J. Fixed Point Theory Appl. (2018) 20:154
https://doi.org/10.1007/s11784-018-0635-1
Published online October 23, 2018

Journal of Fixed Point Theory and Applications

A linearly convergent algorithm for sparse signal reconstruction

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Abstract. For the sparse signal reconstruction problem in compressive sensing, we propose a projection-type algorithm without any backtracking line search based on a new formulation of the problem. Under suitable conditions, global convergence and its linear convergence of the designed algorithm are established. The efficiency of the algorithm is illustrated through some numerical experiments on some sparse signal reconstruction problem.

Mathematics Subject Classifications. 65H10, 90C33, 90C30.

Keywords. Compressive sensing, projection-type algorithm, global convergence, R-linear convergence, sparse signal reconstruction.

1. Introduction

A basic mathematical problem in compressive sensing (CS) is to recover a sparse signal vector $x \in R^n$ from an undetermined linear system $y = Ax$, where $A \in R^{m\times n}$ $(m \ll n)$ is the sensing matrix. A fundamental decoding model in CS is the following basis pursuit denoising problem, which can be mathematically formulated as

$$\min_{x\in R^n} \frac{1}{2}\|Ax - y\|_2^2 + \rho\|x\|_1, \tag{1.1}$$

where $\rho > 0$ is the regularization parameter and $\|x\|_1$ is the ℓ_1-norm of the vector x, i.e., $\|x\|_1 = \sum_{i=1}^n |x_i|$. For more information, see e.g.[6–8,13,18,21,23,27,30,32,33,38,40,47,50,52–55,57,64,69–72].Throughout this paper, we assume that the solution set of (1.1) is nonempty.

Obviously, function $\|x\|_1$ is convex although it is not differential. For convex optimization problem (1.1), there are some standard methods such as smooth Newton-type methods and interior-point methods for solving the

This project is supported by the Natural Science Foundation of China (Grants no.11801309, 11671228, 11601261), and Natural Science Foundation of Shandong Province (Grant no. ZR2016AQ12).

problem [2,15,19,22,25,28,29,37,43,44,46,48,49,51,60,63,65]. Candès et al. [3] developed a novel method for sparse signal recovery for more generic ℓ_1-minimization. Yin et al. [66] proposed an efficient method for solving the ℓ_1-minimization problem based on Bregman iterative regularization. Hale et al. [16] presented a framework for solving the large-scale ℓ_1-regularized convex minimization problem based on operator splitting and continuation. However, these solvers are not tailored for large-scale cases of CS and they become inefficient as dimension n increases. To overcome this drawback, Figueiredo et al. [14] proposed a gradient projection-type algorithm with a backtracking line search for box-constrained quadratic programming formulation of (1.1). A similar algorithm based on conjugate gradient technique is proposed by Xiao and Zhu [61]. For more detail, see [4,5,9–12,17,20,24,26,31,34,36,39, 41,42,56,58,59,62,68]. Due to the high computing cost of the line search procedure, we propose a new type of projection algorithm for problem (1.1) without line search at each iteration in this paper which marginally decrease the computing cost of the algorithm.

The remainder of this paper is organized as follows. Some equivalent reformulations of problem (1.1) are established in Sect. 2. In Sect. 3, we propose a new projection-type algorithm without line search, and establish the global convergence of the new algorithm and its linear convergence rate. In Sect. 4, some numerical experiments on compressive sensing are given to illustrate the efficiency of the proposed method. Some concluding remarks are drawn in Sect. 5.

To end this section, some notations used in this paper are in order. We use R^n_+ to denote the nonnegative quadrant in R^n, and use x_+ to denote the orthogonal projection of vector $x \in R^n$ onto R^n_+, that is, $(x_+)_i := \max\{x_i, 0\},\ 1 \le i \le n$; the norm $\|\cdot\|$ and $\|\cdot\|_1$ denote the Euclidean 2-norm and 1-norm, respectively.

2. New formulation and algorithm

To propose a new projection-type algorithm for problem (1.1), we first establish a new equivalent reformulation. To this end, we define two nonnegative auxiliary variables μ_i and ν_i $(i = 1, 2, \ldots, n)$ such that

$$\mu_i + \nu_i = |x_i|, \quad \mu_i - \nu_i = x_i, \quad i = 1, 2, \ldots, n.$$

Then, problem (1.1) can be reformulated as

$$\begin{aligned} \min_{(\mu;\nu)\in R^{2n}} \ & \tfrac{1}{2}\|(A, -A)(\mu; \nu) - y\|_2^2 + \rho(e^\top, e^\top)(\mu; \nu) \\ \text{s.t. } & (\mu; \nu) \ge 0, \end{aligned} \tag{2.1}$$

where $e \in R^n$ denotes the vector with all entries being 1, i.e., $e = (1, 1, \ldots, 1)^\top$. Based on this, the problem can be simplified as

$$\begin{aligned} &\min f(\mu; \nu) = \tfrac{1}{2}[(\mu; \nu)^\top M(\mu; \nu) - 2p^\top(\mu; \nu) + y^\top y] \\ &\text{s.t.} (\mu; \nu) \quad \in R^{2n}_+, \end{aligned} \tag{2.2}$$

where $M = (A, -A)^\top(A, -A), p = (A, -A)^\top y - \rho(e^\top, e^\top)^\top$.

Obviously, the Hessian matrix M of the quadratic function $f(\mu;\nu)$ is positive semi-definite. By the optimization theory [1], we know that the stationary point of (2.2) coincides with its solution which also coincides with the solution set of the following linear variational inequality problem of finding $(\mu;\nu)^* \in R^{2n}_+$ satisfying

$$((\mu;\nu)-(\mu;\nu)^*)^\top (M(\mu;\nu)^* - p) \geq 0, \quad \forall (\mu;\nu) \in R^{2n}_+. \tag{2.3}$$

Obviously, the solution set of (2.3), denoted by Ω^*, is nonempty provided that the solution of (1.1) is nonempty.

To proceed, we give the definition of projection operator and some related properties. For a nonempty closed convex set $K \subset R^n$ and vector $x \in R^n$, the orthogonal projection of x onto K, i.e., $\arg\min\{\|y-x\| | y \in K\}$, is denoted by $P_K(x)$.

Proposition 2.1 [1,67]. *Let K be a closed convex subset of R^n. For any $x, y \in R^n$ and $z \in K$, the following statements hold.*

(i) $\langle P_K(x) - x, z - P_K(x)\rangle \geq 0$;
(ii) $\|P_K(x) - P_K(y)\|^2 \leq \|x-y\|^2 - \|(P_K(x)-x) - (P_K(y)-y)\|^2$;
(iii) $\|P_K(x) - x\|^2 \leq \|x-z\|^2 - \|P_K(x) - z\|^2$.

For problem (2.3) and $(\mu;\nu) \in R^{2n}$, define the projection residue

$$r((\mu;\nu),\beta) := (\mu;\nu) - P_{R^{2n}_+}((\mu;\nu) - \beta F(\mu;\nu)) = \min\{(\mu;\nu), \beta F(\mu;\nu)\}, \tag{2.4}$$

where $\beta > 0$ is a constant, $F(\mu;\nu) = M(\mu;\nu) - p$.

The projection residue is intimately related to the solution of (2.3) as shown in the following conclusion [35].

Proposition 2.2. *$(\mu;\nu)^*$ is a solution of (2.3) if and only if $r((\mu;\nu)^*, \beta) = 0$ with some $\beta > 0$.*

Proposition 2.3. *For $H = \{(\mu;\nu) \in R^{2n} \mid \alpha^\top (\mu;\nu) - b \leq 0\}$ and any $z \notin H$, it holds that*

$$P_H(z) = z - \frac{\alpha^\top z - b}{\|\alpha\|^2}\alpha, \tag{2.5}$$

where $z, \alpha \in R^{2n}, \alpha \neq 0, b \in R$.

Based on the discussion above, we may formally state our algorithm.

Algorithm 3.1.

Step 0. Select any $0 < \beta < \frac{1}{\|M\|}, t \in [0,1], (\mu;\nu)^0 \in R^{2n}$. Let $k := 0$.

Step 1. Compute

$$z^k = \{(\mu;\nu)^k - \beta F((\mu;\nu)^k)\}_+. \tag{2.6}$$

If $\|r((\mu;\nu)^k, \beta)\| = 0$, stop. Otherwise, go to Step 2.

Step 2. Compute

$$(\mu;\nu)^{k+1} = P_{H_k}((\mu;\nu)^k - \beta d((\mu;\nu)^k)), \tag{2.7}$$

where

$$H_k := \{(\mu;\nu) \in R^{2n} \mid [r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)]^\top [(\mu;\nu) - z^k] \leq 0\}, \tag{2.8}$$

$$d((\mu;\nu)^k) = \frac{t}{\beta}[r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)] + F(z^k). \tag{2.9}$$

Step 3. Go to Step 1 by setting $k := k+1$.

In the algorithm, vector $(\mu;\nu)^{k+1}$ is updated as follows: If

$$[(\mu;\nu)^k - \beta d((\mu;\nu)^k) - z^k]^\top [r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)] \le 0,$$

then $(\mu;\nu)^k - \beta d((\mu;\nu)^k) \in H_k$ and we set

$$(\mu;\nu)^{k+1} = (\mu;\nu)^k - \beta d((\mu;\nu)^k); \tag{2.10}$$

otherwise, $r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k) \neq 0$ and we set

$$\begin{aligned}(\mu;\nu)^{k+1} &= [(\mu;\nu)^k - \beta d((\mu;\nu)^k)]\\ &\quad - \tfrac{[(\mu;\nu)^k - \beta d((\mu;\nu)^k) - z^k]^\top [r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)]}{\|r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)\|^2}[r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)]\\ &= [(\mu;\nu)^k - \beta d((\mu;\nu)^k)]\\ &\quad - \tfrac{[r((\mu;\nu)^k,\beta) - \beta d((\mu;\nu)^k)]^\top [r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)]}{\|r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)\|^2}[r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)].\end{aligned} \tag{2.11}$$

For the half space H_k, we claim that $R_+^{2n} \subseteq H_k$. In fact, for any $(\mu;\nu) \in R_+^{2n}$ and $x = (\mu;\nu)^k - \beta F((\mu;\nu)^k),\ z = (\mu;\nu)$, by Proposition 2.3, one has

$$\begin{aligned}[r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)]^\top [(\mu;\nu) - z^k] &= [(\mu;\nu)^k\\ &\quad - \beta F((\mu;\nu)^k) - z^k]^\top [(\mu;\nu) - z^k] \le 0.\end{aligned}$$

Thus, $(\mu;\nu) \in H_k$.

3. Convergence

To establish the convergence and convergence rate of Algorithm 3.1, we need the following conclusions.

Lemma 3.1. *For z^k and $d((\mu;\nu)^k)$ defined in Algorithm 3.1, it holds that*

$$\langle z^k - (\mu;\nu)^*, d((\mu;\nu)^k)\rangle \ge 0, \tag{3.1}$$

where $(\mu;\nu)^* \in \Omega^*$.

Proof. Since matrix M is positive semi-definite, one has

$$(F(z^k) - F((\mu;\nu)^*))^\top (z^k - (\mu;\nu)^*) = (z^k - (\mu;\nu)^*)^\top M (z^k - (\mu;\nu)^*) \ge 0.$$

Combining this with (2.3) yields

$$F(z^k)^\top (z^k - (\mu;\nu)^*) \ge 0. \tag{3.2}$$

Then, by Proposition 2.1 (i), a direct computation gives

$$\begin{aligned}\langle z^k - (\mu;\nu)^*, d((\mu;\nu)^k)\rangle &= \langle z^k - (\mu;\nu)^*, F(z^k)\rangle\\ &\quad + \left\langle z^k - (\mu;\nu)^*, \frac{t}{\beta}[r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)]\right\rangle\\ &\ge \frac{t}{\beta}\langle z^k - (\mu;\nu)^*, r((\mu;\nu)^k,\beta) - \beta F((\mu;\nu)^k)\rangle\\ &= \frac{t}{\beta}\langle z^k - (\mu;\nu)^*, (\mu;\nu)^k - \beta F((\mu;\nu)^k) - z^k\rangle \ge 0.\end{aligned}$$

□

Lemma 3.2. *Suppose that Algorithm 3.1 generates an infinite sequence* $\{(\mu;\nu)^k\}$*. Then, for any* $(\mu;\nu)^* \in \Omega^*$*, it holds that*

$$\|(\mu;\nu)^{k+1}-(\mu;\nu)^*\|^2 \le \|(\mu;\nu)^k-(\mu;\nu)^*\|^2-(1-\beta^2\|M\|^2)\|r((\mu;\nu)^k,\beta)\|^2. \tag{3.3}$$

Proof. By a direct computation, one has

$$\begin{aligned}
&\|(\mu;\nu)^{k+1}-(\mu;\nu)^*\|^2\\
&\quad=\|P_{H_k}((\mu;\nu)^k-\beta d((\mu;\nu)^k))-(\mu;\nu)^*\|^2\\
&\quad\le \|(\mu;\nu)^k-\beta d((\mu;\nu)^k)-(\mu;\nu)^*\|^2\\
&\qquad-\|P_{H_k}((\mu;\nu)^k-\beta d((\mu;\nu)^k))-[(\mu;\nu)^k-\beta d((\mu;\nu)^k)]\|^2\\
&\quad=\|[(\mu;\nu)^k-(\mu;\nu)^*]-\beta d((\mu;\nu)^k)\|^2\\
&\qquad-\|[(\mu;\nu)^{k+1}-(\mu;\nu)^k]+\beta d((\mu;\nu)^k)]\|^2\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-2\beta\langle(\mu;\nu)^k-(\mu;\nu)^*,d((\mu;\nu)^k\rangle+\beta^2\|d((\mu;\nu)^k\|^2\\
&\qquad-[\|(\mu;\nu)^k-(\mu;\nu)^{k+1}\|^2-2\beta\langle(\mu;\nu)^k-(\mu;\nu)^{k+1},d((\mu;\nu)^k\rangle+\beta^2\|d((\mu;\nu)^k)\|^2\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-2\beta\langle(\mu;\nu)^{k+1}-(\mu;\nu)^*,d((\mu;\nu)^k\rangle-\|(\mu;\nu)^k-(\mu;\nu)^{k+1}\|^2\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2\beta\langle(\mu;\nu)^{k+1}-z^k+z^k-(\mu;\nu)^*,d((\mu;\nu)^k\rangle\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2\beta\langle(\mu;\nu)^{k+1}-z^k,d((\mu;\nu)^k\rangle-\langle z^k-(\mu;\nu)^*,d((\mu;\nu)^k\rangle\\
&\quad\le\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2\beta\langle(\mu;\nu)^{k+1}-z^k,d((\mu;\nu)^k\rangle\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k+z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2\beta\langle(\mu;\nu)^{k+1}-z^k,\frac{t}{\beta}[r((\mu;\nu)^k,\beta)-\beta F((\mu;\nu)^k)]+F(z^k)\rangle\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2-\|z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2\langle(\mu;\nu)^k-z^k,z^k-(\mu;\nu)^{k+1}\rangle\\
&\qquad-2\beta\left\langle(\mu;\nu)^{k+1}-z^k,\frac{t}{\beta}[r((\mu;\nu)^k,\beta)-\beta F((\mu;\nu)^k)]+F(z^k)\right\rangle\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2-\|z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2\beta\left\langle(\mu;\nu)^{k+1}-z^k,\frac{t}{\beta}r((\mu;\nu)^k,\beta)-tF((\mu;\nu)^k)+F(z^k)-\frac{1}{\beta}r((\mu;\nu)^k,\beta)\right\rangle\\
&\quad=\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2-\|z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad-2(t-1)\langle(\mu;\nu)^{k+1}-z^k,r((\mu;\nu)^k,\beta)-\beta F((\mu;\nu)^k)\rangle\\
&\qquad-2\langle(\mu;\nu)^{k+1}-z^k,r((\mu;\nu)^k,\beta)-\beta F((\mu;\nu)^k)\rangle\\
&\qquad-2\langle(\mu;\nu)^{k+1}-z^k,-r((\mu;\nu)^k,\beta)+\beta F(z^k)\rangle\\
&\quad\le\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2-\|z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad+2\langle z^k-(\mu;\nu)^{k+1},\beta(F(z^k)-F((\mu;\nu)^k))\rangle\\
&\quad\le\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2-\|z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad+2\|z^k-(\mu;\nu)^{k+1}\|\|\beta(F(z^k)-F((\mu;\nu)^k))\|\\
&\quad\le\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2-\|z^k-(\mu;\nu)^{k+1}\|^2\\
&\qquad+\|z^k-(\mu;\nu)^{k+1}\|^2+(\beta\|F(z^k)-F((\mu;\nu)^k)\|)^2\\
&\quad\le\|(\mu;\nu)^k-(\mu;\nu)^*\|^2-\|(\mu;\nu)^k-z^k\|^2+\beta^2\|M\|^2\|z^k-(\mu;\nu)^k\|^2
\end{aligned}$$

$$= \|(\mu;\nu)^k - (\mu;\nu)^*\|^2 - (1-\beta^2\|M\|^2)\|r((\mu;\nu)^k,\beta)\|^2,$$

where the first equality follows from (2.7), the first inequality follows from Proposition 2.1, the second inequality follows from (3.1), the third inequality follows from the fact that $(\mu;\nu)^{k+1} \in H_k$, and the fourth inequality uses the Cauchy–Schwarz inequality. □

Now, we are at the position to state our main results in this section.

Theorem 3.1. *Suppose that Algorithm 3.1 generates an infinite sequence $\{(\mu;\nu)^k\}$, and the solution set of (1.1) is nonempty. Then, sequence $\{(\mu;\nu)^k\}$ converges to a solution of (2.3).*

Proof. From (3.3), one has

$$\|(\mu;\nu)^{k+1} - (\mu;\nu)^*\|^2 \leq \|(\mu;\nu)^k - (\mu;\nu)^*\|^2. \tag{3.4}$$

Therefore, the sequence $\{\|(\mu;\nu)^k - (\mu;\nu)^*\|\}$ is non-increasing and bounded. Hence, it converges. Consequently,

$$\begin{aligned} &\lim_{k\to\infty} \|r((\mu;\nu)^k,\beta)\|^2 \\ &\leq \lim_{k\to\infty} \tfrac{1}{(1-\beta^2\|M\|^2)}[\|(\mu;\nu)^k - (\mu;\nu)^*\|^2 - \|(\mu;\nu)^{k+1} - (\mu;\nu)^*\|^2] = 0. \end{aligned} \tag{3.5}$$

Thus, the sequence $\{(\mu;\nu)^k\}$ is bounded. Therefore, there exists convergent subsequence of $\{(\mu;\nu)^k\}$. The subsequence is denoted by $\{(\mu;\nu)^{k_j}\}$ and its limit by $(\hat{\mu};\hat{\nu})$. Then

$$\|r((\hat{\mu};\hat{\nu}),\beta)\|^2 = \lim_{j\to\infty} \|r((\mu;\nu)^{k_j},\beta)\|^2 = 0. \tag{3.6}$$

Hence, $(\hat{\mu};\hat{\nu})$ is a solution of (2.3).

Set $(\mu;\nu)^* = (\hat{\mu};\hat{\nu})$ in (3.3). Then, the sequence $\{\|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|\}$ converges. Since $(\hat{\mu};\hat{\nu})$ is a limit point of subsequence $\{(\mu;\nu)^{k_j}\}$, it follows that $\|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|$ converges to zero, i.e., that $\{(\mu;\nu)^k\}$ converges to $(\hat{\mu};\hat{\nu}) \in \Omega^*$. The desired result follows. □

Theorem 3.2. *The sequence $\{x^k\}$ terminates in a finite number of steps at or converges globally to a solution of (1.1).*

Proof. Assume that the sequence $\{(\mu;\nu)^k\}$ terminates in a finite number of steps at a solution of (2.3). Obviously, the sequence $\{x^k\}$ terminates in a finite number of steps to a solution of (1.1).

In the following analysis, we assume that the sequence $\{(\mu;\nu)^k\}$ is an infinite sequence. From Theorem 3.1, we know that

$$\lim_{k\to\infty} (\mu;\nu)^k = (\hat{\mu};\hat{\nu}). \tag{3.7}$$

Let $\hat{x} = \hat{\mu} - \hat{\nu}$. Then a direct computation gives

$$\begin{aligned} \|x^k - \hat{x}\| &= \|(\mu^k - \nu^k) - (\hat{\mu} - \hat{\nu})\| \\ &\leq \|(\mu^k - \hat{\mu})\| + \|(\nu^k - \hat{\nu})\| \\ &\leq \|(\mu^k - \hat{\mu})\|_1 + \|(\nu^k - \hat{\nu})\|_1 \\ &= \|(\mu^k - \hat{\mu};\nu^k - \hat{\nu})\|_1 \\ &\leq \sqrt{2n}\|(\mu^k - \hat{\mu};\nu^k - \hat{\nu})\| \to 0(\text{as } k \to \infty), \end{aligned} \tag{3.8}$$

where the second and third inequalities use the fact that

$$\|x\| \leq \|x\|_1 \leq \sqrt{n}\|x\|, \forall x \in R^n.$$

Thus, the sequence $\{x^k\}$ converges globally to a solution of (1.1). $\square$

For (2.3), by a similar analysis to the proof of Theorem 4.1 in [45], we can obtain the following result.

Lemma 3.3. *For any $(\mu;\nu) \in R^{2n}$, Then, there exist constant $\hat{\eta} > 0$ and $(\mu;\nu)^* \in \Omega^*$ such that*

$$\|(\mu;\nu) - (\mu;\nu)^*\| \leq \hat{\eta}\{m(\mu;\nu) + m(\mu;\nu)^{\frac{1}{2}}\}, \tag{3.9}$$

where $m(\mu;\nu) = \|[-(\mu;\nu)]_+\| + \|[-\beta F(\mu;\nu)]_+\| + \beta[(\mu;\nu)^\top F(\mu;\nu)]_+$.

Theorem 3.3. *Suppose that $0 < \frac{1-\beta^2\|M\|^2}{\tau^2} < 1$ holds. Then, the sequence $\{(\mu;\nu)^k\}$ converges to a solution of (2.3) linearly, where $(\mu;\nu)^k$ is generated by Algorithm 3.1.*

Proof. From Theorem 3.1, one has

$$\lim_{k\to\infty}(\mu;\nu)^k = (\hat{\mu};\hat{\nu}).$$

Hence, we can take $(\mu;\nu)^* = (\hat{\mu};\hat{\nu})$ in (3.3). Thus,

$$\|(\mu;\nu)^{k+1} - (\hat{\mu};\hat{\nu})\|^2 \leq \|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|^2 - (1-\beta^2\|M\|^2)\|r((\mu;\nu)^k,\beta)\|^2. \tag{3.10}$$

(3.7) yields

$$\|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\| \leq \tau\|r((\mu;\nu),\beta)\|. \tag{3.11}$$

Then by (3.10) and (3.11), one has

$$\begin{aligned}\|(\mu;\nu)^{k+1} - (\hat{\mu};\hat{\nu})\|^2 &\leq \|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|^2 - \tfrac{1-\beta^2\|M\|^2}{\tau^2}\|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|^2 \\ &= [1 - \tfrac{1-\beta^2\|M\|^2}{\tau^2}]\|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|^2.\end{aligned} \tag{3.12}$$

i.e.,

$$\frac{\|(\mu;\nu)^{k+1} - (\hat{\mu};\hat{\nu})\|^2}{\|(\mu;\nu)^k - (\hat{\mu};\hat{\nu})\|^2} \leq 1 - \frac{1-\beta^2\|M\|^2}{\tau^2}.$$

Since $0 < \frac{1-\beta^2\|M\|^2}{\tau^2} < 1$, one has $0 < 1 - \frac{1-\beta^2\|M\|^2}{\tau^2} < 1$. The desired result follows. $\square$

4. Numerical experiments

In this section, we provide some numerical tests to show the efficiency of the proposed method. In our numerical experiment, we set $\rho = 0.01$, $n = 2^{11}$, $m = \text{floor}(n/a)$, $k = \text{floor}(m/b)$, and the measurement matrix A is generated by Matlab scripts:

```
[Q, R]=qr(A',0); A=Q'.
```

Table 1. Comparison of Algorithm 3.1 with CGD for $\sigma = 0.001$

a	b	Algorithm 3.1			CGD		
		Time	Iter	RelErr	Time	Iter	RelErr
4	8	9.5473	416	0.0483	14.7015	220	0.0504
3	9	6.4116	292	0.0308	9.3445	136	0.0361
2	10	3.9780	189	0.0218	8.8141	133	0.0218

Table 2. Comparison of Algorithm 3.1 with CGD for $\sigma = 0.01$

a	b	Algorithm 3.1			CGD		
		Time	Iter	RelErr	Time	Iter	RelErr
4	8	8.3149	403	0.0418	15.9589	228	0.0418
3	9	6.1776	288	0.0283	10.1869	155	0.0283
2	10	4.8360	195	0.0209	5.9124	97	0.0527

The original signal $\bar{x}$ is thus generated by `p=randperm(n); x(p(1:k))=randn(k,1)`, and the observed signal y is generated by $y = A\bar{x} + \bar{n}$, where $\bar{n}$ is generated by a standard Gaussian distribution $N(0,1)$ and then it is normalized to the norm $\sigma = 0.01$ or 0.001. In our numerical experiments, the stopping criterion is

$$\frac{\|f_k - f_{k-1}\|}{\|f_{k-1}\|} < 10^{-5},$$

where f_k denotes the objective value of (1.1) at iteration x_k. For Algorithm 3.1, we set $t = 0.4, \beta = 0.8/\|M\|$. In addition, the initial points $\mu_0 = \max\{0, A^\top y\}$, $\nu_0 = \max\{0, -A^\top y\}$. For the conjugate gradient descent (denoted by CGD) method proposed recently by Xiao and Zhu in [61], we set $\xi = 10, \sigma = 10^{-4}$ and $\rho = 0.5$ in the line search (2.9) of CGD, and the initial points μ_0, ν_0 are set the same as Algorithm 3.1. In each test, we calculate the relative error

$$\text{RelErr} = \frac{\|\tilde{x} - \bar{x}\|}{\|\bar{x}\|},$$

where $\tilde{x}$ denotes the recovery signal.

The numerical results are reported in Tables 1 and 2 from which we can see that Algorithm 3.1 is much better than CGD method for all σ and (a, b).

5. Conclusion

In this paper, we proposed a new projection-type algorithm for solving the compressive sensing (CS) without the backtracking line search. Its global convergence and linear convergence rate were established. Some numerical results were provided to illustrate the efficiency of the proposed method.

Acknowledgements

The authors thank the anonymous reviewers for their valuable comments and suggestions, which helped to improve the paper.

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