Afrika Matematika (2021) 32:969–982
https://doi.org/10.1007/s13370-021-00873-x

On generalizations of quasi-prime ideals of an ordered left almost semigroups

Pairote Yiarayong[1]

Received: 9 May 2019 / Accepted: 2 January 2021 / Published online: 31 January 2021

Abstract
The purposes of this paper are to introduce generalizations of quasi-prime ideals to the context of ϕ-quasi-prime ideals. Let $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ is the set of all left ideals of an ordered $\mathcal{LA}$-semigroup S. A proper left ideal A of an ordered $\mathcal{LA}$-semigroup S is called a ϕ-quasi-prime ideal, if for each $a, b \in S$ with $ab \in A - \phi(A)$, then $a \in A$ or $b \in A$. Some characterizations of quasi-prime and ϕ-quasi-prime ideals are obtained. Moreover, we investigate relationships between weakly quasi-prime, almost quasi-prime, ω-quasi-prime, m-quasi-prime and ϕ-quasi-prime ideals of ordered $\mathcal{LA}$-semigroups. Finally, we obtain necessary and sufficient conditions of ϕ-quasi-prime ideal in order to be a quasi-prime ideal.

Keyword ordered $\mathcal{LA}$-semigroup, quasi-prime ideal, ϕ-quasi-prime ideal, ω-quasi-prime, ϕ-zero.

Mathematics Subject Classification 20M10 · 16Y99.

1 Introduction

In 2010, Shah et al. [35] studied ideals, M-systems, N-systems and I-systems of ordered $\mathcal{LA}$-semigroups and provided that if A is a left ideal of an ordered $\mathcal{LA}$-semigroup with left identity, then A is quasi-prime if and only if $S - A$ is an M-system; A is quasi-semiprime if and only if $S - A$ is an N-system and A is quasi-irreducible if and only if $S - A$ is an I-system. Nowadays many scholars have studied different aspects of ordered $\mathcal{LA}$-semigroups see [4,7,14,33,44,47–49]. In 2012, Faisal et al. [13] introduced the notion of fuzzy ordered Γ-$\mathcal{LA}$-semigroups and studied (2, 2)-regular ordered Γ-$\mathcal{LA}^{**}$-semigroup in terms of fuzzy Γ-left ideals, fuzzy Γ-right ideals, fuzzy Γ-two-sided ideals, fuzzy Γ-generalized bi-ideals, fuzzy Γ-bi-ideals, fuzzy Γ-interior ideals and fuzzy Γ-(1; 2)-ideals. They proved that the set of all fuzzy Γ-two-sided ideals of a (2, 2)-regular ordered Γ-$\mathcal{LA}^{**}$-semigroup S forms a

✉ Pairote Yiarayong
pairote0027@hotmail.com

[1] Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand

semilattice structure with identity S. In 2013, Khan et al. [18] characterized an intra-regular ordered $\mathcal{LA}$-semigroup in terms of interval valued fuzzy left (right, two-sided) ideals. In 2014, Yousafzai et al. [46] introduced the notion of fully regular (V-regular) class of an ordered $\mathcal{LA}$-semigroup and characterized fully regular (V-regular) class of an ordered $\mathcal{LA}$-semigroup in terms of fuzzy (left, right, two-sided, interior, bi-, generalized bi- and quasi) ideals. In 2015, Khan et al. [24] defined $(0, 2)$-ideals and $(1, 2)$-ideals of an ordered $\mathcal{LA}$-semigroups and proved that the ordered $\mathcal{LA}$-semigroup S is 0-$(0, 2)$-bisimple if and only if S is right 0-simple. In 2016, Yousafzai et al. [51] introduced the notion of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, bi-) ideals of an ordered $\mathcal{LA}$-semigroup and characterized intra-regular ordered $\mathcal{LA}$-semigroups in terms of these generalized fuzzy ideals. In 2017, Yiarayong [42] have also studied prime, semiprime, quasi-prime and quasi-semiprime ideals of ordered $\mathcal{LA}$-semigroups. In 2018, Amjid et al. [6] introduced the notion of smallest one-sided ideals in an $\mathcal{LA}$-semigroup.

In 2004, Stevanovic and Proti [37] introduced the notion of a 3-potent element of an $\mathcal{LA}$-semigroup and of AG-3-band. They studied several properties of AG-3-bands and AG-bands. Nowadays many scholars have studied different aspects of $\mathcal{LA}$-semigroups and $\mathcal{LA}$-semihypergroups see [5,10,16,19,22,23,26,29,32,39,40]. In [27,28] Mushtaq and Khan (2006-2007) initiated the study of AG-bands and AG*-groupoids. They proved that an ideal A of an AG-band is prime iff if ideal (S) is totally ordered; it is prime iff it is strongly irreducible. In 2012, Khan and Anis [17] proved that S/γ is a maximal separative semilattice homomorphic image of an $\mathcal{LA}$-semigroup S. In 2013, Shah and Rehman [17] studied several properties of locally associative Γ-$\mathcal{LA}$-semigroups. They proved that for a locally associative Γ-$\mathcal{LA}$-semigroup S with a left identity, S/ρ is a maximal weakly separative homomorphic image of S, where ρ is a relation on S defined by: $a\rho b$ if and only if $a\gamma b^n = b^{n+1}$ and $b\gamma a^n = a^{n+1}$ for some positive integer n and for all $\gamma \in \Gamma$, where $a, b \in S$. In 2014, Abdullah et al. [3] introduced the concept of interval-valued $(\in, \in \vee q)$-fuzzy ideals, interval-valued $(\in, \in \vee q)$-fuzzy bi-ideals and interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideals of an $\mathcal{LA}$-semigroup. Nowadays many scholars have studied different aspects of fuzzy subsets on $\mathcal{LA}$-semigroups and Γ-$\mathcal{LA}$-semigroups see [9,11,12,15,20,30,31,36,38,41,45,50]. In 2015, Abbasi and Basar [1] studied quasi-ideals and bi-ideals of locally associative Γ-$\mathcal{LA}$-semigroups, (m, n) simple Γ-$\mathcal{LA}$-semigroups, minimal bi-Γ-ideal, semiprime Γ-ideal and quasi-regular Γ-$\mathcal{LA}$-semigroups. They proved that the product of two (m, n)-Γ-ideals of a locally associative Γ-$\mathcal{LA}$-semigroup S with left identity is an (m, n)-Γ-ideal of S. In 2016, Khan et al. [25] have also studied (m, n)-ideals and 0-minimal (m, n)-ideals of $\mathcal{LA}$-semigroups and proved that A is a $(0, 2)$-ideal of S if and only if A is a left ideal of some left ideal of S. In 2017, Khan et al. [21] defined (α, β)-fuzzy bi-ideals, (α, β)-fuzzy interior ideals, $(\bar{\beta}, \bar{\alpha})$-fuzzy bi-ideals and $(\bar{\beta}, \bar{\alpha})$-fuzzy interior ideals in $\mathcal{LA}$-semigroups. In 2019, Younas and Mushtaq [43] proved that the set of idempotent elements in a left permutable inverse $\mathcal{LA}$-semigroup is an order ideal. In 2020, Abbasi et al. [2] introduced the notion of soft interior-hyperideals in $\mathcal{LA}$-semihypergroups. They studied several properties of soft interior-hyperideals of LA-semihypergroups.

Motivated and inspired by the above works, the aim of this paper is to extend the concept of quasi-prime ideals in multiplicative hyperrings given by Yiarayong [42] to the context of ϕ-quasi-prime ideals. Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ is the set of all left ideals of M. A proper left ideal A of an ordered $\mathcal{LA}$-semigroup S is called a ϕ-quasi-prime ideal, if for each $a, b \in S$ with $ab \subseteq A - \phi(A)$, then $a \in A$ or $b \in A$. Some characterizations of quasi-prime and ϕ-quasi-prime ideals are obtained. Moreover, we investigate relationships between weakly quasi-prime, almost quasi-prime, ω-quasi-prime, m-quasi-prime and ϕ-

quasi-prime ideals of ordered $\mathcal{LA}$-semigroups. Finally, we obtain necessary and sufficient conditions of ϕ-quasi-prime ideal in order to be a ϕ-quasi-prime ideal.

2 ϕ-quasi-prime ideals of ordered $\mathcal{LA}$-semigroups

In this section, we give some basic properties of ϕ-quasi-prime ideals and investigate ϕ-quasi-prime ideals in several classes of ordered $\mathcal{LA}$-semigroups and give its characterizations corresponding to ϕ-quasi-prime ideals in ordered $\mathcal{LA}$-semigroups.

For the sake of completeness, we state some definitions in the same fashion as found in [8] which are used throughout this paper.

Definition 1 Let S be an ordered $\mathcal{LA}$-semigroup and let $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ be a set of all left ideals of S. A proper left ideal A of S is called a **ϕ-quasi-prime ideal** if for each $a, b \in S$ with $ab \in A - \phi(A)$, then $a \in A$ or $b \in A$.

We now present the following example satisfying above definition.

Example 1 Let $S = \{a, b, c\}$ be an ordered $\mathcal{LA}$-semigroup with following multiplication given by

$\cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

We define a mapping $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ as follows: $\phi(A) = \emptyset$ for every $A \in \mathcal{I}(S)$. Clearly, $\{a\}$ and $\{a, b\}$ are ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S.

Remark 1 It is easy to see that every quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S is a ϕ-quasi-prime ideal of S.

The following example shows that the converse of Remark 1 is not true.

Example 2 Let $S = \{a, b, c\}$ be an ordered $\mathcal{LA}$-semigroup with following multiplication given by

$\cdot$	a	b	c
a	a	a	a
b	c	c	c
c	a	a	c

Consider the proper left ideal $P = \{a, c\}$ of the ordered $\mathcal{LA}$-semigroup S. Define $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ by $\phi(A) = A$ for every $A \in \mathcal{I}(S)$. It is easy to see that P is a ϕ-quasi-prime ideal of S. Notice that $b \cdot b = c \in P$, but $b \notin P$. Therefore P is not a quasi-prime ideal of S.

Let A be a left ideal of an ordered $\mathcal{LA}$-semigroup S and let $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be a function. Since $A - \phi(A) = A - (A \cap \phi(A))$ for $A \in \mathcal{I}(S)$, without loss of generality, we will assume that $\phi(A) \subseteq A$. Throughout this paper, as it is noted earlier, if ϕ is a function, then we always assume that $\phi(A) \subseteq A$.

Theorem 1 *Let A be a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S with left identity. For each element s of $S - A$ if $\phi(A)$ is a quasi-prime ideal of S, then $(A : s]$ is a ϕ-quasi-prime ideal of S with $(\phi(A) : s] \subseteq \phi(A : s]$.*

Proof Obviously, $(A : s]$ is a left ideal of S. Let a and b be any elements of S such that $ab \in (A : s] - \phi(A : s]$. Since $(\phi(A) : s] \subseteq \phi(A : s]$, we have $a(sb) = s(ab) \in A - \phi(A)$. By assumption, $a \in A$ or $bs \in A$. If $bs \in \phi(A)$, then $b \in \phi(A)$ or $s \in \phi(A)$. Now, if $bs \notin \phi(A)$, then $bs \in A - \phi(A)$. Then from hypothesis, $b \in A$ or $s \in A$. In any case, we have $a \in A$ or $b \in A$, which implies that $sa \in (sA] \subseteq (A] = A$ or $sb \in (sA] \subseteq (A] = A$. Consequently, $a \in (A : s]$ or $b \in (A : s]$ and hence $(A : s]$ is a ϕ-quasi-prime ideal of S. □

Remark 2 Let A be a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S and let $\phi(A)$ be a quasi-prime ideal of S with $s_1 \notin A$, $s_2 \notin (A : s_1]$, $s_3 \notin ((A : s_1] : s_2]$, ... and $(\phi(A) : s_1] \subseteq \phi(A : s_1]$, $((\phi(A) : s_1] : s_2] \subseteq \phi((A : s_1] : s_2]$, ... Then $(A : s_1]$, $((A : s_1] : s_2]$, ... are ϕ-quasi-prime ideals of S and $A \subseteq (A : s_1] \subseteq ((A : s_1] : s_2] \subseteq \ldots$.

In the following result, we give an equivalent definition of ϕ-quasi-prime ideals in an ordered $\mathcal{LA}$-semigroup.

Theorem 2 *Let S be an ordered $\mathcal{LA}$-semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*

1. *A is a ϕ-quasi-prime ideal of S.*
2. *For each an element a of S if $a \in S - A$, then $(A : a] = (\phi(A) : a] \cup A$.*

Proof First assume that A is a ϕ-quasi-prime ideal of S. It is easy to see that, $(\phi(A) : a)] \cup A \subseteq (A : a]$. Let b be an element of S such that $b \in (A : a]$. Then we have, $ab \in A$. If $ab \notin \phi(A)$, then $ab \in A - \phi(A)$. Since A is a ϕ-quasi-prime ideal of S, we have $a \in A$ or $b \in A$. By assumption, $b \in A$ that is, $b \in (\phi(A) : a] \cup A$. Now, if $ab \in \phi(A)$, then $b \in (\phi(A) : a] \subseteq (\phi(A) : a] \cup A$. In any case, we have $(A : a] \subseteq (\phi(A) : a] \cup A$ and hence $(A : a] = (\phi(A) : a] \cup A$.

Conversely, assume that 2 holds. Let a and b be any elements of S such that $ab \in A - \phi(A)$. Then we have, $b \in (A : a]$ and $b \notin (\phi(A) : a]$. If $a \in A$, then there is nothing to prove. Now, if $a \notin A$, then $(A : a] = (\phi(A) : a] \cup A$. Since $b \in (A : a]$ and $b \notin (\phi(A) : a]$, we have $b \in A$. Therefore A is a ϕ-quasi-prime ideal of S. □

The following theorem characterize that quasi-prime ideals in terms of ϕ-quasi-prime ideals of an ordered $\mathcal{LA}$-semigroup S.

Theorem 3 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let $\phi(A)$ be a quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S. Then A is a ϕ-quasi-prime ideal of S if and only if A is a quasi-prime ideal of S.*

Proof First assume that A is a quasi-prime ideal of S. Obviously, A is a ϕ-quasi-prime ideal of S.

Conversely, assume that A is a ϕ-quasi-prime ideal of S. Let a and b be any elements of S such that $ab \in A$. If $ab \notin \phi(A)$, then $ab \in A - \phi(A)$. By assumption, $a \in A$ or $b \in A$. Now if $ab \in \phi(A)$, then $a \in A$ or $b \in A$. In any case, we have A is a ϕ-quasi-prime ideal of S. □

Now we introduce the notion of a ϕ-zero in an ordered $\mathcal{LA}$-semigroup.

Definition 2 Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S. An order pair (a, b), where $a, b \in S$ is a **ϕ-zero** if

1. $ab \in \phi(A)$,

2. $a \notin A$ and $b \notin A$.

Remark 3 Note that a proper left ideal A of an ordered $\mathcal{LA}$-semigroup S is a ϕ-quasi-prime ideal of S that is not a quasi-prime ideal of S if and only if A has a ϕ-zero (a, b) for some $a, b \in S$.

Theorem 4 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S. Suppose that B is a left ideal of S and $a \in S$ such that $aB \subseteq A$. If for every an element b of S such that (a, b) is not a ϕ-zero of A, then $a \in A$ or $B \subseteq A$.*

Proof Assume, $a \notin A$ and $B \nsubseteq A$. Then there exists an element $c \in B$ such that $c \notin A$. If $ac \notin \phi(A)$, then $ac \in A - \phi(A)$. Since A is a ϕ-quasi-prime ideal of S, we have $a \in A$ or $c \in A$. Next, let $ac \in \phi(A)$. By hypothesis, $a \in A$ or $c \in A$. In any case, we have $a \in A$ or $c \in A$, which is a contradiction. Hence, $a \in A$ or $B \subseteq A$. □

Theorem 5 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S. For each elements $a, b \in S$ if (a, b) is a ϕ-zero of A, then $aA \subseteq \phi(A)$.*

Proof Let a be an element of S such that $aA \nsubseteq \phi(A)$. Then there exists an element c of A such that $ac \notin \phi(A)$. Thus we have, $a(b \cup c) = (ab) \cup (ac) \nsubseteq \phi(A)$, which implies that $a(b \cup c) \subseteq A - \phi(A)$. Since A is a ϕ-quasi-prime ideal of S, we have $a \in A$ or $b \cup c \subseteq A$. Therefore, $a \in A$ or $b \in A$, which is a contradiction. Consequently, $aA \subseteq \phi(A)$. □

Theorem 6 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function. If A is a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S that is not a quasi-prime ideal, then $A^2 = \phi(A)$.*

Proof Since A is a ϕ-quasi-prime ideal of S that is not a quasi-prime ideal, we have A has a ϕ-zero (a, b) for some $a, b \in S$ by Remark 3. Assume, $cd \notin \phi(A)$ for some $c, d \in A$. Then we have, $(a \cup c)(b \cup d) = ab \cup cb \cup ad \cup cd \nsubseteq \phi(A)$ by Theorem 5. This implies that, $(a \cup c)(b \cup d) \subseteq A - \phi(A)$. By assumption, $a \cup c \subseteq A$ or $b \cup d \subseteq A$. Therefore, $a \in A$ or $b \in A$, which is a contradiction. Hence, $A^2 = \phi(A)$. □

3 ϕ_α-quasi-prime ideals

In this section, we introduce the concept of ϕ-quasi-prime, $\phi_\emptyset$-quasi-prime, $\phi_{n\geq 1}$-quasi-prime and ϕ_ω-quasi-prime ideals of ordered $\mathcal{LA}$-semigroups and study some basic properties of ϕ-quasi-prime, $\phi_\emptyset$-quasi-prime, $\phi_{n\geq 1}$-quasi-prime and ϕ_ω-quasi-prime ideals of ordered $\mathcal{LA}$-semigroups. Our starting points are the following definitions:

Definition 3 Let $\alpha \in \mathbf{Z}^+ \cup \{\omega\} \cup \{\emptyset\}$ and let $\phi_\alpha : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ is a set of all left ideals of an ordered $\mathcal{LA}$-semigroup S. A proper left ideal A of S is called a ϕ_α-**quasi-prime ideal** if for each $a, b \in S$ with $ab \in A - \phi_\alpha(A)$, then $a \in A$ or $b \in A$.

Let A be a ϕ_α-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S.

- If $\phi_\alpha(A) = \emptyset$ for every $A \in \mathcal{I}(S)$, then we say that $\phi_\alpha = \phi_\emptyset$ and A is called a $\phi_\emptyset$-**quasi-prime ideal** of S and hence A is a **quasi-prime ideal** of S.
- If $\phi_\alpha(A) = A$ for every $A \in \mathcal{I}(S)$, then we say that $\phi_\alpha = \phi_1$ and A is called a ϕ_1-**quasi-prime ideal** of S.

- If $\phi_\alpha(A) = A^2$ for every $A \in \mathcal{I}(S)$, then we say that $\phi_\alpha = \phi_2$ and A is called a ϕ_n-**quasi-prime ideal** of S, and hence A is an **almost quasi-prime ideal** of S.
- If $\phi_\alpha(A) = A^m$ for every $A \in \mathcal{I}(S)$, then we say that $\phi_\alpha = \phi_{m\geq 3}$ and A is called a ϕ_m-**quasi-prime ideal** of S, and hence A is a m-**quasi-prime ideal** of S.
- If $\phi_\alpha(A) = \bigcap_{i=1}^{\infty} A^i$ for every $A \in \mathcal{I}(S)$, then we say that $\phi_\alpha = \phi_\omega$ and A is called a ϕ_ω-**quasi-prime ideal** of S, and hence A is an ω-**quasi-prime ideal** of S.

Remark 4 Let $\alpha \in \mathbf{Z}^+ \cup \{\omega\} \cup \{\emptyset\}$ and let $\phi_\alpha : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function.

1. A left ideal A of an ordered $\mathcal{LA}$-semigroup S is a $\phi_\emptyset$-quasi-prime ideal of S if and only if A is a quasi-prime ideal of S.
2. A left ideal A of an ordered $\mathcal{LA}$-semigroup S is a ϕ_1-quasi-prime ideal of S if and only if A is a proper left ideal of S.
3. If A is a quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S, then A is a ϕ_α-quasi-prime ideal of S.

We start with our main result in which we give a characterization of ϕ_α-quasi-prime ideals in ordered $\mathcal{LA}$-semigroups. For that, we need the following proposition.

Proposition 1 *Let S be an ordered $\mathcal{LA}$-semigroup and let $\phi, \varphi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be two functions. Then the following properties hold:*

1. *If A is a ϕ-quasi-prime ideal of S such that $\phi \leq \varphi$, then A is a φ-quasi-prime ideal of S.*
2. *If A is a quasi-prime ideal of S, then A is a ϕ_ω-quasi-prime ideal of S.*
3. *If A is a ω-quasi-prime ideal of S, then A is a m-quasi-prime ideal of S.*
4. *If A is an almost quasi-prime ideal of S, then A is a ϕ_1-quasi-prime ideal of S.*

Proof 1. Let a and b be any elements of S such that $ab \in A - \varphi(A)$. Since $\phi \leq \varphi$, we have $\phi(A) \subseteq \varphi(A)$. Then we have, $ab \in A - \varphi(A) \subseteq A - \phi(A)$. Since A is a ϕ-quasi-prime ideal of S, we have $a \in A$ or $b \in A$. Hence A is a φ-quasi-prime ideal of S.

2 - 4. It are obvious. □

Remark 5 Let S be an ordered $\mathcal{LA}$-semigroup and let $\mathcal{I}(S)$ be a set of all left ideals of S. It is easy to see that, $\phi_\emptyset \leq \phi_\omega \leq \ldots \leq \phi_{n+1} \leq \phi_n \leq \ldots \leq \phi_2 \leq \phi_1$.

Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups. Then $S_1 \times S_2$ is an ordered $\mathcal{LA}$-semigroup and for each left ideal of $S_1 \times S_2$ is of the form $A_1 \times A_2$ for some left ideals A_1 and A_2 of S_1 and S_2, respectively.

Next we show that, $S_1 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times \ldots \times S_k$ if and only if A_i is a ψ_i-quasi-prime ideal of S_i. First, we would like to show that, $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ if and only if A_1 is a ψ_1-quasi-prime ideal of S_1.

Theorem 7 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left identities and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. Then the following conditions are equivalent:*

1. *$A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$.*
2. (a) *A_1 is a ψ_1-quasi-prime ideal of S_1 where $\psi_2(S_2) \neq S_2$.*
   (b) *For each elements $(a_1, b_1), (a_2, b_2)$ of $S_1 \times S_2$ such that $a_1a_2 \in \psi_1(A_1)$ if $b_1 \in S_2 - (\psi_2(S_2) : S_2]$, then $a_1 \in A_1$ or $a_2 \in A_1$.*

Proof First assume that $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$.

(a) Let a_1 and a_2 be any elements of S_1 such that $a_1a_2 \in A_1 - \psi_1(A_1)$. Then we have, $(a_1, e)(a_2, e) = (a_1a_2, e) \in A_1 \times S_2 - \psi_1(A_1) \times \psi_2(S_2) = A_1 \times S_2 - \phi(A_1 \times S_2)$. By assumption, $(a_1, e) \in A_1 \times S_2$ or $(a_2, e) \in A_1 \times S_2$. Therefore, $a_1 \in A_1$ or $a_2 \in A_1$ and hence A_1 is a ψ_1-quasi-prime ideal of S_1.

(b) Let (a_1, b_1), and (a_2, b_2) be any elements of $S_1 \times S_2$ be such that $a_1a_2 \in \psi_1(A_1)$ and $a_1, a_2 \notin A_1$. In fact, since $b_1 \in S_2 - (\psi_2(S_2) : S_2]$, there exists an element b_2 of S_2 such that $b_2b_1 \notin \psi_2(S_2)$. Thus, $(a_1, b_2)(a_2, b_1) = (a_1a_2, b_2b_1) \in A_1 \times S_2 - \psi_1(A_1) \times \psi_2(S_2) = A_1 \times S_2 - \phi(A_1 \times S_2)$. Then by part (a), i.e., $(a_1, b_2) \in A_1 \times S_2$ or $(a_2, b_1) \in A_1 \times S_2$. Therefore, $a_1 \in A_1$ or $a_2 \in A_1$, which is a contradiction. Consequently, $b_1 \in (\psi_2(S_2) : S_2]$.

Assume that 2 holds. Let (a_1, b_1) and (a_2, b_2) be any elements of $S_1 \times S_2$ be such that $(a_1a_2, b_1b_2) = (a_1, b_1)(a_2, b_2) \in A_1 \times S_2 - \phi(A_1 \times S_2) = A_1 \times S_2 - \psi_1(A_1) \times \psi_2(S_2)$. If $a_1a_2 \notin \psi_1(A_1)$, then $a_1a_2 \in A_1 - \psi_1(S_1)$. Then by part (a), $a_1 \in A_1$ or $a_2 \in A_1$. Thus, $(a_1, b_1) \in A_1 \times S_2$ or $(a_2, b_2) \in A_1 \times S_2$ and thus we are done. If $a_1a_2 \in \psi_1(A_1)$, then $b_1b_2 \notin \psi_2(S_2)$, which implies that $b_2 \notin (\psi_2(S_2) : S_2]$. Hence by part (b), $a_1 \in A_1$ or $a_2 \in A_1$. Therefore, $(a_1, b_1) \in A_1 \times S_2$ or $(a_2, b_2) \in A_1 \times S_2$ and hence $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$. □

The following theorem can be seen in a similar way as in the proof of Theorem 7.

Theorem 8 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left identities and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. Then the following conditions are equivalent:*

1. *$S_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$.*
2. *(a) A_2 is a ψ_2-quasi-prime ideal of S_2 where $\psi_1(S_1) \neq S_1$.*
   (b) For each elements $(a_1, b_1), (a_2, b_2)$ of $S_1 \times S_2$ such that $b_1b_2 \in \psi_2(A_2)$ if $a_1 \in S_1 - (\psi_1(S_1) : S_1]$, then $b_1 \in A_2$ or $b_2 \in A_2$.

The proof of the next result is similar to that of Theorem 7.

Theorem 9 *Let S_i be a ordered $\mathcal{LA}$-semigroup with left identity and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \ldots \times \psi_k$. Then the following conditions are equivalent:*

1. *$S_1 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times \ldots \times S_k$.*
2. (a) *A_i is a ψ_i-quasi-prime deal of S_i where $\psi_j(S_j) \neq S_j$.*
   (b) *For each elements $(a_{(1,1)}, \ldots, a_{(k,1)}), (a_{(1,2)}, \ldots, a_{(k,2)})$ of $S_1 \times \ldots \times S_k$ such that $a_{(1,i)}a_{(2,i)} \in \psi_i(A_i)$ if $a_{(j,1)} \in S_j - (\psi_j(S_j) : S_j)$ for all $j \in \{1, \ldots, k\} - \{i\}$, then $a_{(1,i)} \in A_i$ or $a_{(2,i)} \in A_i$.*

Recall that an element 0 of an ordered $\mathcal{LA}$-semigroup S is called a **left zero element** of S if $0s \leq 0$ for any $s \in S$.

Let S be an ordered $\mathcal{LA}$-semigroup with left zero. If $\phi_\alpha(A) = \{0\}$ for every $A \in \mathcal{I}(S)$, then we say that $\phi_\alpha = \phi_0$ and A is called a ϕ_0**-quasi-prime ideal** of S, and hence A is a **weakly quasi-prime ideal** of A.

As a simple consequence of Theorem 6, we give the following result.

Theorem 10 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a left ideal of an ordered $\mathcal{LA}$-semigroup S with left zero that is not a quasi-prime ideal. If A is a weakly quasi-prime ideal of S, then $A^2 = \{0\}$.*

Next we show that, if A_i is a $(\psi_i)_0$-quasi-prime ideal of S_i, then $S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup $S_1 \times S_2 \times \ldots \times S_k$ if $S_1 \times \ldots \times S_{i-1} \times \{0\} \times S_{i+1} \times \ldots \times S_k \subseteq \phi(S_1 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k)$. First, we would like to show that, A_1 is a $(\psi_1)_0$-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_1, then $A_1 \times S_2$ is a ϕ-quasi-prime ideal if $\{0\} \times S_2 \subseteq \phi(A_1 \times S_2)$.

Theorem 11 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left zeroes and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. If A_1 is a weakly quasi-prime ideal of S_1 such that $\{0\} \times S_2 \subseteq \phi(A_1 \times S_2)$, then $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$.*

Proof Let (a_1, b_1) and (a_2, b_2) be any elements of $S_1 \times S_2$ be such that

$$(a_1, b_1)(a_2, b_2) \in A_1 \times S_2 - \phi(A_1 \times S_2).$$

In fact, since $\{0\} \times S_2 \subseteq \phi(A_1 \times S_2)$, we have $(a_1a_2, b_1b_2) = (a_1, b_1)(a_2, b_2) \notin \{0\} \times S_2$, which means that $a_1a_2 \neq 0$. Then we have, $a_1a_2 \in A_1 - (\psi_1)_0(A_1)$. Since A_1 is a weakly quasi-prime ideal of S_1, we have $a_1 \in A_1$ or $a_2 \in A_1$. Therefore, $(a_1, b_1) \in A_1 \times S_2$ or $(a_2, b_2) \in A_1 \times S_2$ and hence $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$. □

From Theorem 11 we can easily obtain the following theorem.

Theorem 12 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left zeroes and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. If A_2 is a weakly quasi-prime ideal of S_2 such that $S_1 \times \{0\} \subseteq \phi(S_1 \times A_2)$, then $S_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$.*

From Theorems 11, 12 we can easily obtain the following theorem.

Theorem 13 *Let S_i be an ordered $\mathcal{LA}$-semigroup with left zero and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \ldots \times \psi_k$ and $S_1 \times \ldots \times S_{i-1} \times \{0\} \times S_{i+1} \times \ldots \times S_k \subseteq \phi(S_1 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k)$. Then A_i is a weakly quasi-prime ideal of S_i if and only if $S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times S_2 \times \ldots \times S_k$.*

Next, let S be an ordered $\mathcal{LA}$-semigroup. Clearly, every quasi-prime ideal of S is ϕ-quasi-prime ideal, but the converse does not necessarily hold. In Theorem 14 and Corollary 1 provide some conditions under which a ϕ-quasi-prime ideal is a quasi-prime ideal in an ordered $\mathcal{LA}$-semigroup.

Theorem 14 *Let $\phi, \phi_3 : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be two functions and let A be a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S. If $\phi_2 \nleq \phi$, then A is a quasi-prime ideal of S.*

Proof Let a_1 and a_2 be any elements of S such that $a_1a_2 \in A$. If $a_1a_2 \notin \phi(A)$, then $a_1a_2 \in A - \phi(A)$. Since A is a ϕ-quasi-prime ideal of S, we have $a_1 \in A$ or $a_2 \in A$. Next, let a_1a_2 be an element of $\phi(A)$. Since $\phi_2 \nleq \phi$, we have $A^2 \nsubseteq \phi(A)$. Then there exist elements b_1 and b_2 of A such that $b_1b_2 \notin \phi(A)$, which means that $(a_1 \cup b_1)(a_2 \cup b_2) = a_1b_1 \cup a_2b_1 \cup a_1b_2 \cup a_2b_2 \subseteq A - \phi(A)$. By hypothesis, $a_1 \cup b_1 \subseteq A$ or $a_2 \cup b_2 \subseteq A$. Therefore, $a_1 \in A$ or $a_2 \in A$ and hence A is a quasi-prime ideal of S. □

In the following theorem, we give a sort of consequences whose proof is similar to those of quasi-prime ideals in ordered $\mathcal{LA}$-semigroups.

Corollary 1 *Let $\phi_n : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a weakly quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S with left zero. If $\phi_2 \neq \phi_0$, then A is a quasi-prime ideal of S.*

Proof Similar to the proof of Theorem 14. □

Let S_i be an ordered $\mathcal{LA}$-semigroup. For each elements k, n of $\mathbf{Z}^+$ such that $k \geq 2, n \geq 1$, $(\psi_i)_n : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ and let $\phi_{(k,n)} = (\psi_1)_n \times (\psi_2)_n \times \ldots \times (\psi_k)_n$.

Theorem 15 *If A_1 is a weakly quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_1 with left zero such that $(\psi_2)_2(S_2) = S_2$, then $A_1 \times S_2$ is a $\phi_{(2,2)}$-quasi-prime ideal of $S_1 \times S_2$.*

Proof If A_1 is a quasi-prime ideal of S_1, then $A_1 \times S_2$ is a quasi-prime ideal of $S_1 \times S_2$. Obviously, $A_1 \times S_2$ is a $\phi_{(2,2)}$-quasi-prime ideal of $S_1 \times S_2$. Assume that A_1 is not a quasi-prime ideal of S_1. Then by Corollary 1, $(\psi_1)_2 \leq (\psi_1)_0$, which implies that $(A_1)^2 = \{0\}$. By assumption,

$$\begin{aligned}\phi_{(2,2)}(A_1 \times S_2) &= (\psi_1)_2 \times (\psi_2)_2(A_1 \times S_2)\\ &= (\psi_1)_2(A_1) \times (\psi_2)_2(S_2)\\ &= \{0\} \times S_2.\end{aligned}$$

It follows from Theorem 11 that $A_1 \times S_2$ is a $\phi_{(2,2)}$-quasi-prime ideal of $S_1 \times S_2$. □

From Theorem 15 we can easily obtain the following theorem.

Theorem 16 *If A_2 is a weakly quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_2 with left zero such that $(\psi_1)_2(S_1) = S_1$, then $S_1 \times A_2$ is a $\phi_{(2,2)}$-quasi-prime ideal of $S_1 \times S_2$.*

From Theorems 15, 16 we can easily obtain the following theorem.

Theorem 17 *If A_i is a weakly quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_i with left zero such that $(\psi_j)_2(S_j) = S_j$, then $S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a $\phi_{(k,2)}$-quasi-prime ideal of $S_1 \times \ldots \times S_k$.*

Next we show that, if A_i is a quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_i, then $S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times \ldots \times S_2$ if $\psi_j(S_j) \neq S_j$. First, we would like to show that, A_1 is a quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_1, then $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ if $\psi_2(S_2) \neq S_2$.

Theorem 18 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left identities and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\phi = \psi_1 \times \psi_2$. Then the following conditions are equivalent:*

1. *A_1 is a quasi-prime ideal of S_1.*
2. *$A_1 \times S_2$ is a quasi-prime ideal of $S_1 \times S_2$.*
3. *$A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ where $\psi_2(S_2) \neq S_2$.*

Proof $(1 \Rightarrow 2)$. Assume that A_1 is a quasi-prime ideal of S_1. Let (a_1, b_1) and (a_2, b_2) be any elements of $S_1 \times S_2$ be such that $(a_1a_2, b_1b_2) = (a_1, b_1)(a_2, b_2) \in A_1 \times S_2$, which implies that $a_1a_2 \in A_1$. By assumption, $a_1 \in A_1$ or $a_2 \in A_1$. Therefore, $(a_1, b_1) \in A_1 \times S_2$ or $(a_2, b_2) \in A_1 \times S_2$. Consequently, $A_1 \times S_2$ is a quasi-prime ideal of $S_1 \times S_2$.

$(2 \Rightarrow 3)$. It is obvious.

$(3 \Rightarrow 1)$. Assume that 3 holds. Let a_1 and a_2 be any elements of S_1 be such that $a_1a_2 \in A_1$. Since $\psi_2(S_2) \neq S_2$, there exists an element c of S_2 such that $c \notin \psi_2(S_2)$. In fact, since $(a_1, e)(a_2, c) = (a_1a_2, c) \notin A_1 \times \psi_2(S_2)$ and $\phi(A_1 \times S_2) = (\psi_1 \times \psi_2)(A_1 \times S_2) \subseteq A_1 \times \psi_2(S_2)$, we have $(a_1, e)(a_2, c) \notin \phi(A_1 \times S_2)$, which means that $(a_1, e)(a_2, c) \in A_1 \times S_2 - \phi(A_1 \times S_2)$. Since $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$, we have $(a_1, e) \in A_1 \times S_2$ or $(a_2, c) \in A_1 \times S_2$. Therefore, $a_1 \in A_1$ or $a_2 \in A_1$ and hence A_1 is a quasi-prime ideal of S_1. □

From Theorem 18 we can easily obtain the following theorem.

Theorem 19 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left identities and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\phi = \psi_1 \times \psi_2$. Then the following conditions are*

1. *A_2 is a quasi-prime ideal of S_2.*
2. *$S_1 \times A_2$ is a quasi-prime ideal of $S_1 \times S_2$.*
3. *$S_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$, where $\psi_1(S_1) \neq S_1$.*

From Theorems 18, 19 we can easily obtain the following theorem.

Theorem 20 *Let S_i be a ordered $\mathcal{LA}$-semigroup with left identity and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\phi = \psi_1 \times \ldots \times \psi_k$. Then the following conditions are equivalent:*

1. *A_i is a quasi-prime ideal of S_i.*
2. *$S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a quasi-prime ideal of $S_1 \times \ldots \times S_k$.*
3. *$S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times \ldots \times S_2$ with $\psi_j(S_j) \neq S_j$.*

Next, we show that if A_i is a ψ_i-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_i, then $S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times \ldots \times S_k$ if $\psi_j(S_j) = S_j$. First, we would like to show that, A_1 is a ψ_1-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup S_1, then $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$, if $\psi_2(S_2) = S_2$.

Theorem 21 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left identities and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_2(S_2) = S_2$ and $\phi = \psi_1 \times \psi_2$. Then $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ if and only if A_1 is a ψ_1-quasi-prime ideal of S_1.*

Proof First assume that $A_1 \times S_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$. The proof is trivial and hence omitted.

Conversely, assume that A_1 is a ψ_1-quasi-prime ideal of S_1. Let (a_1, b_1) and (a_2, b_2) be any elements of $S_1 \times S_2$ be such that

$$\begin{aligned}(a_1a_2, b_1b_2) &= (a_1, b_1)(a_2, b_2) \in A_1 \times S_2 - \phi(A_1 \times S_2)\\ &= A_1 \times S_2 - (\psi_1 \times \psi_2)(A_1 \times S_2)\\ &= A_1 \times S_2 - \psi_1(A_1) \times S_2.\end{aligned}$$

Obviously, $a_1a_2 \in A_1 - \psi_1(A_1)$. By assumption, $a_1 \in A_1$ or $a_2 \in A_1$. Consequently, A_1 is ψ_1-quasi-prime ideal of S_1. □

From Theorem 21 we can easily obtain the following theorem.

Theorem 22 *Let S_1 and S_2 be two ordered $\mathcal{LA}$-semigroups with left identities and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_1(S_1) = S_1$ and $\phi = \psi_1 \times \psi_2$. Then $S_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ if and only if A_2 is a ψ_2-quasi-prime ideal of S_2.*

From Theorems 21,22 we can easily obtain the following theorem.

Theorem 23 *Let S_i be an ordered $\mathcal{LA}$-semigroup with left identity and let $\psi_i : \mathcal{I}(S_i) \rightarrow \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_j(S_j) = S_j$ and $\phi = \psi_1 \times \ldots \times \psi_k$. Then $S_1 \times S_2 \times \ldots \times S_{i-1} \times A_i \times S_{i+1} \times \ldots \times S_k$ is a ϕ-quasi-prime ideal of $S_1 \times \ldots \times S_k$ if and only if A_i is a ψ_i-quasi-prime ideal of S_i.*

Next, we show that if $A_1 \times A_2$ is a ϕ-quasi-prime ideal of an ordered $\mathcal{LA}$-semigroup $S_1 \times S_2$, then A_i is a ψ_1-quasi-prime ideal of S_i for all $i = 1, 2$.

Theorem 24 *Let A_1 and A_2 be any proper left ideals of ordered $\mathcal{LA}$-semigroups with left identities S_1 and S_2, respectively and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\phi = \psi_1 \times \psi_2$. Then the following properties hold:*

1. *If $A_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ such that $A_2 \neq \psi_2(A_2)$, then A_1 is a ψ_1-quasi-prime ideal of S_1.*
2. *If $A_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$ such that $A_1 \neq \psi_1(A_1)$, then A_2 is a ψ_2-quasi-prime ideal of S_2.*

Proof 1. Let a_1 and a_2 be any elements of S_1 be such that $a_1a_2 \in A_1 - \psi(A_1)$. If $A_2 \neq \psi_2(A_2)$, then there exists an element c of S_2 such that $c \notin \psi_2(A_2)$. This implies that, $(a_1, e)(a_2, c) = (a_1a_2, c) \in A_1 \times A_2 - \psi_1(A_1) \times \psi_2(A_2) = A_1 \times A_2 - \phi(A_1 \times A_2)$. Since $A_1 \times A_2$ is a ϕ-quasi-prime ideal of $S_1 \times S_2$, we have $(a_1, e) \in A_1 \times S_2$ or $(a_2, c) \in A_1 \times S_2$. Therefore, $a_1 \in A_1$ or $a_2 \in A_1$ and hence A_1 is a ψ_1-quasi-prime ideal of S_1.

2. This follows from part 1. □

From Theorem 24 we can easily obtain the following theorem.

Theorem 25 *Let A_i be a proper left ideal of an ordered $\mathcal{LA}$-semigroup S_i with left identity and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\phi = \psi_1 \times \psi_2 \times \ldots \times \psi_k$. If $A_1 \times A_2 \times \ldots \times A_k$ is a ϕ-quasi-prime ideal of $S_1 \times S_2 \times \ldots \times S_k$ such that $A_j \neq \psi_j(A_j)$, then A_i is a ψ_i-quasi-prime ideal of S_i.*

The next theorem gives conditions for a ϕ-quasi-prime ideal to be quasi-prime ideal in an ordered $\mathcal{LA}$-semigroup.

Theorem 26 *Let S_i be an ordered $\mathcal{LA}$-semigroup with left identity and left zero and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_i(S_i) \neq S_i$ and $\phi = \psi_1 \times \psi_2$. If A is a ϕ-quasi-prime ideal of $S_1 \times S_2$, then $A = \phi(A)$ or A is a quasi-prime ideal of $S_1 \times S_2$.*

Proof Suppose that A is a ϕ-quasi-prime ideal of $S_1 \times S_2$ that is not a quasi-prime ideal of $S_1 \times S_2$. To show that $A \neq \phi(A)$. First assume, $A_1 \times A_2 = A \neq \phi(A) = \phi(A_1 \times A_2) = \psi_1(A_1) \times \psi_2(A_2)$. Then there exists an element i of $\{1, 2\}$ such that $A_i \neq \psi_i(A_i)$. We may assume that $A_1 \neq \psi_1(A_1)$, there exists an element c_1 of A_1 such that $c_1 \notin \psi_1(A_1)$. We will to show that $A_2 = S_2$. Next, assume, $A_2 \neq S_2$, it follows that there exists an element c_2 of S_2 such that $c_2 \notin A_2$. In fact, since $(e, c_2)(c_1, e) = (c_1, c_2e) \notin \psi_1(A_1) \times \psi_2(A_2) = \phi(A)$, we have $(e, c_2)(c_1, e) \in A - \phi(A)$. Thus, $(e, c_2)) \in A$ or $(c_1, e) \in A$. Obviously, $c_2 \in A_2$, which is a contradiction. Therefore, $A = A_1 \times S_2$, which means that $(0, e) \in A$. By Theorem 14,

$$\begin{aligned}(0, e) &= (0, e)^2 \in A^2 \\ &= \phi_2(A) \\ &\subseteq \phi(A) \\ &= \psi_1(A_1) \times \psi_2(A_2),\end{aligned}$$

which is a contradiction. Hence, $A = \phi(A)$. □

From Theorem 26 we can easily obtain the following theorem.

Theorem 27 *Let S_i be an ordered $\mathcal{LA}$-semigroup with left identity and left zero and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_i(S_i) \neq S_i$ and $\phi = \psi_1 \times \psi_2 \times \ldots \times \psi_{k\geq 2}$. If A is a ϕ-quasi-prime ideal of $S_1 \times S_2 \times \ldots \times S_k$, then $A = \phi(A)$ or A is a quasi-prime ideal of $S_1 \times S_2 \times \ldots \times S_k$.*

The above theorem shows the relationship between quasi-prime ideals and ϕ-quasi-prime ideals in an ordered $\mathcal{LA}$-semigroup $S_1 \times S_2$. From the above theorem, we have the following theorem.

Theorem 28 *Let S_i be an ordered $\mathcal{LA}$-semigroup with left identity and left zero and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_i(S_i) \neq S_i$, $\phi = \psi_1 \times \psi_2$ and $A \neq \phi(A)$. Then A is a ϕ-quasi-prime ideal of $S_1 \times S_2$ if and only if A is a quasi-prime ideal of $S_1 \times S_2$.*

Proof This follows from Theorem 26. □

From Theorem 27 we can easily obtain the following theorem.

Theorem 29 *Let S_i be an ordered $\mathcal{LA}$-semigroup with left identity and left zero and let $\psi_i : \mathcal{I}(S_i) \to \mathcal{I}(S_i) \cup \{\emptyset\}$ be a function such that $\psi_i(S_i) \neq S_i$, $\phi = \psi_1 \times \psi_2 \times \ldots \times \psi_{k\geq 2}$ and $A \neq \phi(A)$. Then A is a ϕ-quasi-prime ideal of $S_1 \times S_2 \times \ldots \times S_k$ if and only if A is a quasi-prime ideal of $S_1 \times S_2 \times \ldots \times S_k$.*

As a simple consequence of Theorem 6, we give the following result.

Lemma 1 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$*

be a function and let A be a left ideal of an ordered $\mathcal{LA}$-semigroup S that is not a quasi-prime ideal. If A is a ϕ-quasi-prime ideal of S such that $\phi \leq \phi_2$, then $A^2 = A^{n+1}$.

The next theorem gives conditions for a ϕ-quasi-prime ideal to be ω-quasi-prime ideal in a commutative semigroup.

Theorem 30 *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\phi \leq \phi_{n+1}$. Then A is a ϕ-quasi-prime ideal of S if and only if A is an ω-quasi-prime ideal of S.*

Proof First assume that A is a ϕ-quasi-prime ideal of S. If A is a quasi-prime ideal of S, then it is ω-quasi-prime ideal. Now assume that A is not a quasi-prime ideal of S. Then by Lemma 1, $A^2 = A^{n+1}$. By assumption, A is a ϕ-quasi-prime ideal of S and $\phi \leq \phi_{n+1}$, which implies that A is a ϕ_{n+1}-quasi-prime ideal of S. On the other hand, $\phi_\omega(A) = A^{n+1} = \phi_{n+1}(A)$. Therefore A is an ω-quasi-prime ideal of S.

Conversely, assume that A is a ϕ-quasi-prime ideal of S. The proof is trivial and hence omitted. □

4 Conclusion

In study the structure of ordered $\mathcal{LA}$-semigroups, we notice that the quasi-prime ideals with special properties always play an important role. The purposes of this paper are to introduce generalizations of quasi-prime ideals to the context of ϕ-quasi-prime ideals. Some characterizations of quasi-prime and ϕ-quasi-prime ideals are obtained. Moreover, we investigate relationships between weakly quasi-prime, almost quasi-prime, ω-quasi-prime, m-quasi-prime and ϕ-quasi-prime ideals of ordered $\mathcal{LA}$-semigroups. Finally, we obtain necessary and sufficient conditions of ϕ-quasi-prime ideal in order to be a quasi-prime ideal.

References

1. Abbasi, M.Y., Basar, A.: On generalizations of ideals in LA-Γ-semigroups. Southeast Asian Bull. Math. **39**, 1–12 (2015)
2. Abbasi, M.Y., Khan, S.A., Talee, A.F., Khan, A.: Soft interior-hyperideals in left regular LA-semihypergroups. Kragujevac J. Math. **44**(2), 217–236 (2020)
3. Abdullah, S., Aslam, S., Amin, N.: LA-semigroups characterized by the properties of interval valued (α, β)-fuzzy ideals. J. Appl. Math. Informatics **32**(3–4), 405–426 (2014)
4. Ali, A., Shi, F.G., Khan, M.: Characterizations of ordered AG-groupoids in terms of soft sets. Italian J. Pure Appl. Math. **35**, 415–432 (2015)
5. Amjad, V., Hila, K., Yousafzai, F.: Generalized hyperideals in locally associative left almost semihypergroups. N. Y. J. Math. **20**, 1063–1076 (2014)
6. Amjid, V., Yousafzai, F., Hila, K.: A study of ordered AG-groupoids in terms of semilattices via smallest (fuzzy) ideals. Adv. Fuzzy Syst. **2018**, 8 (2018)
7. Amjad, V., Yousafzai, F., Iampan, A.: On generalized fuzzy ideals of ordered LA-semigroups. Boletin de Mat. **22**(1), 1–19 (2015)
8. Anderson, D.D., Bataineh, M.: Generalizations of prime ideals. Commun. Algebra **36**(2), 686–696 (2015)
9. Davvaz, B., Khan, M., Anis, S., Haq, S.: Generalized fuzzy quasi-ideals of an intraregular Abel-Grassmann's groupoid. J. Appl. Math. **2012**, 16 (2012)
10. Dudek, W.A., Gigon, R.S.: Completely inverse AG**-groupoids. Semigroup Forum **87**, 201–229 (2013)
11. Faisal, A.K., Davvaz, B.: On fully regular AG-groupoids. Afr. Mat. **25**(2), 449–459 (2014)
12. Faisal, N.Y., Ghareeb, A.: Left regular AG-groupoids in terms of fuzzy interior ideals. Afr. Mat. **24**(4), 577–587 (2013)
13. Faisal, Y.N., Hila, K.: On fuzzy(2, 2)-regular ordered Γ-AG** -groupoids. UPB Sci. Bull. Ser. A **74**(2), 87–104 (2012)
14. Faisal, N.Y., Saeid, A.B.: Some results in bipolar-valued fuzzy ordered AG-groupoids. Discuss. Math. General Algebra Appl. **32**, 55–76 (2012)
15. Feng, F., Khan, M., Fotea, V.L., Anis, V., Ajaib, N.: Fuzzy soft set approach to ideal theory of regular AG-groupoids. An. St. Univ. Ovidius Constanta **24**(1), 263–288 (2016)
16. Iqbal, M., Ahmad, I.: Ideals in CA-AG-groupoids. Indian J. Pure Appl. Math. **49**(2), 265–284 (2018)
17. Khan, M., Anis, S.: On semilattice decomposition of an Abel-Grassmann's Groupoid. Acta Math. Sin. Engl. Ser. **28**(7), 1461–1468 (2012)
18. Khan, A., Faisal, W.K., Yaqoob, N.: Ordered LA-semigroups in terms of interval valued fuzzy ideals. J. Adv. Res. Pure Math. **5**(1), 100–117 (2013)
19. Khan, A., Farooq, M., Izhar, M., Davvaz, B.: Fuzzy hyperideals of left almost semihypergroups. Int. J. Anal. Appl. **15**(2), 155–171 (2018)
20. Khan, M., Jun, Y.B., Yousafzai, F.: Fuzzy ideals in right regular LA-semigroups. Hacettepe J. Math. Stat. **44**(3), 569–586 (2015)
21. Khan, F.M., Khan, H.U., Mukhtar, S., Khan, A., Sarmin, N.H.: Some innovative types of fuzzy ideals in AG-groupoids. J. Intell. Syst. **1–19**, 5 (2013)
22. Khan, M., Mushtaq, Q., Anis, S.: A note on Abel-Grassmann's groupoids. Res. J. Appl. Sci. Eng. Technol. **7**(9), 1705–1709 (2014)
23. Khan, M., Shum, K.P., Iqbal, M.F.: On minimal ideals of AG**-groupoids. Southeast Asian Bull. Math. **38**, 389–399 (2014)
24. Khan, W., Yousafzai, F., Khan, A.: On ordered Abel-Grassmann's groupoids. Probl. Phys. Math. Technics **2**(23), 40–47 (2015)
25. Khan, W., Yousafzai, F., Khan, M.: On generalized ideals of left almost semigroups. Eur. J. Pure Appl. Math. **9**(3), 277–291 (2016)
26. Mushtaq, Q.: Zeroids and idempoids in AG-groupoids. Quasigroups Relat. Syst. **11**, 79–84 (2004)

27. Mushtaq, Q., Khan, M.: Decomposition of AG*-groupoids. Quasigroups Relat. Syst. **15**, 303–308 (2007)
28. Mushtaq, Q., Khan, M.: Ideals in AG-band and AG*-groupoid. Quasigroups Relat. Syst. **14**, 207–215 (2006)
29. Rehman, N., Park, C., Shah, S.I.A., Ali, A.: On generalized roughness in LA-semigroups. Mathematics **6**(7), 1–8 (2018)
30. Rehman, N., Shah, N., Ali, M.I., Ali, A.: Generalised roughness in $(\in, \in \vee q)$-fuzzy substructures of LA-semigroups. J. Natn. Sci. Found. Sri Lanka **46**(3), 465–473 (2018)
31. Sezer, A.S.: A new approach to LA-semigroup theory via the soft sets. J. Intell. Fuzzy Syst. **26**(5), 2483–2495 (2014)
32. Sezgin, A.: A new view on AG-groupoid theory via soft sets for uncertainty modeling. Filomat **32**(8), 2995–3030 (2018)
33. Shah, T., Kausar, N.: Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals. Theor. Comput. Sci. **529**, 96–110 (2014)
34. Shah, T., Rehman, I.: Decomposition of locally associative Γ-AG-groupoids. Novi Sad. J. Math. **43**(1), 1–8 (2013)
35. Shah, T., Rehman, I., Chinram, R.: On M-systems in ordered AG-groupoids. Far East J. Math. Sci. **47**(1), 13–21 (2010)
36. Shah, T., Rehman, I., Khan, A.: Fuzzy Γ-ideals in Γ-AG-groupoids. Hacettepe J. Math. Stat. **43**(4), 625–634 (2014)
37. Stevanovic, N., Proti, P.V.: Composition of Abel-Grassmann's 3-bands. Novi Sad J. Math. **34**(2), 175–182 (2004)
38. Yaqoob, N., Chinram, R., Ghareeb, A., Aslam, M.: Left almost semigroups characterized by their interval valued fuzzy ideals. Afr. Mat. **24**(2), 231–245 (2013)
39. Yaqoob, N., Corsini, P., Yousafzai, F.: On intra-regular left almost semihypergroups with pure left identity. J. Math. **2013**, 10 (2013)
40. Yiarayong, P.: On semiprime and quasi semiprime ideals in AG-groupoids. KKU Res. J. **18**(6), 893–896 (2013)
41. Yiarayong, P.: On product of fuzzy semiprime ideals in Γ-LA-semigroups. GU J. Sci. **29**(2), 491–502 (2016)
42. Yiarayong, P.: On prime and semiprime ideals in ordered AG-groupoids. Iran. J. Math. Sci. Informatics **12**(1), 1–11 (2017)
43. Younas, I., Mushtaq, Q.: On left permutable inverse LA-semigroups. Maejo Int. J. Sci. Technol. **13**(01), 10–18 (2019)
44. Yousafzai, F., Asif, T., Khan, A., Davvaz, B.: A study of non-associative ordered semigroups in terms of semilattices via smallest (double-framed soft) ideals. Int. J. Anal. Appl. **16**(4), 484–502 (2018)
45. Yousafzai, F., Iampan, A., Tang, J.: Study on smallest (fuzzy) ideals of LA-semigroups. Thai J. Math. **16**(2), 549–561 (2018)
46. Yousafzai, F., Khan, A., Amjad, V., Zeb, A.: On fuzzy fully regular ordered AG-groupoids. J. Intell. Fuzzy Syst. **26**, 2973–2982 (2014)
47. Yousafzai, F., Khan, A., Khan, W., Aziz, T.: $(\in, \in \vee q_k)$-fuzzy ideals in left regular ordered LA-semigroups. Afr. Mat. **35**(4), 583–606 (2013)
48. Yousafzai, F., Khan, W., Khan, A., Hila, K.: I-V fuzzy set theory applied on po-LA-semigroups. Afr. Mat. **27**(1–2), 23–35 (2016)
49. Yousafzai, F., Khan, A., Iampan, A.: On (m, n)-ideals of an ordered Abel-Grassmann groupoid. Korean J. Math. **23**(3), 357–370 (2015)
50. Yousafzai, F., Yaqoob, N., Haq, S., Manzoor, R.: A note on intuitionistic fuzzy Γ-LA-semigroups. World Appl. Sci. J. **19**(12), 1710–1720 (2012)
51. Yousafzai, F., Yaqoob, N., Zeb, A.: On generalized fuzzy ideals of ordered AG-groupoids. Int. J. Mach. Learn. Cyber. **7**(6), 995–1004 (2016)