Beitr Algebra Geom (2020) 61:151–155
https://doi.org/10.1007/s13366-019-00455-w

ORIGINAL ARTICLE

Equilateral sets in the ℓ_1 sum of Euclidean spaces

Aaron Lin[1]

Received: 16 May 2019 / Accepted: 4 June 2019 / Published online: 11 June 2019

Abstract
Let E^n denote the (real) n-dimensional Euclidean space. It is not known whether an equilateral set in the ℓ_1 sum of E^a and E^b, denoted here as $E^a \oplus_1 E^b$, has maximum size at least $\dim(E^a \oplus_1 E^b) + 1 = a + b + 1$ for all pairs of a and b. We show, via some explicit constructions of equilateral sets, that this holds for all $a \leqslant 27$, as well as some other instances.

Keywords Equilateral sets · Normed spaces · Regular simplices

Mathematics Subject Classification 46B20 · 52A21 · 52C10

1 The problem

An equilateral set in a normed space $(X, \|\cdot\|)$ is a subset $S \subset X$ such that for all distinct $x, y \in S$, we have $\|x - y\| = \lambda$ for some fixed λ. Since X is a normed space, the maximum size of an equilateral set in X is independent of λ, and we denote it by $e(X)$. When $\dim(X) = n$, we have the tight upper bound $e(X) \leqslant 2^n$, proved by Petty (1971) nearly 50 years ago. However, the following conjecture concerning a lower bound on $e(X)$, formulated also by Petty (amongst others), remains open for $n \geqslant 5$. (The $n = 2$ case is easy; see Petty (1971) and Väisälä (2012) for the $n = 3$ case, and Makeev (2005) for the $n = 4$ case.)

Conjecture 1 *Let X be an n-dimensional normed space. Then $e(X) \geqslant n + 1$.*

We wish to verify this conjecture for the Cartesian product $\mathbb{R}^a \times \mathbb{R}^b$, equipped with the norm $\|\cdot\|$ given by

$$\|(x, y)\| = \|x\|_2 + \|y\|_2,$$

where $x \in \mathbb{R}^a$, $y \in \mathbb{R}^b$, and $\|\cdot\|_2$ denotes the Euclidean norm. We denote this space by $E^a \oplus_1 E^b$, and refer to it as the ℓ_1 sum of the Euclidean spaces E^a and E^b. This

✉ Aaron Lin
aaronlinhk@gmail.com

1 Department of Mathematics, The London School of Economics and Political Science, London, UK

was considered originally by Roman Karasev of the Moscow Institute of Physics and Technology, as a possible counterexample to Conjecture 1. See Swanepoel (2016, Section 3) for more background on equilateral sets.

2 The results

Observe that we need only construct $a+b+1$ points in $E^a \oplus_1 E^b$ which form an equilateral set to show that $e(E^a \oplus_1 E^b) \geqslant \dim(E^a \oplus_1 E^b) + 1 = a+b+1$. We will work with these points in the form $(x_i, y_i) \in \mathbb{R}^a \times \mathbb{R}^b$, since we can then examine the x_i's and y_i's separately when necessary. By abuse of notation, we will denote the origin of any Euclidean space by o.

Let d_n denote the circumradius of a regular n-simplex ($n \geqslant 1$) with unit side length. Note that

$$d_n = \left(\sqrt{2 + \frac{2}{n}}\right)^{-1}$$

is a strictly increasing function of n, and we have $1/2 \leqslant d_n < 1/\sqrt{2}$.

The $a = 1$ case is easy.

Proposition 2 $e(E^1 \oplus_1 E^b) \geqslant b + 2$.

Proof Let $y_1, \ldots, y_{b+1}$ be the vertices of a regular b-simplex with unit side length centred on the origin. Then the points $(o, y_1), \ldots, (o, y_{b+1}), (1 - d_b, o)$ are pairwise equidistant. □

We next deal with the case where $b = a$.

Proposition 3 $e(E^a \oplus_1 E^a) \geqslant 2a + 1$.

Proof We first describe an equilateral set of size $2a$ in $E^a \oplus_1 E^a$: consider the set of points $\{(v_i, \frac{1}{2}e_i) : i = 1, \ldots, a\} \cup \{(v_i, -\frac{1}{2}e_i) : i = 1, \ldots, a\}$, where $v_1, \ldots, v_a$ are the vertices of a regular simplex of codimension one, centred on the origin with side length $1 - 1/\sqrt{2}$, and $e_1, \ldots, e_a$ are the standard basis vectors. Note that the $2a$ vectors $\pm\frac{1}{2}e_i$ for $i = 1, \ldots, a$ form a cross-polytope in E^a, centred on the origin.

We now want to add a point of the form (x, o) to the above set, a unit distance away from every other point. Note that we must have $\|x - v_i\|_2 = 1/2$ for $i = 1, \ldots, a$, and x must lie on the one-dimensional subspace orthogonal to the $(a-1)$-dimensional subspace spanned by the v_i's. This is realisable if $\|x - v_i\|_2 \geqslant (1 - 1/\sqrt{2})d_{a-1}$ (note that the $(a-1)$-simplex formed by the v_i's has side length $1 - 1/\sqrt{2}$), in which case we have an equilateral set of size $2a + 1$ in $E^a \oplus_1 E^a$. But we have

$$\frac{1}{2} > \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\right) > \left(1 - \frac{1}{\sqrt{2}}\right) d_{a-1}$$

for all $a \geqslant 2$. □

In the remaining case and our main result, we have $b > a \geqslant 2$, and we find sufficient conditions for an equilateral set of size $a + b + 1$ to exist in $E^a \oplus_1 E^b$.

Theorem 4 *Let $b > a \geqslant 2$. Write $b = (c-1)(a+1) + \beta = c(a+1) - \alpha$ with $\beta \in \{0, \dots, a\}$ and $\alpha \in \{1, \dots, a+1\}$. If either of the conditions $\beta = 0$, $\beta = 1$, $\beta = a$ is satisfied, or the inequality*

$$\frac{\alpha-1}{2\alpha}\left(1-\sqrt{\frac{c-1}{c}}\right)^2 + \frac{\beta-1}{2\beta}\left(1-\sqrt{\frac{c}{c+1}}\right)^2 \leqslant \left(1-\sqrt{\frac{1}{2}\left(\frac{c-1}{c}+\frac{c}{c+1}\right)}\right)^2 \tag{1}$$

holds, then $e(E^a \oplus_1 E^b) \geqslant a+b+1$.

Note that if inequality (1) is satisfied by all pairs of a and b with $b > a \geqslant 2$ and $b \neq 0, 1$, or $a \pmod{a+1}$, then Proposition 2, Proposition 3, and Theorem 4 cover all possible cases, as $E^a \oplus_1 E^b$ is isometrically isomorphic to $E^b \oplus_1 E^a$. Unfortunately, this is not true, and we explore its limitations after the proof of Theorem 4.

Proof of Theorem 4 We are going to describe an equilateral set of size $a+b+1$ with unit distances between points. Noting that $\alpha \cdot (c-1) + \beta \cdot c = b$, consider the following decomposition of E^b into pairwise orthogonal subspaces:

$$E^b = U_1 \oplus \cdots U_\alpha \oplus V_1 \oplus \cdots \oplus V_\beta,$$

where $\dim U_i = c-1$ for $i = 1, \dots, \alpha$ and $\dim V_j = c$ for $j = 1, \dots, \beta$. Let $u_1^{(i)}, \dots, u_c^{(i)}$ be the vertices of a regular $(c-1)$-simplex with unit side length centred on the origin in U_i, and let $v_1^{(j)}, \dots, v_{c+1}^{(j)}$ be the vertices of a regular c-simplex with unit side length centred on the origin in V_j.

The $a+b+1$ points of our equilateral set will be

$$\left\{\left(w_i, u_k^{(i)}\right) : 1 \leqslant i \leqslant \alpha, 1 \leqslant k \leqslant c\right\} \cup \left\{\left(z_j, v_\ell^{(j)}\right) : 1 \leqslant j \leqslant \beta, 1 \leqslant \ell \leqslant c+1\right\}.$$

Note here that $\alpha \cdot c + \beta \cdot (c+1) = a+b+1$, and we have $\|u_k^{(i)} - u_{k'}^{(i)}\|_2 = \|v_\ell^{(j)} - u_{\ell'}^{(j)}\|_2 = 1$ for $k \neq k'$ and $\ell \neq \ell'$. All that remains is then to calculate how far apart the w_i's and z_j's should be in E^a, and see if such a configuration is realisable.

We only have three non-trivial distances to calculate:

- the distance between $\left(z_j, v_\ell^{(j)}\right)$ and $\left(z_{j'}, v_{\ell'}^{(j')}\right)$ for $j \neq j'$ should be one, and so

$$\|z_j - z_{j'}\|_2 = 1 - \sqrt{d_c^2 + d_c^2} = 1 - \sqrt{\frac{c}{c+1}} =: f(c),$$

- the distance between $\left(w_i, u_k^{(i)}\right)$ and $\left(w_{i'}, u_{k'}^{(i')}\right)$ for $i \neq i'$ should be one, and so

$$\|w_i - w_{i'}\|_2 = 1 - \sqrt{d_{c-1}^2 + d_{c-1}^2} = 1 - \sqrt{\frac{c-1}{c}} = f(c-1),$$

- finally, the distance between $\left(w_i, u_k^{(i)}\right)$ and $\left(z_j, v_\ell^{(j)}\right)$ should also be one, and so

$$\|w_i - z_j\|_2 = 1 - \sqrt{d_{c-1}^2 + d_c^2} = 1 - \sqrt{\frac{1}{2}\left(\frac{c-1}{c} + \frac{c}{c+1}\right)} =: g(c).$$

What we need in E^a is thus a regular $(\alpha - 1)$-simplex with side length $f(c-1)$ and a regular $(\beta - 1)$-simplex with side length $f(c)$, with the distance between any point from one simplex and any point from the other being $g(c)$. Note that here we consider the (-1)-simplex to be empty. We now show that this configuration is realisable (in E^a) if the conditions in the statement of the theorem are satisfied.

We first consider the special cases $\beta = 0$ and $\beta = 1$ or a, and then the main case $2 \leqslant \beta \leqslant a-1$. It is trivial if $\beta = 0$: then $\alpha = a+1$ and we only need to find a regular a-simplex with side length $f(c-1)$ in E^a.

If $\beta = 1$, in which case $\alpha = a$, consider the decomposition $E^a = E^{a-1} \oplus E^1$. Consider the points $(p_1, o), \ldots, (p_a, o)$, where $p_1, \ldots, p_a$ are the vertices of a regular $(a-1)$-simplex with side length $f(c-1)$, centred on the origin in E^{a-1}. We want to add a point (o, ζ) for some $\zeta \in E^1$ such that, for any $i = 1, \ldots, a$, we have

$$\|(p_i, o) - (o, \zeta)\|_2 = g(c),$$

or equivalently,

$$d_{a-1}^2 f(c-1)^2 + \zeta^2 = g(c)^2.$$

Noting that $d_{a-1} < 1/\sqrt{2}$, it suffices to show, for all $c \geqslant 2$, that

$$f(c-1)^2 < 2g(c)^2.$$

But this is easily verifiable to be true, and so the desired a-simplex exists in E^a. By symmetry and the fact that $f(c)^2 < f(c-1)^2$, the desired a-simplex also exists if $\beta = a$.

Now suppose $2 \leqslant \beta \leqslant a-1$ so that $\alpha, \beta \geqslant 2$. Consider this time, the decomposition $E^a = E^{\alpha-1} \oplus E^{\beta-1} \oplus E^1$, noting that $\alpha + \beta = a + 1$. Suppose $p_1, \ldots, p_\alpha$ are the vertices of a regular $(\alpha - 1)$-simplex with side length $f(c-1)$, centred on the origin in $E^{\alpha-1}$, and $q_1, \ldots, q_\beta$ are the vertices of a regular $(\beta - 1)$-simplex with side length $f(c)$, centred on the origin in $E^{\beta-1}$. Consider then the set of points $\{(p_i, o, o) : i = 1, \ldots, \alpha\} \cup \{(o, q_j, \zeta) : j = 1, \ldots, \beta\}$, where $\zeta \in E^1$ is to be determined. As before, we want a ζ such that for all i and j, we have

$$\|(p_i, o, o) - (o, q_j, \zeta)\|_2 = g(c),$$

or equivalently

$$(d_{\alpha-1} f(c-1))^2 + \left(d_{\beta-1} f(c)\right)^2 \leqslant g(c)^2. \tag{2}$$

But this is exactly inequality (1). □

As mentioned above, inequality (1), and thus inequality (2), does not hold for all pairs of a and b. However, we have the following result.

Lemma 5 *If $b \geqslant a^2 + a$, then inequality (2) holds.*

Proof Since $f(n)$ is a decreasing function of n, inequality (2) holds if a and b satisfy

$$\left(d_{\alpha-1}^2 + d_{\beta-1}^2\right) f(c-1)^2 < g(c)^2.$$

Using the fact that $\alpha = a+1-\beta$ implies $d_{\alpha-1}^2 + d_{\beta-1}^2 \leqslant (a-1)/(a+1)$, we therefore just need a and b to satisfy

$$\frac{a-1}{a+1} < \left(\frac{g(c)}{f(c-1)}\right)^2.$$

But the latter expression is an increasing function of c, and so if $c \geqslant a$, or equivalently, when $b \geqslant a^2 + a$, we need only consider the inequality

$$\frac{a-1}{a+1} < \left(\frac{g(a)}{f(a-1)}\right)^2,$$

which is then easily verifiable to be true. □

It can be checked (by computer) that inequality (2) holds for all $a \leqslant 27$, but does not hold for $a = 28$ and $b = 40$, $a = 29$ and $39 \leqslant b \leqslant 44$, and $a = 30$ and $40 \leqslant b \leqslant 47$. The spaces of smallest dimension where we could not find an equilateral set of size $a + b + 1$ are $E^{28} \oplus_1 E^{40}$ and $E^{29} \oplus_1 E^{39}$.

Acknowledgements The author would like to thank Konrad Swanepoel for introducing him to this problem, and for the numerous helpful suggestions in writing this up.

References

Makeev, V.V.: Equilateral simplices in a four-dimensional normed space. Zap. Nauchn. Sem. St. Peterburg. Otdel. Mat. Inst. Steklov. (POMI) Geom. I Topol **329**(9), 88–91, 197 (2005)

Petty, C.M.: Equilateral sets in Minkowski spaces. Proc. Am. Math. Soc. **29**, 369–374 (1971)

Swanepoel, K.: Combinatorial distance geometry in normed spaces. arXiv:1702.00066

Väisälä, J.: Regular simplices in three-dimensional normed spaces. Beitr. Algebra Geom. **53**(2), 569–570 (2012)