap N \in \text{sm}_\pi(N)$ and $\text{sm}_\pi(N) = \text{Hall}_\pi(N)$ by Lemma 2.6. \square

- (3) $HN/N \in \text{sm}_\pi(G/N)$.

Proof Denote by ϕ the natural epimorphism $G \rightarrow G/N$. We need to show that $H^\phi \in \text{sm}_\pi(G^\phi)$. One can assume that there exists a group X such that $G \trianglelefteq\trianglelefteq X$ and $H = G \cap K$ for some $K \in \text{m}_\pi(X)$. Denote by Y the normal closure $\langle N^X \rangle$ of N in X . Since $N \trianglelefteq G \trianglelefteq\trianglelefteq X$, the Fitting Theorem [4, Chapter A, Theorem 8.8] implies that Y is nilpotent. Consequently, $G \cap Y$ is also nilpotent. Since $G \cap Y \trianglelefteq G$, we have

$$N \leq G \cap Y \leq F(G) = N.$$

Hence $G \cap Y = N$. Consider the restriction $\tau : G \rightarrow X/Y$ to G of the natural epimorphism $X \rightarrow X/Y$. Then the kernel of τ coincides with N . By the homomorphism Theorem, there exists an injective homomorphism $\psi : G^\phi = G/N \rightarrow X/Y$ such that the following diagram is commutative:



Then

$$G^{\phi\psi} = G^\tau = GY/Y \trianglelefteq\trianglelefteq X/Y.$$

Moreover, we have

$$H^{\phi\psi} = H^\tau = HY/Y = (G \cap K)Y/Y = (GY/Y) \cap (KY/Y) = G^{\phi\psi} \cap (KY/Y),$$

where $KY/Y \in \text{m}_\pi(X/Y)$ in view of (1). Thus $H^\phi \in \text{sm}_\pi(G^\phi)$ by the definition of a π -submaximal subgroup. □

In order to prove next statements (4) and (5), we need the following lemma.

Lemma 3.1 *In the above notation, HN is π -separable and $H \in \text{Hall}_\pi(HN)$.*

Proof The group HN has the subnormal series

$$HN \supseteq N \supseteq H \cap N \supseteq 1$$

such that HN/N is a π -group, $N/(H \cap N)$ is a π' -group by (2) and $H \cap N$ is a π -group. Thus HN is π -separable. Moreover, $H \in \text{Hall}_\pi(HN)$ in view of

$$|HN|_\pi = |HN/N||H \cap N| = |H/(H \cap N)||H \cap N| = |H|.$$

□

(4) HN/N is pronormal in G/N if and only if H is pronormal in G .

Proof If H is pronormal in G , then the pronormality of HN/N in G/N is evident.

Conversely, assume that HN/N is pronormal in G/N . Let $g \in G$. We need to prove that H and H^g are conjugate in $\langle H, H^g \rangle$.

Firstly, consider the case when $g \in N_G(HN)$. Then $H^g \leq HN$ and $H^g \in \text{Hall}_\pi(HN)$ by Lemma 3.1. Clearly, the subgroup $\langle H, H^g \rangle$ of the π -separable group HN is also π -separable. Lemma 2.6 implies that the π -Hall subgroups H and H^g of $\langle H, H^g \rangle$ are conjugate in $\langle H, H^g \rangle$.

Now consider the general case for $g \in G$. Since HN/N is pronormal in G/N , there is $y \in \langle H, H^g \rangle$ such that $(HN)^y = (HN)^g$. Hence $gy^{-1} \in N_G(HN)$ and, in view of above, there exist some $z \in \langle H, H^{gy^{-1}} \rangle \leq \langle H, H^g \rangle$ such that $H^z = H^{gy^{-1}}$. Hence H and H^g are conjugate by $x = zy \in \langle H, H^g \rangle$. \square

- (5) H is intravariant in G if and only if the conjugacy class of HN/N in G/N is invariant under $\overline{\text{Aut}(G)}$.

Proof Recall that $\overline{\text{Aut}(G)}$ is the image in $\text{Aut}(G/N)$ of $\text{Aut}(G)$ under the map $\phi \mapsto \bar{\phi}$ where $\bar{\phi} : Ng \mapsto Ng^\phi$ for $\phi \in \text{Aut}(G)$.

If H is intravariant in G , then for every $\phi \in \text{Aut}(G)$, there is $g \in G$ such that $H^\phi = H^g$. Hence

$$(HN/N)^\phi = \{(Nh)^\phi \mid h \in H\} = \{Nh^\phi \mid h \in H\} = H^\phi N/N = H^g N/N$$

and so the conjugacy class of HN/N in G/N is invariant under $\overline{\text{Aut}(G)}$.

Conversely, assume that the conjugacy class of HN/N in G/N is invariant under $\overline{\text{Aut}(G)}$. We need to show that for every $\phi \in \text{Aut}(G)$ there exists $g \in G$ such that $H^\phi = H^g$. It is clear that $H^\phi \in \text{sm}_\pi(G)$. By the hypothesis, there is $x \in G$ such that $H^\phi N = H^x N$, and in view of Lemma 3.1, $H^\phi, H^x \in \text{Hall}_\pi(H^\phi N)$ and $H^\phi N$ is π -separable. Hence, $H^\phi = H^{xy}$ for some $y \in H^\phi N$ by Lemma 2.6. \square

4 Proof of Theorem 1.1 and Corollary 1.1

We divide our proof of Theorem 1.1 onto three parts. In the first part (Proposition 4.1 in Sect. 1.1), we prove that if H is a π -submaximal subgroup of a minimal simple group S , then H can be found in that of Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 which corresponds to given S and π . In the second part (Proposition 4.2 in Sect. 1.2), we prove that every H in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 are π -submaximal for corresponding S and prove that the information on the π -maximality, conjugacy, intravariancy and the action of $\text{Aut}(S)$ on the conjugacy classes for this subgroups is true. Finally, in the third part (Proposition 4.3 in Sect. 1.3) we prove the pronormality of the π -submaximal subgroups in the minimal simple groups and, as a consequence, in the minimal nonsolvable groups (Corollary 1.1).

1.1 The classification of π -submaximal subgroups in minimal simple groups

In this section, S is a group in the Thompson list \mathcal{T} , that is, S is one of the following groups:

- (1) $L_2(q)$ where $q = 2^p$, p is a prime;
- (2) $L_2(q)$ where where $q = 3^p$, p is an odd prime;
- (3) $L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$;
- (4) $Sz(q)$ where $q = 2^p$, p is an odd prime;
- (5) $L_3(3)$.

We identify S with $\text{Inn}(S) \cong S$. Let π be a set of primes such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$. The following statement gives a classification of π -submaximal subgroups in the minimal simple groups.

Proposition 4.1 *If $H \in \text{sm}_\pi(S)$ where $S \in \mathcal{T}$, then H appears in the corresponding column in that of Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 which corresponds to S and π .*

Proof Let $H \in \text{sm}_\pi(S)$. Lemma 2.8 implies that $H = K \cap S$ for some $K \in \text{m}_\pi(G)$ where $S \leq G \leq \text{Aut}(S)$ and $G = KS$. By Lemma 2.9, $N_S(H)/H$ is a π' -group and, in particular, $H \neq 1$. Since S is not a π -group, we have $K < G$ and $K \leq M$ for some maximal subgroup M of G . Note that $M \cap S < S$ (in fact, if $M \cap S = S$ it would be $G = KS \leq MS \leq M$, which contradicts $M < G$), so M is solvable. Moreover, $K \in \text{m}_\pi(M) = \text{Hall}_\pi(M)$ in view of solvability of M . Hence by Lemma 2.4,

$$H = K \cap S = K \cap (M \cap S) \in \text{Hall}_\pi(M \cap S).$$

It follows from Lemma 2.2 that one of the following cases holds:

- (I) $M \cap S < S$;
- (II) $S \cong L_2(7)$, $G = \text{Aut}(S)$, $M \cong D_{12}$ and $M \cap S \cong D_6$;
- (III) $S \cong L_2(7)$, $G = \text{Aut}(S)$, $M \cong D_{16}$ and $M \cap S \cong D_8$;
- (IV) $S \cong L_3(3)$, $G = \text{Aut}(S)$, $M \cong GL_2(3) : 2$ and $M \cap S \cong GL_2(3)$;
- (V) $S \cong L_3(3)$, $G = \text{Aut}(S)$, $M \cong 3_+^{1+2} : D_8$ and $M \cap S \cong 3_+^{1+2} : C_2^2$.

Firstly, we consider Cases (II)–(V) keeping in mind that $1 < H \in \text{Hall}_\pi(M \cap S)$.

Assume that Case (II) holds. Then $S = L_2(q)$, where $q = 7$ and $M \cap S \cong D_6$.

If $2 \notin \pi$, then $|H| = 3$ and H appears in the 2-nd row of Table 5 for $\pi_+ = \{3\}$.

If $3 \notin \pi$, then $|H| = 2$ and $2 \in \pi$. In this case, H is contained in a Sylow 2-subgroup P of S . But $1 < N_P(H)/H \leq N_S(H)/H$, so $N_S(H)/H$ is not a π' -group, which contradicts Lemma 2.9. Thus, this case is impossible.

If $2, 3 \in \pi$, then $H = M \cap S$ and H appears in the 2-nd row of Table 6 for $\pi_+ \neq \{2\}$.

Assume that Case (III) holds. Then $S = L_2(q)$, where $q = 7$ and $M \cap S \cong D_8$. In this case, $H \in \text{Hall}_\pi(M \cap S)$ implies that $2 \in \pi$, $H = M \cap S$ is a Sylow 2-subgroup of S , and H appears in the 3-rd row of Table 6 for $q \equiv -1 \pmod{4}$.

Assume that Case (IV) holds. Then $S = L_3(3)$ and $M \cap S \cong GL_2(3)$.

If $2 \notin \pi$, then $|H| = 3$ and H is contained in a Sylow 3-subgroup P of S . But $1 < N_P(H)/H \leq N_S(H)/H$, so $N_S(H)/H$ is not a π' -group, which contradicts Lemma 2.9. Thus, this case is impossible.

If $3 \notin \pi$, then $H \cong SD_{16}$, H coincides with a Sylow 2-subgroup of S , and $\pi \cap \pi(S) = \{2, 13\}$. Hence, H appears in the 2-nd row of Table 10.

If $2, 3 \in \pi$, then $H = M \cap S$ and H appears in the 3-rd row of Table 11 for $\pi \cap \pi(S) = \{2, 3\}$.

Assume that Case (V) holds. Then $S = L_3(3)$ and $M \cap S \cong 3_+^{1+2} : C_2^2$.

If $2 \notin \pi$, then $\pi \cap \pi(S) = \{3, 13\}$, $H \cong 3_+^{1+2}$ and H coincides with a Sylow 3-subgroup of S . In this case, H appears in the 2-nd row of Table 9.

If $3 \notin \pi$, then $|H| = 4$ and $2 \in \pi$. In this case, H is contained in a Sylow 2-subgroup P of S . But $1 < N_P(H)/H \leq N_S(H)/H$. Hence $N_S(H)/H$ is not a π' -group, contrary to Lemma 2.9. Thus, this case is impossible.

If $2, 3 \in \pi$, then $H = M \cap S$ and H appears in the 2-nd row of Table 11 for $\pi \cap \pi(S) = \{2, 3\}$.

Now we consider Case (I). In this case, H coincides with some nontrivial π -Hall subgroup of a maximal subgroup $U = M \cap S$ of $S \in \mathcal{T}$. By using Lemma 2.1, we consider the nontrivial π -Hall subgroups of maximal subgroups U of all groups $S \in \mathcal{T}$.

Case (I)(1) $S = L_2(q)$ where $q = 2^p$, p is a prime. In this case, the numbers $2, q - 1$ and $q + 1$ are pairwise coprime and hence

$$\pi(S) = \{2\} \dot{\cup} \pi(q - 1) \dot{\cup} \pi(q + 1)$$

(the dot over the symbol of union means that we have a union of pairwise disjoint sets). Let

$$\pi_\varepsilon = \pi \cap \pi(q - \varepsilon) \quad \text{for } \varepsilon \in \{+, -\}.$$

By Lemma 2.1, U is one of the following groups: either $E_q : C_{(q-1)}$, or $D_{2(q-\varepsilon)}$, where $\varepsilon \in \{+, -\}$.

If $2 \notin \pi$, then

$$\pi \cap \pi(S) = \pi_+ \dot{\cup} \pi_-$$

and H must be a nontrivial cyclic group of order $(q - \varepsilon)_\pi$ and so $\pi_\varepsilon \neq \emptyset$. Thus, H appears in Table 1.

If $2 \in \pi$, then

$$\pi \cap \pi(S) = \{2\} \dot{\cup} \pi_+ \dot{\cup} \pi_-$$

and H must be either the Frobenius group in the 1-st row of Table 2 or the dihedral group $D_{2(q-\varepsilon)_\pi}$, where $\varepsilon \in \{+, -\}$. Moreover, in the last case, $\pi_\varepsilon \neq \emptyset$ since otherwise $|H| = 2$ and H is contained as a proper subgroup in a Sylow 2-subgroup P of S , contrary to Lemma 2.9. Thus H appears in Table 2.

Case (I)(2) $S = L_2(q)$ where $q = 3^p$, p is an odd prime. In this case,

$$\pi(S) = \{3\} \cup \pi(q - 1) \cup \pi(q + 1).$$

Moreover, 3 is coprime with both $q - 1$ and $q + 1$ and $(q - 1, q + 1) = 2$. As in above, we set

$$\pi_\varepsilon = \pi \cap \pi(q - \varepsilon) \quad \text{for } \varepsilon \in \{+, -\}.$$

By Lemma 2.1, U is one of the following groups: either $E_q : C_{\frac{1}{2}(q-1)}$, or $D_{q-\varepsilon}$, where $\varepsilon \in \{+, -\}$, or A_4 .

If $2, 3 \notin \pi$, then

$$\pi \cap \pi(S) = \pi_+ \dot{\cup} \pi_-$$

and H must be a nontrivial cyclic group of order $(q - \varepsilon)_\pi$, and so $\pi_\varepsilon \neq \emptyset$. Thus, H appears in Table 3.

If $2 \notin \pi$ and $3 \in \pi$, then

$$\pi \cap \pi(S) = \{3\} \dot{\cup} \pi_+ \dot{\cup} \pi_-$$

and H must be either the Frobenius group in the 1-st row of Table 3 or a nontrivial cyclic group of order $(q - \varepsilon)_\pi$ and hence $\pi_\varepsilon \neq \emptyset$.

Assume that $\varepsilon = +$ and H is cyclic of order $(q - 1)_\pi$. We claim that $H \notin \text{sm}_\pi(G)$. Indeed, if $H \in \text{sm}_\pi(G)$, then $H = K \cap S$ for some $K \in \mathfrak{m}_\pi(\text{Aut}(S))$ by Lemma 2.8. It is known that $|\text{Aut}(S)/S| = 2p$ (see [1, Table 8.1], for example). Consider two cases: $p \notin \pi$ and $p \in \pi$.

If $p \notin \pi$, then $K \leq S$ and $H = K$. In particular, in this case, $H \in \mathfrak{m}_\pi(S)$. But it is easy to see that $H \in \text{Hall}_{\pi_+}(S)$ and $S \in \mathcal{D}_{\pi_+}$ by Lemma 2.5. A maximal subgroup of the type $E_q : C_{\frac{1}{2}(q-1)}$ of S contains some π_+ -Hall subgroup of S and H is conjugate to this subgroup. It means that H normalizes a Sylow 3-subgroup of S , contrary to $H \in \mathfrak{m}_\pi(S)$.

If $p \in \pi$, then, in view of $(p, q + 1) = (p, 3^p + 1) = 1$, we have by Lemma 2.10 that

$$\text{Aut}(S) \in \mathcal{D}_\tau \quad \text{where } \tau = \pi_+ \cup \{3, p\}.$$

Since $K \in \mathfrak{m}_\pi(\text{Aut}(S))$ and K is a τ -group, we obtain that

$$K \in \mathfrak{m}_\tau(\text{Aut}(S)) = \text{Hall}_\tau(\text{Aut}(S)).$$

But $|K| = p|H|$ is not divisible by 3, a contradiction again.

Thus, if $2 \notin \pi$ and $3 \in \pi$, then H is not a cyclic group of order $(q - 1)_\pi$. In the remaining cases, H appears in Table 3.

Suppose that $2 \in \pi$. Then the maximal subgroups containing a Sylow 2-subgroup of S are D_{q+1} and A_4 . Hence H is a 2-group if and only if $\pi_- = \{2\}$ and $H = D_{(q+1)_2} = D_{(q+1)_\pi}$. In particular, such H can not be contained in $E_q : C_{\frac{1}{2}(q-1)}$ and D_{q-1} . Now it is easy to see that if $3 \notin \pi$, then H coincides with a π -Hall subgroup of $D_{q-\varepsilon}$, where $\varepsilon \in \{+, -\}$; and if $3 \in \pi$, then H can not coincide with a Sylow 2-subgroup P by Lemma 2.9 in view of $N_S(P) \cong A_4$. Hence H must coincides with one of the following groups $E_q : C_{\frac{1}{2}(q-1)_\pi}, D_{(q-1)_\pi}, D_{(q+1)_\pi}$ or A_4 . In all these cases, H appears in Table 4.

Case (I)(3) $S = L_2(q)$, where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$. In this case,

$$\pi(S) = \{q\} \cup \pi(q - 1) \cup \pi(q + 1),$$

q is coprime with both $q - 1$ and $q + 1$ and $(q - 1, q + 1) = 2$. Let

$$\pi_\varepsilon = \pi \cap \pi(q - \varepsilon) \quad \text{for } \varepsilon \in \{+, -\}$$

as usual.

In this case, $\text{Aut}(S) \cong PGL_2(q)$. By Lemma 2.1, U is one of the following groups: or $C_q : C_{\frac{1}{2}(q-1)}$, or $D_{q-\varepsilon}$ where $\varepsilon \in \{+, -\}$, or one of A_4 and S_4 for $q \equiv \pm 3 \pmod{8}$ and $q \equiv \pm 1 \pmod{8}$, respectively. It follows from [1, Table 8.1] that, with the exception of the case $U \cong S_4$, we have $U = V \cap S$ for some $V < \text{Aut}(S)$ such that $\text{Aut}(S) = SV$ and $|V : U| = 2$.

Suppose that $2 \notin \pi$. Since $|\text{Aut}(S) : S| = 2$, H must be a π -maximal subgroup of S by Lemma 2.8. If H is contained in one of A_4 or S_4 , then $|H| = 3$ and, since any Sylow 3-subgroup of S is cyclic, H is contained in $D_{q-\varepsilon}$ with $q \equiv \varepsilon \pmod{3}$. Moreover, if $q \in \pi$, then the unique π -Hall subgroup of D_{q-1} is not π -maximal in S . Indeed, this subgroup is a cyclic π_+ -Hall subgroup of S . Lemma 2.5 implies that this subgroup is conjugate to a subgroup in some Frobenius group $C_q : C_{\frac{1}{2}(q-1)} \leq S$ and normalizes a Sylow q -subgroup of S . Hence, if $2 \notin \pi$, then H appears in Table 5.

Suppose that $2 \in \pi$. In order to show that H appears in Table 6, it is sufficient to show that a π -Hall subgroup H of $D_{q-\varepsilon}$ is not π -submaximal in S if $\pi_\varepsilon = \{2\}$, $3 \in \pi$ and $q \not\equiv \varepsilon \pmod{8}$. Indeed, if $q \equiv -\varepsilon \pmod{8}$, then $|H| = (q - \varepsilon)_2 = 2 < |S|_2$ and $H < P$ for some Sylow 2-subgroup P of S , which contradicts Lemma 2.9; if $q \equiv \pm 3 \pmod{8}$, then H is an elementary abelian Sylow 2-subgroup of S , and $|H| = 4$, which contradicts Lemma 2.9 since, by the Sylow Theorem, H is contained as a normal subgroup in a maximal subgroup of S isomorphic to A_4 and this subgroups is a π -group.

Case (I)(4) $S = Sz(q)$ where $q = 2^p$, p is an odd prime. In this case,

$$\pi(S) = \{2\} \dot{\cup} \pi(q - 1) \dot{\cup} \pi(q - r + 1) \dot{\cup} \pi(q + r + 1)$$

where $r = \sqrt{2q} = 2^{(p+1)/2}$. Let

$$\pi_0 = \pi \cap \pi(q - 1) \text{ and } \pi_\varepsilon = \pi \cap \pi(q - \varepsilon r + 1) \text{ for } \varepsilon \in \{+, -\}.$$

By Lemma 2.1, U is one of the following groups: or $E_q^{1+1} : C_{(q-1)}$, or $C_{(q-1)\pi} : C_4$, or $D_{2(q-\varepsilon r+1)\pi}$ where $\varepsilon \in \{+, -\}$.

Note that S has exactly one conjugacy class of cyclic subgroups of order $q - 1$ (see the character table of S in [27, Theorem 13]). Hence, if $2 \notin \pi$, then any π -Hall subgroup of $E_q^{1+1} : C_{(q-1)}$ is contained as a π -Hall subgroup in some maximal subgroup of kind $C_{(q-1)\pi} : C_4$. Thus, in the case when $2 \notin \pi$, H appears in Table 7.

Now assume that $2 \in \pi$. If one of the sets π_0, π_+, π_- is empty, then any π -Hall subgroup of the respective subgroups: or $C_{(q-1)\pi} : C_4$, or $D_{2(q-r+1)}$, or $D_{2(q-r+1)}$, is a 2-group, but is not a Sylow 2-subgroup. Then Lemma 2.9 implies that case where H is a π -Hall subgroup in one of these maximal subgroup of S is impossible. In the remaining cases, H appears in Table 8.

Case (I)(5) $S = L_3(3)$. In this case, $\pi(S) = \{2, 3, 13\}$. By Lemma 2.1, U is one of the following groups: or $E_{3^2} : GL_2(3)$, or $C_{13} : C_3$, or S_4 . Since $\pi(S) \not\subseteq \pi$ and $|\pi \cap \pi(S)| > 1$, the intersection $\pi \cap \pi(S)$ coincides with one of sets $\{3, 13\}$, $\{2, 13\}$, and $\{2, 3\}$.

Suppose that $\pi \cap \pi(S) = \{3, 13\}$. Then a π -Hall subgroup H of $E_{3^2} : GL_2(3)$ coincides with a Sylow 3-subgroup 3_+^{1+2} of S and appears in Table 9. A π -Hall subgroup H of $C_{13} : C_3$ is $C_{13} : C_3$ itself and H appears in Table 9. A π -Hall subgroup H of S_4 is of order 3 and Lemma 2.9 implies that this case is impossible.

Suppose that $\pi \cap \pi(S) = \{2, 13\}$. Then a π -Hall subgroup H of $E_{3^2} : GL_2(3)$ coincides with a semi-dihedral Sylow 2-subgroup SD_{16} of both $E_{3^2} : GL_2(3)$ and S , and H appears in Table 10. A π -Hall subgroup H of $C_{13} : C_3$ is cyclic of order 13 and appears in Table 10. A π -Hall subgroup H of S_4 is of order 8 and is not a Sylow 2-subgroup of S . We may exclude this case by Lemma 2.9.

Finally, suppose that $\pi \cap \pi(S) = \{2, 3\}$. Then a π -Hall subgroup H of $E_{3^2} : GL_2(3)$ is $E_{3^2} : GL_2(3)$ itself and H appears in Table 11. A π -Hall subgroup H of $C_{13} : C_3$ is cyclic of order 3 and Lemma 2.9 implies that this case is impossible. A π -Hall subgroup H of S_4 is S_4 itself and it appears in Table 11. □

1.2 The π -submaximality, π -maximality, conjugacy, and intravariancy for subgroups in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10

In this section, as in above, S is a group in the Thompson list \mathcal{T} .

We prove that the converse of Proposition 4.1 holds, that is, the subgroups placed in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 are π -submaximal under corresponding conditions. Moreover, we prove that the information about the π -maximality, the conjugacy, the action of $\text{Aut}(S)$ on the conjugacy classes of this subgroups, and the intravariancy in this tables is correct.

Proposition 4.2 *Let $S \in \mathcal{T}$, π be a set of primes such that $|\pi \cap \pi(S)| > 1$ and $\pi(S) \not\subseteq \pi$, and let H be a subgroup of S placed in corresponding column of one of Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 which corresponds to S . Then*

- (1) $H \in \text{sm}_\pi(S)$.
- (2) $H \in \text{Hall}_\pi(M)$ for every $M \triangleleft S$ such that $H \leq M$ (and so $H \in \text{m}_\pi(S)$) with the exception of the following cases:
 - (2a) $S = L_2(q)$ for some prime q such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$; $2, 3 \in \pi$, $H = D_{(q-\varepsilon)\pi}$, $\varepsilon \in \{+, -\}$, $\pi \cap \pi(q - \varepsilon) = \{2\}$ and

$$q \equiv -\varepsilon 7 \pmod{48}.$$

In this case, H is a Sylow 2-subgroup of S , $H \cong D_8$, and $H < S_4 \leq S$.

- (2b) $S = L_2(q)$ for some prime q such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$; $2, 3 \in \pi$, $H = D_{(q-\varepsilon)\pi}$, $\varepsilon \in \{+, -\}$, $\pi \cap \pi(q - \varepsilon) = \{2, 3\}$ and

$$q \equiv \varepsilon 7, \varepsilon 31 \pmod{72}.$$

In this case, $H \cong S_3 \cong D_6$, H is a $\{2, 3\}$ -Hall subgroup of the normalizer in S of a Sylow 3-subgroup and $H < S_4 \leq S$.

- (2c) $S = L_3(3)$, $\pi \cap \pi(S) = \{2, 3\}$, and $H = 3_+^{1+2} : C_2^2$. In this case, $H < E_{3^2} : GL_2(3) \leq S$.
- (2d) $S = L_3(3)$, $\pi \cap \pi(S) = \{2, 3\}$, and $H = GL_2(3)$. In this case, $H < E_{3^2} : GL_2(3) \leq S$.
- (3) S has a unique conjugacy class of π -submaximal subgroups isomorphic to H with the exception of the following cases when S has exactly two conjugacy classes of π -submaximal subgroups isomorphic to H that are fused in $\text{Aut}(S)$:
- (3a) $S = L_2(q)$ where q is a prime such that $q > 3$, $q^2 + 1 \equiv 0 \pmod{5}$, $q \equiv \pm 1 \pmod{8}$, $2, 3 \in \pi$, and $H = S_4$;
- (3b) $S = L_3(3)$, $\pi \cap \pi(S) = \{2, 3\}$, and $H = E_{3^2} : GL_2(3)$.
- (4) H is intravariant in S excepting the cases determined in (3) where S has two conjugacy classes of π -submaximal subgroups isomorphic to H .

Proof Firstly, we prove (2).

Non- π -maximality of H and corresponding inclusions in cases (2b) or (2c) follow from [2].

Suppose case (2a) holds. Then $S = L_2(q)$, $2, 3 \in \pi$, $H = D_{(q-\varepsilon)\pi}$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$, $\varepsilon \in \{+, -\}$.

Consider the case where $\pi \cap \pi(q - \varepsilon) = \{2\}$ and $q \equiv -\varepsilon 7 \pmod{48}$. Then $q \equiv \varepsilon \pmod{8}$. This means that H coincides with a Sylow 2-subgroup of S and, moreover, S contains a maximal subgroup isomorphic to S_4 by Lemma 2.1. Since $q \equiv -7\varepsilon \pmod{16}$, we have $|H| = (q - \varepsilon)\pi = 8 = |S_4|_2$. The Sylow Theorem implies that H is conjugate to a Sylow 2-subgroup of S_4 . Thus, H is not π -maximal in S in view of $2, 3 \in \pi$ and S_4 is a π -group.

Now consider the case where $\pi \cap \pi(q - \varepsilon) = \{2, 3\}$ and $q \equiv \varepsilon 7, \varepsilon 31 \pmod{72}$. Then $q \equiv -\varepsilon \pmod{8}$. It means that S contains a maximal subgroup M isomorphic to S_4 by Lemma 2.1. But it is easy to calculate that $|H| = (q - \varepsilon)\pi = 6$ and so $H \cong S_3$. Moreover, H contains a Sylow 3-subgroup P of S since $|S|_3 = 3$. It follows that H is contained in $N_S(P)$. By considering the maximal subgroups of S given in Lemma 2.1, it is easy to see that $N_S(P) = D_{q-\varepsilon}$, so H is a $\{2, 3\}$ -Hall subgroup of $N_S(P)$. By the Sylow Theorem, we can assume that $P < M$. Since $N_M(P) \cong S_3$, we obtain $N_M(P) \in \text{Hall}_{\{2,3\}}(N_S(P))$. The Hall Theorem and the solvability of $N_S(P)$ imply that H is conjugate with $N_M(P)$ and H is not π -maximal in S since $2, 3 \in \pi$ and S_4 is a π -group.

In order to complete the proof of (2), it is sufficient to show that, in the remaining cases, H is a π -Hall subgroup in every maximal subgroup M of S containing H . By Lemma 2.1, an easy calculation show that one of the following cases holds:

- (I) up to isomorphism, there is a unique maximal subgroup M of S such that $|H|$ divides $|M|$, $H \leq M$ and H is a π -Hall subgroup of M .

- (II) $S = L_2(q)$ where $q = 2^p$ for a prime p , $2 \notin \pi$, $H = C_{(q-1)\pi}$, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = E_q : C_{(q-1)}$ or $M = D_{2(q-1)}$. In this case, H is a π -Hall subgroup of every maximal subgroup containing H .
- (III) $S = L_2(q)$ where $q = 2^p$ for a prime p , $2 \in \pi$, $H = D_{2(q-1)\pi}$, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = E_q : C_{(q-1)}$ or $M = D_{2(q-1)}$. In this case H is not isomorphic to a subgroup of $E_q : C_{(q-1)}$ and is contained only in $M = D_{2(q-1)}$. Hence H is a π -Hall subgroup of every maximal subgroup containing H .
- (IV) $S = L_2(q)$ where either $q = 3^p$ for odd prime or $q > 3$ is a prime, $2 \notin \pi$, $H = C_{(q-1)\pi}$ and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = E_q : C_{\frac{1}{2}(q-1)}$ or $M = D_{q-1}$. In this case, H is a π -Hall subgroup of every maximal subgroup containing H .
- (V) $S = Sz(q)$ where $q = 2^p$ for a prime p , $2 \notin \pi$, $H = C_{(q-1)\pi}$, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = E_q^{1+1} : C_{(q-1)}$ or $M = C_{q-1} : C_4$. In this case, H is a π -Hall subgroup of every maximal subgroup containing H .
- (VI) $S = Sz(q)$ where $q = 2^p$ for a prime p , $2 \in \pi$, $H = C_{(q-1)\pi} : C_4$, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = E_q^{1+1} : C_{(q-1)}$ or $M = C_{q-1} : C_4$. In this case H is not isomorphic to a subgroup of $E_q^{1+1} : C_{(q-1)}$ and is contained only in $M = C_{q-1} : C_4$. Hence H is a π -Hall subgroup of every maximal subgroup containing H .
- (VII) $S = L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$, $2 \in \pi$, $H = D_{(q-\varepsilon)\pi}$ for some $\varepsilon \in \{+, -\}$, $|H|$ divides 24, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = D_{q-\varepsilon}$ or $M \in \{A_4, S_4\}$. This case will be separately considered below.
- (VIII) $S = L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$, $2, 3 \in \pi$, $H \in \{A_4, S_4\}$, $|H|$ divides $q - \varepsilon$ for some $\varepsilon \in \{+, -\}$ and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = D_{q-\varepsilon}$ or $M \in \{A_4, S_4\}$. In this case H is not isomorphic to any subgroup of $D_{q-\varepsilon}$, so H itself is maximal and coincide with its π -Hall subgroup.
- (IX) $S = L_3(3)$, $\pi = \{2, 3\}$, $H = S_4$, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = E_{3^2} : GL_2(3)$ or $M = S_4$. In this case, H itself is maximal and coincide with its π -Hall subgroup.

Thus, in order to complete the proof of (2), we need to consider Case (VII): $S = L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$, $2 \in \pi$, $H = D_{(q-\varepsilon)\pi}$ for some $\varepsilon \in \{+, -\}$, $|H|$ divides 24, and, up to isomorphism, there are exactly two maximal subgroup M of S such that $|H|$ divides $|M|$, namely either $M = D_{q-\varepsilon}$ or $M \in \{A_4, S_4\}$. It is clear that H is a π -Hall subgroup of $D_{q-\varepsilon}$. We need to show that

either H is not contained in $M \in \{A_4, S_4\}$, or H is a π -Hall subgroup in such M , or one of the exceptional cases (2a) and (2b) holds.

Note that we have the condition in Table 6 for H that either $\pi_\varepsilon \neq \{2\}$, or $3 \notin \pi$, or $q \equiv \varepsilon \pmod{8}$.

Suppose that $\pi_\varepsilon \neq \{2\}$. Since $|H| = (q - \varepsilon)_\pi$ divides 24, the assumption $\pi_\varepsilon \neq \{2\}$ means that $3 \in \pi$, 3 divides H , and $M \in \{A_4, S_4\}$ is a π -group. Since H must contain a dihedral group of order 6 in this case, $M \neq A_4$ and so $M = S_4$. This implies that $q \equiv \pm 1 \pmod{8}$. But S_4 does not contain a dihedral group of order 12 and we have that $|H| = 6$. In particular,

$$2 = |H|_2 = (q - \varepsilon)_2$$

and $q \equiv -\varepsilon \pmod{4}$. Thus, $q \equiv -\varepsilon \pmod{8}$. Moreover, $|H|$ divides 24 implies that $q - \varepsilon$ is not divisible by 9 and hence $q \equiv \pm 3 + \varepsilon \pmod{9}$. It is not difficult to prove that

$$\begin{aligned} q &\equiv -\varepsilon \pmod{8} \text{ and } q \equiv \pm 3 + \varepsilon \pmod{9} \\ &\text{if and only if } q \equiv \varepsilon 7, \varepsilon 31 \pmod{72}. \end{aligned}$$

Thus, exceptional case (2b) holds.

Thus, we can consider that $\pi_\varepsilon = \{2\}$. Suppose that $3 \notin \pi$. Since $|H|$ divides 24, we have $H = D_8$ or $H = D_4 = C_2 \times C_2$. In the first case, $M = S_4$ and H is a π -Hall subgroup of both $D_{q-\varepsilon}$ and S_4 , and in the second case, $M = A_4$ and H is π -Hall in both $D_{q-\varepsilon}$ and A_4 .

Consider the last case when $\pi_\varepsilon = \{2\}$ and $3 \in \pi$. Then $q \equiv \varepsilon \pmod{8}$ and condition $|H|$ divides 24 implies that $|H| = 8$. Hence $q \not\equiv \varepsilon \pmod{16}$. This implies that $q \equiv 8 + \varepsilon \pmod{16}$. Moreover, $q \equiv -\varepsilon \pmod{3}$ since otherwise $3 \in \pi_\varepsilon = \{2\}$. It is not difficult to prove that

$$q \equiv 8 + \varepsilon \pmod{16} \text{ and } q \equiv -\varepsilon \pmod{3} \text{ if and only if } q \equiv -\varepsilon 7 \pmod{48}.$$

Thus, the exceptional case (2a) holds.

This completes the proof of (2).

Now we prove (1). In view of (2), it is sufficient to prove that if one of the exceptional cases (2a)–(2d) in (2) holds, then $H = K \cap S$ for some $K \in \mathfrak{m}_\pi(G)$ where $G = \text{Aut}(S)$.

Suppose one of cases (2a)–(2b) holds. Then $S = L_2(q)$, $2, 3 \in \pi$, $H = D_{(q-\varepsilon)_\pi}$ where q is a prime such that $q > 3$, $q^2 + 1 \equiv 0 \pmod{5}$, $\varepsilon \in \{+, -\}$, and $G = PGL_2(q)$. It follows from [1, Table 8.1] that $H \leq U < G$ where $U \cong D_{2(q-\varepsilon)}$. Let $K \cong D_{2(q-\varepsilon)_\pi}$ be a π -Hall subgroup of U . Then $G = SK$. Assume that $K \notin \mathfrak{m}_\pi(G)$. Then $K < L$ for some π -maximal subgroup L of G . It follows from $\pi(S) \not\subseteq \pi$ that $S \not\leq L$. Let V be a maximal subgroup of G such that $L \leq V$. In view of Lemma 2.2, either $V \cap S < S$ or $q = 7$ and $V \cap S \in \{D_6, D_8\}$. It is easy to see that the both arithmetic conditions $q \equiv -\varepsilon 7 \pmod{48}$ in (2a) and $q \equiv \varepsilon 7, \varepsilon 31 \pmod{72}$ in (2b) imply that $q \equiv \pm 1 \pmod{8}$, so S has no maximal subgroup isomorphic to A_4 . Moreover, $VS = G = PGL_2(q)$. If we assume that $V \cap S < S$ and $V \cap S \cong$

S_4 , then the conjugacy class in S of maximal subgroups isomorphic to S_4 would be invariant under $G = \text{Aut}(S)$. But it is not so, see Table 14. Thus, $V \cap S$ is one of the following groups: $C_q : C_{\frac{1}{2}(q-1)}$, $D_{q+\varepsilon}$, and $D_{q-\varepsilon}$. Since $V \leq N_G(V \cap S)$, we have $V = N_G(V \cap S)$ and V coincides with one of $C_q : C_{q-1}$, $D_{2(q+\varepsilon)}$, and $D_{2(q-\varepsilon)}$. But V contains $K \cong D_{2(q-\varepsilon)\pi}$ and $K \notin \text{Hall}_\pi(V)$. We can exclude the case where $V = C_q : C_{q-1}$ since in this case every dihedral subgroup of V is isomorphic to D_{2q} but $((q - \varepsilon)\pi, q) = 1$. If $V = D_{2(q+\varepsilon)}$, then $(q - \varepsilon)\pi$ divide $(q - \varepsilon, q + \varepsilon) = 2$, but $(q - \varepsilon)\pi = |H| \in \{6, 8\}$; a contradiction. The last case when $V = D_{2(q-\varepsilon)}$ is impossible in view of $K \notin \text{Hall}_\pi(V)$. Thus, $K \in \text{m}_\pi(G)$ and $H = K \cap S \in \text{sm}_\pi(S)$.

In the cases (2c) and (2d), $\pi \cap \pi(S) = \{2, 3\}$. Let $G = \text{Aut}(S) \cong L_3(3) : C_2$. Lemma 2.2 implies that $H = V \cap S$ where $V \triangleleft G$ and $V = GL_2(3) : C_2$ in the case (2c) and $V \cong 3_+^{1+2} : D_8$ in the case (2d). Thus $V \in \text{m}_\pi(G)$ and $H \in \text{sm}_\pi(S)$. Therefore, (1) is proved.

Prove (3). By Lemma 2.1 (Tables 14 and 16), if one of the exceptional cases (3a) or (3b) holds, then H is a maximal subgroup of corresponding S , any maximal subgroup of S which is not isomorphic to H does not contain a subgroup isomorphic to H , and S has exactly two conjugacy classes of (maximal) subgroups isomorphic to H interchanged by every non-inner automorphism of S .

We need to prove that in the remaining cases, every $K \in \text{sm}_\pi(S)$ isomorphic to H is conjugate to H . By (2), if there is a maximal subgroup M of S containing both H and K , then $H, K \in \text{Hall}_\pi(M)$. The solvability of M implies that H and K are conjugate in M in view the Hall Theorem. Hence we can consider that H and K are contained in non-conjugate maximal subgroups M and N , respectively.

If M and N are isomorphic, then Lemma 2.1 implies that either (a) $S = L_2(q)$, $q > 3$ is a prime, and $M \cong N \cong S_4$, or (b) $S = L_3(3)$ and $M \cong N \cong E_{3^2} : GL_2(3)$. Since $H \in \text{Hall}_\pi(M)$ and $K \in \text{Hall}_\pi(N)$ in view of (2) and $H \neq M$ and $K \neq N$ (if not, one of the exceptional cases (3a) or (3b) holds), H and K are Sylow 2- or 3-subgroups of M and N , respectively. Now Lemma 2.9 and $H, K \in \text{sm}_\pi(S)$ imply that H, K are Sylow subgroups in S and they are conjugate by the Sylow Theorem.

Consider the case where M and N are non-isomorphic maximal subgroups of S . In this case, $|H| = |K|$ divides $(|M|, |N|)$ and, by the arguments similar above, we can consider that the numbers $|H| = |K|$ and $(|M|, |N|)$ are not powers of primes. Consider all possibilities for S .

Let $S = L_2(q)$ where $q = 2^p$, p is a prime. Since $(|M|, |N|)$ is not a power of prime, by the information in Table 12 we can consider that $M = E_q : C_{(q-1)}$ and $N = D_{2(q-1)}$. If $2 \in \pi$, then the π -Hall subgroups of M and N are non-isomorphic. Hence $2 \notin \pi$. In this case, both H and K are abelian π -Hall subgroups of S and they are conjugate by Lemma 2.5.

Let $S = L_2(q)$, where $q = 3^p$ and p is an odd prime. Note that the order of $E_q : C_{\frac{1}{2}(q-1)}$ is odd and the orders of D_{q-1} and D_{q+1} are not divisible by 3. Now it easy to see from Table 13 that the condition that $(|M|, |N|)$ is not a power of prime implies that we can consider that $M = E_q : C_{\frac{1}{2}(q-1)}$, $N = D_{q-1}$, and $2 \notin \pi$. In this case, both H and K are abelian π -Hall subgroups of S and they are conjugate by Lemma 2.5.

Let $S = L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$. The case when one of M and N is $E_q : C_{\frac{1}{2}(q-1)}$ and other one is D_{q-1} can be argued similarly as the previous case, and as $(|M|, |N|)$ is not a power of prime, we can consider that $M \in \{A_4, S_4\}$. Hence $2, 3 \in \pi$. If $M = A_4$, then $H = M$ (if not, $|H|$ is a power of 2 or 3), but the maximal subgroups of S of the other types contain no subgroups isomorphic to A_4 . Hence $M = S_4$. Since the other maximal subgroups of S do not contain subgroups isomorphic to S_4 and H is not a power of a prime, we have that $H \cong S_3$. Since a maximal subgroup N of S contains a subgroup K isomorphic to $H \cong S_3$, it is easy to see from Table 14 that $N = D_{q-\varepsilon}$ for some $\varepsilon \in \{+, -\}$ and in view of $K \in \text{Hall}_\pi(N)$ we have that $|S|_3 = |N|_3 = 3$. Hence N is the normalizer of a Sylow 3-subgroup P of S and $K \in \text{Hall}_\pi(N_S(P))$. But $H \cong K$ means that H is also a π -Hall subgroup of some (solvable) normalizer of a Sylow 3-subgroup Q of S . Hence by the Sylow and Hall Theorems, H and K are conjugate.

Let $S = Sz(q)$, where $q = 2^p$, p is an odd prime. This case can be investigated without essential changes as the case $S = L_2(q)$, $q = 2^p$.

Finally, let $S = L_3(3)$. Since $(|M|, |N|)$ is not a power of prime and in view of the information in Table 16, we can consider that $M = E_{3^2} : GL_2(3)$, $N = S_4$ and $\pi = \{2, 3\}$. But $H \in \text{Hall}_\pi(M)$ and $K \in \text{Hall}_\pi(N)$ implies that $H = M \not\cong N = K$; a contradiction.

Statement (4) is a straightforward consequence of (3). □

1.3 The pronormality of the π -submaximal subgroups of minimal nonsolvable groups

In order to complete the proof of Theorem 1.1, we need to establish the pronormality of the π -maximal subgroups in minimal simple groups, that is, we need to show that every subgroup H appearing in one of Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 is pronormal in the corresponding group $S \in \mathcal{T}$.

Proposition 4.3 *Let S be a minimal simple group and $H \in \text{sm}_\pi(S)$. Then H is pronormal in S .*

Proof Let $g \in S$. We need to show that H and H^g are conjugate in $\langle H, H^g \rangle$. It is trivial if $\langle H, H^g \rangle = S$. Hence we can consider that $\langle H, H^g \rangle \leq M$ for some $M < S$. If $H \in \text{Hall}_\pi(M)$, then $H, H^g \in \text{Hall}_\pi(\langle H, H^g \rangle)$ and the solvability of M implies that H and H^g are conjugate in $\langle H, H^g \rangle$.

Thus, in view of the statement (2) of Proposition 4.2, we only need to consider the cases (2a)–(2d) in this statement.

(2a) $S = L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$, $2, 3 \in \pi$, $H = D_{(q-\varepsilon)\pi}$, $\varepsilon \in \{+, -\}$, $\pi \cap \pi(q - \varepsilon) = \{2\}$, $q \equiv -\varepsilon 7 \pmod{48}$, and $H \cong D_8$ is a Sylow 2-subgroup of S .

In this case, H is pronormal as a Sylow subgroup of S .

(2b) $S = L_2(q)$ where q is a prime such that $q > 3$ and $q^2 + 1 \equiv 0 \pmod{5}$, $2, 3 \in \pi$, $H = D_{(q-\varepsilon)\pi}$, $\varepsilon \in \{+, -\}$, $\pi \cap \pi(q - \varepsilon) = \{2, 3\}$, $q \equiv \varepsilon 7, \varepsilon 31 \pmod{72}$, and

$H \cong S_3 \cong D_6$ is a $\{2, 3\}$ -Hall subgroup of the normalizer of a Sylow 3-subgroup of S .

By the Sylow Theorem, one can consider that H and H^g are $\{2, 3\}$ -Hall subgroups of the same (solvable) normalizer of a Sylow 3-subgroup of S . Then by the Hall Theorem, H and H^g are conjugate in $\langle H, H^g \rangle$.

(2c) $S = L_3(3)$, $\pi \cap \pi(S) = \{2, 3\}$, and $H = 3_+^{1+2} : C_2^2$ is the normalizer of a Sylow 3-subgroup of S .

In this case, H and H^g are conjugate in $K = \langle H, H^g \rangle$ since both H and H^g are normalizers of Sylow 3-subgroups of K .

(2d) $S = L_3(3)$, $\pi \cap \pi(S) = \{2, 3\}$, and $H = GL_2(3)$.

It is easy to see that in this case H contains a Sylow 2-subgroup P of S . One can take some $x \in \langle H, H^g \rangle$ such that $P^x = P^g$. Then $gx^{-1} \in N_S(P)$. But $N_S(P) = P$ by [19, Corollary] and $g \in Px \subseteq \langle H, H^g \rangle$. Hence H and H^g are conjugate in $K = \langle H, H^g \rangle$. \square

PROOF of Corollary 1.1. The corollary is a straightforward consequence of Propositions 1.2(4) and 4.3.

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