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CORRECTION

Correction to: Quasimodular forms and $s\ell(m|m)^\wedge$ characters

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This note corrects the proof of Theorem 1.1 of [1], and extends the statement of the result to odd m and also furnishes the missed statement with regard to the funding obtained from the European Research council and that provided to A. Folsom in the article note.

1 Introduction and statement of results

Let for $m \in \mathbb{N}$

$$\varphi_m(z) = \varphi_m(z;\tau) := \left(\frac{\vartheta\left(z+\frac{1}{2}\right)}{\vartheta(z)}\right)^m,$$

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where ($q := e^{2\pi i \tau}$, $\zeta := e^{2\pi i z}$ with $\tau \in \mathbb{H}$, $z \in \mathbb{C}$)

$$\vartheta(z) = \vartheta(z;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} e^{\pi i \nu \tau + 2\pi i \nu\left(z+\frac{1}{2}\right)}$$

is the *Jacobi theta function*. Note that in contrast to [1], we write φ_m in order to highlight the dependence on m. Denote the coefficients of the Fourier expansion (in z) by χ_r, so that

$$\varphi_m(z;\tau) =: \sum_{r\in\mathbb{Z}} \chi_r(\tau)\zeta^r. \tag{1.1}$$

Define the Nebentypus character ψ_m for matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ by

$$\psi_m(\gamma) := e^{\frac{\pi i m}{2}\left(\frac{c}{2}d+d-1\right)}. \tag{1.2}$$

Moreover, we require the well-known Eisenstein series $E_{2j}(\tau)$. For $j \geq 2$, they are holomorphic modular forms, while $E_2(\tau)$ is a *quasimodular* form. The Bernoulli numbers B_ℓ are defined for non-negative integers ℓ by the generating function

$$\frac{t}{e^t-1} = \sum_{\ell\geq 0} B_\ell \frac{t^\ell}{\ell!}.$$

Theorem 1.1 *For $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have*

$$\chi_r(\tau) = \frac{q^r}{1+(-1)^{m+1}q^r} \sum_{0\leq \ell < \frac{m}{2}} r^{m-2\ell-1} \frac{D_{m-2\ell}(\tau)}{(m-2\ell-1)!}$$

$$(\textit{assuming that } r \neq 0 \textit{ if } m \textit{ is even}),$$

$$\chi_0(\tau) = D_0(\tau) + \sum_{1\leq j\leq \frac{m}{2}} \frac{B_{2j}}{(2j)!} D_{2j}(\tau)E_{2j}(\tau) \qquad \textit{for } m \textit{ even},$$

where for each $0 \leq j \leq m$ such that $j \equiv m \pmod 2$, the function D_j is a modular form of weight $-j$ on $\Gamma_0(2)$ with Nebentypus character ψ_m, as defined in (1.2).

Remark Theorem 1.1 was given for even m in [1]; above, we have extended the statement to hold for odd m. Moreover, the proof in [1] had a mistake: the second displayed formula in the proof of Proposition 3.3 was incorrect. We thank Sander Zwegers for pointing out the mistake and for fruitful discussion.

2 Proof of Theorem 1.1

Using that, for $\lambda, \mu \in \mathbb{Z}$, we have

$$\vartheta(z+\lambda\tau+\mu) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z}\vartheta(z),$$

$$\vartheta\left(z+\frac{1}{2}+\lambda\tau+\mu\right)=(-1)^{\mu}q^{-\frac{\lambda^2}{2}}e^{-2\pi i\lambda z}\vartheta\left(z+\frac{1}{2}\right),$$

we obtain that

$$\varphi_m(z+\lambda\tau+\mu)=(-1)^{m\lambda}\varphi_m(z). \tag{2.1}$$

Let for $z_0\in\mathbb{C},\ \tau\in\mathbb{H}$

$$P_{z_0}:=\{z_0+r\tau+s:0\le r,s\le 1\}.$$

Then, with z_0 such that no pole of φ_m lies at the boundary of P_{z_0}, we compute

$$\begin{aligned}\int_{\partial P_{z_0}}\varphi_m(w)e^{-2\pi irw}\mathrm{d}w&=\left(\int_{z_0}^{z_0+1}+\int_{z_0+1}^{z_0+1+\tau}+\int_{z_0+1+\tau}^{z_0+\tau}+\int_{z_0+\tau}^{z_0}\right)\varphi_m(w)e^{-2\pi irw}\mathrm{d}w\\&=\int_0^1\varphi_m(z_0+t)e^{-2\pi ir(z_0+t)}\mathrm{d}t\\&\quad+\tau\int_0^1\varphi_m(z_0+1+t\tau)e^{-2\pi ir(z_0+t\tau)}\mathrm{d}t\\&\quad-\int_0^1\varphi_m(z_0+\tau+t)e^{-2\pi ir(z_0+\tau+t)}\mathrm{d}t\\&\quad-\tau\int_0^1\varphi_m(z_0+t\tau)e^{-2\pi ir(z_0+t\tau)}\mathrm{d}t.\end{aligned}\tag{2.2}$$

Using (2.1) gives

$$\varphi_m(z_0+1+t\tau)=\varphi_m(z_0+t\tau),\quad \varphi_m(z_0+t+\tau)=(-1)^m\varphi_m(z_0+t).$$

Thus (2.2) becomes

$$e^{-2\pi irz_0}\left(1+(-1)^{m+1}e^{-2\pi ir\tau}\right)\int_0^1\varphi_m(z_0+t)e^{-2\pi irt}\mathrm{d}t.$$

Inserting the Fourier expansion of φ_m yields

$$\int_0^1\varphi_m\left(z_0+t\right)e^{-2\pi irt}\mathrm{d}t=\sum_{\ell\in\mathbb{Z}}\chi_\ell(\tau)e^{2\pi i\ell z_0}\int_0^1e^{2\pi i(\ell-r)t}\mathrm{d}t=\chi_r(\tau)e^{2\pi irz_0}.$$

So (assuming $r\neq 0$ if m is even)

$$\chi_r(\tau)=\frac{(-1)^{m+1}q^r}{1+(-1)^{m+1}q^r}\int_{\partial P_{z_0}}\varphi_m(w)e^{-2\pi irw}\mathrm{d}w. \tag{2.3}$$

We now compute (2.2) in another way, picking $z_0 = -\frac{1}{2} - \frac{\tau}{2}$. Then the only pole of φ_m in P_{z_0} is at $z = 0$. So, using the Residue Theorem, (2.2) equals

$$2\pi i \operatorname{Res}_{z=0}\left(\varphi_m(z)e^{-2\pi i r z}\right). \tag{2.4}$$

Write (noting that φ_m is even or odd, depending on the parity of m)

$$\varphi_m(z) = \sum_{m-2\ell>0} \frac{D_{m-2\ell}(\tau)}{(2\pi i z)^{m-2\ell}} + O(1). \tag{2.5}$$

Inserting the series expansion of $e^{-2\pi i r z}$, (2.4) becomes

$$(-1)^{m+1} \sum_{0\leq\ell<\frac{m}{2}} r^{m-2\ell-1} \frac{D_{m-2\ell}(\tau)}{(m-2\ell-1)!}.$$

Thus, for $r \in \mathbb{Z}$ (with the restriction that $r \neq 0$ if m is even) we obtain by comparing with (2.3),

$$\chi_r(\tau) = \frac{q^r}{1+(-1)^{m+1}q^r} \sum_{0\leq\ell<\frac{m}{2}} r^{m-2\ell-1} \frac{D_{m-2\ell}(\tau)}{(m-2\ell-1)!}.$$

This gives the first equation in Theorem 1.1.

To determine χ_0 (for m even), we plug in to (1.1), which implies

$$\varphi_m(z) = \sum_{1\leq\ell\leq\frac{m}{2}} \frac{D_{2\ell}(\tau)}{(2\ell-1)!} \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{r^{2\ell-1}q^r\zeta^r}{1-q^r} + \chi_0(\tau). \tag{2.6}$$

We now insert the Laurent expansions around $z = 0$ on both sides. We write the sum on r as

$$\sum_{r\geq1} \frac{r^{2\ell-1}q^r\zeta^r}{1-q^r} + \sum_{r\geq1} \frac{r^{2\ell-1}\zeta^{-r}}{1-q^r}. \tag{2.7}$$

It is not hard to see that both sums converge absolutely for $-v < y < 0$, where $v := \operatorname{Im}(\tau)$, $y := \operatorname{Im}(z)$. We write the second summand in (2.7) as

$$\sum_{r\geq1} \frac{r^{2\ell-1}\zeta^{-r}}{1-q^r} = \sum_{r\geq1} r^{2\ell-1}\zeta^{-r} + \sum_{r\geq1} \frac{r^{2\ell-1}\zeta^{-r}q^r}{1-q^r}. \tag{2.8}$$

The first summand equals

$$\left(-\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2\ell-1}\sum_{r\geq 1}\zeta^{-r}=\left(-\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2\ell-1}\frac{1}{\zeta-1}$$
$$=\left(-\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2\ell-1}\left(\frac{B_0}{2\pi i z}+\frac{B_{2\ell}(2\pi i z)^{2\ell-1}}{(2\ell)!}\right)+O\left(z^2\right)$$
$$=\frac{(2\ell-1)!}{(2\pi i z)^{2\ell}}-\frac{B_{2\ell}}{2\ell}+O\left(z^2\right).$$

The second summand combines with the first summand in (2.7) as using that φ_m is an even function of z,

$$2\sum_{r\geq 1}\frac{r^{2\ell-1}q^r}{1-q^r}+O\left(z^2\right).$$

Thus the right hand side in (2.6) becomes

$$\sum_{1\leq\ell\leq\frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell-1)!}\left(2\sum_{r\geq 1}\frac{r^{2\ell-1}q^r}{1-q^r}-\frac{B_{2\ell}}{2\ell}+\frac{(2\ell-1)!}{(2\pi i z)^{2\ell}}\right)+\chi_0(\tau)+O\left(z^2\right)$$
$$=-\sum_{1\leq\ell\leq\frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell)!}B_{2\ell}E_{2\ell}(\tau)+\sum_{1\leq\ell\leq\frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\pi i z)^{2\ell}}+\chi_0(\tau)+O\left(z^2\right).$$

Picking off the constant term on both sides of (2.5) then gives

$$\chi_0(\tau)=D_0(\tau)+\sum_{1\leq\ell\leq\frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell)!}B_{2\ell}E_{2\ell}(\tau),$$

as claimed.

The proof of the modularity follows from the fact that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2)$, we have that

$$\varphi_m\left(\frac{z}{c\tau+d};\gamma\tau\right)=\psi_m(\gamma)\varphi_m(z;\tau).$$

Reference

1. Bringmann, K., Folsom, A., Mahlburg, K.: Quasimodular forms and $s\ell(m|m)\hat{}$ characters. Ramanujan J. **36**, 103–116 (2015)