

Colombeau algebras without asymptotics

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Received: 30 May 2017 / Revised: 31 October 2017 / Accepted: 7 November 2017 /
Published online: 23 November 2017
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Abstract We present a construction of algebras of generalized functions of Colombeau-type which, instead of asymptotic estimates with respect to a regularization parameter, employs only topological estimates on certain spaces of kernels for its definition.

Keywords Nonlinear generalized functions · Colombeau algebras · Asymptotic estimates · Elementary Colombeau algebra · Diffeomorphism invariance

Mathematics Subject Classification 46F30 · 46F05

1 Introduction

Colombeau algebras, as introduced by Colombeau [1, 2], today represent the most widely studied approach to embedding the space of Schwartz distributions into an algebra of generalized functions such that the product of smooth functions as well as partial derivatives of distributions are preserved. These algebras have found numerous applications in situations involving singular objects, differentiation and nonlinear operations (see, e.g., [9, 12, 15]).

All constructions of Colombeau algebras so far incorporate certain *asymptotic estimates* for the definition of the spaces of moderate and negligible functions, the quotient of which constitutes the algebra. There is a certain degree of freedom in the asymp-

Dedicated to the memory of Prof. Todor Gramchev.

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otic scale employed for these estimates; while commonly a polynomial scale is used, generalizations in several directions are possible. For an overview we refer to works on asymptotic scales [3, 7], $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras [5], sequence spaces with exponent weights [6] and asymptotic gauges [8].

In this article we will present an algebra of generalized functions which instead of asymptotic estimates employs only topological estimates on certain spaces of kernels for its definition. This is a direct generalization of the usual seminorm estimates valid for distributions.

We will first develop the most general setting in the local scalar case, namely that of diffeomorphism invariant full Colombeau algebras. We will then derive a simpler variant, similar to Colombeau’s elementary algebra. Finally, we give canonical mappings into the most important Colombeau algebras, which points to a certain universality of the construction offered here.

2 Preliminaries

\mathbb{N} and \mathbb{N}_0 denote the sets of positive and non-negative integers, respectively, and \mathbb{R}^+ the set of nonnegative real numbers. Concerning distribution theory we use the notation and terminology of L. Schwartz [18].

Given any subsets $K, L \subseteq \mathbb{R}^n$ (with $n \in \mathbb{N}$) the relation $K \Subset L$ means that K is compact and contained in the interior L° of L .

Let $\Omega \subseteq \mathbb{R}^n$ be open. $C^\infty(\Omega)$ is the space of complex-valued smooth functions on Ω . For any $K, L \Subset \Omega$, $m, l \in \mathbb{N}_0$ and any bounded subset $B \subseteq C^\infty(\Omega)$ we set

$$\begin{aligned} \|f\|_{K,m} &:= \sup_{x \in K, |\alpha| \leq m} |\partial^\alpha f(x)| && (f \in C^\infty(\Omega)), \\ \|\vec{\varphi}\|_{K,m;L,l} &:= \sup_{\substack{x \in K, |\alpha| \leq m \\ y \in L, |\beta| \leq l}} \left| \partial_x^\alpha \partial_y^\beta \vec{\varphi}(x)(y) \right| && (\vec{\varphi} \in C^\infty(\Omega, \mathcal{D}(\Omega))), \\ \|\vec{\varphi}\|_{K,m;B} &:= \sup_{\substack{x \in K, |\alpha| \leq m \\ f \in B}} \left| \langle f(y), \partial_x^\alpha \vec{\varphi}(x)(y) \rangle \right| && (\vec{\varphi} \in C^\infty(\Omega, \mathcal{E}'(\Omega))). \end{aligned}$$

Note that $\|\cdot\|_{K,m}$, $\|\cdot\|_{K,m;L,l}$ and $\|\cdot\|_{K,m;B}$ are continuous seminorms on the respective spaces.

We define $\vec{\delta} \in C^\infty(\Omega, \mathcal{E}'(\Omega))$ by $\vec{\delta}(x) := \delta_x$ for $x \in \Omega$, where δ_x is the delta distribution at x .

$\mathcal{D}_L(\Omega)$ is the space of test functions on Ω with support in L . For two locally convex spaces E and F , $\mathcal{L}(E, F)$ denotes the space of linear continuous mappings from E to F , endowed with the topology of bounded convergence. By $\mathcal{U}_x(\Omega)$ we denote the filter base of open neighborhoods of a point x in Ω , and by $\mathcal{U}_K(\Omega)$ the filter base of open neighborhoods of K . By $\text{csn}(E)$ we denote the set of continuous seminorms of a locally convex space E . $B_r(x) := \{y \in \mathbb{R}^n : \|y-x\| < r\}$ is the open Euclidean ball of radius $r > 0$ at $x \in \mathbb{R}^n$, and for any subset $K \subseteq \mathbb{R}^n$ we define $B_r(K) := \bigcup_{x \in K} B_r(x)$.

Our notion of smooth functions between arbitrary locally convex spaces is that of convenient calculus [11]. In particular, $d^k f$ denotes the k -th differential of a smooth mapping f .

3 Construction of the algebra

Throughout this section let $\Omega \subseteq \mathbb{R}^n$ be a fixed open set. Let \mathcal{C} be the category of locally convex spaces with smooth mappings in the sense of convenient calculus as morphisms.

Definition 1 Consider $C^\infty(-, \mathcal{D}(\Omega))$ and $C^\infty(-)$ as sheaves with values in \mathcal{C} . We define the *basic space* of nonlinear generalized functions on Ω to be the set of sheaf homomorphisms

$$\mathcal{B}(\Omega) := \text{Hom}(C^\infty(-, \mathcal{D}(\Omega)), C^\infty(-)).$$

Hence, an element of $\mathcal{B}(\Omega)$ is given by a family $(R_U)_U$ of mappings

$$R_U \in C^\infty(C^\infty(U, \mathcal{D}(\Omega)), C^\infty(U)) \quad (U \subseteq \Omega \text{ open})$$

satisfying $R_U(\vec{\varphi})|_V = R_V(\vec{\varphi}|_V)$ for all open subsets $V \subseteq U$ and $\vec{\varphi} \in C^\infty(U, \mathcal{D}(\Omega))$. We will casually write R in place of R_U .

Remark 2 The basic space $\mathcal{B}(\Omega)$ can be identified with the set of all mappings $R \in C^\infty(C^\infty(\Omega, \mathcal{D}(\Omega)), C^\infty(\Omega))$ such that for any open subset $U \subseteq \Omega$ and $\vec{\varphi}, \vec{\psi} \in C^\infty(\Omega, \mathcal{D}(\Omega))$ the equality $\vec{\varphi}|_U = \vec{\psi}|_U$ implies $R(\vec{\varphi})|_U = R(\vec{\psi})|_U$ (cf. [10]).

$\mathcal{B}(\Omega)$ is a $C^\infty(\Omega)$ -module with multiplication

$$(f \cdot R)_U(\vec{\varphi}) = f|_U \cdot R_U(\vec{\varphi})$$

for $R \in \mathcal{B}(\Omega)$, $f \in C^\infty(\Omega)$, $U \subseteq \Omega$ open and $\vec{\varphi} \in C^\infty(U, \mathcal{D}(\Omega))$. Moreover, it is an associative commutative algebra with product $(R \cdot S)_U(\vec{\varphi}) := R_U(\vec{\varphi}) \cdot S_U(\vec{\varphi})$.

A distribution $u \in \mathcal{D}'(\Omega)$ defines a sheaf morphism from $C^\infty(-, \mathcal{D}(\Omega))$ to $C^\infty(-)$. In fact, for $U \subseteq \Omega$ open and $\vec{\varphi} \in C^\infty(U, \mathcal{D}(\Omega))$ the function $x \mapsto \langle u, \varphi(x) \rangle$ is an element of $C^\infty(U)$ (see [18, Chap. IV, §1, Th. II, p. 105] or [20, Theorem 40.2, p. 416]). More abstractly, this can be seen using the theory of topological tensor products [16, 17, 20] as follows:

$$C^\infty(U, \mathcal{D}(\Omega)) \cong C^\infty(U) \widehat{\otimes} \mathcal{D}(\Omega) \cong \mathcal{L}(\mathcal{D}'(\Omega), C^\infty(U)),$$

where $C^\infty(U) \widehat{\otimes} \mathcal{D}(\Omega)$ denotes the completed projective tensor product of $C^\infty(U)$ and $\mathcal{D}(\Omega)$. The assignment $\vec{\varphi} \mapsto \langle u, \vec{\varphi} \rangle$ is smooth, being linear and continuous [11, 1.3, p. 9]. Hence, we have the following embeddings of distributions and smooth functions into $\mathcal{B}(\Omega)$:

Definition 3 We define $\iota : \mathcal{D}'(\Omega) \rightarrow \mathcal{B}(\Omega)$ and $\sigma : C^\infty(\Omega) \rightarrow \mathcal{B}(\Omega)$ by

$$\begin{aligned} (uu)(\vec{\varphi})(x) &:= \langle u, \vec{\varphi}(x) \rangle \quad (u \in \mathcal{D}'(\Omega)) \\ (\sigma f)(\vec{\varphi})(x) &:= f(x) \quad (f \in C^\infty(\Omega)) \end{aligned}$$

for $\vec{\varphi} \in C^\infty(U, \mathcal{D}(\Omega))$ with $U \subseteq \Omega$ open and $x \in U$.

Clearly ι is linear and σ is an algebra homomorphism. Directional derivatives on $\mathcal{B}(\Omega)$ then are defined as follows:

Definition 4 Let $X \in C^\infty(\Omega, \mathbb{R}^n)$ be a smooth vector field and $R \in \mathcal{B}(\Omega)$. We define derivatives $\widetilde{D}_X : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$ and $\widehat{D}_X : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$ by

$$\begin{aligned} (\widetilde{D}_X R)(\vec{\varphi}) &:= D_X(R_U(\vec{\varphi})) \\ (\widehat{D}_X R)(\vec{\varphi}) &:= -dR_U(\vec{\varphi})(D_X^{\text{SK}} \vec{\varphi}) + D_X(R_U(\vec{\varphi})) \end{aligned}$$

for $\vec{\varphi} \in C^\infty(U, \mathcal{D}(\Omega))$ with $U \subseteq \Omega$ open, where we set

$$D_X^{\text{SK}} \vec{\varphi} := D_X \vec{\varphi} + D_X^w \circ \vec{\varphi}.$$

Here, $(D_X \vec{\varphi})(x)$ is the directional derivative of $\vec{\varphi}$ at x in direction $X(x)$ and $(D_X^w \circ \vec{\varphi})(x)$ is the Lie derivative of $\vec{\varphi}(x)$ considered as a differential form, given by $D_X^w(\vec{\varphi}(x)) = D_X(\vec{\varphi}(x)) + (\text{Div } X)(x) \cdot \vec{\varphi}(x)$.

Note that both \widetilde{D}_X and \widehat{D}_X satisfy the Leibniz rule. We have

$$\widetilde{D}_x \circ \sigma = \sigma \circ \widetilde{D}_X, \quad \widehat{D}_X \circ \sigma = \sigma \circ \widehat{D}_X, \quad \widehat{D}_X \circ \iota = \iota \circ \widehat{D}_X.$$

While \widetilde{D}_X is $C^\infty(\Omega)$ -linear in X , \widehat{D}_X is only \mathbb{C} -linear in X . We refer to [13, 14] for a discussion of the role of these derivatives in differential geometry.

Definition 5 For $k \in \mathbb{N}_0$ we set

$$\begin{aligned} \mathcal{P}_k &:= \mathbb{R}^+[y_0, \dots, y_k], \\ \mathcal{I}_k &:= \{\lambda \in \mathbb{R}^+[y_0, \dots, y_k, z_0, \dots, z_k] \mid \lambda(y_0, \dots, y_k, 0, \dots, 0) = 0\}. \end{aligned}$$

More explicitly, \mathcal{P}_k is the commutative semiring of polynomials in the $k+1$ commuting variables y_0, \dots, y_k with coefficients in \mathbb{R}^+ . Similarly, \mathcal{I}_k is the commutative semiring in the $2(k+1)$ commuting variables $y_0, \dots, y_k, z_0, \dots, z_k$ with coefficients in \mathbb{R}^+ and such that, if $\lambda \in \mathcal{I}_k$ is given by the finite sum

$$\lambda = \sum_{\alpha, \beta \in \mathbb{N}_0^{k+1}} \lambda_{\alpha\beta} y^\alpha z^\beta,$$

then $\lambda_{\alpha 0} = 0$ for all α . Note that \mathcal{P}_k is a subsemiring of \mathcal{P}_{k+1} and \mathcal{I}_k a subsemiring of \mathcal{I}_{k+1} . Furthermore, \mathcal{I}_k is an ideal in \mathcal{P}_k if \mathcal{P}_k is considered as a subsemiring of $\mathbb{R}^+[y_0, \dots, y_k, z_0, \dots, z_k]$. Given $\lambda \in \mathcal{P}_k$ and $y_i \leq y'_i$ for $i = 0 \dots k$ we have

$\lambda(y) \leq \lambda(y')$. For $\lambda, \mu \in \mathcal{P}_k$ we write $\lambda \leq \mu$ if $\lambda(y) \leq \mu(y)$ for all $y \in (\mathbb{R}^+)^{k+1}$, and similarly for $\lambda, \mu \in \mathcal{I}_k$.

We can now formulate the following definitions of moderateness and negligibility, not involving any asymptotic estimates:

Definition 6 An element $R \in \mathcal{B}(\Omega)$ is called *moderate* if

$$\begin{aligned} & (\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U) (\forall m, k \in \mathbb{N}_0) \\ & (\exists c, l \in \mathbb{N}_0) (\exists \lambda \in \mathcal{P}_k) (\forall \vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(U, \mathcal{D}_L(U))) : \\ & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \leq \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}). \end{aligned}$$

The subset of all moderate elements of $\mathcal{B}(\Omega)$ is denoted by $\mathcal{M}(\Omega)$.

Definition 7 An element $R \in \mathcal{B}(\Omega)$ is called *negligible* if

$$\begin{aligned} & (\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U) (\forall m, k \in \mathbb{N}_0) (\exists c, l \in \mathbb{N}_0) \\ & (\exists \lambda \in \mathcal{I}_k) (\exists B \subseteq C^\infty(\Omega) \text{ bounded}) (\forall \vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(U, \mathcal{D}_L(U))) : \\ & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|\vec{\varphi}_1\|_{K,c;B}, \dots, \|\vec{\varphi}_k\|_{K,c;B}). \end{aligned}$$

The subset of all negligible elements of $\mathcal{B}(\Omega)$ is denoted by $\mathcal{N}(\Omega)$.

It is worthwhile to discuss possible simplifications of these definitions, which at this stage should be considered more as a proof of concept than as the definite form they should have. First, we note that we cannot replace $(\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U)$ by $(\forall K, L \Subset \Omega)$. In fact, in the second case K and L can be distant from each other, while in the first case it suffices to control the situation where K and L are close to each other. However, the following result shows that we can always assume $K \Subset L$ and that the $\vec{\varphi}_0, \dots, \vec{\varphi}_k$ are given merely on an arbitrary open neighborhood of K , i.e., as elements of the direct limit $C^\infty(K, \mathcal{D}_L(\Omega)) := \varinjlim_{V \in \mathcal{U}_K(\Omega)} C^\infty(V, \mathcal{D}_L(\Omega))$:

Proposition 8 Let $R \in \mathcal{B}(\Omega)$. Then R is moderate if and only if

$$\begin{aligned} & (\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U : K \Subset L) (\forall m, k \in \mathbb{N}_0) \\ & (\exists c, l \in \mathbb{N}_0) (\exists \lambda \in \mathcal{P}_k) (\forall \vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(K, \mathcal{D}_L(U))) : \\ & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \leq \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}). \end{aligned}$$

Similarly, R is negligible if and only if

$$\begin{aligned} & (\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U : K \Subset L) (\forall m, k \in \mathbb{N}_0) (\exists c, l \in \mathbb{N}_0) \\ & (\exists \lambda \in \mathcal{I}_k) (\exists B \subseteq C^\infty(U) \text{ bounded}) (\forall \vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(K, \mathcal{D}_L(U))) : \\ & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|\vec{\varphi}_1\|_{K,c;B}, \dots, \|\vec{\varphi}_k\|_{K,c;B}). \end{aligned}$$

Proof Obviously each of these conditions is weaker than the corresponding one of Definition 6 or Definition 7.

Suppose we are given $R \in \mathcal{B}(\Omega)$ such that the condition stated for moderateness holds. Given $x \in \Omega$ there hence exists some $U \in \mathcal{U}_x(\Omega)$. Now given arbitrary $K, L \Subset U$ we choose a set $L' \Subset U$ such that $K \cup L \Subset L'$. Fixing $m, k \in \mathbb{N}_0$ for the moderateness test, for (K, L') we hence obtain $c, l \in \mathbb{N}_0$ and $\lambda \in \mathcal{P}_k$. Now fix some $\vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(U, \mathcal{D}_L(U))$; each of those represents an element of $C^\infty(K, \mathcal{D}_{L'}(U))$, whence we have the estimate

$$\begin{aligned} \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} &\leq \lambda(\|\vec{\varphi}_0\|_{K,c;L',l}, \dots, \|\vec{\varphi}_k\|_{K,c;L',l}) \\ &= \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}) \end{aligned}$$

where the last equality follows because the $\vec{\varphi}_0, \dots, \vec{\varphi}_k$ take values in $\mathcal{D}_L(U)$. This shows that R is moderate.

For the case of negligibility we proceed similarly until we obtain $c, l \in \mathbb{N}_0, \lambda \in \mathcal{I}_k$ and $B \subseteq C^\infty(U)$. Let $\chi \in \mathcal{D}(U)$ be such that $\chi \equiv 1$ on a neighborhood of L' and set $B' := \{\chi f \mid f \in B\} \subseteq C^\infty(\Omega)$, which is bounded. For any $\vec{\varphi}_0, \dots, \vec{\varphi}_k$ we then obtain

$$\begin{aligned} \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} &\leq \lambda(\|\vec{\varphi}_0\|_{K,c;L',l}, \dots, \|\vec{\varphi}_k\|_{K,c;L',l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|\vec{\varphi}_1\|_{K,c;B}, \dots, \|\vec{\varphi}_k\|_{K,c;B}) \\ &= \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B'}, \|\vec{\varphi}_1\|_{K,c;B'}, \dots, \|\vec{\varphi}_k\|_{K,c;B'}) \end{aligned}$$

which proves negligibility of R . □

If the test of Definition 6, Definition 7 or Definition 8 holds on some U then clearly it also holds on any open subset of U . The following characterization of moderateness and negligibility is obtained by applying polarization identities to the differentials of R :

Lemma 9 *Let $R \in \mathcal{B}(\Omega)$.*

(i) *R is moderate if and only if*

$$\begin{aligned} &(\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U) (\forall m, k \in \mathbb{N}_0) \\ &(\exists c, l \in \mathbb{N}_0) (\exists \lambda \in \mathcal{P}_{\min(1,k)}) (\forall \vec{\varphi}, \vec{\psi} \in C^\infty(U, \mathcal{D}_L(U))) : \\ &\|d^k R(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi})\|_{K,m} \leq \begin{cases} \lambda(\|\vec{\varphi}\|_{K,c;L,l}) & \text{if } k = 0, \\ \lambda(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\psi}\|_{K,c;L,l}) & \text{if } k \geq 1. \end{cases} \end{aligned}$$

(ii) *R is negligible if and only if*

$$\begin{aligned} &(\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U) (\forall m, k \in \mathbb{N}_0) (\exists c, l \in \mathbb{N}_0) \\ &(\exists \lambda \in \mathcal{I}_{\min(1,k)}) (\exists B \subseteq C^\infty(\Omega) \text{ bounded}) (\forall \vec{\varphi}, \vec{\psi} \in C^\infty(U, \mathcal{D}_L(U))) : \end{aligned}$$

$$\|d^k R(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi})\|_{K,m} \leq \begin{cases} \lambda(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\varphi} - \vec{\delta}\|_{K,c;B}) & \text{if } k = 0, \\ \lambda(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\psi}\|_{K,c;L,l}, \|\vec{\varphi} - \vec{\delta}\|_{K,c;B}, \|\vec{\psi}\|_{K,c;B}) & \text{if } k \geq 1. \end{cases}$$

Proof We assume $k \geq 1$, as for $k = 0$ the statements are identical. (i) “ \Rightarrow ”: One obtains $\lambda \in \mathcal{P}_k$ such that

$$\|d^k R(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi})\|_{K,m} \leq \lambda(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\psi}\|_{K,c;L,l}, \dots, \|\vec{\psi}\|_{K,c;L,l}) \\ = \lambda'(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\psi}\|_{K,c;L,l})$$

with $\lambda' \in \mathcal{P}_1$ given by $\lambda'(y_0, y_1) = \lambda(y_0, y_1, \dots, y_1)$.

“ \Leftarrow ”: One obtains $\lambda \in \mathcal{P}_1$. We then use the polarization identity [19, eq. (7), p. 471]

$$d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k) = \frac{1}{n!} \sum_{a=1}^k (-1)^{k-a} \sum_{\substack{J \subseteq \{1..k\} \\ |J|=a}} \Delta^*(d^k R(\vec{\varphi}_0))(S_J)$$

where $S_J := \sum_{i \in J} \vec{\varphi}_i$ and we have set $\Delta^*(d^k R(\vec{\varphi}_0))(\vec{\psi}) = d^k R(\vec{\varphi}_0)(\vec{\psi}, \dots, \vec{\psi})$.

Hence,

$$\|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \leq \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \|\Delta^*(d^k R(\vec{\varphi}_0))(S_J)\|_{K,m} \\ \leq \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \|S_J\|_{K,c;L,l}) \\ \leq \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \sum_{i \in J} \|\vec{\varphi}_i\|_{K,c;L,l}) \\ = \lambda'(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l})$$

with $\lambda' \in \mathcal{P}_k$ given by

$$\lambda'(y_0, \dots, y_k) = \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \lambda(y_0, \sum_{i \in J} y_i).$$

(ii) “ \Rightarrow ”: We have $\lambda \in \mathcal{I}_k$ such that

$$\|d^k R(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi})\|_{K,m} \leq \lambda(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\psi}\|_{K,c;L,l}, \dots, \|\vec{\psi}\|_{K,c;L,l}, \\ \|\vec{\varphi} - \vec{\delta}\|_{K,c;B}, \|\vec{\psi}\|_{K,c;B}, \dots, \|\vec{\psi}\|_{K,c;B}) \\ = \lambda'(\|\vec{\varphi}\|_{K,c;L,l}, \|\vec{\psi}\|_{K,c;L,l}, \|\vec{\varphi} - \vec{\delta}\|_{K,c;B}, \|\vec{\psi}\|_{K,c;B})$$

with $\lambda' \in \mathcal{I}_k$ given by

$$\lambda'(y_0, y_1, z_0, z_1) = \lambda(y_0, y_1, \dots, y_1, z_0, z_1, \dots, z_1).$$

“ \Leftarrow ”: We obtain $\lambda \in \mathcal{I}_1$ such that, as above,

$$\begin{aligned} & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \|S_J\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|S_J\|_{K,c;B}) \\ & \leq \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \sum_{i \in J} \|\vec{\varphi}_i\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \sum_{i \in J} \|\vec{\varphi}_i\|_{K,c;B}) \\ & = \lambda'(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|\vec{\varphi}_1\|_{K,c;B}, \dots, \|\vec{\varphi}_k\|_{K,c;B}) \end{aligned}$$

with $\lambda' \in \mathcal{I}_k$ given by

$$\lambda'(y_0, \dots, y_k, z_0, \dots, z_k) = \frac{1}{n!} \sum_{a=1}^k \sum_{|J|=a} \lambda \left(y_0, \sum_{i \in J} y_i, z_0, \sum_{i \in J} z_i \right).$$

□

Note that the polarization identities could be applied also in the formulation of Proposition 8.

Proposition 10 $\mathcal{N}(\Omega) \subseteq \mathcal{M}(\Omega)$.

Proof Let $R \in \mathcal{N}(\Omega)$ and fix $x \in \Omega$ for the moderateness test. By negligibility of R there exists $U \in \mathcal{U}_x(\Omega)$ as in Definition 7. Let $K, L \Subset U$ and $m, k \in \mathbb{N}_0$ be arbitrary. Then there exist c, l, λ and B such that the estimate of Definition 7 holds. We know that $\lambda \in \mathcal{I}_k$ is given by a finite sum

$$\lambda(y_0, \dots, y_k, z_0, \dots, z_k) = \sum_{\alpha, \beta} \lambda_{\alpha\beta} y^\alpha z^\beta.$$

It suffices to show that there are $\lambda_1, \lambda_2 \in \mathcal{P}_0$ such that for any $\vec{\varphi} \in C^\infty(U, \mathcal{D}_L(U))$ we have the estimates

$$\|\vec{\varphi} - \vec{\delta}\|_{K,c;B} \leq \lambda_1(\|\vec{\varphi}\|_{K,c;L,l}), \quad (1)$$

$$\|\vec{\varphi}\|_{K,c;B} \leq \lambda_2(\|\vec{\varphi}\|_{K,c;L,l}). \quad (2)$$

In fact, these inequalities imply

$$\begin{aligned} & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \sum_{\alpha, \beta} \lambda_{\alpha\beta} \|\vec{\varphi}_0\|_{K,c;L,l}^{\alpha_0} \cdots \|\vec{\varphi}_k\|_{K,c;L,l}^{\alpha_k} \end{aligned}$$

$$\begin{aligned}
& \cdot \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}^{\beta_0} \cdot \|\vec{\varphi}_1\|_{K,c;B}^{\beta_1} \cdot \cdots \cdot \|\vec{\varphi}_k\|_{K,c;B}^{\beta_k} \\
& \leq \sum_{\alpha,\beta} \lambda_{\alpha\beta} \|\vec{\varphi}_0\|_{K,c;L,l}^{\alpha_0} \cdot \cdots \cdot \|\vec{\varphi}_k\|_{K,c;L,l}^{\alpha_k} \\
& \quad \cdot \lambda_1(\|\vec{\varphi}_0\|_{K,c;L,l})^{\beta_0} \cdot \lambda_2(\|\vec{\varphi}_1\|_{K,c;L,l})^{\beta_1} \cdots \lambda_2(\|\vec{\varphi}_k\|_{K,c;L,l})^{\beta_k} \\
& = \lambda'(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l})
\end{aligned}$$

with $\lambda' \in \mathcal{P}_k$ given by

$$\lambda'(y_0, \dots, y_k) = \sum \lambda_{\alpha\beta} y^\alpha \lambda_1(y_0)^{\beta_0} \lambda_2(y_1)^{\beta_1} \cdots \lambda_2(y_k)^{\beta_k}.$$

Inequality (1) is seen as follows:

$$\begin{aligned}
\|\vec{\varphi} - \vec{\delta}\|_{K,c;B} &= \sup_{\substack{x \in K, |\alpha| \leq c \\ f \in B}} \left| \int_L f(y) \partial_x^\alpha \vec{\varphi}(x)(y) \, dy - \partial^\alpha f(x) \right| \\
&\leq |L| \cdot \sup_{f \in B} \|f\|_{L,0} \cdot \|\vec{\varphi}\|_{K,c;L,l} + \sup_{f \in B} \|f\|_{K,c} \\
&= \lambda_1(\|\vec{\varphi}\|_{K,c;L,l})
\end{aligned}$$

with $\lambda_1(y_0) = |L| \cdot \sup_{f \in B} \|f\|_{L,0} \cdot y_0 + \sup_{f \in B} \|f\|_{K,c}$, where $|L|$ denotes the Lebesgue measure of L . Similarly, inequality (2) results from

$$\begin{aligned}
\|\vec{\varphi}\|_{K,c;B} &= \sup_{\substack{x \in K, |\alpha| \leq c \\ f \in B}} \left| \int_L f(y) \partial_x^\alpha \vec{\varphi}(x)(y) \, dy \right| \\
&\leq |L| \cdot \sup_{f \in B} \|f\|_{L,0} \cdot \|\vec{\varphi}\|_{K,c;L,l} \\
&= \lambda_2(\|\vec{\varphi}\|_{K,c;L,l})
\end{aligned}$$

with $\lambda_2(y_0) = |L| \cdot \sup_{f \in B} \|f\|_{L,0} \cdot y_0$. □

Proposition 11 $\mathcal{M}(\Omega)$ is a subalgebra of $\mathcal{B}(\Omega)$ and $\mathcal{N}(\Omega)$ is an ideal in $\mathcal{M}(\Omega)$.

Proof This is evident from the definitions. □

Theorem 12 Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. Then

- (i) u is moderate,
- (ii) σf is moderate,
- (iii) $\iota f - \sigma f$ is negligible, and
- (iv) if u is negligible then $u = 0$.

Proof (i): Fix x for the moderateness test and let $U \in \mathcal{U}_x(\Omega)$ be arbitrary. Fix any $K, L \Subset U$ and $m \in \mathbb{N}_0$. Then there are constants $C = C(L) \in \mathbb{R}^+$ and $l = l(L) \in \mathbb{N}_0$ such that $|\langle u, \varphi \rangle| \leq C \|\varphi\|_{L,l}$ for all $\varphi \in \mathcal{D}_L(\Omega)$. Hence, we see that

$$\begin{aligned} \|(u)(\vec{\varphi}_0)\|_{K,m} &= \|\langle u, \vec{\varphi}_0 \rangle\|_{K,m} = \sup_{x \in K, |\alpha| \leq m} |\langle u, \partial_x^\alpha \vec{\varphi}_0(x) \rangle| \\ &\leq C \cdot \sup_{\substack{x \in K, |\alpha| \leq m \\ y \in L, |\beta| \leq l}} \left| \partial_x^\alpha \partial_y^\beta \vec{\varphi}_0(x)(y) \right| = C \|\vec{\varphi}_0\|_{K,m;L,l} = \lambda(\|\vec{\varphi}_0\|_{K,m;L,l}). \end{aligned}$$

with $\lambda(y_0) = Cy_0$. Moreover, we have

$$\|\mathbf{d}(u)(\vec{\varphi}_0)(\vec{\varphi}_1)\|_{K,m} \leq C \|\vec{\varphi}_1\|_{K,m;L,l} = \lambda(\|\vec{\varphi}_0\|_{K,m;L,l}, \|\vec{\varphi}_1\|_{K,m;L,l})$$

with $\lambda(y_0, y_1) = Cy_1$. Higher differentials of u vanish and the moderateness test is satisfied with $\lambda = 0$ for $k \geq 2$.

(ii): Fix x and let $U \in \mathcal{U}_x(\Omega)$ be arbitrary. For any $K, L \Subset U$ and $m \in \mathbb{N}_0$ we have

$$\|(\sigma f)(\vec{\varphi}_0)\|_{K,m} = \|f\|_{K,m} = \lambda(\|\vec{\varphi}_0\|_{K,0;L,0})$$

with $\lambda(y_0) = \|f\|_{K,m}$. Differentials of σf vanish, i.e., $\lambda = 0$ for $k \geq 1$.

(iii): Fix x and let $U \in \mathcal{U}_x(\Omega)$ be arbitrary. For any $K, L \Subset U$ and $m, k \in \mathbb{N}_0$ we have

$$\begin{aligned} (uf - \sigma f)(\vec{\varphi}_0) &= \langle f, \vec{\varphi}_0 - \vec{\delta} \rangle, \\ \mathbf{d}(uf - \sigma f)(\vec{\varphi}_0)(\vec{\varphi}_1) &= \langle f, \vec{\varphi}_1 \rangle, \\ \mathbf{d}^k(uf - \sigma f)(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k) &= 0 \quad \text{for } k \geq 2. \end{aligned}$$

Hence, with $c = m$, $l = 0$ and $B = \{f\}$ the negligibility test is satisfied with $\lambda(y_0, z_0) = z_0$ for $k = 0$, $\lambda(y_0, y_1, z_0, z_1) = z_1$ for $k = 1$ and $\lambda = 0$ for $k \geq 2$.

(iv): We show that every point $x \in \Omega$ has an open neighborhood V such that $u|_V = 0$, which implies $u = 0$.

Given $x \in \Omega$, let $U \in \mathcal{U}_x(\Omega)$ be as in the characterization of negligibility in Proposition 8. Choose an open neighborhood V of x such that $K := \overline{V} \Subset U$ and $r > 0$ such that $L := \overline{B_r(K)} \Subset U$. With $k = m = 0$, Proposition 8 gives $c, l \in \mathbb{N}_0$, $\lambda \in \mathcal{I}_0$ and $B \subseteq C^\infty(U)$, where λ has the form

$$\lambda(y, z) = \sum_{\alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}} \lambda_{\alpha\beta} y^\alpha z^\beta.$$

Choose $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } \varphi \subseteq B_1(0)$, $\int \varphi(x) dx = 1$ and $\int x^\gamma \varphi(x) dx = 0$ for $\gamma \in \mathbb{N}_0^n$ with $0 < |\gamma| \leq q$, where q is chosen such that $\beta(q+1) > \alpha(n+c+l)$ for all α, β with $\lambda_{\alpha\beta} \neq 0$ (e.g., take $q = (n+c+l) \deg_y \lambda$, where $\deg_y \lambda$ is the degree of λ with respect to y). For $\varepsilon > 0$ set $\varphi_\varepsilon(y) = \varepsilon^{-n} \varphi(y/\varepsilon)$. Then for $\varepsilon < r$, $\vec{\varphi}_\varepsilon(x)(y) := \varphi_\varepsilon(y-x)$ defines an element $\vec{\varphi}_\varepsilon \in C^\infty(K, \mathcal{D}_L(\Omega))$ because $\text{supp } \varphi_\varepsilon(-x) = x + \text{supp } \varphi_\varepsilon \subseteq B_\varepsilon(x) \subseteq B_r(K) \subseteq L$ for $x \in B_{r-\varepsilon}(K)$. Consequently, we have

$$\|(u)(\vec{\varphi}_\varepsilon)\|_{K,0} \leq \lambda(\|\vec{\varphi}_\varepsilon\|_{K,c;L,l}, \|\vec{\varphi}_\varepsilon - \vec{\delta}\|_{K,c;B}).$$

Because of the estimates

$$\begin{aligned}\|\vec{\varphi}_\varepsilon\|_{K,c;L,l} &= O(\varepsilon^{-(n+l+c)}) \\ \|\vec{\varphi}_\varepsilon - \vec{\delta}\|_{K,c;B} &= O(\varepsilon^{q+1}),\end{aligned}$$

which may be verified by a direct calculation, we have

$$\|(u)(\vec{\varphi}_\varepsilon)\|_{K,0} \leq \sum_{\alpha,\beta} \lambda_{\alpha,\beta} \cdot O(\varepsilon^{-\alpha(n+c+l)}) \cdot O(\varepsilon^{\beta(q+1)}) \rightarrow 0$$

by the choice of q , which means that $(u)(\vec{\varphi}_\varepsilon)|_V \rightarrow 0$ in $C(V)$ and hence also in $\mathcal{D}'(V)$. On the other hand, we have

$$\langle u, \vec{\varphi}_\varepsilon \rangle|_V \rightarrow u|_V$$

in $\mathcal{D}'(V)$, as is easily verified. This completes the proof. \square

Theorem 13 For $X \in C^\infty(\Omega, \mathbb{R}^n)$ we have

- (i) $\widetilde{D}_X(\mathcal{M}(\Omega)) \subseteq \mathcal{M}(\Omega)$ and $\widehat{D}_X(\mathcal{M}(\Omega)) \subseteq \mathcal{M}(\Omega)$,
- (ii) $\widetilde{D}_X(\mathcal{N}(\Omega)) \subseteq \mathcal{N}(\Omega)$ and $\widehat{D}_X(\mathcal{N}(\Omega)) \subseteq \mathcal{N}(\Omega)$.

Proof The claims for \widetilde{D}_X are clear because

$$\begin{aligned}\|d^k(\widetilde{D}_X R)(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi})\|_{K,m} &= \|D_X(d^k R(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi}))\|_{K,m} \\ &\leq C \|d^k R(\vec{\varphi})(\vec{\psi}, \dots, \vec{\psi})\|_{K,m+1}\end{aligned}$$

for some constant C depending on X . As to \widehat{D}_X , we have to deal with terms of the form

$$d^{k+1} R(\vec{\varphi})(D_X^{\text{SK}} \vec{\varphi}, \vec{\psi}, \dots, \vec{\psi}) \quad \text{and} \quad d^k R(\vec{\varphi})(D_X^{\text{SK}} \vec{\psi}, \vec{\psi}, \dots, \vec{\psi})$$

for which we use the estimate

$$\|D_X^{\text{SK}} \vec{\varphi}\|_{K,c;L,l} \leq C \|\vec{\varphi}\|_{K,c,+1;L,l+1}$$

for some constant C depending on X . \square

We now come to the quotient algebra.

Definition 14 We define the Colombeau algebra of generalized functions on Ω by $\mathcal{G}(\Omega) := \mathcal{M}(\Omega)/\mathcal{N}(\Omega)$.

$\mathcal{G}(\Omega)$ is a $C^\infty(\Omega)$ -module and an associative commutative algebra with unit $\sigma(1)$. ι is a linear embedding of $\mathcal{D}'(\Omega)$ and σ an algebra embedding of $C^\infty(\Omega)$ into $\mathcal{G}(\Omega)$ such that $\iota f = \sigma f$ in $\mathcal{G}(\Omega)$ for all smooth functions $f \in C^\infty(\Omega)$. Furthermore, the derivatives \widehat{D}_X and \widetilde{D}_X are well-defined on $\mathcal{G}(\Omega)$.

Finally, we establish sheaf properties of \mathcal{G} . Note that for $\Omega' \Subset \Omega$ open, the restriction $R|_{\Omega'}(\vec{\varphi}) := R(\vec{\varphi})$ is well-defined because for $U \subseteq \Omega'$ open we have $C^\infty(U, \mathcal{D}(\Omega')) \subseteq C^\infty(U, \mathcal{D}(\Omega))$.

Proposition 15 *Let $R \in \mathcal{B}(\Omega)$ and $\Omega' \subseteq \Omega$ be open. If R is moderate then $R|_{\Omega'}$ is moderate; if R is negligible then $R|_{\Omega'}$ is negligible.*

Proof Suppose that $R \in \mathcal{M}(\Omega)$. Fix $x \in \Omega'$, which gives $U \in \mathcal{U}_x(\Omega)$. Set $U' := U \cap \Omega' \in \mathcal{U}_x(\Omega')$ and let $K, L \Subset U'$ and $m, k \in \mathbb{N}_0$ be arbitrary. Then there are c, l, λ as in Definition 6. Let now $\vec{\varphi}'_0, \dots, \vec{\varphi}'_k \in C^\infty(U', \mathcal{D}_L(U'))$ be given. Choose $\rho \in \mathcal{D}(U')$ such that $\rho \equiv 1$ on a neighborhood of K . Then $\rho \cdot \vec{\varphi}'_i \in C^\infty(U, \mathcal{D}_L(U))$ ($i = 0 \dots k$) and

$$\begin{aligned} \|\mathrm{d}^k R|_{\Omega'}(\vec{\varphi}'_0)(\vec{\varphi}'_1, \dots, \vec{\varphi}'_k)\|_{K,m} &= \|\mathrm{d}^k R|_{\Omega'}(\rho \vec{\varphi}'_0)(\rho \vec{\varphi}'_1, \dots, \rho \vec{\varphi}'_k)\|_{K,m} \\ &= \|\mathrm{d}^k R(\rho \vec{\varphi}'_0)(\rho \vec{\varphi}'_1, \dots, \rho \vec{\varphi}'_k)\|_{K,m} \\ &\leq \lambda(\|\rho \vec{\varphi}'_0\|_{K,c;L,l}, \dots, \|\rho \vec{\varphi}'_k\|_{K,c;L,l}) \\ &= \lambda(\|\vec{\varphi}'_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}'_k\|_{K,c;L,l}). \end{aligned}$$

Hence, the moderateness test is satisfied for $R|_{\Omega'}$.

Now suppose that $R \in \mathcal{N}(\Omega)$. For the negligibility test fix $x \in \Omega'$, which gives $U \in \mathcal{U}_x(\Omega)$. Set $U' := U \cap \Omega'$ and let $K, L \Subset U'$ and $m, k \in \mathbb{N}_0$ be arbitrary. Then $\exists c, l, B, \lambda$ as in Definition 7. Let now $\vec{\varphi}'_0, \dots, \vec{\varphi}'_k \in C^\infty(U', \mathcal{D}_L(U'))$ be given. Choose $\rho \in \mathcal{D}(U')$ such that $\rho \equiv 1$ on a neighborhood of K . Then $\rho \cdot \vec{\varphi}'_i \in C^\infty(U, \mathcal{D}_L(U))$ ($i = 0 \dots k$) and

$$\begin{aligned} \|\mathrm{d}^k R|_{\Omega'}(\vec{\varphi}'_0)(\vec{\varphi}'_1, \dots, \vec{\varphi}'_k)\|_{K,m} &= \|\mathrm{d}^k R|_{\Omega'}(\rho \vec{\varphi}'_0)(\rho \vec{\varphi}'_1, \dots, \rho \vec{\varphi}'_k)\|_{K,m} \\ &= \|\mathrm{d}^k R(\rho \vec{\varphi}'_0)(\rho \vec{\varphi}'_1, \dots, \rho \vec{\varphi}'_k)\|_{K,m} \\ &\leq \lambda(\|\rho \vec{\varphi}'_0\|_{K,c;L,l}, \dots, \|\rho \vec{\varphi}'_k\|_{K,c;L,l}, \|\rho \vec{\varphi}'_0 - \vec{\delta}\|_{K,c;B}, \dots, \|\rho \vec{\varphi}'_k\|_{K,c;B}) \\ &= \lambda(\|\vec{\varphi}'_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}'_k\|_{K,c;L,l}, \|\vec{\varphi}'_0 - \vec{\delta}\|_{K,c;B}, \dots, \|\vec{\varphi}'_k\|_{K,c;B}) \end{aligned}$$

which shows negligibility of $R|_{\Omega'}$. □

Proposition 16 $\mathcal{G}(-)$ is a sheaf of algebras on Ω .

Proof Let $X \subseteq \Omega$ be open and $(X_i)_i$ be a family of open subsets of Ω such that $\bigcup_i X_i = X$.

We first remark that if $R \in \mathcal{B}(X)$ satisfies $R|_{X_i} \in \mathcal{N}(X_i)$ for all i then $R \in \mathcal{N}(X)$, as is evident from the definition of negligibility.

Suppose now that we are given $R_i \in \mathcal{M}(X_i)$ such that $R_i|_{X_i \cap X_j} - R_j|_{X_i \cap X_j} \in \mathcal{N}(X_i \cap X_j)$ for all i, j with $X_i \cap X_j \neq \emptyset$. Let $(\chi_i)_i$ be a partition of unity subordinate to $(X_i)_i$, i.e., a family of mappings $\chi_i \in C^\infty(X)$ such that $0 \leq \chi_i \leq 1$, $(\text{supp } \chi_i)_i$ is locally finite, $\sum_i \chi_i(x) = 1$ for all $x \in X$ and $\text{supp } \chi_i \subseteq X_i$. Choose functions $\rho_i \in C^\infty(X_i, \mathcal{D}(X_i))$ which are equal to 1 on an open neighborhood of the diagonal in $X_i \times X_i$ for each i . For $V \subseteq X$ open and $\vec{\varphi} \in C^\infty(V, \mathcal{D}(X))$ we define $R_V(\vec{\varphi}) \in C^\infty(V)$ by

$$R_V(\vec{\varphi}) := \sum_i \chi_i|_V \cdot (R_i)_{V \cap X_i}(\rho_i|_{V \cap X_i} \cdot \vec{\varphi}|_{V \cap X_i}). \quad (3)$$

For showing smoothness of R_V consider a curve $c \in C^\infty(\mathbb{R}, C^\infty(V, \mathcal{D}(X)))$. We have to show that $t \mapsto R_V(c(t))$ is an element of $C^\infty(\mathbb{R}, C^\infty(V))$. By [11, 3.8, p. 28] it suffices to show that for each open subset $W \subseteq V$ which is relatively compact in V the curve $t \mapsto R_V(c(t))|_W = R_W(c(t)|_W)$ is smooth, but this holds because the sum in (3) then is finite. Hence, $(R_V)_V \in \mathcal{B}(\Omega)$.

Fix $x \in X$ for the moderateness test. There is a finite index set F and an open neighborhood $W \in \mathcal{U}_x(X)$ such that $W \cap \text{supp } \chi_i \neq \emptyset$ implies $i \in F$. We can also assume that $x \in \bigcap_{i \in F} X_i$. Let Y be a neighborhood of x such that $\rho_i \equiv 1$ on $Y \times Y$ for all $i \in F$. For each $i \in F$ let $U_i \in \mathcal{U}_x(X_i)$ be obtained from moderateness of R_i as in Definition 6. Set $U := \bigcap_{i \in F} U_i \cap W \cap Y \in \mathcal{U}_x(X)$, and let $K, L \Subset U$ as well as $m, k \in \mathbb{N}_0$ be arbitrary. For each $i \in F$ there are c_i, l_i, λ_i such that for any $\vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(U, \mathcal{D}_L(U))$ we have

$$\|d^k R_i(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \leq \lambda_i (\|\vec{\varphi}_0\|_{K,c_i;L,l_i}, \dots, \|\vec{\varphi}_k\|_{K,c_i;L,l_i}).$$

Now we have, for $\vec{\varphi} \in C^\infty(U, \mathcal{D}_L(U))$,

$$R(\vec{\varphi})|_W = \sum_{i \in F} \chi_i|_W \cdot (R_i)_{W \cap X_i}(\rho_i \vec{\varphi}|_{W \cap X_i})$$

and hence, for $\vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(U, \mathcal{D}_L(U))$,

$$\begin{aligned} & d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)|_W \\ &= \sum_{i \in F} \chi_i|_W \cdot d^k((R_i)_{W \cap X_i})(\rho_i \vec{\varphi}_0|_{W \cap X_i})(\rho_i \vec{\varphi}_1|_{W \cap X_i}, \dots, \rho_i \vec{\varphi}_k|_{W \cap X_i}). \end{aligned}$$

We see that

$$\begin{aligned} & \|d^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \sum_{i \in F} C(m) \cdot \|\chi_i\|_{K,m} \cdot \lambda_i (\|\vec{\varphi}_0\|_{K,c_i;L,l_i}, \dots, \|\vec{\varphi}_k\|_{K,c_i;L,l_i}) \\ & = \lambda (\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_k\|_{K,c;L,l}) \end{aligned}$$

with $c = \max_{j \in F} c_j$, $l = \max_{j \in F} l_j$, some constant $C(m)$ coming from the Leibniz rule, and $\lambda \in \mathcal{P}_k$ given by

$$\lambda = \sum_{i \in F} C(m) \|\chi_i\|_{K,m} \cdot \lambda_i.$$

This shows that R is moderate. Finally, we claim that $R|_{X_j} - R_j \in \mathcal{N}(X_j)$ for all j . For this we first note that

$$(R|_{X_j} - R_j)(\vec{\varphi}) = \sum_i \chi_i|_{X_j} \cdot (R_i(\rho_i \vec{\varphi}|_{X_i \cap X_j}) - R_j(\vec{\varphi}))$$

for $\vec{\varphi} \in C^\infty(X_j, \mathcal{D}(X_j))$. Again, for $x \in X_j$ there is a finite index set F and an open neighborhood $W \in \mathcal{U}_x(X)$ such that $W \cap \text{supp } \chi_i \neq \emptyset$ implies $i \in F$, and we can assume that $x \in \bigcap_{i \in F} X_i$. Let Y be a neighborhood of x such that $\rho_i \equiv 1$ on $Y \times Y$ for all $i \in F$ and let $U_i \in \mathcal{U}_x(X_i \cap X_j)$ be given by the negligibility test of $R_i|_{X_i \cap X_j} - R_j|_{X_i \cap X_j}$ according to Definition 7. Set $U := \bigcap_{i \in F} U_i \cap W \cap Y$. Fix any $K, L \Subset U$ and $m, k \in \mathbb{N}_0$. For each $i \in F$ there are c_i, l_i, λ_i, B_i such that for $\vec{\varphi}_0, \dots, \vec{\varphi}_k \in C^\infty(U, \mathcal{D}_L(U))$ we have

$$\begin{aligned} & \|d^k(R_i|_{X_i \cap X_j} - R_j|_{X_i \cap X_j})(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \lambda_i(\|\vec{\varphi}_0\|_{K,c_i;L,l_i}, \dots, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c_i;B_i}, \|\vec{\varphi}_1\|_{K,c_i;B_i}, \dots, \|\vec{\varphi}_k\|_{K,c_i;B_i}). \end{aligned}$$

As above, we then have

$$\begin{aligned} & \|d^k(R|_{X_j} - R_j)(\vec{\varphi}_0)(\vec{\varphi}_1, \dots, \vec{\varphi}_k)\|_{K,m} \\ & \leq \sum_{i \in F} C(m) \cdot \|\chi_i\|_{K,m} \cdot \lambda_i(\|\vec{\varphi}_0\|_{K,c_i;L,l_i}, \dots, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c_i;B_i}, \|\vec{\varphi}_1\|_{K,c_i;B_i}, \dots) \\ & \leq \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \dots, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|\vec{\varphi}_1\|_{K,c;B}, \dots) \end{aligned}$$

with $c = \max_{i \in F} c_i, l = \max_{i \in F} l_i, B = \bigcup_{i \in F} B_i$, and $\lambda \in \mathcal{I}_k$ given by

$$\lambda = \sum_{i \in F} C(m) \|\chi_i\|_{K,m} \cdot \lambda_i.$$

This completes the proof. □

4 An elementary version

We will now give a variant of the construction of Sect. 3 similar in spirit to Colombeau’s elementary algebra [2]: if we only consider derivatives along the coordinate lines of \mathbb{R}^n we can replace the smoothing kernels $\vec{\varphi} \in C^\infty(U, \mathcal{D}_L(\Omega))$ by convolutions. This way, one can use a simpler basic space which does not involve calculus on infinite dimensional locally convex spaces anymore:

Definition 17 Let $\Omega \subseteq \mathbb{R}^n$ be open. We set

$$U(\Omega) := \{(\varphi, x) \in \mathcal{D}(\mathbb{R}^n) \times \Omega \mid \text{supp } \varphi + x \subseteq \Omega\}$$

and define $\mathcal{B}^c(\Omega)$ to be the set of all mappings $R: U(\Omega) \rightarrow \mathbb{C}$ such that $R(\varphi, \cdot)$ is smooth for fixed φ .

Note that this is almost the basic space used originally by Colombeau (see [2, 1.2.1, p. 18] or [9, Definition 1.4.3, p. 59]) but with $\mathcal{D}(\mathbb{R}^n)$ in place of the space of test functions whose integral equals one. We now introduce a notation for the convolution kernel determined by a test function.

Definition 18 For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we define $\check{\varphi} \in C^\infty(\mathbb{R}^n, \mathcal{D}(\mathbb{R}^n))$ by

$$\check{\varphi}(x)(y) := \varphi(y - x).$$

In fact, with this definition we have $\langle u, \check{\varphi} \rangle = u * \check{\varphi}$, where as usually we set $\check{\varphi}(y) := \varphi(-y)$. Furthermore, for $c \in \mathbb{N}_0$ we write

$$\|\varphi\|_c := \sup_{x \in \mathbb{R}^n, |\alpha| \leq c} |\partial^\alpha \varphi(x)| \quad (\varphi \in \mathcal{D}(\mathbb{R}^n)).$$

The direct adaptation of Definition 6,7 then looks as follows:

Definition 19 Let $R \in \mathcal{B}^c(\Omega)$. Then R is called *moderate* if

$$\begin{aligned} & (\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U : K \Subset L) (\forall m \in \mathbb{N}_0) \\ & (\exists c \in \mathbb{N}_0) (\exists \lambda \in \mathcal{P}_0) (\forall \varphi \in \mathcal{D}(\mathbb{R}^n) : K + \text{supp } \varphi \subseteq L) : \\ & \|R(\varphi, \cdot)\|_{K,m} \leq \lambda(\|\varphi\|_c). \end{aligned}$$

The subset of all moderate elements of $\mathcal{B}^c(\Omega)$ is denoted by $\mathcal{M}^c(\Omega)$.

Similarly, R is called *negligible* if

$$\begin{aligned} & (\forall x \in \Omega) (\exists U \in \mathcal{U}_x(\Omega)) (\forall K, L \Subset U : K \Subset L) (\forall m \in \mathbb{N}_0) (\exists c \in \mathbb{N}_0) \\ & (\exists \lambda \in \mathcal{I}_0) (\exists B \subseteq C^\infty(U) \text{ bounded}) (\forall \varphi \in \mathcal{D}(\mathbb{R}^n) : K + \text{supp } \varphi \subseteq L) : \\ & \|R(\varphi, \cdot)\|_{K,m} \leq \lambda(\|\varphi\|_c, \|\check{\varphi} - \vec{\delta}\|_{K,c;B}). \end{aligned}$$

The subset of all negligible elements of $\mathcal{B}^c(\Omega)$ is denoted by $\mathcal{N}^c(\Omega)$.

It is convenient to work with the following simplification of these definitions.

Proposition 20 $R \in \mathcal{B}^c(\Omega)$ is moderate if and only if

$$\begin{aligned} & (\forall K \Subset \Omega) (\exists r > 0 : \overline{B_r(K)} \Subset \Omega) (\forall m \in \mathbb{N}_0) (\exists c \in \mathbb{N}_0) \\ & (\exists \lambda \in \mathcal{P}_0) (\forall \varphi \in \mathcal{D}(\mathbb{R}^n) : \text{supp } \varphi \subseteq B_r(0)) : \\ & \|R(\varphi, \cdot)\|_{K,m} \leq \lambda(\|\varphi\|_c). \end{aligned}$$

Similarly, $R \in \mathcal{B}^c(\Omega)$ is negligible if and only if

$$\begin{aligned} & (\forall K \Subset \Omega) (\exists r > 0 : \overline{B_r(K)} \Subset \Omega) (\forall m \in \mathbb{N}_0) (\exists c \in \mathbb{N}_0) \\ & (\exists \lambda \in \mathcal{I}_0) (\exists B \subseteq C^\infty(\Omega) \text{ bounded}) (\forall \varphi \in \mathcal{D}(\mathbb{R}^n) : \text{supp } \varphi \subseteq B_r(0)) : \\ & \|R(\varphi, \cdot)\|_{K,m} \leq \lambda(\|\varphi\|_c, \|\check{\varphi} - \vec{\delta}\|_{K,c;B}). \end{aligned}$$

Proof Suppose R is moderate and fix $K \Subset \Omega$. We can cover K by finitely many open sets U_i obtained from Definition 19 and write $K = \bigcup_i K_i$ with $K_i \Subset U_i$. Choose $r > 0$ such that $L_i := \overline{B_r(K_i)} \Subset U_i$ for all i . Fixing m , by moderateness there exist c_i and λ_i for each i . Set $c = \max_i c_i$ and choose λ with $\lambda \geq \lambda_i$ for all i . Now given $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } \varphi \subseteq B_r(0)$ we also have $K_i + \text{supp } \varphi \subseteq L_i$ and we can estimate

$$\|R(\varphi, \cdot)\|_{K,m} \leq \sup_i \|R(\varphi, \cdot)\|_{K_i,m} \leq \sup_i \lambda_i (\|\varphi\|_{c_i}) \leq \lambda (\|\varphi\|_c).$$

Conversely, suppose the condition holds and fix $x \in \Omega$ for the moderateness test. Choose $a > 0$ such that $\overline{B_a(x)} \Subset \Omega$. By assumption there is $r > 0$ with $\overline{B_{r+a}(x)} \Subset \Omega$. Set $U := B_{r/2}(x)$. Then, fix $K \Subset L \Subset U$ and m for the moderateness test. There are c and λ by assumption. Now given φ with $K + \text{supp } \varphi \subseteq L$, we see that for $y \in \text{supp } \varphi$ and an arbitrary point $z \in K$ we have $|y| \leq |y + z - x| + |z - x| < r$, which means that $\text{supp } \varphi \subseteq B_r(0)$. But then $\|R(\varphi, \cdot)\|_{K,m} \leq \lambda (\|\varphi\|_c)$ as desired.

If R is negligible we proceed similarly until the choice of $K_i \Subset L_i \Subset U_i$ and m gives c_i , λ_i and B_i . Choose $\chi_i \in \mathcal{D}(U_i)$ with $\chi_i \equiv 1$ on a neighborhood of L_i , and define $B := \bigcup_i \{\chi_i f \mid f \in B_i\}$, which is bounded in $C^\infty(\Omega)$. Then with $c = \max_i c_i$ and $\lambda \geq \lambda_i$ for all i we have

$$\|R(\varphi, \cdot)\|_{K,m} \leq \sup_i \lambda_i (\|\varphi\|_{c_i}, \|\varphi^* - \bar{\delta}\|_{K_i, c_i; B_i}) \leq \lambda (\|\varphi\|_c, \|\varphi^* - \bar{\delta}\|_{K, c; B}).$$

The converse is seen as for moderateness by restricting the elements of $B \subseteq C^\infty(\Omega)$ to U . \square

The embeddings now take the following form.

Definition 21 We define $\iota^c: \mathcal{D}'(\Omega) \rightarrow \mathcal{B}^c(\Omega)$ and $\sigma^c: C^\infty(\Omega) \rightarrow \mathcal{B}^c(\Omega)$ by

$$\begin{aligned} (\iota^c u)(\varphi, x) &:= \langle u, \varphi(\cdot - x) \rangle & (u \in \mathcal{D}'(\Omega)) \\ (\sigma^c f)(\varphi, x) &:= f(x) & (f \in C^\infty(\Omega)). \end{aligned}$$

Partial derivatives on $\mathcal{B}^c(\Omega)$ then can be defined via differentiation in the second variable:

Definition 22 Let $R \in \mathcal{B}^c(\Omega)$. We define derivatives $D_i: \mathcal{B}^c(\Omega) \rightarrow \mathcal{B}^c(\Omega)$ ($i = 1, \dots, n$) by

$$(D_i R)(\varphi, x) := \frac{\partial}{\partial x_i} (x \mapsto R(\varphi, x)).$$

Theorem 23 We have $D_i(\mathcal{M}^c(\Omega)) \subseteq \mathcal{M}^c(\Omega)$ and $D_i(\mathcal{N}^c(\Omega)) \subseteq \mathcal{N}^c(\Omega)$.

Proof This is evident from the definitions. \square

Proposition 24 We have $D_i \circ \iota = \iota \circ \partial_i$ and $D_i \circ \sigma = \sigma \circ \partial_i$.

Proof $D_i(uu)(\varphi, x) = \frac{\partial}{\partial x_i} \langle u(y), \varphi(y-x) \rangle = \langle u(y), -(\partial_i \varphi)(y-x) \rangle = \langle \partial_i u(y), \varphi(y-x) \rangle = \iota(\partial_i u)(\varphi, x)$. The second claim is clear. \square

Proposition 25 $\mathcal{N}^c(\Omega) \subseteq \mathcal{M}^c(\Omega)$.

Proof The result follows from

$$\|\check{\varphi} - \check{\delta}\|_{K,c;B} \leq \lambda_1(\|\varphi\|_{c_1})$$

for suitable λ_1 and c_1 , which is seen as in the proof of Proposition 10. \square

Similarly to Proposition 11 we have:

Proposition 26 $\mathcal{M}^c(\Omega)$ is a subalgebra of $\mathcal{B}^c(\Omega)$ and $\mathcal{N}^c(\Omega)$ is an ideal in $\mathcal{M}^c(\Omega)$.

Theorem 27 Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. Then

- (i) $\iota^c u$ is moderate,
- (ii) $\sigma^c f$ is moderate,
- (iii) $\iota^c f - \sigma^c f$ is negligible, and
- (iv) if $\iota^c u$ is negligible then $u = 0$.

The proof is almost identical to that of Theorem 12 and hence omitted.

Definition 28 We define the elementary Colombeau algebra of generalized functions on Ω by $\mathcal{G}^c(\Omega) := \mathcal{M}^c(\Omega)/\mathcal{N}^c(\Omega)$.

As before, one may show that \mathcal{G}^c is a sheaf.

5 Canonical mappings

In this section we show that the algebra \mathcal{G} constructed above is near to being universal in the sense that there exist canonical mappings from it into most of the classical Colombeau algebras which are compatible with the embeddings.

We begin by constructing a mapping $\mathcal{G}(\Omega) \rightarrow \mathcal{G}^c(\Omega)$.

Definition 29 Given $R \in \mathcal{B}(\Omega)$ we define $\tilde{R} \in \mathcal{B}^c(\Omega)$ by

$$\tilde{R}(\varphi, x) := R(\vec{\varphi})(x) \quad ((\varphi, x) \in U(\Omega))$$

where $\vec{\varphi} \in C^\infty(\Omega, \mathcal{D}(\Omega))$ is chosen such that $\vec{\varphi} = \check{\varphi}$ in a neighborhood of x .

This definition is meaningful: given (φ, x) in $U(\Omega)$ we have $\text{supp } \varphi(\cdot - x') \subseteq \Omega$ for x' in a neighborhood V of x . Choosing $\rho \in \mathcal{D}(\Omega)$ with $\text{supp } \rho \subseteq V$ and $\rho \equiv 1$ in a neighborhood of x , we can take $\vec{\varphi}(x) := \rho \check{\varphi}$. Obviously, $\tilde{R}(\varphi, x)$ does not depend on the choice of $\vec{\varphi}(x)$ and $\tilde{R}(\varphi, \cdot)$ is smooth, so indeed we have $\tilde{R} \in \mathcal{B}^c(\Omega)$.

Proposition 30 Let $R \in \mathcal{B}(\Omega)$. Then the following holds:

- (i) $\tilde{u} = \iota^c u$ for $u \in \mathcal{D}'(\Omega)$.

- (ii) $\widetilde{\sigma}f = \sigma^c f$ for $f \in C^\infty(\Omega)$.
- (iii) $\widetilde{R} \in \mathcal{M}^c(\Omega)$ for $R \in \mathcal{M}(\Omega)$.
- (iv) $\widetilde{R} \in \mathcal{N}^c(\Omega)$ for $R \in \mathcal{N}(\Omega)$.

Proof (i): For $u \in \mathcal{D}'(\Omega)$ we have

$$\widetilde{u}(\varphi, x) = (u)(\vec{\varphi})(x) = \langle u, \vec{\varphi}(x) \rangle = \langle u, \vec{\varphi}^*(x) \rangle = \langle u(y), \varphi(y-x) \rangle = (l^c u)(\varphi, x).$$

(ii) is clear.

(iii): Suppose that $R \in \mathcal{M}(\Omega)$. Fixing $x \in \Omega$, we obtain U as in Proposition 8. Let $K \Subset L \Subset U$ and m be given, set $k = 0$, and choose L' such that $L \Subset L' \Subset U$. Then Proposition 8 gives c, l, λ such that for $\vec{\varphi} \in C^\infty(K, \mathcal{D}_{L'}(U))$,

$$\|R(\vec{\varphi})\|_{K,m} \leq \lambda(\|\vec{\varphi}\|_{K,c;L',l}).$$

Now for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $K + \text{supp } \varphi \subseteq L$ we have $\vec{\varphi} \in C^\infty(K, \mathcal{D}_{L'}(U))$, which gives

$$\|\widetilde{R}(\varphi, \cdot)\|_{K,m} = \|R(\vec{\varphi})\|_{K,m} \leq \lambda(\|\vec{\varphi}\|_{K,c;L',l}) \leq \lambda(\|\varphi\|_{c+l})$$

which proves that $\widetilde{R} \in \mathcal{M}^c(\Omega)$.

(iv): Similarly, if $R \in \mathcal{N}(\Omega)$ then for $x \in \Omega$ we have U as in Proposition 8. For $K \Subset L \Subset U$, m given, $k = 0$, and L' such that $L \Subset L' \Subset U$, we obtain c, l, λ, B as in Proposition 8 such that

$$\|R(\vec{\varphi})\|_{K,m} \leq \lambda(\|\vec{\varphi}\|_{K,c;L',l}, \|\vec{\varphi} - \vec{\delta}\|_{K,c;B})$$

and hence

$$\begin{aligned} \|\widetilde{R}(\varphi, \cdot)\|_{K,m} &= \|R(\vec{\varphi})\|_{K,m} \\ &\leq \lambda(\|\vec{\varphi}\|_{K,c;L',l}, \|\vec{\varphi} - \vec{\delta}\|_{K,c;B}) \\ &\leq \lambda(\|\varphi\|_{c+l}, \|\vec{\varphi} - \vec{\delta}\|_{K,c;B}) \end{aligned}$$

which gives negligibility of \widetilde{R} . □

5.1 The special algebra

We define the special Colombeau algebra \mathcal{G}^s with the embedding as in [4]: fix a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ with

$$\int \rho(x) dx = 1, \quad \int x^\alpha \rho(x) dx = 0 \quad \forall \alpha \in \mathbb{N}_0^n \setminus \{0\}.$$

Choosing $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B_1(0)$ and $\text{supp } \chi \subseteq B_2(0)$ we set

$$\rho_\varepsilon(y) := \varepsilon^{-n} \rho(y/\varepsilon), \quad \theta_\varepsilon(y) := \rho_\varepsilon(y) \chi(y |\ln \varepsilon|) \quad (\varepsilon > 0).$$

Moreover, with

$$K_\varepsilon = \{x \in \Omega \mid d(x, \mathbb{R}^n \setminus \Omega) \geq \varepsilon\} \cap B_{1/\varepsilon}(0) \Subset \Omega \quad (\varepsilon > 0)$$

we choose functions $\kappa_\varepsilon \in \mathcal{D}(\Omega)$ such that $0 \leq \kappa_\varepsilon \leq 1$ and $\kappa_\varepsilon \equiv 1$ on K_ε . Then the special algebra $\mathcal{G}^s(\Omega)$ is given by

$$\begin{aligned} \mathcal{E}^s(\Omega) &:= C^\infty(\Omega)^I \text{ with } I := (0, 1], \\ \mathcal{E}_M^s(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \Subset \Omega \forall m \in \mathbb{N}_0 \exists N \in \mathbb{N} : \|u_\varepsilon\|_{K,m} = O(\varepsilon^{-N})\}, \\ \mathcal{N}^s(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \Subset \Omega \forall m \in \mathbb{N}_0 \forall N \in \mathbb{N} : \|u_\varepsilon\|_{K,m} = O(\varepsilon^N)\}, \\ \mathcal{G}^s(\Omega) &:= \mathcal{E}_M^s(\Omega) / \mathcal{N}^s(\Omega), \\ (\iota^s u)_\varepsilon &:= \langle u, \vec{\psi}_\varepsilon \rangle \quad (u \in \mathcal{D}'(\Omega)), \\ (\sigma^s f)_\varepsilon &:= f \quad (f \in C^\infty(\Omega)), \\ \vec{\psi}_\varepsilon(x)(y) &:= \theta_\varepsilon(x - y)\kappa_\varepsilon(y). \end{aligned}$$

Definition 31 For $R \in \mathcal{B}(\Omega)$ we define $R^s = (R_\varepsilon^s)_\varepsilon \in \mathcal{E}^s(\Omega)$ by

$$R_\varepsilon^s(x) := R(\vec{\psi}_\varepsilon)(x).$$

Proposition 32 (i) $(\iota u)^s = \iota^s u$ for $u \in \mathcal{D}'(\Omega)$.

(ii) $(\sigma f)^s = \sigma^s f$ for $f \in C^\infty(\Omega)$.

(iii) $R^s \in \mathcal{E}_M^s(\Omega)$ for $R \in \mathcal{M}(\Omega)$.

(iv) $R^s \in \mathcal{N}^s(\Omega)$ for $R \in \mathcal{N}(\Omega)$.

Proof (i) and (ii) are clear.

For (iii) it suffices to show the needed estimate locally. Fix $x \in \Omega$, which gives $U \in \mathcal{U}_x(\Omega)$ as in Proposition 8. Choose any K, L such that $x \in K \Subset L \Subset U$, fix m , and set $k = 0$. Then there are c, l, λ as in Proposition 8. Because $\text{supp } \vec{\psi}_\varepsilon(x) \subseteq B_{2|\ln \varepsilon|^{-1}}(x)$ we have $\vec{\psi}_\varepsilon \in C^\infty(K, \mathcal{D}_L(U))$ for ε small enough, which gives

$$\|R_\varepsilon^s\|_{K,m} \leq \lambda(\|\vec{\psi}_\varepsilon\|_{K,c;L,l}).$$

Consequently, $(R_\varepsilon^s)_\varepsilon \in \mathcal{E}_M^s(\Omega)$ follows from

$$\|\vec{\psi}_\varepsilon\|_{K,c;L,l} = \sup_{x,\alpha,y,\beta} \left| \partial_x^\alpha \partial_y^\beta (\rho_\varepsilon(x - y)\chi((x - y)|\ln \varepsilon|)\kappa_\varepsilon(y)) \right| = O(\varepsilon^{-n-c-l}).$$

For negligibility we proceed similarly; the claim then follows by using that for a bounded subset $B \subseteq C^\infty(U)$ we have $\|\vec{\psi}_\varepsilon - \delta\|_{K,c;B} = O(\varepsilon^N)$ for all $N \in \mathbb{N}$, which is seen as in [4, Prop. 12, p. 38] and actually merely a restatement of the fact that $\iota^s f - \sigma^s f = O(\varepsilon^N)$ for all N uniformly for $f \in B$. \square

5.2 The diffeomorphism invariant algebra

There are several variants of the diffeomorphism invariant algebra \mathcal{G}^d ; we will employ the following formulation [10, 13, 14]:

$$\begin{aligned}\mathcal{E}^d(\Omega) &:= C^\infty(\mathcal{D}(\Omega), C^\infty(\Omega)) \\ \mathcal{E}_M^d(\Omega) &:= \{R \in C^\infty(\mathcal{D}(\Omega)) \mid \forall K \Subset \Omega \forall k, m \in \mathbb{N}_0 \forall (\vec{\varphi}_\varepsilon)_\varepsilon \in S(\Omega) \forall (\vec{\psi}_{1,\varepsilon})_\varepsilon, \dots, \\ &\quad (\vec{\psi}_{k,\varepsilon})_\varepsilon \in S^0(\Omega) \exists N \in \mathbb{N} : \|\mathrm{d}^k R(\vec{\varphi}_\varepsilon)(\vec{\psi}_{1,\varepsilon}, \dots, \vec{\psi}_{k,\varepsilon})\|_{K,m} = O(\varepsilon^{-N})\}, \\ \mathcal{N}^d(\Omega) &:= \{R \in C^\infty(\mathcal{D}(\Omega)) \mid \forall K \Subset \Omega \forall k, m \in \mathbb{N}_0 \forall (\vec{\varphi}_\varepsilon)_\varepsilon \in S(\Omega) \forall (\vec{\psi}_{1,\varepsilon})_\varepsilon, \dots, \\ &\quad (\vec{\psi}_{k,\varepsilon})_\varepsilon \in S^0(\Omega) \forall N \in \mathbb{N} : \|\mathrm{d}^k R(\vec{\varphi}_\varepsilon)(\vec{\psi}_{1,\varepsilon}, \dots, \vec{\psi}_{k,\varepsilon})\|_{K,m} = O(\varepsilon^N)\}, \\ \mathcal{G}^d(\Omega) &:= \mathcal{E}_M^d(\Omega) / \mathcal{N}^d(\Omega), \\ (\iota^d u)(\varphi)(x) &:= \langle u, \varphi \rangle, \\ (\sigma^d f)(\varphi)(x) &:= f(x).\end{aligned}$$

The spaces $S(\Omega)$ and $S^0(\Omega)$ employed in this definition are given as follows:

Definition 33 Let a net of smoothing kernels $(\vec{\varphi}_\varepsilon)_\varepsilon \in C^\infty(\Omega, \mathcal{D}(\Omega))^l$ be given and denote the corresponding net of smoothing operators by $(\Phi_\varepsilon)_\varepsilon \in \mathcal{L}(\mathcal{D}'(\Omega), C^\infty(\Omega))^l$. Then $(\vec{\varphi}_\varepsilon)_\varepsilon$ is called a *test object* on Ω if

- (i) $\Phi_\varepsilon \rightarrow \mathrm{id}$ in $\mathcal{L}(\mathcal{D}'(\Omega), \mathcal{D}'(\Omega))$,
- (ii) $\forall p \in \mathrm{csn}(\mathcal{L}(\mathcal{D}'(\Omega), C^\infty(\Omega))) \exists N \in \mathbb{N} : p(\Phi_\varepsilon) = O(\varepsilon^{-N})$,
- (iii) $\forall p \in \mathrm{csn}(\mathcal{L}(C^\infty(\Omega), C^\infty(\Omega))) \forall m \in \mathbb{N} : p(\Phi_\varepsilon|_{C^\infty(\Omega)} - \mathrm{id}) = O(\varepsilon^m)$,
- (iv) $\forall x \in \Omega \exists V \in \mathcal{U}_x(\Omega) \forall r > 0 \exists \varepsilon_0 > 0 \forall y \in V \forall \varepsilon < \varepsilon_0 : \mathrm{supp} \varphi_\varepsilon(y) \subseteq B_r(y)$.

We denote the set of test objects on Ω by $S(\Omega)$. Similarly, $(\vec{\varphi}_\varepsilon)_\varepsilon$ is called a *0-test object* if it satisfies these conditions with (i) and (iii) replaced by the following conditions:

- (i') $\Phi_\varepsilon \rightarrow 0$ in $\mathcal{L}(\mathcal{D}'(\Omega), \mathcal{D}'(\Omega))$,
- (iii') $\forall p \in \mathrm{csn}(\mathcal{L}(C^\infty(\Omega), C^\infty(\Omega))) \forall m \in \mathbb{N} : p(\Phi_\varepsilon|_{C^\infty(\Omega)}) = O(\varepsilon^m)$.

The set of all 0-test objects on Ω is denoted by $S^0(\Omega)$.

Definition 34 For $R \in \mathcal{B}(\Omega)$ we define $R^d \in \mathcal{E}^d(\Omega)$ by

$$R^d(\varphi)(x) := R([x' \mapsto \varphi])(x).$$

Proposition 35 (i) $(\iota u)^d = \iota^d u$ for $u \in \mathcal{D}'(\Omega)$.

(ii) $(\sigma f)^d = \sigma^d u$ for $f \in C^\infty(\Omega)$.

(iii) $R^d \in \mathcal{E}_M^d(\Omega)$ for $R \in \mathcal{M}(\Omega)$.

(iv) $R^d \in \mathcal{N}^d(\Omega)$ for $R \in \mathcal{N}(\Omega)$.

Proof (i) and (ii) are clear from the definition. (iii) and (iv) follow directly from the estimates

$$\begin{aligned}\|\vec{\varphi}_\varepsilon\|_{K,c;L,l} &= O(\varepsilon^{-N}) \quad \text{for some } N, \\ \|\vec{\varphi}_\varepsilon - \vec{\delta}\|_{K,c;B} &= O(\varepsilon^N) \quad \text{for all } N,\end{aligned}$$

which hold by definition of the spaces $S(\Omega)$ and $S^0(\Omega)$. □

5.3 The elementary algebra

For Colombeau's elementary algebra we employ the formulation of [9, Section 1.4], Sect. 1.4. For $k \in \mathbb{N}_0$ we let $\mathcal{A}_k(\mathbb{R}^n)$ be the set of all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with integral one such that, if $k \geq 1$, all moments of φ order up to k vanish.

$$\begin{aligned} U^e(\Omega) &:= \{(\varphi, x) \in \mathcal{A}_0(\mathbb{R}^n) \times \Omega \mid x + \text{supp } \varphi \subseteq \Omega\} \\ \mathcal{E}^e(\Omega) &:= \{R: U^e(\Omega) \rightarrow \mathbb{C} \mid \forall \varphi \in \mathcal{A}_0(\mathbb{R}^n) : R(\varphi, \cdot) \text{ is smooth}\} \\ \mathcal{E}_M^e(\Omega) &:= \{R \in \mathcal{E}^e(\Omega) \mid \forall K \Subset \Omega \forall m \in \mathbb{N}_0 \exists N \in \mathbb{N} \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : \\ &\quad \|R(S_\varepsilon \varphi, \cdot)\|_{K,m} = O(\varepsilon^{-N})\} \\ \mathcal{N}^e(\Omega) &:= \{R \in \mathcal{E}^e(\Omega) \mid \forall K \Subset \Omega \forall m \in \mathbb{N}_0 \forall N \in \mathbb{N} \exists q \in \mathbb{N} \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n) : \\ &\quad \|R(S_\varepsilon \varphi, \cdot)\|_{K,m} = O(\varepsilon^N)\} \\ \mathcal{G}^e(\Omega) &:= \mathcal{E}_M^e(\Omega) / \mathcal{N}^e(\Omega) \\ (\iota^e u)(\varphi, x) &:= \langle u, \varphi(\cdot - x) \rangle \\ (\sigma^e f)(\varphi, x) &:= f(x) \end{aligned}$$

Definition 36 For $R \in \mathcal{B}^c(\Omega)$ we define $R^e \in \mathcal{E}^e(\Omega)$ by $R^e(\varphi, x) := R(\varphi, x)$.

Proposition 37 (i) $(\iota^e u)^e = \iota^e u$ for $u \in \mathcal{D}'(\Omega)$.

(ii) $(\sigma^e f)^e = \sigma^e u$ for $f \in C^\infty(\Omega)$.

(iii) $R^e \in \mathcal{E}_M^e(\Omega)$ for $R \in \mathcal{M}^c(\Omega)$.

(iv) $R^e \in \mathcal{N}^e(\Omega)$ for $R \in \mathcal{N}^c(\Omega)$.

Proof Again, (i) and (ii) are clear from the definition. For (iii), fix $K \Subset \Omega$ and $m \in \mathbb{N}_0$. From Proposition 20 we obtain r, c and λ such that for $\text{supp } \varphi \subseteq B_r(0)$, $\|R(\varphi, \cdot)\|_{K,m} \leq \lambda(\|\varphi\|_c)$. For $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ and ε small enough, $\text{supp } S_\varepsilon \varphi \subseteq B_r(0)$, so we only have to take into account that $\|S_\varepsilon \varphi\|_c = O(\varepsilon^{-N})$ for some $N \in \mathbb{N}$. Similarly, (iv) is obtained from the fact that given any N , for q large enough we have $\|(S_\varepsilon \varphi)^* - \delta\|_{K,c;B} = O(\varepsilon^N)$ for all $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$. \square

Acknowledgements Open access funding provided by Austrian Science Fund (FWF). This research was supported by project P26859-N25 of the Austrian Science Fund (FWF). The author expresses his thanks to D. Scarpalázos for a remark inciting the present study.

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