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On the Lower Bound of the Inner Radius of Nodal Domains

Bogdan Georgiev[1]

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Abstract
We discuss the asymptotic lower bound on the inner radius of nodal domains that arise from Laplacian eigenfunctions φ_λ on a closed Riemannian manifold (M, g). In the real-analytic case, we present an improvement of the currently best-known bounds, due to Mangoubi (Commun Partial Differ Equ 33:1611–1621, 2008; Can Math Bull 51(2):249–260, 2008). Furthermore, using recent results of Hezari (P Am Math Soc, 2016, https://doi.org/10.1090/proc/13766; Anal PDE 11(4):855–871, 2018), we obtain log-type improvements in the case of negative curvature and improved bounds for (M, g) possessing an ergodic geodesic flow.

Keywords Laplace eigenfunctions · Nodal domains · Inner radius · Concentration of eigenfunctions

Mathematics Subject Classification 35P20 · 58J50 · 58J05

1 Introduction

Let M be a closed Riemannian manifold of dimension $n \geq 3$ with metric g and denote by φ_λ an eigenfunction of the Laplacian Δ of M, corresponding to the eigenvalue λ. We are interested in the geometry of nodal domains in the high-energy limit, i.e. as $\lambda \to \infty$. For a readable and far-reaching survey, we refer to [13] and [14].

By a result of Dan Mangoubi [10] (see also [11]), it is known that for a nodal domain Ω_λ, corresponding to φ_λ, the following asymptotic estimate holds:

$$\frac{c_1}{\lambda^{\frac{n-1}{4}+\frac{1}{2n}}} \leq \operatorname{inrad}(\Omega_\lambda) \leq \frac{c_2}{\sqrt{\lambda}}, \tag{1}$$

where $c_{1,2}$ depend on (M, g) and where $\operatorname{inrad}(\Omega)$ denotes the inner radius of Ω, i.e. the radius of the largest geodesic ball fully contained in Ω.

✉ Bogdan Georgiev
bogeor@mpim-bonn.mpg.de

1 Max-Planck-Institut fur Mathematik, Bonn, Nordrhein-Westfalen, Germany

In particular, the asymptotic estimates are sharp in the case of a Riemannian surface, i.e. the inner radius of a nodal domain is comparable to the wavelength $\frac{1}{\sqrt{\lambda}}$. A natural question is whether the mentioned lower bound is optimal also for higher dimensions.

We present the following improvement in the real-analytic case.

Theorem 1.1 *Let (M, g) be a real-analytic closed manifold of dimension at least 3. Let φ_λ be an eigenfunction of the Laplace operator Δ and Ω_λ be a nodal domain of φ_λ. Then, there exists $r > 0$ and a ball $B_{\frac{r}{\sqrt{\lambda}}} \subset M$ of radius $\frac{r}{\sqrt{\lambda}}$ with the following property: An initially given proportion (say, 10%) of $\mathrm{Vol}(\Omega_\lambda \cap B_{\frac{r}{\sqrt{\lambda}}})$ is occupied by a collection of inscribed balls $\{B^i_{c_1\lambda^{-1}}\}_{i=0}^{i_0}$, $B^i_{c_1\lambda^{-1}} \subset \Omega_\lambda \cap B_{\frac{r}{\sqrt{\lambda}}}$ of radius $c_1\lambda^{-1}$, where $c_1 = c_1(M, g)$.*

In particular, there exist constants c_1 and c_2 which depend only on (M, g), such that

$$\frac{c_1}{\lambda} \leq \mathrm{inrad}(\Omega_\lambda) \leq \frac{c_2}{\sqrt{\lambda}} \tag{2}$$

We note that Theorem 1.1 improves Mangoubi's estimates for dimensions $n \geq 5$. Moreover, we remark that the initially given proportion of inscribed balls is referred to as 10% only for the ease of presentation. In fact, one has the freedom to select it; however, the constants r, c_1 will be different. As this is not crucial for our present discussion, we do not pursue the investigation of the precise relation between the constants in this note.

Further, the present lower bound on the inner radius appears to be unoptimal, and it seems that a combinatorial argument can lead to a further improvement. This is also reasonable in the smooth setting, having in mind the recent progress on Yau's conjecture (cf. [9]). We address these in a forthcoming note.

1.1 Outline

We present the proof of Theorem 1.1 in Sect. 2.

Roughly speaking, the argument consists of two ingredients.

First, we observe that one can almost inscribe a wavelength ball in the nodal domain up to a small in volume error set. In fact, a well-known result due to Lieb [8] states that for arbitrary domains Ω in $\mathbb{R}^n$, one can find almost inscribed balls of radius $\frac{1}{\sqrt{\lambda_1(\Omega)}}$. Furthermore, we refer to [12] for a result in this spirit stated in terms of capacities.

Second, one would like to somehow rule out the error set that may enter in the almost inscribed ball near a point of maximum $x_0 \in \Omega_\lambda$. One way to argue is as follows. Being in the real-analytic setting, eigenfunctions resemble polynomials of degree $\sqrt{\lambda}$. This observation was utilized in the works of Donnelly–Fefferman ([2]) and Jakobson–Mangoubi ([7]). What is more, if one takes the unit cube and subdivide it into wavelength-sized small cubes, then these polynomials will be close to their average on most of the small cubes. This implies that the growth of eigenfunctions is controlled on most wavelength-sized smaller cubes. Now, roughly speaking, we start from a wavelength cube at x_0 and rescale to the unit cube I^n. Further, I^n is subdivided into wavelength cubes Q^ν, and hence, most of them will be good. But, if the error set intersects the majority of Q^ν deeply it will gain sufficient volume to contradict the volume decay of the first step. This means that there is a sufficient proportion of the Q^{ν} which is not deeply intersected by the error set.

Finally, utilizing some recent results of Hezari [4], we get that, if one assumes in addition that (M, g) is negatively curved, then the inradius improves by a factor of $\log \lambda$. A similar argument works also for (M, g) with ergodic geodesic flow.

2 The Lower Bound on the Inner Radius of Nodal Domains

We first gather the necessary preliminary statements (Sects. 2.1, 2.2) and then introduce the proof of Theorem 1.1 in Sect. 2.3.

2.1 Existence of an "Almost" Inscribed Ball

We start with the following observation, which does not require the real-analyticity of (M, g).

Proposition 2.1 *Let Ω_λ be a nodal domain as above and let $\epsilon > 0$ be a given small number. There exists $r = r(\epsilon)$ and a point x_0, such that*

$$\frac{\operatorname{Vol}\left(B_{\frac{r}{\sqrt{\lambda}}}(x_0) \cap \Omega_\lambda\right)}{\operatorname{Vol}\left(B_{\frac{r}{\sqrt{\lambda}}}(x_0)\right)} > 1 - \epsilon. \tag{3}$$

We refer to the ball $B_{\frac{r}{\sqrt{\lambda}}}(x_0)$ as an almost inscribed ball into the nodal domain Ω_λ.

The statement from Proposition 2.1 is inferred from Corollary 2 [8] and a partition of unity argument.

2.2 A Toolbox of a Few Technical Lemmas Concerning "Good" Cubes

We consider the case of a real analytic manifold (M, g) of dimension at least 3.

As our present discussion is focused on (M, g) being a real-analytic manifold, let us first attempt to briefly motivate the role of real-analyticity towards eigenfunctions and their nodal geometry.

As the eigenequation possesses real-analytic coefficients, a main insight in this situation is that polynomials approximate eigenfunctions sufficiently well, i.e. an eigenfunction φ_λ exhibits a behaviour, which is similar to that of a polynomial of degree $\sqrt{\lambda}$. The analogy exhibits itself when it comes to local growth, vanishing orders at the zero set, etc. A celebrated work of Donnelly–Fefferman [2], addressing Yau's conjecture for nodal sets, is a vivid example of these heuristics (cf. also [7]).

On the other hand, if (M, g) is assumed to be only smooth, then formal results mimicking certain facts of real-analytic case (Lemmas 2.2, 2.5, for instance) are still not known. Roughly, the difficulty arises from the lack of good polynomial approximation and appropriate holomorphic extensions.

Now let us start describing the real-analytic tools that we will need: we make use of four auxiliary Lemmas (2.2, 2.4, 2.5 and 2.6), which are explicitly stated below. The Lemmas originate from [2] and [7].

First, we have the following:

Lemma 2.2 *Let (M, g) be real-analytic and let us take a sufficiently small number $r > 0$ (to be determined later), and consider an arbitrary ball $B_{\frac{r}{\sqrt{\lambda}}}$ of radius $\frac{r}{\sqrt{\lambda}}$. Furthermore, rescale the ball $B_{\frac{r}{\sqrt{\lambda}}}$ to the unit ball $B_1 \subset \mathbb{R}^n$ and denote the corresponding rescaled eigenfunction on the unit ball by $\varphi_\lambda^{\mathrm{loc}}$. There exists a cube $Q \subseteq B_1$, which does not depend on $\varphi_\lambda^{\mathrm{loc}}$ and λ, and has the following property: suppose $\delta > 0$ is taken, so that $\delta \le \frac{C_1}{\sqrt{\lambda}}$. We decompose Q into smaller cubes $\{Q_s^\nu\}_\nu$ with sides of size $s \in (\delta, 2\delta)$. Then, for a small number $\epsilon > 0$, there exists a subset $E_\epsilon \subseteq Q$ of measure $|E_\epsilon| \le C_2 \epsilon \sqrt{\lambda} \delta$, so that*

$$\frac{1}{C_3(\epsilon)} \le \frac{\left(\varphi_\lambda^{\mathrm{loc}}(x)\right)^2}{Av_{(Q_\nu)_x}\left(\varphi_\lambda^{\mathrm{loc}}\right)^2} \le C_3(\epsilon), \quad \forall x \in Q \backslash E_\epsilon, \tag{4}$$

with $C_3(\epsilon) \to \infty$ as $\epsilon \to 0$. The constants C_1, C_2, C_3 do not depend on φ_λ and λ. The notation $Av_{(Q_\nu)_x} F$ denotes the average of F over a cube Q_s^ν which contains x.

We first remark that Lemma 2.2 is a direct adaptation of Proposition 4.1 [7], where instead of working in a wavelength ball $B_{\frac{r}{\sqrt{\lambda}}}$ (identified with B_1 as above), Jakobson and Mangoubi are working on an arbitrary small open set V (again identified with a ball) in which the metric can be expanded in power series. A further remark is that rescaling back to the manifold, the cube Q, which is prescribed by the Lemma, is identified with a small wavelength cube inside $B_{\frac{r}{\sqrt{\lambda}}}$, side of which is comparable to $\frac{r}{\sqrt{\lambda}}$ and that the cubes $\{Q_s^\nu\}_\nu$ are identified to even smaller subcubes of size comparable to λ^{-1}.

Now, let us briefly sketch the arguments behind Lemma 2.2.

Proof of Lemma 2.2 As already stated above, we essentially follow Proposition 4.1 [7].

First, we observe that $\varphi_\lambda^{\mathrm{loc}}|_{B_1}$ has an analytic continuation F on a complex ball $B^{\mathbb{C}}(0, \rho_1)$ (complex balls will be denoted by an upper index $\mathbb{C}$) for some $\rho_1 < 1$, and moreover the function F is bounded as follows:

$$\sup_{B^{\mathbb{C}}(0,\rho_1)} |F| \le \mathrm{e}^{C\sqrt{\lambda}} \sup_{B_1} |\varphi_\lambda^{\mathrm{loc}}|. \tag{5}$$

We observe that the size ρ_1 does not depend on λ (Lemma 7.1 [2], where one uses the fact that on a wavelength scale, $\varphi_\lambda^{\mathrm{loc}}$ is almost harmonic, i.e. it is a solution to slight perturbation of the standard Laplace equation).

Now, we select a fixed $\rho_2 = \rho_2(\rho_1)$ such that the polydisc $B_{2\rho_2}^n := D_{2\rho_2} \times \cdots \times D_2 \subseteq B^{\mathbb{C}}(0, \rho_1) \subset \mathbb{C}^n$. The well-known Donnelly–Fefferman growth bound (cf. [2]) gives that

$$\sup_{B_1} |\varphi_\lambda^{\mathrm{loc}}| \leq \mathrm{e}^{C\sqrt{\lambda}/\rho_2} \sup_{B(0,\rho_2)} |\varphi_\lambda^{\mathrm{loc}}|. \tag{6}$$

In particular, we obtain

$$\sup_{B^n_{2\rho_2}} |F| \leq \mathrm{e}^{C\sqrt{\lambda}} \sup_{B(0,\rho_2)} |\varphi_\lambda^{\mathrm{loc}}|. \tag{7}$$

By shifting the coordinate system to a point $x \in B(0, \rho_2)$ such that $\varphi_\lambda^{\mathrm{loc}}(x) = \sup_{B(0,\rho_2)} |\varphi_\lambda^{\mathrm{loc}}|$, we have

$$\sup_{B^n_{\rho_2}} |F| \leq \mathrm{e}^{C\sqrt{\lambda}} |F(0)|. \tag{8}$$

We now invoke Proposition 3.7 [7], applied to the function F^2, thus inferring Lemma 2.2. □

We now address the notion of "good" cubes.

Let us take the cube Q prescribed by Lemma 2.2 and subdivide it into small cubes Q_s^ν for which the statement of the Lemma holds.

Definition 2.3 Q_s^ν is called E_ϵ-*good*, if

$$\frac{|E_\epsilon \cap Q_s^\nu|}{|Q_s^\nu|} < 10^{-2n} \omega_n, \tag{9}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Otherwise, Q_s^ν is E_ϵ-*bad*.

It turns out that the E_ϵ-good cubes Q_s^ν are characterized also as places where the eigenfunction possesses controlled growth (cf. also Lemma 5.3 [7]). We have

Lemma 2.4 *Let Q_s^ν be an E_ϵ-good cube. Let $B \subseteq 2B \subseteq Q_s^\nu$ be a ball centered somewhere in $\frac{1}{2} Q_s^\nu$, size of which is comparable to the size of Q_s^ν. Then*

$$\frac{\int_{2B} (\varphi_\lambda^{\mathrm{loc}})^2}{\int_B (\varphi_\lambda^{\mathrm{loc}})^2} \leq \tilde{C}_1 C_3(\epsilon), \tag{10}$$

where $C_3(\epsilon)$ comes from Lemma 2.2 and $\tilde{C}_1$ depends only on the dimension n.

Lemma 2.5 *The proportion of bad cubes to all cubes is smaller than $\tilde{C}_2 |E_\epsilon|$, where $\tilde{C}_2$ depends only on the dimension.*

Finally, let us recall a reason why the good cubes of bounded growth are important from the point of view of nodal geometry. We have

Lemma 2.6 *Suppose that a cube Q_s^ν from the collection above is good and suppose that φ_λ vanishes somewhere in $\frac{1}{2} Q_s^\nu$ (here $\frac{1}{2} Q$ denotes a concentric cube of half-sized side length). Then assuming that λ is sufficiently large, one has*

$$\frac{\mathrm{Vol}(\{\varphi_\lambda > 0\} \cap Q_s^\nu)}{\mathrm{Vol}(Q_s^\nu)} \geq C, \tag{11}$$

where C depends on $n, \rho, (M, g)$, as well as the control on the doubling number, that is $\tilde{C}_1 C_3(\epsilon)$ from Lemma 2.4. The same statement holds for the negativity set.

A proof of the last Lemma 2.6 for Q_s^ν replaced by a small ball can be found, for example, in Proposition 1 [1]. An adaptation for cubes is yielded by essentially following the same argument and using that at small scales

$$B_{\frac{r}{4}}(p) \subseteq Q_r(p) \subseteq B_{\sqrt{n}r}(p), \tag{12}$$

where $Q_r(p)$, $B_r(p)$ denote a cube, resp. a ball, of size r and centered at a point p.

2.3 Proof of Theorem 1.1

We now put all of the tools above together and prove our main result.

Proof of Theorem 1.1 Let us assume without loss of generality that φ_λ is positive on Ω_λ.

First, let $\epsilon > 0$ be a sufficiently small number to be determined below and let us invoke Proposition 2.1 with this ϵ in order to find $r = r(\epsilon)$ and an almost inscribed ball $B_{\frac{r}{\sqrt{\lambda}}}(x_0)$.

Further, we apply the machinery outlined in Sect. 2.2 inside the ball $B_{\frac{r}{\sqrt{\lambda}}}(x_0)$. More precisely, by Lemmas 2.2, 2.4 and 2.5, we can find a cube $Q_{\frac{r_1}{\sqrt{\lambda}}} \subset B_{\frac{r}{\sqrt{\lambda}}}(x_0)$ of comparable side length $\frac{r_1}{\sqrt{\lambda}}$ which, using the above notation, is subdivided into a collection $\mathcal{Q} = \{Q^\nu_{c\lambda^{-1}}\}_\nu$ of cubes of side length $c\lambda^{-1}$. For these, we know that there is a subset $\mathcal{Q}^g \subseteq \mathcal{Q}$ of E_ϵ-good cubes that consists of a large proportion (say, at least 90%) of all of the small cubes.

Now, let us define the error set (or "spike") $S := B_{\frac{r}{\sqrt{\lambda}}}(x_0) \backslash \Omega_\lambda$, which by our selection of an almost inscribed ball satisfies (3)

$$\frac{\mathrm{Vol}(S)}{\mathrm{Vol}\left(B_{\frac{r}{\sqrt{\lambda}}}(x_0)\right)} \leq \epsilon. \tag{13}$$

Let us also define a subcollection of the good cubes inner half of which is intersected by S, i.e.

$$U := \left\{ Q^\nu_{c\lambda^{-1}} \in \mathcal{Q}^g \middle| \frac{1}{2} Q^\nu_{c\lambda^{-1}} \cap S \neq \emptyset \right\}. \tag{14}$$

In order to get a contradiction, let us suppose that U occupies a very large proportion of $\mathcal{Q}^g$. Otherwise, there will be a sufficient proportion of cubes $\mathcal{Q}^g / U$, which all possess inscribed (in the nodal domain Ω_λ) balls of radius $\frac{C}{\lambda}$—this implies the claim of Theorem 1.1.

Now for each cube $Q^\nu_{c\lambda^{-1}} \in U$, we distinguish two cases:

1. Suppose that in a E_ϵ-good cube $Q^\nu_{c\lambda^{-1}}$ the nodal set does not intersect $\frac{1}{2} Q^\nu_{c\lambda^{-1}}$. This means that $\frac{1}{2} Q^\nu_{c\lambda^{-1}} \subseteq S$; hence,

$$\frac{\mathrm{Vol}(S \cap Q^{\nu}_{c\lambda^{-1}})}{\mathrm{Vol}(Q^{\nu}_{c\lambda^{-1}})} \geq \frac{1}{2^n}. \tag{15}$$

2. Suppose that the nodal set intersects $\frac{1}{2}Q^{\nu}_{c\lambda^{-1}}$. Since $Q^{\nu}_{c\lambda^{-1}}$ is E_ϵ-good, we can then invoke Lemma 2.6 which implies that

$$\frac{\mathrm{Vol}(\{\varphi_\lambda < 0\} \cap Q^{\nu}_{c\lambda^{-1}})}{\mathrm{Vol}(Q^{\nu}_{c\lambda^{-1}})} \geq C. \tag{16}$$

By definition $\{\varphi_\lambda < 0\} \cap Q^{\nu}_{c\lambda^{-1}} \subseteq S \cap Q^{\nu}_{c\lambda^{-1}}$, so we get

$$\frac{\mathrm{Vol}(S \cap Q^{\nu}_{c\lambda^{-1}})}{\mathrm{Vol}(Q^{\nu}_{c\lambda^{-1}})} \geq C. \tag{17}$$

Summing up the two cases over all cubes in U, we see that

$$\frac{\mathrm{Vol}\left(S \cap Q_{\frac{r_1}{\sqrt{\lambda}}}\right)}{\mathrm{Vol}\left(Q_{\frac{r_1}{\sqrt{\lambda}}}\right)} \geq C. \tag{18}$$

By using the estimate (13) and selecting ϵ sufficiently small, we arrive at a contradiction to (18). This means that U does not occupy a too large proportion of Q^g. The proof is finished. □

Let us conclude by mentioning a few remarks.

Remark 2.7 Concerning the location of the wavelength ball prescribed in Theorem 1.1, Theorem 1.3 [3] indicates a refinement of Lieb's result, specifying the location where a ball of wavelength size can almost be inscribed, as well as the way the error set grows in volume nearby. More precisely, wavelength balls can almost be inscribed at points where φ_λ achieves $\|\varphi_\lambda\|_{L^\infty(\Omega_\lambda)}$.

We note that the statement extends also to points x_0 at which the eigenfunction almost reaches its maximum on Ω_λ in the sense, that

$$C\varphi_\lambda(x_0) \geq \|\varphi_\lambda\|_{L^\infty(\Omega_\lambda)}, \tag{19}$$

for some fixed constant $C > 0$. In particular, if there are multiple "almost-maximum" points x_0, there should be an inscribed ball of radius $\frac{1}{\lambda}$ near each of them.

Remark 2.8 Let us observe that the estimates essentially depend on the growth of φ_λ at x_0. We have used the upper bound $C\sqrt{\lambda}$ on the doubling exponent in the worst possible scenario as shown by Donnelly–Fefferman. It is believed that φ_λ rarely exhibits such an extremal growth. If the growth is better, this allows to take larger cubes Q^{ν}_{s} and the bound on the inner radius improves. In particular, a constant growth implies the existence of a wavelength-inscribed ball.

2.4 The Quantum Ergodic Case

First, we mention some recent results of Hezari [4,5], addressing quantum ergodic sequences of eigenfunctions. Let us assume that (M, g) is a closed Riemannian manifold with negative sectional curvature. Let (φ_{λ_i}) be any orthonormal basis of $L^2(M)$, where (φ_{λ_i}) are eigenfunctions with eigenvalues λ_i. Then, for a given $\epsilon > 0$, there exists a density-one subsequence S_ϵ, so that

$$a_1(\log \lambda_j)^{\frac{(n-1)(n-2)}{4n^2}-\epsilon} \lambda_j^{-\frac{1}{2}-\frac{(n-1)(n-2)}{4n}} \leq \operatorname{inrad}(\Omega_\lambda) \tag{20}$$

We refer to [4] for further details.

The heart of Hezari's argument lies in observing that growth exponents (i.e. doubling exponents) improve, provided that eigenfunctions equidistribute at small scales (cf. [5]). More precisely, if we assume that for some small $r > \frac{1}{\sqrt{\lambda}}$, then we have

$$K_1 r^n \leq \int_{B_r(x)} |\varphi_\lambda|^2 \leq K_2 r^n, \tag{21}$$

for K_1, K_2 fixed constants and all geodesic balls $B_r(x)$, then

$$\log\left(\frac{\sup_{B_{2s}(x)} |\varphi_\lambda|^2}{\sup_{B_s(x)} |\varphi_\lambda|^2}\right) \leq Cr\sqrt{\lambda}. \tag{22}$$

Here the statement holds for all s smaller than $10r$. In particular, in the negatively curved setting, results of [6] give that r above could be taken as $(\log \lambda)^{-k}$ for any $k \in (0, \frac{1}{2n})$.

We have the following observation:

Corollary 2.9 *Let (M, g) be a negatively-curved real-analytic closed manifold of dimension at least 3. Then the collection of inscribed balls from Theorem 1.1 can be taken with radius $\frac{C(\log \lambda)^k}{\lambda}$, where k can be selected as any number in $(0, \frac{1}{2n})$. In particular,*

$$\operatorname{inrad}(\Omega_\lambda) \geq \frac{C(\log \lambda)^k}{\lambda}. \tag{23}$$

Proof We note the improvement by a factor of r of Hezari's growth bound (22) over the Donnelly–Fefferman growth estimate (6), which holds for all wavelength and smaller balls. The discussion after Lemma 2.2 indicates that ϕ_λ admits a holomorphic continuation with improved growth control. Hence, Lemma 2.2 holds with $\delta \leq \frac{C_1}{\sqrt{\lambda} r}$, so while going through the arguments above, we can actually take collections $\{Q_s^\nu\}_\nu$ consisting of cubes, side length of which is larger by a factor of $\frac{1}{r}$.

As remarked above r could be taken as $(\log \lambda)^{-k}$ for any $k \in (0, \frac{1}{2n})$. □

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