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A counting invariant for maps into spheres and for zero loci of sections of vector bundles

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Abstract
The set of unrestricted homotopy classes $[M, S^n]$ where M is a closed and connected spin $(n+1)$-manifold is called the n-th cohomotopy group $\pi^n(M)$ of M. Using homotopy theory it is known that $\pi^n(M) = H^n(M;\mathbb{Z}) \oplus \mathbb{Z}_2$. We will provide a geometrical description of the $\mathbb{Z}_2$ part in $\pi^n(M)$ analogous to Pontryagin's computation of the stable homotopy group $\pi_{n+1}(S^n)$. This $\mathbb{Z}_2$ number can be computed by counting embedded circles in M with a certain framing of their normal bundle. This is a similar result to the mod 2 degree theorem for maps $M \to S^{n+1}$. Finally we will observe that the zero locus of a section in an oriented rank n vector bundle $E \to M$ defines an element in $\pi^n(M)$ and it turns out that the $\mathbb{Z}_2$ part is an invariant of the isomorphism class of E. At the end we show that if the Euler class of E vanishes this $\mathbb{Z}_2$ invariant is the final obstruction to the existence of a nowhere vanishing section.

1 Introduction

Pontryagin computed in [16] the (stable) homotopy group $\pi_{n+1}(S^n)$ $(n \geq 3)$ using differential topology. Let us describe briefly his construction, since this paper will generalize his idea.

He showed that $\pi_{n+1}(S^n)$ is isomorphic to the bordism group of closed 1-dimensional submanifolds of $\mathbb{R}^{n+1}$ furnished with a framing on its normal bundle (a *framing* is a homotopy class of trivializations, see Sect. 2). We denote this bordism group by $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$. Let (C, φ) be a representative of an element of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$, i.e. C is a union of embedded circles in $\mathbb{R}^{n+1}$ and there are maps $\varphi_1, \dots, \varphi_n : C \to \mathbb{R}^{n+1}$ such that $(\varphi_1(x), \dots, \varphi_n(x))$ is a basis of $\nu(C)_x$ for every $x \in C$. Let φ_{n+1} be a trivialization of the tangent bundle of C such that $(\varphi_1(x), \dots, \varphi_{n+1}(x))$ is a positive oriented basis of $\mathbb{R}^{n+1}$ for every $x \in C$. Without loss of generality we may assume that $\varphi_1, \dots, \varphi_{n+1}$ is pointwise an orthonormal basis. If $(e_1, \dots, e_{n+1})$

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denotes the standard basis of $\mathbb{R}^{n+1}$, then consider the map $A = (a_{ij}) : C \to \mathrm{SO}(n+1)$ such that

$$\varphi_i(x) = \sum_{j=1}^{n+1} a_{ij}(x) e_j$$

for $x \in C$. Let $\pi_1(\mathrm{SO}(n+1))$ be identified with $\mathbb{Z}_2$, then Pontryagin defines [16, Theorem 20]

$$\delta(C, \varphi) := [A] + (n(C) \mod 2)$$

where $[A]$ denotes the homotopy class of A in $\pi_1(\mathrm{SO}(n+1))$ and $n(C)$ is the number of connected components of C. He showed that δ is well-defined on $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ and is an isomorphism of groups.

From a different point of view, one may consider his computation not as a computation of a homotopy group of S^n but rather of a *cohomotopy group* of S^{n+1}. If X is a CW space then the cohomotopy set of X is defined as the set of (unrestricted) homotopy classes $\pi^n(X) := [X, S^n]$, cf. [3, 18]. The set $\pi^n(X)$ for X a finite CW complex of dimension $n+1$ carries naturally a group structure, which is described in the beginning of Sect. 4. *Steenrod* showed [19, Theorem 28.1, p. 318] that $\pi^n(X)$ fits into a short exact sequence

$$0 \longrightarrow H^{n+1}(X;\mathbb{Z}_2)/\mathrm{Sq}^2\mu(H^{n-1}(X;\mathbb{Z})) \longrightarrow \pi^n(X) \longrightarrow H^n(X;\mathbb{Z}) \longrightarrow 0,$$

where $\mu : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_2)$ is the mod 2 reduction homomorphism (see also [17] for a more geometric proof of this sequence). Here the surjective map is the Hurewicz homomorphism which assigns to every $f \in \pi^n(X)$ the cohomology class $f^*(\sigma) \in H^n(X;\mathbb{Z})$ where $\sigma \in H^n(S^n, \mathbb{Z})$ is a fixed generator.

Moreover using methods of *Larmore* and *Thomas* [11] *Taylor* showed in [20, Theorem 6.2, Example 6.3] that the short exact sequence splits, provided the images of $\mathrm{Sq}^2 : H^{n-1}(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z}_2)$ and $\mathrm{Sq}^2 \circ \mu : H^{n-1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}_2)$ coincide.

If $X = M$ is an oriented manifold then the *second Wu class* [24] is equal to the second Stiefel-Whitney class $w_2(M)$, hence $\mathrm{Sq}^2(x) = w_2(M) \smile x$ for $x \in H^{n-1}(M;\mathbb{Z}_2)$. Therefore if M is spin then $\pi^n(M)$ fits into the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi^n(M) \longrightarrow H^n(M;\mathbb{Z}) \longrightarrow 0. \tag{ST}$$

and (ST) splits by [20, Example 6.3] thus

$$\pi^n(M) \cong H^n(M;\mathbb{Z}) \oplus \mathbb{Z}_2$$

as abelian groups. However the splitting map is constructed in a purely homotopy theoretic way and an aim of this article is to provide a geometric description in case M is a spin manifold.

This splitting map $\kappa : \pi^n(M) \to \mathbb{Z}_2$ (see Definition 3.8) for (ST) will be constructed similarly to Pontryagin's invariant δ from above. An important ingredient in Pontryagin's construction was the canonical *background framing* given by the standard basis of $\mathbb{R}^{n+1}$, which allowed him to define the map $A : C \to \mathrm{SO}(n+1)$. In general if we replace S^{n+1} or $\mathbb{R}^{n+1}$ by M, this background framing is not available any more. But this can be circumvented by using the spin structure of M, since over a circle every vector bundle with a spin structure defines a certain framing, cf. Lemma 3.1. Section 4 is devoted to determine geometrically the kernel of the Hurewicz map $\pi^n(M) \to H^n(M;\mathbb{Z})$. Finally we show that the splitting map possesses a naturality property, cf. Proposition 4.3 and that for a map $f : M \to S^n$ the number $\kappa(f)$ can be described by a counting formula, cf. Corollary 4.4. This is an analogous result to the mod 2 Hopf theorem, see [15, §4]. It should be mentioned that in [8] the authors discuss the case $n = 3$ and in [10] a similar construction of a $\mathbb{Z}_2$ invariant was used to classify quaternionic line bundles over closed spin 5-manifolds.

In Sect. 5 we will apply the results of Sects. 3 and 4 to the theory of vector bundles. Suppose $E \to M$ is an oriented vector bundle of rank n over a closed spin $(n+1)$-manifold M. Then any section of E which is transverse to the zero section defines by of its zero locus an element of $\Omega_1^{\mathrm{fr}}(M)$ and this element is independent of the transverse section. Thus using κ one defines an invariant $\kappa(E) \in \Omega_1^{\mathrm{fr}}$ of the isomorphism class of the bundle $E \to M$. In Theorem 5.5 it is shown that $\kappa(E)$ can be regarded as the secondary obstruction to the existence of a nowhere vanishing section. As an application we provide in Example 5.6 a simple proof of the well-known fact, that the maximal number of linear independent vector fields on S^{4k+1} is equal to 1. Finally we show that $\pi^n(M)$ can be mapped injectively into the set of isomorphism classes of oriented rank n vector bundles over spin $(n+1)$-manifolds for $n = 4$ and $n = 8$, cf. Proposition 5.8.

2 Preliminaries

If not otherwise stated we denote by M an $(n+1)$-dimensional oriented, closed and connected manifold, where $n \geq 3$. Let N be a arbitrary manifold and $E \to N$ a trivial vector bundle over N of rank r. A *trivialization* of $E \to N$ are r sections $s_1, \dots, s_r : N \to E$, such that $(s_1(q), \dots, s_r(q))$ is a basis of the fiber E_q for all $q \in N$. A *framing* φ of $E \to N$ is a homotopy class of trivializations.

We recall now the notion of bordism classes of normally framed submanifold in M of dimension k (cf. [15, §7]). Let C be a k-dimensional closed submanifolds of M. We say that C is *normally framed* if the normal bundle of C is trivial and possesses a framing φ. Two such normally framed submanifolds (C_0, φ_0) and (C_1, φ_1) are *framed bordant* if there is a $(k+1)$-dimensional submanifold $\Sigma \subset M \times [0,1]$ such that

(a) $\partial\Sigma \cap (M \times i) = C_i$ for $i = 0, 1$,
(b) $\partial\Sigma = C_0 \cup C_1$,
(c) Σ is normally framed in $M \times [0,1]$ such that the framing restricted to the $\partial\Sigma \cap (M \times i)$ coincides with φ_i.

To be framed bordant is an equivalence relation and the set of equivalence classes is called the *bordism classes of normally framed k-dimensional submanifolds in M* denoted by $\Omega_k^{\mathrm{fr}}(M)$. If (C, φ) is a normally framed submanifold then we denote by $[C, \varphi]$ its bordism class in $\Omega_k^{\mathrm{fr}}(M)$.

The *Pontryagin-Thom map* provides a bijection between $\pi^{n+1-k}(M)$ and $\Omega_k^{\mathrm{fr}}(M)$ as follows (cf. [15, §7]): Fix an orientation on S^n and let $f : M \to S^{n+1-k}$ represent an element of $\pi^{n+1-k}(M)$. Choose a regular value $x_0 \in S^{n+1-k}$ and set $C_{x_0} := f^{-1}(x_0)$. Moreover choosing an oriented basis of the tangent space $T_{x_0}S^{n+1-k}$ endows the normal bundle with a framing φ_{x_0} by means of the derivative of f. The bordism class $[C_{x_0}, \varphi_{x_0}] \in \Omega_k^{\mathrm{fr}}(M)$ is well defined and the map

$$\pi^{n+1-k}(M) \longrightarrow \Omega_k^{\mathrm{fr}}(M), \quad [f] \mapsto [C_{x_0}, \varphi_{x_0}].$$

is a bijection, see [15, Theorem B and A].

A *stable framing* of a real vector bundle $E \to C$ of rank r is an equivalence class of trivializations of

$$E \oplus \varepsilon^l$$

for some $l \in \mathbb{N}$ where two trivializations

$$\tau_1 : E \oplus \varepsilon^{l_1} \to \varepsilon^{r+l_1} \quad \text{and} \quad \tau_2 : E \oplus \varepsilon^{l_2} \to \varepsilon^{r+l_2},$$

are considered to be equivalent if there exists some $L > l_1, l_2$ such that the isomorphisms

$$\tau_1 \oplus \mathrm{id} : E \oplus \varepsilon^{l_1} \oplus \varepsilon^{L-l_1} \to \varepsilon^{L+r}$$

and

$$\tau_2 \oplus \mathrm{id} : E \oplus \varepsilon^{l_2} \oplus \varepsilon^{L-l_2} \to \varepsilon^{L+r}$$

are homotopic, cf. [5, Section 8.3]. If E is the tangent bundle of C, then a stable framing of TC is called a *stable tangential framing*. If E is the normal bundle of an embedding of C into a sphere whose dimension is big enough, then we call a stable framing a *stable normal framing*.

We define Ω_k^{fr} to be the bordism classes of stably (tangential) framed k-dimensional manifolds. More precisely two stably framed manifolds (C_0, φ_0) and (C_1, φ_1) where $\varphi_i : TC_i \oplus \varepsilon^l \to \varepsilon^{k+l}$ is an isomorphism are equivalent if there is a bordism Σ between C_0 and C_1 such that the tangent bundle of Σ possesses a stable framing and the restriction on C_0 and C_1 coincides with the framing φ_0 and φ_1 respectively. Note that Ω_k^{fr} is isomorphic to π_k^S, the k-stable homotopy group of spheres (cf. [5, Theorem 8.17]) and by the Pontryagin-Thom construction we have $\Omega_k^{\mathrm{fr}} = \varinjlim_l \Omega_k^{\mathrm{fr}}(S^l)$ where we use the equatorial embeddings $S^{l_1} \hookrightarrow S^{l_2}$ if $l_1 < l_2$ to construct well-defined maps $\Omega_k^{\mathrm{fr}}(S^{l_1}) \to \Omega_k^{\mathrm{fr}}(S^{l_2})$.

For this article the case $k = 1$ will be of importance. In this case we have $\Omega_1^{\mathrm{fr}} \cong \pi_1^S \cong \mathbb{Z}_2$. Consider a connected and closed 1-dimensional manifold S_0 and stable tangential framing

$$\varphi_0 : TS_0 \oplus \varepsilon^n \xrightarrow{\sim} \varepsilon^{n+1}.$$

From the discussion above, (S_0, φ_0) defines a class in Ω_1^{fr} and can be realized as follows: Let $S_0 = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 = 1, x_i = 0, i = 3, \dots, n+1\}$. Denote by $e_1, \dots, e_{n+1}$ the canonical basis of $\mathbb{R}^{n+1}$ and $E_i(x) = e_i$ for $x \in \mathbb{R}^{n+1}$. Moreover let $V(x) = x$ for $x \in \mathbb{R}^n$. The normal bundle $\nu(S_0)$ of S_0 is trivialized by $V, E_3, \dots, E_{n+1}$ restricted to S_0. Using this normal framing we obtain a stable framing

$$TS_0 \oplus \varepsilon^n \cong TS_0 \oplus \nu(S_0) \cong (T\mathbb{R}^{n+1})|_{S_0} \cong \varepsilon^{n+1}$$

where the latter framing is induced by $E_1, \dots, E_{n+1}$. Hence this defines an element in $\Omega_1^{\mathrm{fr}}(S^{n+1})$ which represents the framed null bordism, since the framing of $\nu(S_0)$ can be extended to a properly embedded stably framed disc in $S^{n+1} \times [0, 1]$. Clearly the non-trivial element of $\Omega_1^{\mathrm{fr}}(S^{n+1})$ can be represented by twisting the normal framing $V, E_3, \dots, E_{n+1}$ with a map $S_0 \to \mathrm{SO}(n)$ such that its homotopy class in $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ is not zero. Every stable tangential framing of a closed and connected 1-dimensional manifold can be obtained in this way.

Let $E \to N$ be an oriented vector bundle over a manifold N. After choosing a Euclidean bundle metric on E one obtains a vector bundle with structure group $\mathbf{SO}(n)$. Since the space of such Euclidean bundle metrics is contractible, every construction which depends up to homotopy from a metric is independent thereof.

We say that E is *spinnable* if the second Stiefel–Whitney class $w_2(E)$ is zero. This means that E can carry a spin structure, that is a lift of the classifying map $N \to B\mathrm{SO}(n)$ to a map $N \to B\mathrm{Spin}(n)$ in the fibration $K(\mathbb{Z}_2, 1) \to B\mathrm{Spin}(n) \to B\mathrm{SO}(n)$. Consequently E is a *spin bundle* if it is spinnable and a spin structure is fixed. If a spin structure is fixed on $E \to N$ then any other spin structure is in $1:1$ correspondence with elements in $H^1(N;\mathbb{Z}_2)$, cf. [12, Chapter II, Theorem 1.7]

We write $F(N)$ for the orthonormal frame bundle of an oriented manifold N, where the frames are consistent with the given orientation on N. If $V \subset N$ is a submanifold such that its normal bundle is framed then we obtain an embedding $F(V) \subset F(N)$. Thus a spin structure on N induces a spin structure on V, cf. [14]. In particular if V is the boundary of a spin manifold N, then V inherits a spin structure from N. Finally if $E \to N$ is a vector bundle with a spin structure and $V \subset N$ a submanifold, then clearly $E|_V \to V$ also inherits a spin structure from $E \to N$.

Let $E \to S^1$ be a spinnable vector bundle of rank $r \geq 3$ over the unit circle S^1. Then E has exactly two non-isomorphic spin structures. Clearly $E \to S^1$ can be extended to $E \to D^2$, where D^2 denotes the closed unit disc in $\mathbb{R}^2$. Since D^2 is contractible $E \to D^2$ admits a unique spin structure. Restricting this structure to the boundary of D^2 gives a spin structure on $E \to S^1$, which will be called the *standard spin structure* (D^2 is equipped with the standard orientation of $\mathbb{R}^3$ and the orientation on S^1 is induced by the outward pointing normal of S^1). The other should be called the *non-standard spin structure*. In other words, the standard spin structure on $E \to S^1$ can be extended to D^2 in contrary to the non-standard one.

3 The index of framed circles

In this section we define the key invariant of this article. For its construction the following basic lemma is the crucial observation.

Lemma 3.1 *Let $E \to S^1$ be a spinnable vector bundle of rank ≥ 3. Then E is isomorphic to the trivial bundle and a choice of a spin structure on E determines a framing on E.*

Proof E is isomorphic to the trivial bundles since it is an orientable vector bundle over a circle. Fix a spin structure on E, i.e. let $F'(E)$ be a $\mathbf{Spin}(n)$-principal bundle over S^1 which is a two-sheeted cover over the frame bundle $F(E)$ of E. Let $\pi : F'(E) \to F(E)$ be the projection which is equivariant with respect to the two-sheeted covering $\mathbf{Spin}(n) \to \mathbf{SO}(n)$. Clearly $F'(E)$ is the trivial $\mathbf{Spin}(n)$-principal bundle over S^1 and denote by $\sigma : S^1 \to F'(E)$ a global section. Then $\pi \circ \sigma$ is a global section of $F(E)$ hence a trivialization of $E \to S^1$. Any other such global section $\widetilde{\sigma} : S^1 \to F(E)$ differs from σ by a map $\varphi : S^1 \to \mathbf{Spin}(n)$. Since $\pi_1(\mathbf{Spin}(n)) = 1$ the map φ has to be null-homotopic which means that the two trivializations $\pi \circ \sigma$ and $\pi \circ \widetilde{\sigma}$ have to be homotopic, thus they define the same framing on E. □

In the same way one proves

Corollary 3.2 *Let Σ be a 1-dimensional CW-complex (not necessarily connected) and $E \to \Sigma$ a vector bundle of rank ≥ 3 endowed with a spin structure. Then E is isomorphic to the trivial bundle and the spin structure induces a framing on E.*

Definition 3.3 Let $E \to S^1$ be a spinnable vector bundle. The framing induced by the standard spin structure on E is called the *standard framing* and the one induced by the non-standard spin structure the *non-standard framing*.

Example 3.4 The spheres S^{n+1} admit a unique spin structure which can be constructed as mentioned in the preliminaries, i.e. S^{n+1} is the boundary of the closed unit ball D^{n+2} in $\mathbb{R}^{n+2}$ which admits a unique spin structure.

Let $S_0 \subset S^{n+1}$ be the intersection of a 2-dimensional linear subspace $W \subset \mathbb{R}^{n+2}$ with S^{n+1} and denote by $D_0^2 = W \cap D^{n+2}$. Thus after Lemma 3.1$TS^{n+1}|_{S_0}$ inherits a framing from the spin structure. Denoting by $\varphi_1, \dots, \varphi_{n+1}$ a trivialization of this framing, the framing

$$\overline{\varphi} : S_0 \to \mathrm{SO}(n+2), \quad x \mapsto (x, \varphi_1(x), \dots, \varphi_{n+1}(x))$$

must be null homotopic in $\mathrm{SO}(n+2)$ by the definition of the spin structures of S^{n+1} and $TS^{n+1}|_{S_0}$ (such that it lifts to $\mathrm{Spin}(n+2)$). Thus $\overline{\varphi}$ must be homotopic the constant framing $x \mapsto (e_1, \dots, e_{n+2})$, where $e_1, \dots, e_{n+2}$ denotes the canonical basis of $\mathbb{R}^{n+2}$. In particular this means that $TS^{n+1}|_{S_0}$ inherits the standard framing from the spin structure of S^{n+1}.

$\Omega_1^{\mathrm{fr}}(M)$ possesses a group structure which can be expressed as follows: Having two 1-dimensional closed submanifolds C and C' of M which are normally framed, they are framed bordant in M to framed submanifolds $\tilde{C}$ and $\tilde{C}'$ with empty intersection. Taking the equivalence class of the disjoint union $\tilde{C} \cup \tilde{C}'$ with the respective framings yields an abelian group structure on $\Omega_1^{\mathrm{fr}}(M)$, cf. [15, Problem 17 and p. 50].

Next, we construct a homomorphism $\kappa : \Omega_1^{\mathrm{fr}}(M) \to \Omega_1^{\mathrm{fr}}$ for M a spin manifold. Therefore let (C, φ_C) be a closed submanifold of dimension 1, such that its normal bundle $\nu(C)$ is framed by φ_C (thus representing an element in $\Omega_1^{\mathrm{fr}}(M)$). From Lemma 3.1 the bundle $TM|_C$ inherits a framing φ_σ from the spin structure of M. Using also the framing of φ_C we obtain a stable tangential framing

$$\varepsilon^{n+1} \cong TM|_C \cong TC \oplus \nu(C) \cong TC \oplus \varepsilon^n$$

which we denote by φ_{st}.

Proposition 3.5 *The bordism class $[C, \varphi_{\mathrm{st}}] \in \Omega_1^{\mathrm{fr}}$ depends only on the bordism class $[C, \varphi_C] \in \Omega_1^{\mathrm{fr}}(M)$.*

Proof Let $(C', \varphi_{C'})$ be another normally framed closed 1-dimensional submanifold framed bordant to (C, φ_C). Thus there is a bordism $\Sigma \subset M \times I$ between C and C' such that the normal bundle of Σ in $M \times I$ possess a framing φ_Σ. By definition restricting φ_Σ to C and C' yields φ_C and $\varphi_{C'}$ respectively. Since Σ is homotopy equivalent to a 1-dimensional CW-complex and since $M \times I$ inherits a unique spin structure from M we obtain a framing $\varphi_{\Sigma,\sigma}$ on $T(M \times I)|_\Sigma$. Of course the framings $\varphi_{\Sigma,\sigma}$ restricted to C and C' are just the framings φ_σ and φ'_σ respectively (i.e. induced by the spin structure of $TM|_C$ and $TM|_{C'}$). Since

$$T(M \times I)|_\Sigma \cong T\Sigma \oplus \nu(\Sigma)$$

the framings $\varphi_{\Sigma,\sigma}$ and φ_Σ determine a stable framing $\varphi_{\Sigma,\mathrm{st}}$ of $T\Sigma$. Then $(\Sigma, \varphi_{\Sigma,\mathrm{st}})$ is a stably framed bordism between $(C, \varphi_{\mathrm{st}})$ and $(C', \varphi'_{\mathrm{st}})$. ◻

Remark 3.6 As described above, the group structure of $\Omega_1^{\mathrm{fr}}(M)$ is given by disjoint union of submanifolds and their respective normal framings. Let (C, φ) be a framed 1-dimensional closed submanifold of M and denote by $C = S_1 \cup \ldots \cup S_k$ the connected components of C. We may assume that the union is always disjoint. Thus S_i is an embedded circle and $\varphi_i := \varphi|_{S_i}$ a normal framing of S_i. Consequently we have

$$[C, \varphi] = \sum_{i=1}^{k} [S_i, \varphi_i]$$

in $\Omega_1^{\mathrm{fr}}(M)$.

Definition 3.7 Let $S \subset M$ be an embedded circle and φ a framing of $\nu(S)$. We call the bordism class $[S, \varphi] \in \Omega_1^{\mathrm{fr}}(M)$ a *framed circle of* M. The corresponding stable class $[S, \varphi_{\mathrm{st}}] \in \Omega_1^{\mathrm{fr}}$ will be called *the index of* $[S, \varphi]$ *(with respect to the spin structure of* M) and will be denoted by $\mathrm{ind}(S, \varphi)$.

Definition 3.8 Let M be an $(n+1)$-dimensional closed spin manifold. Then we define a map

$$\kappa : \Omega_1^{\mathrm{fr}}(M) \to \Omega_1^{\mathrm{fr}}, \quad [C, \varphi] \mapsto \kappa([C, \varphi]) := \sum_{\substack{S \subset C, \\ S \text{ connected}}} \mathrm{ind}(S, \varphi|_S) = [C, \varphi_{\mathrm{st}}].$$

We call κ the *degree map* of M with respect to the chosen spin structure.

Remark 3.9 It follows from the construction that κ is a homomorphism.

Examples 3.10

(a) Consider S^{n+1} with the induced spin structure from the unit disc of $\mathbb{R}^{n+2}$, cf. Example 3.4 respectively [14]. Let S_0 be the intersection of S^{n+1} with a 2-dimensional linear subspace W of $\mathbb{R}^{n+2}$. We argued in Example 3.4 that $TS^{n+1}|_{S_0}$ inherits the standard framing. Choose the standard framing φ_0 on $\nu(S_0)$, then

$$\kappa([S_0, \varphi_0]) = 0.$$

Consequently the non-standard framing φ_1 of $\nu(S_0)$ yields

$$\kappa([S_0, \varphi_1]) \neq 0.$$

(b) Let N be a closed, simply connected, spin manifold of dimension n. Then $M := S^1 \times N$ admits two different spin structures since $H^1(S^1 \times N; \mathbb{Z}_2) \cong H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. M is the boundary of $D^2 \times N$ which has up to isomorphism a unique spin structure. The two different spin structures on M can be described as follows: One can be extended from M to $D^2 \times N$ and the other not. We call the former one the *standard spin structure* and the latter one the *non-standard spin structure* of $S^1 \times N$.

For $q_0 \in N$ consider the circle $S_0 := S^1 \times q_0 \subset S^1 \times N$. Clearly we have a canonical isomorphism

$$\nu(S_0) \cong S_0 \times T_{q_0} N.$$

Thus choosing a basis in $T_{q_0}N$ gives a framing φ_0 on $\nu(S_0)$ which extends to a framing of $(D^2 \times q_0) \times T_{q_0}N$. This implies

$$\kappa_0([S_0, \varphi_0]) = 0$$

for the standard spin structure and

$$\kappa_1([S_0, \varphi_0]) \neq 0$$

for the non-standard spin structure.

For $q_1 \in N$ with $q_0 \neq q_1$ we consider $C = S^1 \times q_0 \cup S^1 \times q_1$ with fixed normal framing on $S^1 \times q_i$ which gives a framing φ on C. Then $\kappa([C, \varphi])$ is independent of the chosen spin structure of M. This shows that in general κ will depend on the spin structure. The next proposition will show how it depends on it.

Suppose $C \subset M$ is a closed 1-dimensional submanifold. Then C defines a $\mathbb{Z}_2$ fundamental homology class $[C] \in H_1(M;\mathbb{Z}_2)$. We denote by $w(C) \in H^n(M;\mathbb{Z}_2)$ the cohomology class which is the Poincaré dual of $[C]$.

Proposition 3.11 *Fix a spin structure σ on M and denote by κ the degree map of M with respect to the chosen spin structure. Choose another spin structure of M, which is represented by $\alpha \in H^1(M;\mathbb{Z}_2)$ and denote by κ^α the corresponding degree map. Then we have*

$$\kappa([C, \varphi]) = \kappa^\alpha([C, \varphi]) + \delta(\alpha \smile w(C)),$$

where $\delta : H^{n+1}(M;\mathbb{Z}_2) \to \Omega_1^{\mathrm{fr}}$ is the unique isomorphism. Thus if $w(C) \smile \alpha = 0$ then $\kappa([C, \varphi]) = \kappa^\alpha([C, \varphi])$.

Proof Assume first that (S, φ) is a framed circle and $i : S \to M$ is the inclusion. The spin structure σ induces a spin structure on $TM|_S = i^*(TM)$ and the spin structure induced by α is represented by $i^*(\alpha) \in H^1(S;\mathbb{Z}_2)$. Of course $TM|_S$ can have at most two different spin structures. From the definition of the index we have

$$\mathrm{ind}(S, \varphi) = \mathrm{ind}_\alpha(S, \varphi) + \delta(i^*(\alpha))$$

where ind is defined by σ, ind_α by α and $\delta : H^1(S;\mathbb{Z}_2) \to \Omega_1^{\mathrm{fr}}$ the unique isomorphism.

Let $[S] \in H_1(S;\mathbb{Z}_2)$ be the $\mathbb{Z}_2$ fundamental class of S, then $i^*(\alpha) \frown [S] \in H_0(S;\mathbb{Z}_2)$, which is mapped under i_* to $\alpha \frown i_*([S]) \in H_0(M;\mathbb{Z}_2)$. Let $[M] \in H_n(M;\mathbb{Z}_2)$ denote the $\mathbb{Z}_2$ fundamental class of M. Then

$$\alpha \frown i_*([S]) = \alpha \frown (w(S) \frown [M]) = (\alpha \smile w(S)) \frown [M],$$

where we used that $i_*([S])$ is Poincaré dual to $w(S)$. Since $\cdot \frown [S]$ and $\cdot \frown [M]$ are isomorphisms by Poincaré duality as well as $i_* : H_0(S;\mathbb{Z}_2) \to H_0(M;\mathbb{Z}_2)$ because S and M are connected we infer

$$\mathrm{ind}(S, \varphi) = \mathrm{ind}_\alpha(S, \varphi) + \delta(\alpha \smile w(S))$$

where now $\delta : H^{n+1}(M;\mathbb{Z}_2) \to \Omega_1^{\mathrm{fr}}$ is again the unique isomorphism.

Consider now (C, φ) with the disjoint union $C = S_1 \cup \ldots S_k$ and $\varphi_j := \varphi|_{S_j}$, such that S_j is connected. With the previous computations we have

$$\kappa([C,\varphi]) = \sum_{j=1}^{k} \left(\mathrm{ind}_\alpha(S_j,\varphi_j) + \delta(\alpha \smile w(S_j))\right) = \kappa^\alpha([C,\varphi]) + \delta(\alpha \smile w(C)).$$

and the proposition follows. □

We continue with the description of the „dual" short exact sequence to (ST). There is a natural group homomorphism $\Omega_1^{\mathrm{fr}}(M) \to \Omega_1^{\mathrm{SO}}(M)$, which assigns to every framed 1-submanifold $[C,\varphi]$ the oriented bordism class induced by the orientation of framing φ. This is well-defined since every normally framed bordism in M is also an oriented bordism (M is oriented). By the seminal work of *Thom* [21] we have an isomorphism

$$\Omega_1^{\mathrm{SO}}(M) \to H_1(M;\mathbb{Z})$$

which assigns every oriented submanifold its fundamental class in $H_1(M;\mathbb{Z})$. Thus we obtain a group homomorphism

$$\Phi : \Omega_1^{\mathrm{fr}}(M) \to H_1(M;\mathbb{Z}) \tag{1}$$

which is clearly surjective. The kernel of Φ is at most isomorphic to $\mathbb{Z}_2$ and elements of the kernel are represented by framed circles (S,φ) such that S is oriented null-bordant, i.e. there is an embedded oriented disc $D \subset M \times I$ with the properties $\partial D = S$ and the orientations of ∂D and S agree. We may equip the normal bundle of S with two framings. If both framings can be extended over D then the kernel is trivial and otherwise $\mathbb{Z}_2$.

Lemma 3.12 *The restricted degree map* $\kappa|_{\ker\Phi} : \ker\Phi \to \Omega_1^{\mathrm{fr}}$ *is an isomorphism.*

Proof Since κ is a homomorphism it will map the neutral element of $\ker\Phi$ to that of Ω_1^{fr}. Thus it suffices to show the following: Let (S,φ) be a framed circle such that S is oriented null-bordant in M but φ cannot be extended over the nullbordism. We have to show $\kappa([S,\varphi]) \neq 0$, where 0 denotes the neutral element of Ω_1^{fr}. We may assume that S lies in a chart of M.[1] Thus we may embed S into $\mathbb{R}^{n+1}$ endowed with a normal framing, which cannot be extended over a nullbordism in $\mathbb{R}^{n+1}$. Hence the index of (S,φ) defines a non-trivial element in Ω_1^{fr} (note that since $w(S)=0$ the element $\kappa[(S,\varphi)]$ does not depend on the spin structure of M, cf. Lemma 3.11). □

Thus we may identify $\ker\Phi$ with Ω_1^{fr} via $(\kappa|_{\ker\Phi})^{-1}$ and we obtain a short exact sequence

$$0 \longrightarrow \Omega_1^{\mathrm{fr}} \longrightarrow \Omega_1^{\mathrm{fr}}(M) \longrightarrow H_1(M;\mathbb{Z}) \longrightarrow 0$$

and from Lemma 3.12 κ is a splitting map. Therefore we showed

Theorem 3.13 *Let M be an $(n+1)$-dimensional closed spin manifold. Choose a spin structure on M. Then*

$$\Omega_1^{\mathrm{fr}}(M) \longrightarrow H_1(M;\mathbb{Z}) \oplus \Omega_1^{\mathrm{fr}}, \quad [C,\varphi] \mapsto ([C], \kappa([C,\varphi]))$$

[1] Take a small embedded closed disc and choose a framing on the circle bounding the disc which cannot be extended over a proper embedded disc in $M \times I$.

is an isomorphism of abelian groups.

We finish this section by giving an alternative way to compute the index of a framed circle in the spirit of Pontryagin [16]. Suppose $[S, \varphi]$ is a framed circle, thus there are trivializations of $\nu(S)$ and $TM|_S$ such that we obtain the stable framing

$$\varepsilon^{n+1} \cong TS \oplus \varepsilon^n$$

(where we can assume that the isomorphism is orientation preserving). Denote by $v_1, \ldots, v_{n+1}$ and by $w_2, \ldots, w_{n+1}$ the trivializations of $TM|_S$ and $\nu(S)$ respectively. Let w_1 be a trivialization of TS. Let $\Phi : TS \oplus \varepsilon^n \to \varepsilon^{n+1}$ be the isomorphism of the stable framing, then there is a matrix $A = (A_{ij}) : S \to \mathrm{GL}^+(n+1)$ (where $\mathrm{GL}^+(n+1)$ is the set of all invertible real matrices of size $(n+1) \times (n+1)$ with positive determinant) such that

$$\Phi(w_i) = \sum_{j=1}^{n+1} A_{ij} \cdot v_j.$$

Since $\mathrm{SO}(n+1)$ is a strong deformation retract of $\mathrm{GL}^+(n+1)$ we have $\pi_1\big(\mathrm{GL}^+(n+1)\big) \cong \mathbb{Z}_2$. The map $A : S \to \mathrm{GL}^+(n+1)$ defines an element $[A] \in \pi_1\big(\mathrm{GL}^+(n+1)\big)$. Changing the homotopy classes of trivializations of $TM|_S$ and $\nu(S)$ does not change $[A]$. Furthermore $[A]$ is also independent of the choice of trivializations of TS.

According to the Preliminaries (Section 2) any stable framing $\mathrm{ind}(S, \varphi)$ can be represented by a framed circle S_0 in $\mathbb{R}^{n+1}$ such that

$$TS_0 \oplus \varepsilon^n \cong TS_0 \oplus \nu(S_0) \cong (T\mathbb{R}^{n+1})|_{S_0} \cong \varepsilon^{n+1}$$

recovers the stable framing of (S, φ). It follows that

$$\mathrm{ind}(S, \varphi) = \delta(S_0, \varphi_0),$$

where δ is the invariant constructed by Pontryagin, [16, Theorem 20]. We will use a different notation: Let us denote by $[A]$ the homotopy class constructed above from the stable framing and by $\overline{[A]}$ the element $[A] + 1 \in \Omega_1^{\mathrm{fr}}(S^n) \cong \mathbb{Z}_2$ where 1 is the non-trivial element. Thus we proved

Lemma 3.14 *We identify $\pi_1\big(\mathrm{GL}^+(n+1)\big)$ with Ω_1^{fr} by the unique isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$. Then*

$$\overline{[A]} = \mathrm{ind}(S, \varphi).$$

4 Computation of $\pi^n(M)$

We start this section to explain the group structure of $\pi^n(M)$. Let $j : S^n \vee S^n \to S^n \times S^n$ be the inclusion of the $(2n-1)$-skeleton of $S^n \times S^n$ (endowed with the standard CW structure) then, since M is $n+1$-dimensional CW complex, the induced map $j_\# : [M, S^n \vee S^n] \to [M, S^n \times S^n]$ is an isomorphism. For $f, g \in \pi^n(M)$ the group structure is defined by

$$f + g := (\mathrm{id}_{S^n} \vee \mathrm{id}_{S^n})_\# \circ (j_\#)^{-1}(f \times g).$$

This makes $\pi^n(M)$ to an abelian group.

Now, let $f : M \to S^n$ be a differentiable map and $x_0 \in S^n$ a regular value. We orient S^n by the normal vector field pointing outwards and the standard orientation of $\mathbb{R}^{n+1}$.

Let $\Psi : \pi^n(M) \to H^n(M;\mathbb{Z})$ be the map $\Psi([f]) := f^*\sigma$ where $\sigma \in H^n(S^n;\mathbb{Z})$ is a fixed generator. We define the analogous degree map $\kappa : \pi^n(M) \to \pi_1^S$, where π_1^S is the first stable homotopy group of spheres, as follows: κ is the composition of

$$\pi^n(M) \xrightarrow{\sim} \Omega_1^{\mathrm{fr}}(M) \xrightarrow{\kappa} \Omega_1^{\mathrm{fr}} \xrightarrow{\sim} \pi_1^S.$$

where the first and the last isomorphism is again induced by the Pontryagin-Thom isomorphism.

Theorem 4.1 *Let M be a closed $(n+1)$-dimensional spin manifold. Then*

(a) *The generator of* $\ker \Psi \cong \mathbb{Z}_2$ *is given by the homotopy class of the map* $\eta \circ \omega : M \to S^n$, *where η represents a generator of $\pi_{n+1}(S^n)$ and $\omega : M \to S^{n+1}$ is a map of odd degree. Thus* $\ker \Psi \cong \pi_{n+1}(S^n)$.

(b) *Identifying π_1^S with $\pi_{n+1}(S^n)$ the degree map $\kappa : \pi^n(M) \to \pi_1^S$ splits the short exact sequence* (ST). *Thus we have*

$$\pi^n(M) \longrightarrow H^n(M;\mathbb{Z}) \oplus \pi_{n+1}(S^n), \quad [f] \mapsto (f^*\sigma, \kappa([f])).$$

is an isomorphism of abelian groups.

Proof Clearly we have $[\eta\circ\omega] \in \ker \Psi$. For (a) it is enough to check that $\kappa([\eta\circ\omega])$ is non-zero in π_1^S. We choose an odd degree map $\omega : M \to S^{n+1}$ as follows: Let $\{p_1, \dots, p_l\}$ be the preimage of a regular value y_0 and choose open sets $U_1, \dots, U_l \subset M$ as well as $V \subset S^{n+1}$ such that for all $i = 1, \dots, l$

(a) U_i and V are contractible,
(b) $p_i \in U_i$ and $y_0 \in V$,
(c) there are charts $\psi_i : U_i \to \mathbb{R}^{n+1}$, $\psi : V \to \mathbb{R}^{n+1}$,
(d) $\omega_i := \omega|_{U_i}$ is an orientation preserving diffeomorphism onto V.

Since ω has odd degree, l has to be an odd number (such maps exists e.g. using the Pontryagin-Thom construction). Furthermore let $x_0 \in S^n$ be a regular value of η and $S_0 = \eta^{-1}(x_0)$. We may assume that S_0 is connected (e.g. see [15, Theorem C]) and $S_0 \subset V$. Let φ_0 be the framing of $\nu(S_0)$ induced by η, then $0 \neq [S_0, \varphi_0] \in \Omega_1^{\mathrm{fr}}(S^{n+1}) \cong \pi_{n+1}(S^n) \cong \mathbb{Z}_2$ and therefore by definition we have $\mathrm{ind}(S_0, \varphi_0) \neq 0$.

Denote by $S_i := \omega_i^{-1}(S_0)$ and frame $\nu(S_i)$ by φ_0 and $d\omega_i$. Then $C = S_1 \cup \dots \cup S_l$ together with the framings φ_i is a Pontryagin manifold for $\eta\circ\omega$ to the regular value x_0. Note that $w(S_i) = 0$ for $i = 1, \dots, l$, since they are contained in a chart of M. By Proposition 3.11 this means that their indices do not depend on the spin structure of M. Clearly we deduce $\mathrm{ind}(S_i, \varphi_i) = \mathrm{ind}(S_0, \varphi_0) \neq 0$ for all $i = 1, \dots, l$ and from that we infer

$$\kappa([\eta\circ\omega]) = \sum_{i=1}^{l} \operatorname{ind}(S_i, \varphi_i) = l \cdot \operatorname{ind}(S_0, \varphi_0) \neq 0$$

since l is odd, which proves (a).

Part (b) follows directly from part (a). □

Corollary 4.2 *Suppose M is simply connected, then, up to homotopy, there are exactly two maps $M \to S^n$ and one of them is the constant map. The homotopy class of the non-trivial map is represented by $\eta\circ\omega : M \to S^n$, see Theorem 4.1.*

Finally we would like to show that κ is natural with respect to maps between manifolds which preserve the spin structure

Proposition 4.3 *Suppose $\Phi : M_1 \to M_2$ is a map between two closed and connected spin manifolds of dimension $(n+1)$. We assume that the spin structure of M_1 coincides with the pull-back spin structure by Φ* of *M_2. Then for the natural homomorphism $\Phi^\# : \pi^n(M_2) \to \pi^n(M_1)$, $f \mapsto \Phi\circ f$ we have*

$$\kappa\left(\Phi^\#(f)\right) = \deg_2 \Phi \cdot \kappa(f).$$

where $\deg_2 \Phi$ is the mod 2 degree of Φ. Therefore using the isomorphism

$$\pi^n(M) \cong H^n(M;\mathbb{Z}) \oplus \pi_{n+1}(S^n)$$

we have

$$\Phi^\# : \pi^n(M_2) \to \pi^n(M_1), \quad (\alpha, \nu) \mapsto (\Phi^*(\alpha), \deg_2 \Phi \cdot \nu)$$

Proof First note that $\Phi^\#$ is well-defined on the homotopy class of Φ. For $f \in \pi^n(M_2)$ there is a decomposition $f = f_\alpha + f_\nu$ with $\kappa(f_\alpha) = 0$, $f_\alpha^*(\sigma) = \alpha$ and $\kappa(f_\nu) = \nu$ as well as $f_\nu^*(\sigma) = 0$.

Let us show first $\Phi^\#(f_\alpha) = f_{\Phi^*(\alpha)}$. Clearly we have $\Phi^\#(f_\alpha)(\sigma) = \Phi^*(\alpha)$ thus it remains to show $\kappa(\Phi^\#(f_\alpha)) = 0$. Let C_2 be the preimage of a regular value of f_α with a normal framing φ_0 such that $\kappa([C_2, \varphi_0]) = 0$. Moreover we may choose f_α such that each framed circle of (C_2, φ_0) has index 0. Deform Φ to be transversal to C_2, thus $C_1 := \Phi^{-1}(C_2)$ is a closed 1-dimensional submanifold of M_1. The normal bundle to C_1 is isomorphic to the pull back of the normal bundle of C_2 by Φ. This induces a framing on C_2 such that every framed circle thereof has index 0 (note that the spin structure of M_1 is the pulled back by Φ from M_2) which is also the framing induced by the map $\Phi\circ f_\alpha$. But this means $\kappa(\Phi^\#(f_\alpha)) = 0$.

On the other hand we may assume a preimage of a regular point in S^n under f_ν is a contractible circle S_2 in M_2 with normal framing φ such that the index of the framed circle (S_2, φ) is $\nu \in \pi_{n+1}(S^n)$. Then making again Φ transverse to S_2 we obtain a normally framed submanifold (C_1, φ) such that the index of each framed circle in C_1 has index ν. As in the proof of Theorem 4.1 the degree of (C_1, φ) is just $\deg_2 \Phi \cdot \nu$. Therefore $\Phi^\#(f_\nu) = f_{\deg_2 \Phi\cdot\nu}$ and the proposition follows. □

Corollary 4.4 *Let $f : M \to S^n$* and *$x_0 \in S^n$ a regular value. Write $S_1 \cup \ldots \cup S_k = f^{-1}(x_0)$ such that S_i is a connected component of $f^{-1}(x_0)$ and denote by φ_i the induced framing from f. Then the number*

$$\#\{i : \kappa([S_i, \varphi_i]) \neq 0\} \mod 2$$

does not depend on x_0 and is a homotopy invariant. Using the notation of Lemma 3.14 we can rewrite the above number as

$$\#\{i : \overline{[A_i]} \neq 0\} \mod 2$$

where $[A_i]$ is the homotopy class of matrices corresponding to $[S_i, \varphi_i]$.

5 Application to vector bundles

In this section $\pi : E \to M$ should denote an oriented vector bundle of rank n endowed with a spin structure. Let $s : M \to E$ be a section. If not otherwise stated, we say s is *transversal* if s is transversal to the zero section 0_E of E. For a transversal section s the zero locus C is a smooth 1-dimensional closed submanifold of M. The differential $ds : TM \to TE$ restricted to $\nu(C)$ is an isomorphism of the vector bundles $\nu(C) \to E|_C$. Since E possesses a spin structure, by Lemma 3.1 $E|_C$ has a framing and with ds this endows $\nu(C)$ with the framing φ of $E|_C$. Note that the homology class $[C] \in H_1(M;\mathbb{Z})$ is the Poincaré dual of the Euler class of E.

Proposition 5.1 *The class $[C, \varphi] \in \Omega_1^{\mathrm{fr}}(M)$ does not depend on the section s.*

Proof Let $s' : M \to E$ be another transversal section and denote the corresponding normally framed zero locus by (C', φ'). Let $s^* : M \times I \to \mathrm{pr}^*(E)$ be a section of $\mathrm{pr}^*(E) \to M \times I$ (where $\mathrm{pr} : M \times I \to M$) such that $s^*|_{M\times 0} = s$ and $s^*|_{M\times 1} = s'$. We may deform s^* to a section $\hat{s}$ which is transverse to the zero section of $\mathrm{pr}^*(E) \to M \times I$ and agrees with s and s' on the boundary of $M \times I$. The zero locus of $\hat{s}$, call it $\Sigma \subset M \times I$ is a bordism between C and C' by construction. Moreover by Lemma 3.1 $T(M \times I)|_\Sigma$ inherits a framing from the spin structure of M as well as $\nu(\Sigma)$ from $d\hat{s}$ and the spin structure of $\mathrm{pr}^*(E)|_\Sigma$. Thus Σ is a normally framed bordism between (C, φ) and (C', φ'). □

Definition 5.2 The bordism class $[C, \varphi] \in \Omega_1^{\mathrm{fr}}(M)$ constructed above is called the *framed divisor of $E \to M$*. Furthermore we define the *degree $\kappa(E)$ of E* as $\kappa([C, \varphi])$

For $[C, \varphi] \in \Omega_1^{\mathrm{fr}}(M)$ we denoted by $w(C) \in H^n(M;\mathbb{Z}_2)$ the Poincaré dual of the $\mathbb{Z}_2$ fundamental class $[C] \in H_1(M;\mathbb{Z}_2)$. If $[C, \varphi]$ is the framed divisor of $E \to M$ then $w(C)$ is the n-th Stiefel-Whitney class $w_n(E)$ (the zero locus of a generic section of E represents the Poincaré dual of the Euler class of E, cf. [4, Proposition 12.8]. Therefore, since n-th Stiefel-Whitney class of E is the Euler class modulo 2, we have that $[C]$ is Poincaré dual to $w_n(E)$)). Moreover if one changes the spin structure of E by an element $\alpha \in H^1(M;\mathbb{Z}_2)$ then for the framed divisor it is the same as keeping the spin structure of E and changing that of M by α. Hence if $w_n(E) = 0$ then the degree of E does not depend on the spin structure (see Lemma 3.1 and Proposition 3.11).

Proposition 5.3 *If $w_n(E) = 0$ then the framed divisor is independent of the spin structures on M and E.*

For the next theorem we will need a technical Lemma. Let D^m denote the closed unit ball in $\mathbb{R}^m$ and consider a smooth map $f : D^{n+k+1} \to \mathbb{R}^{n+1}$. Assume that $0 \in \mathbb{R}^{n+1}$ is a regular value for f and $\Sigma_f^k := f^{-1}(0)$ does not intersect the boundary of D^{n+k+1}. Denote by φ_f the induced framing on $\nu(\Sigma_f^k)$. Since Σ_f^k is a submanifold of $\mathbb{R}^{n+k+1}$ the trivialization φ_f defines a stable tangential framing of Σ_f^k thus the pair (Σ_f^k, φ_f) defines an element in Ω_k^{fr}. On the other side, consider

$$g : S^{n+k} = \partial D^{n+k+1} \to S^n, \quad g(x) := \frac{f(x)}{|f(x)|}$$

and choose a regular value $y \in S^n$. Denote by (Σ_g^k, φ_g) the induced stably framed manifold.

Lemma 5.4 *With the notation above we have that (Σ_f^k, φ_f) and (Σ_g^k, φ_g) are stably framed bordant, thus they define the same element in Ω_k^{fr}.*

Proof There is an $\varepsilon > 0$ such that the closed ball D_ε centered in $0 \in \mathbb{R}^{n+1}$ with radius ε contains only regular values of f. The preimage of D_ε under f is a disc bundle $D(\Sigma_f^k)$ of the normal bundle $\nu(\Sigma_f^k \hookrightarrow \mathbb{R}^{n+k+1})$. Denote by $S(\Sigma_f^k)$ its sphere bundle. Then $f|_{S(\Sigma_f^k)}$ has image $S_\varepsilon = \partial D_\varepsilon$. Thus for $y' \in S_\varepsilon$, $\Sigma_{y'} = \left(f|_{S(\Sigma_f^k)}\right)^{-1}(y')$ lies completely in $S(\Sigma_f^k)$. Moreover the Pontryagin manifold $(\Sigma_{y'}, \varphi_{y'})$ is framed bordant to (Σ_f^k, φ_f). Thus we would like to show that $(\Sigma_{y'}, \varphi_{y'})$ represents the same element in Ω_k^{fr} as (Σ_g^k, φ_g). Since the normal bundle of $S(\Sigma_f^k)$ is trivial the framing $\varphi_{y'}$ induces a framing $\varphi'_{y'}$ on $\nu(\Sigma_{y'} \hookrightarrow S(\Sigma_f^k))$ such that $(\Sigma_{y'}, \varphi_{y'})$ is stably framed bordant to $(\Sigma_{y'}, \varphi'_{y'})$. But the latter normally framed manifold is the Pontryagin manifold to the map $f|_{S(\Sigma_f^k)} : S(\Sigma_f^k) \to S_\varepsilon$ at the point $y' \in S_\varepsilon$.

Let N be the complement of the interior of $D(\Sigma_f^k)$ in D^{n+k+1}. Then N is a framed cobordism between $S^{n+k} = \partial D^{n+k+1}$ and $S(\Sigma_f^k)$. The restriction of the map

$$F : N \to S^n, \quad F(x) := \frac{f(x)}{|f(x)|}$$

to S^{n+k} is equal to g and F restricted to $S(\Sigma^k)$ is equal to $\varepsilon^{-1}\hat{f}$. Hence F defines a framed bordism between (Σ_g^k, φ_g) and $(\Sigma_{y'}, \varphi'_{y'})$ which proves the lemma. □

Theorem 5.5 *Let $E \to M$ be an oriented vector bundle of rank n with $w_2(E) = 0$ over a closed spin manifold M of dimension $n + 1$. Then E admits a nowhere vanishing section if and only if the Euler class is zero and $\kappa(E) = 0$.*

Proof Suppose there is a nowhere vanishing section of E then clearly this section is transverse and has an empty framed divisor. Thus from Theorem 3.13 we have that the Euler class must be zero and $\kappa(E) = 0$.

Assume now that $e(E) = 0$ and $\kappa(E) = 0$. Consider the fibration

$$S^{n-1} \longrightarrow B\mathrm{SO}(n-1) \longrightarrow B\mathrm{SO}(n).$$

where $B\mathrm{SO}(k)$ denotes the classifying space to the special orthogonal group $\mathrm{SO}(k)$. Consider the classifying map $g : M \to B\mathrm{SO}(n)$ for $E \to M$. There exists a nowhere vanishing section if and only if there is a lift $\hat{g} : M \to B\mathrm{SO}(n-1)$ of g up to homotopy.

First we put a CW-structure on M (e.g. induced by a Morse function) then over the $(n-1)$-skeleton of M there exists such a lift $\hat{g}$ of g. The obstruction to extend the lift over the n-skeleton lies in $H^n(M;\pi_{n-1}(S^{n-1})) = H^n(M;\mathbb{Z})$ which is given by the Euler class $e(E)$. Since this is assumed to be zero $\hat{g}$ extends over the n-skeleton of M. The obstruction to extend $\hat{g}$ over the top cell of M lies in $H^{n+1}(M;\pi_n(S^{n-1})) \cong \pi_n(S^{n-1}) \cong \mathbb{Z}_2$. Let e_{n+1} be the top cell of M and $\psi : \partial e_{n+1} \cong S^n \to M$ the corresponding attaching map. The bundle $E|_{e_{n+1}}$ is canonical isomorphic to $e_{n+1} \times \mathbb{R}^n$. Let $\sigma : M \to E$ be a section which has no zeroes over the n-skeleton of M and which is transverse to the zero section of E. Then consider the map

$$g : \partial e_{n+1} \cong S^n \to S^{n-1}, \quad g(x) := \frac{\sigma \circ \psi(x)}{|\sigma \circ \psi(x)|}$$

(where the norm is taken with respect to a Euclidean bundle metric on E). The homotopy class of g in $\pi_n(S^{n-1})$ is the obstruction to extend a no where vanishing section over the n-skeleton to the $(n+1)$-skeleton of M. Since $\pi_n(S^{n-1})$ is isomorphic to the stable homotopy group π_1^S we consider the homotopy class of g as an element therein.

From Lemma 5.4 we infer that $[g] \in \pi_1^S \cong \Omega_1^{\mathrm{fr}}$ is equal to the framed divisor $\kappa(E)$ of E defined by σ, thus E admits a nowhere vanishing section in case $e(E) = 0$ and $\kappa(E) = 0$.

□

Example 5.6 As an application of our theory we will reprove the following fact due to Whitehead [23] and Eckmann [6]: The number of linear independent vector fields on S^{4k+1} is equal to 1 (see also [1] and in [22]).

Denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean product in $\mathbb{R}^{4k+2}$. The vector field

$$v : \mathbb{R}^{4k+2} \to \mathbb{R}^{4k+2}, \quad v(x_1, x_2, \ldots, x_{4k+2}) = (-x_2, x_1, \ldots, -x_{4k+2}, x_{4k+1})$$

defines a nowhere vanishing vector field on S^{4k+1} since $\langle v(x), x \rangle = 0$ for $x \in S^{4k+1}$. Let E be the subbundle of TS^{4k+1} orthogonal to the line bundle spanned by v. For any vector field on S^{4k+1} which is in every point linear independent to v there is a nowhere vanishing section of E[2]. Since the Euler class of E vanishes, it suffices to show that $\kappa(E)$ is not zero by Theorem 5.5 (note that the spin structures of S^{4k+1} and that of E are unique up to homotopy).

Consider now the vector field

$$w : \mathbb{R}^{4k+2} \to \mathbb{R}^{4k+2}, \quad w(x) = (0, 0, -x_5, x_6, x_3, -x_4, -x_9, x_{10}, x_7, -x_8, \ldots)$$

Since $\langle w(x), x \rangle = \langle w(x), v(x) \rangle = 0$ we have that w is a section of E. Furthermore w is transverse to the zero section of E and the zero locus is given by

$$S = \{(x_1, x_2, 0, \ldots, 0) \in S^{4k+1} : x_1^2 + x_2^2 = 1\}.$$

In Example 3.10 we saw that $TS^{4k+1}|_S$ inherits the standard framing from the spin structure. But the induced framing on $E|_S$ cannot be the standard framing. To see this assume it inherits the standard framing and let $\tau_1, \ldots, \tau_n$ be a trivialization of $E|_S$, then, since the spin structure on E is induced by TS^{4k+1} and v, the map $S \to \mathrm{SO}(4k+2)$, $x \mapsto (x, v(x), \tau_1(x), \ldots, \tau_n(x))$ has to be nullhomotopic cf. Example 3.10 (note that $v|_S$ is tangent to S) which is a contradiction. Thus from Example 3.10 we deduce that the index of the framed divisor is not zero, hence $\kappa(E) = 1$ and therefore E does not admit a nowhere vanishing section from Theorem 5.5.

[2] For any pair of orthonormal vector fields v_1, v_2 of S^{4k+1} one can choose a new pair of orthonormal vector fields which consists of v and a section of E.

Remark 5.7 In [7, Theorem 1.6] the authors show that for any n-dimensional CW-complex of dimension X and any k-dimensional integral cohomology class $a \in H^k(X;\mathbb{Z})$ there exists an oriented vector bundle over X whose Euler class equals $2 \cdot N(n,k) \cdot a$.

Suppose $\dim X = 2k+1$. By Steenrod's exact sequence (ST) it follows that the Hurewicz map $\pi^n(X) \to H^n(X;\mathbb{Z})$ is surjective. Then for every $a \in H^n(X;\mathbb{Z})$ there is a map $f_a \in \pi^n(X)$ such that $f_a^*(\sigma) = a$, where $\sigma \in H^n(S;\mathbb{Z})$ denotes the generator such that 2σ equals to the Euler class of the tangent bundle TS^n of S^n. Clearly the vector bundle $f_a^*(TS^n)$ has Euler class $2 \cdot a$ and therefore $N(2k, 2k+1) = 1$ in the notation of [7].

Note that any vector bundle over S^n for $n \neq 2,4,8$ has an Euler class divisible by 2, cf. [2, 13]. In the cases $n = 2,4,8$ there are real vector bundles whose Euler class is a generator of $H^n(S^n;\mathbb{Z})$, namely the associated bundles to the Hopf fibrations $S^{2n-1} \to S^n$. We deduce

Proposition 5.8 *Suppose $n = 4$ or $n = 8$ and let M be a $(n+1)$-dimensional closed spin manifold. Denote by $\mathrm{Vect}_n(M)$ the set oriented vector bundles over M of rank n up to isomorphism. Let $E_0 \to S^n$ denote the oriented rank n vector bundle such that the Euler class of E_0 is a generator of $H^n(S^n;\mathbb{Z})$. Then the map*

$$\pi^n(M) \to \mathrm{Vect}_n(M), \quad f \mapsto f^*(E_0)$$

is injective.

Proof We consider $f_1, f_2 \in \pi^n(M)$ such that $E_1 := f_1^*(E_0) \cong f_2^*(E_0) =: E_2$ since they represent the Euler class the respective bundles. This implies $f_1^*(\sigma) = f_2^*(\sigma)$ for a generator in $H^n(S^n;\mathbb{Z})$. Thus it remains to show that $\kappa(f_1) = \kappa(f_2)$. Let $x_i \in S^n$ be a regular value for f_i for $i = 1,2$. There is a section $\sigma_{0,i} : S^n \to E_0$ which is transverse to the zero section with an isolated zero in x_i (note that the Poincaré dual of x_i in S^n represents the Euler class of E_0. Therefore $\sigma_{0,i}$ can only exist since if the Euler class is a generator, since the index of transverse sections is always ± 1). Then $\sigma_i := f^*(\sigma_{0,i})$ is a transverse section of E_i. Note that from the Pontryagin-Thom construction we may assume that $f_i^{-1}(x_i)$ is connected, hence the zero locus of σ_i coincides with $f_i^{-1}(x_i)$. Moreover the framed divisor of E_i coincides with the degree of f_i (cf. Definitions 3.8 and 5.2). Since $E_1 \cong E_2$ we have $\kappa(E_1) \cong \kappa(E_2)$ by construction of the framed divisor and Proposition 5.1. From $f_1^*(\sigma) = f_2^*(\sigma)$ and $\kappa(f_1) = \kappa(E_1) = \kappa(E_2) = \kappa(f_2)$ it follows from Theorem 4.1 that f_1 is homotopic to f_2. □

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