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# Markov chains on $\mathbb{Z}^+$: analysis of stationary measure via harmonic functions approach

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**Abstract**
We suggest a method for constructing a positive harmonic function for a wide class of transition kernels on $\mathbb{Z}^+$. We also find natural conditions under which this harmonic function has a positive finite limit at infinity. Further, we apply our results on harmonic functions to asymptotically homogeneous Markov chains on $\mathbb{Z}^+$ with asymptotically negative drift which arise in various queueing models. More precisely, assuming that the Markov chain satisfies Cramér's condition, we study the tail asymptotics of its stationary distribution. In particular, we clarify the impact of the rate of convergence of chain jumps towards the limiting distribution.

**Keywords** Transition kernel · Harmonic function · Markov chain · Stationary distribution · Renewal function · Exponential change of measure · Queues

**Mathematics Subject Classification** 60J10 · 60J45 · 60K25 · 60F10 · 31C05

## 1 Introduction

Let us consider a positive recurrent Markov chain $X_n$ on $\mathbb{Z}^+$ with stationary probabilities $\pi(i) > 0$, that is,

$$\sum_{i=0}^{\infty} \pi(i) = 1 \quad \text{and} \quad \pi(i) = \sum_{j=0}^{\infty} \pi(j) P(j, i) \quad \text{for all } i \in \mathbb{Z}^+,$$

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where $P(j, i)$ is the transition probability from $j$ to $i$. In terms of infinite matrices, $\pi$ is a positive left eigenvector of $P$ corresponding to the eigenvalue 1. We are interested in the asymptotic behaviour of $\pi(i)$ for large values of $i$. Markov chains on $\mathbb{Z}^+$ and their stationary probabilities naturally arise in various queueing models. A classical example is given by the queue length process at service completion epochs in the M/G/1 queue, which goes back to Kendall [10]; see Boxma and Lotov [3] for further analysis. A natural modification of this process is a system where the service rate depends on the current state of the system.

We study the case where the distribution of $X_n$ has some positive exponential moments finite, more precisely, the so-called Cramér case. Let $\xi(i)$ denote a random variable distributed as the jump of the chain $X_n$ from the state $i$, that is,

$$\mathbb{P}\{\xi(i) = j\} = P(i, i + j), \quad j \geq -i.$$

We shall always assume that $X_n$ is *asymptotically homogeneous in space*, that is,

$$\xi(i) \Rightarrow \xi \quad \text{as } i \to \infty. \tag{1}$$

We assume $\mathbb{E}\xi < 0$ and that $\mathbb{Z}$ is the lattice with minimal span for the distribution of $\xi$. By the Cramér case, we mean the case where

$$\mathbb{E}e^{\beta\xi} = 1 \quad \text{and} \quad \mathbb{E}\xi e^{\beta\xi} < \infty \quad \text{for some } \beta > 0.$$

The simplest and one of the most important examples of an asymptotically homogeneous Markov chain is a random walk with delay at zero—the *Lindley recursion* [11]:

$$W_{n+1} = (W_n + \zeta_{n+1})^+, \ n \geq 0,$$

where $\{\zeta_k\}$ are independent copies of $\xi$. As is well known, the stationary measure of $W_n$, say $\pi_W$, coincides with the distribution of $\sup_{n\geq 0} \sum_{k=1}^n \zeta_k$. Then, by the classical result due to Cramér and Lundberg, for some $c > 0$,

$$\pi_W(i) \sim ce^{-\beta i} \quad \text{as } i \to \infty.$$

Since the jumps of the chains $X_n$ and $W_n$ are asymptotically equivalent, one could expect that their stationary distributions are asymptotically equivalent too. This is true on the logarithmic scale only: Borovkov and Korshunov have shown, see Theorem 3 in [2], that if $\sup_{i\geq 0} \mathbb{E}e^{\beta\xi(i)} < \infty$, then

$$\frac{1}{i} \log \pi(i) \to -\beta \quad \text{as } i \to \infty.$$

It turns out that the exact (without logarithmic scaling) asymptotic behaviour of $\pi$ depends not only on the distribution of $\xi$, but also on the speed of weak convergence in (1); see Theorems 2, 3 and Corollaries 2, 3. Our aim in this contribution is to show how such results may be proved via an exponential change of measure, similarly to how it is done for sums of iid random variables, where an exponential change of measure is used to change the sign of the drift. The issue with transition probabilities is that if we apply an exponential change of measure with parameter $\beta$, then we get a nonstochastic transition kernel instead of a transition probability. The good news is that the transition kernel obtained, say $Q(i, j)$, is asymptotically homogeneous in space, that is, $Q(i, i+j) \to e^{\beta j}\mathbb{P}\{\xi = j\}$ as $i \to \infty$, where the limit corresponds to the jump distribution of a random walk with positive mean $\mathbb{E}\xi e^{\beta\xi}$.

So, we need to alter this transition kernel a little in order to get a transition probability which makes it possible to analyse the corresponding stochastic object, which is a Markov chain with asymptotically positive drift. We perform such an alteration via Doob's $h$-transform with a suitably chosen harmonic function. To make this approach possible, we first need to understand how to construct harmonic functions for transition kernels and how to analyse the harmonic functions constructed.

Let $Q$ be a nonnegative finite transition kernel on $\mathbb{Z}^+$, that is, $Q(i,j) \ge 0$ and

$$Q(i, \mathbb{Z}^+) = \sum_{j=0}^{\infty} Q(i,j) < \infty \quad \text{for all } i \in \mathbb{Z}^+.$$

We additionally assume that $Q(i, \mathbb{Z}^+) > 0$ for all $i \in \mathbb{Z}^+$. This kernel is also assumed irreducible in the sense that, for all $i$ and $j$, there exists $n$ such that $Q^n(i,j) > 0$.

A function $u(i)$ is called *harmonic* if $Qu = u$, which means

$$\sum_{j=0}^{\infty} Q(i,j)u(j) = u(i) \quad \text{for all } i \in \mathbb{Z}^+.$$

Then $u$ is a right eigenvector of $Q$ corresponding to the eigenvalue 1. In this paper, we only consider *nonnegative* harmonic functions. Pruitt [14] has found sufficient and necessary conditions for the existence of such a function, but these conditions are quite hard to verify. Furthermore, his results do not give any information on the limiting behaviour of harmonic functions as $i \to \infty$. Since this information is important for the study of asymptotic properties of Markov chains (see, for example, Foley and McDonald [8]), we are interested in a constructive approach to harmonic functions which would allow us to determine their asymptotics. This is the first problem we solve here; see Sect. 2.

As mentioned above, our primary motivation for studying harmonic functions of transition kernels comes from asymptotic tail analysis of stationary measures of Markov chains. The standard tool for studying large deviations is an exponential change of measure (Cramér transform). If we follow this approach in the context of Markov chains, then we usually get a positive transition kernel which is not stochastic, in general. In Sect. 5, we show how the results on asymptotics for harmonic functions obtained in Sect. 3 can be helpful in the study of stationary measures of asymptotically homogeneous Markov chains.

If the jumps of a positive recurrent Markov chain are bounded, then the equation for its invariant measure can be considered as a system of linear difference equations. The asymptotics of a fundamental solution to these equations can be found using refinements of the Poincaré–Perron theorem; see Elaydi [6] for details. However, one can not apply these results directly as we are only interested in a positive solution.

## 2 Construction of harmonic function

Any transition kernel can be seen as a combination of a stochastic transition kernel and of a total mass evolution. Indeed, let us consider the following stochastic transition kernel:

$$P^Q(i,j) := \frac{Q(i,j)}{Q(i,\mathbb{Z}^+)},$$

and a Markov chain $X_n^Q$ with transition probabilities $P^Q(i,j)$. Then we have, for all $n \geq 1$ and $i, j \in \mathbb{Z}^+$, that

$$Q^n(i,j) = \mathbb{E}_i\left[\prod_{k=0}^{n-1} Q(X_k^Q, \mathbb{Z}^+);\, X_n^Q = j\right];$$

hereinafter, $\mathbb{E}_i$ means the expectation given $X_0^Q = i$. We call $X_n^Q$ the *underlying* Markov chain.

Assume that

$$\mathbb{E}_i \prod_{n=0}^{\infty} \max(Q(X_n^Q, \mathbb{Z}^+), 1) < \infty \quad \text{for all states } i. \tag{2}$$

This condition ensures that the following function is well defined:

$$f(i) := \mathbb{E}_i \prod_{n=0}^{\infty} Q(X_n^Q, \mathbb{Z}^+) \in [0, \infty). \tag{3}$$

Under the condition (2), the function $f$ is harmonic for the kernel $Q$. Indeed, it follows by conditioning on $X_1^Q$ that

$$\begin{aligned} f(i) &= Q(i,\mathbb{Z}^+) \sum_{j=0}^{\infty} P^Q(i,j)\mathbb{E}\left\{\prod_{n=1}^{\infty} Q(X_n^Q, \mathbb{Z}^+)\middle| X_1^Q = j\right\} \\ &= \sum_{j=0}^{\infty} Q(i,j)f(j). \end{aligned}$$

The expression (3) for a harmonic function $f(i)$ originates from what we observe in the following two particular cases: The first simple case is provided by a stochastic kernel $Q$ where we have the harmonic function $f(i) \equiv 1$ which is the unique (up to a constant factor) bounded harmonic function for recurrent Markov kernels; see Meyn and Tweedie ([12], Theorem 17.1.5).

The second case concerns a kernel $Q$ which is obtained from some stochastic kernel $P$ of a Markov chain $Y_n$ by killing it in some set $B \subset \mathbb{Z}^+$, that is,

$$Q(i,j) = P(i,j)\mathbb{I}\{j \notin B\},$$

which is only defined for those $i$ where $P(i, \mathbb{Z}^+ \setminus B) > 0$. In this case, $Q(i, \mathbb{Z}^+) = \mathbb{P}_i\{Y_1 \notin B\}$ and (3) reads as

$$f(i) = \mathbb{P}_i\{\tau_B = \infty\}, \tag{4}$$

where $\tau_B := \min\{n \ge 1 : Y_n \in B\}$. So, if the original Markov chain $Y_n$ with transition probabilities $P$ is transient and $B$ is finite, then the probability of not returning to $B$ is a harmonic function for this Markov chain killed in $B$. It was proved by Doney [5, Theorem 1] that there is a unique harmonic function for a transient (towards $\infty$) random walk on $\mathbb{Z}$ killed at leaving $\mathbb{Z}^+$. This harmonic function equals the renewal function generated by descending ladder heights, which in turn is equal to $\mathbb{P}_i\{\tau_B = \infty\}$ with $B = \{-1, -2, \dots\}$; see Sect. 4.

An equivalent way to introduce the condition (2) is as follows:

$$\mathbb{E}_i e^{\sum_{n=0}^{\infty} \delta^+(X_n^Q)} < \infty, \tag{5}$$

where $\delta(j) := \log Q(j, \mathbb{Z}^+)$; hereinafter, $\delta^+ := \max(\delta, 0)$ and $\delta^- := (-\delta)^+$ so that $\delta = \delta^+ - \delta^-$. Then the function $f(i)$ may be also defined as

$$f(i) \ := \ \mathbb{E}_i e^{\sum_{j=0}^{\infty} \ell(j)\delta(j)}, \tag{6}$$

where $\ell(j)$ is the *local time* at state $j$,

$$\ell(j) \ := \ \sum_{n=0}^{\infty} \mathbb{I}\{X_n^Q = j\}.$$

## 3 On the limiting behaviour of the harmonic function for a transient kernel

In this section, we answer, in particular, the following question: What are natural conditions sufficient for (2) in the case when $Q$ is transient? These sufficient conditions are presented in Proposition 3, and they guarantee that $\limsup_{i\to\infty} f(i) \le 1$. Another question is what conditions guarantee that $f(i)$ is a positive function and, moreover, $\liminf_{i\to\infty} f(i) \ge 1$. This is answered in Proposition 2. Combining these two statements, we find sufficient conditions for the existence of a harmonic function satisfying $f(i) \to 1$, which is our main result in this section, given in the following theorem.

**Theorem 1** Suppose that the chain $X_n^Q$ is transient,

$$\sum_{i=0}^{\infty} |\delta(i)| < \infty, \tag{7}$$

and that

$$\sup_{i\in\mathbb{Z}^+} \mathbb{E}_i e^{\delta\ell(i)} < \infty, \tag{8}$$

where

$$\delta := \sum_{i=0}^{\infty} \delta^+(i). \tag{9}$$

Then the function $f$ is harmonic and $f(i) \to 1$ as $i \to \infty$.

We split the proof of this result into two parts: the lower and upper bounds, given in Propositions 2 and 3, respectively.

Our construction of a harmonic function is alternative to the construction of Foley and McDonald; see [8, Proposition 2.1]. Their analysis is based on the assumption that the series

$$\sum_{n=1}^{\infty} Q^n(i,j)z^n$$

has the (common for all $i$ and $j$) radius of convergence $R$ bigger than 1. It seems to be quite difficult to compare this assumption with our condition (8). Clearly, the condition (8) is ready for verification in particular cases because the total masses and the embedded Markov chain are factorised there. Also, our condition (7) is weaker than the 'closeness' condition in [8].

### 3.1 Lower bound for the harmonic function *f*

We start with a solidarity property for the kernel $Q$ related to positivity of the function $f$, which may be considered as a generalisation of the Harnack inequality well known for sub-stochastic matrices.

**Proposition 1** If $f(i) > 0$ for some $i \in \mathbb{Z}^+$, then $f(j) > 0$ for all $j \in \mathbb{Z}^+$.

**Proof** Irreducibility of $Q$ implies that

$$\mathbb{P}_j\{\nu_i < \infty\} > 0 \quad \text{for all } j \in \mathbb{Z}^+, \tag{10}$$

where $\nu_i := \min\{n \geq 1 : X_n^Q = i\}$ is a stopping time. By the strong Markov property,

$$\begin{aligned} f(j) &\geq \mathbb{E}_j \prod_{n=0}^{\nu_i - 1} Q(X_n^Q, X_{n+1}^Q) \times \mathbb{E}_i \prod_{k=0}^{\infty} Q(X_k^Q, \mathbb{Z}^+) \\ &= f(i)\mathbb{E}_j \prod_{n=0}^{\nu_i - 1} P^Q(X_n^Q, X_{n+1}^Q) Q(X_n^Q, \mathbb{Z}^+), \end{aligned}$$

which is a positive quantity by the hypothesis and (10), so, hence, the proof follows. □

**Proposition 2** Suppose that the chain $X_n^Q$ is transient,

$$\sum_{i=0}^{\infty} \delta^-(i) < \infty, \tag{11}$$

and that the local times satisfy

$$\sup_{i \in \mathbb{Z}^+} \mathbb{E}_i \ell(i) < \infty. \tag{12}$$

Then

$$\liminf_{i \to \infty} f(i) \geq 1.$$

In particular,

$$\inf_{i \in \mathbb{Z}^+} f(i) > 0.$$

**Proof** First notice that, since $X_n^Q$ is irreducible due to irreducibility of the kernel $Q$, transience of $X_n^Q$ is equivalent to the following: for any fixed $N$,

$$\mathbb{P}_i\{X_n^Q > N \text{ for all } n \geq 0\} \to 1 \quad \text{as } i \to \infty. \tag{13}$$

Further notice that

$$\begin{aligned} f(i) &\geq \mathbb{E}_i e^{-\sum_{j=0}^{\infty} \delta^-(j)\ell(j)} \\ &\geq e^{-\sum_{j=0}^{\infty} \delta^-(j)\mathbb{E}_i \ell(j)}, \end{aligned} \tag{14}$$

by Jensen's inequality applied to the convex function $e^x$. Fix an $N$. By the Markov property, for every $j \leq N$,

$$\mathbb{E}_i \ell(j) \leq \mathbb{P}_i\{X_n^Q \leq N \text{ for some } n \geq 0\} \mathbb{E}_j \ell(j).$$

Therefore, by (13) and the condition (12),

$$\sum_{j=0}^{N} \delta^-(j) \mathbb{E}_i \ell(j) \to 0 \quad \text{as } i \to \infty. \tag{15}$$

On the other hand,

$$\sum_{j=N+1}^{\infty} \delta^{-}(j)\mathbb{E}_i\ell(j) \le \sum_{j=N+1}^{\infty} \delta^{-}(j)\mathbb{E}_j\ell(j) \to 0 \quad \text{as } N \to \infty,$$

due to the conditions (11) and (12). Together with (15), this implies that

$$\sum_{j=0}^{\infty} \delta^{-}(j)\mathbb{E}_i\ell(j) \to 0 \quad \text{as } i \to \infty,$$

which being substituted into (14) completes the proof. □

### 3.2 Upper bound for the harmonic function *f*

**Proposition 3** Suppose that the sequence $\delta^{+}(j)$ is summable, that is, the $\delta$ defined in (9) is finite. Then, for every $i \in \mathbb{Z}^{+}$,

$$f(i) \le \sup_{j\in\mathbb{Z}^{+}} \mathbb{E}_i e^{\delta\ell(j)} \le \infty.$$

If, in addition, the chain $X_n^Q$ is transient and

$$\sup_{i\in\mathbb{Z}^{+}} \mathbb{E}_i e^{\delta\ell(i)} < \infty, \tag{16}$$

then

$$\limsup_{i\to\infty} f(i) \le 1.$$

**Proof** Take $r(j) := \frac{\delta}{\delta^{+}(j)}$ so that $\sum_{j=0}^{\infty} \frac{1}{r(j)} = 1$. We have

$$f(i) \le \mathbb{E}_i \prod_{j=0}^{\infty} e^{\delta^{+}(j)\ell(j)} = \mathbb{E}_i \prod_{j=0}^{\infty} e^{\frac{\delta}{r(j)}\ell(j)}.$$

Applying Hölder's inequality,

$$\mathbb{E}_i \prod_{j=0}^{\infty} e^{\frac{\delta}{r(j)}\ell(j)} \le \prod_{j=0}^{\infty} \left(\mathbb{E}_i e^{\delta\ell(j)}\right)^{1/r(j)}.$$

Therefore,

$$
\begin{aligned}
f(i) &\le \prod_{j=0}^{\infty} (\mathbb{E}_i e^{\delta \ell(j)})^{\delta^+(j)/\delta} \\
&\le \Big( \sup_{j \in \mathbb{Z}^+} \mathbb{E}_i e^{\delta \ell(j)} \Big)^{\sum_{j=0}^{\infty} \delta^+(j)/\delta},
\end{aligned}
\tag{17}
$$

which yields the upper bound for $f(i)$ in terms of exponential moments of local times, by (9).

Now let us turn to a more precise upper bound. Fix some $N$ and rewrite (17) as follows:

$$
f(i) \le \prod_{j=0}^{N} (\mathbb{E}_i e^{\delta \ell(j)})^{\delta^+(j)/\delta} \times \prod_{j=N+1}^{\infty} (\mathbb{E}_i e^{\delta \ell(j)})^{\delta^+(j)/\delta}.
$$

By the Markov property, for any fixed $j \le N$,

$$
\begin{aligned}
\mathbb{E}_i e^{\delta \ell(j)} \le\; & \mathbb{P}_i\{X_n^Q > N \text{ for all } n \ge 0\} \\
& + \mathbb{P}_i\{X_n^Q \le N \text{ for some } n \ge 0\} \mathbb{E}_j e^{\delta \ell(j)} \to 1 \quad \text{as } i \to \infty,
\end{aligned}
$$

by (13) and the condition (16). Therefore, for any fixed $N$,

$$
\prod_{j=0}^{N} (\mathbb{E}_i e^{\delta \ell(j)})^{\delta^+(j)/\delta} \to 1 \quad \text{as } i \to \infty.
\tag{18}
$$

The second product possesses the following upper bound:

$$
\begin{aligned}
\prod_{j=N+1}^{\infty} (\mathbb{E}_i e^{\delta \ell(j)})^{\delta^+(j)/\delta} &\le \prod_{j=N+1}^{\infty} (\mathbb{E}_j e^{\delta \ell(j)})^{\delta^+(j)/\delta} \\
&\le \Big( \sup_{j \in \mathbb{Z}^+} \mathbb{E}_j e^{\delta \ell(j)} \Big)^{\sum_{j=N+1}^{\infty} \delta^+(j)/\delta}.
\end{aligned}
$$

Since $\sum_{j=N+1}^{\infty} \delta^+(j)/\delta \to 0$ as $N \to \infty$, by (16),

$$
\prod_{j=N+1}^{\infty} (\mathbb{E}_i e^{\delta \ell(j)})^{\delta^+(j)/\delta} \to 1
$$

as $N \to \infty$ uniformly for all $i \in \mathbb{Z}^+$, which together with (18) yields the required upper bound for $f(i)$. $\square$

### 3.3 On exponential moments for local times

**Example 1** Let $Q$ be a local perturbation at the origin of the transition kernel of a simple random walk on $\mathbb{Z}^+$ as follows:

$$\begin{aligned}&Q(0,1)=\alpha>0,\\&Q(i,i+1)=p>1/2,\quad Q(i,i-1)=1-p=:q,\quad i\geq 1.\end{aligned}$$

Then we have $Q(0,\mathbb{Z}^+)=\alpha$ and $Q(i,\mathbb{Z}^+)=1$ for all $i\geq 1$. In other words, $\delta(0)=\log\alpha$ and $\delta(i)=0$ for $i\geq 1$, so that $\delta=\log\alpha$. The underlying Markov chain $X_n^Q$ is a simple random walk with reflection at zero. More precisely, its transition kernel is given by

$$\begin{aligned}&P^Q(0,1)=1,\\&P^Q(i,i+1)=p,\quad P^Q(i,i-1)=q,\quad i\geq 1.\end{aligned}$$

According to Theorem 1, the condition $\mathbb{E}_0\alpha^{\ell(0)}<\infty$ implies that the function $f(i)=\mathbb{E}_i\alpha^{\ell(0)}$ is a positive harmonic function with $f(i)\to 1$ as $i\to\infty$. The local time $\ell(0)$ is geometrically distributed with parameter $q/p$, that is,

$$\mathbb{P}_0\{\ell(0)=k\}=(1-q/p)(q/p)^{k-1},\quad k\geq 1.$$

Therefore,

$$f(0)=\mathbb{E}_0\alpha^{\ell(0)}=\alpha\frac{1-q/p}{1-\alpha q/p}<\infty\quad\text{if and only if } \alpha<p/q. \tag{19}$$

Moreover, for all $i\geq 1$,

$$\begin{aligned}f(i)&=\mathbb{E}_i\alpha^{\ell(0)}\\&=\mathbb{P}_i\{X_n^Q\neq 0\text{ for all } n\geq 1\}+\mathbb{P}_i\{X_n^Q=0\text{ for some } n\geq 1\}\mathbb{E}_0\alpha^{\ell(0)}\\&=1-(q/p)^i+(q/p)^i f(0).\end{aligned} \tag{20}$$

Since any harmonic function $f(i)$ for $Q$ is a solution to the system of equations

$$\begin{aligned}&f(0)=\alpha f(1),\\&f(i)=pf(i+1)+qf(i-1),\quad i\geq 1,\end{aligned}$$

we may determine it using standard methods in the theory of difference equations. Indeed, equations for $i\geq 1$ can be rewritten as

$$f(i+1)-f(i)=\frac{q}{p}(f(i)-f(i-1)).$$

Consequently,

$$\begin{aligned}f(i)-f(0)&=\sum_{j=0}^{i-1}(f(j+1)-f(j))=(f(1)-f(0))\sum_{j=0}^{i-1}(q/p)^j\\&=(f(1)-f(0))\frac{1-(q/p)^i}{1-q/p}.\end{aligned}$$

Noting that $f(1) = f(0)/\alpha$, we get

$$f(i) = f(0)\left[1 + (1/\alpha - 1)\frac{1-(q/p)^i}{1-q/p}\right], \quad i \geq 1. \tag{21}$$

Choosing $f(0)$ as in (19), we conclude that the expressions in (20) and (21) are equal for all $\alpha < p/q$. Further, for every $\alpha > p/q$ and every $f(0) > 0$, the function $f(i)$ from (21) becomes negative for $i$ large enough. Therefore, there is no positive harmonic function for $\alpha > p/q$. Finally, in the critical case $\alpha = p/q$, we have $f(i) = f(0)(q/p)^i \to 0$. □

**Example 2** Consider the following transition kernel:

$$\begin{aligned} &Q(i, i+1) = \alpha_i > 1, \quad i = 0, \dots, N-1, \\ &Q(N, N+1) = p, \quad Q(N, 0) = q, \\ &Q(i, i+1) = p, \quad Q(i, i-1) = q, \quad i > N. \end{aligned}$$

Aggregating the states 0,..., $N-1$ into a new state, we obtain the transition kernel from Example 1 with $\alpha = \alpha_0 \dots \alpha_{N-1}$. Therefore, there exists a positive harmonic function $f$ with $f(i) \to 1$ as $i \to \infty$ if and only if $p/q > \alpha_0 \dots \alpha_{N-1}$. This is equivalent to

$$\sup_{i<N} e^{\delta \ell(i)} < \infty, \quad \text{where } \delta = \sum_{i=0}^{N-1} \log \alpha_i.$$

This shows that exponential moment assumption on the local times in Theorem 1 is quite close to a necessary one. □

Next let us give simple sufficient conditions that guarantee finiteness of some exponential moment of local times for a Markov chain $X_n$ valued in $\mathbb{Z}^+$.

**Proposition 4** Suppose that there exist a level $N \in \mathbb{Z}^+$ and a random variable $\eta$ such that $\mathbb{E}\eta > 0$ and, for all $i \geq N$ and $j \in \mathbb{Z}$,

$$\mathbb{P}_i\{X_1 - X_0 > j\} \geq \mathbb{P}\{\eta > j\}, \tag{22}$$

that is, $\eta$ is a stochastic minorant for the family of jumps of $X_n$ above the level $N$. Then

$$\sup_{i\in\mathbb{Z}^+} \mathbb{E}_i e^{\gamma \ell(i)} < \infty \quad \text{for some } \gamma > 0. \tag{23}$$

**Proof** Let $\nu_i := \min\{n \geq 1 : X_n = i\}$. By the Markov property, the distribution of the local time $\ell(i)$ given $X_0 = i$ is geometric with parameter $\mathbb{P}_i\{\nu_i < \infty\}$. Thus, the statement on exponential moments of local times is equivalent to the following:

$$\sup_{i\in\mathbb{Z}^+} \mathbb{P}_i\{\nu_i < \infty\} < 1. \tag{24}$$

Due to irreducibility of the transition kernel, the solidarity property says that either all states are transient or recurrent. Therefore, (24) is equivalent to

$$\sup_{i\geq N}\mathbb{P}_i\{\nu_i<\infty\}<1, \tag{25}$$

and it suffices to check that

$$\inf_{i\geq N}\mathbb{P}_i\{X_n\geq i+1 \text{ for all } n\geq 0\}>0. \tag{26}$$

Indeed, by the condition (22), for any initial state $i\geq N$, we may construct the chain $X_0=i$, $X_1$,... and independent copies $\eta_1$, $\eta_2$,... of $\eta$ on some probability space in such a way that

$$X_n\geq i+\eta_1+\cdots+\eta_n \quad \text{for all } n\geq 1,$$

on the event where $\eta_1+\cdots+\eta_n\geq 1$ for all $n\geq 1$. Since $\mathbb{E}\eta>0$, owing to the strong law of large numbers,

$$p_\eta := \mathbb{P}\{\eta_1+\cdots+\eta_n\geq 1 \text{ for all } n\geq 1\}>0,$$

which implies (26) and the proof is complete. □

As follows from the proof, (23) holds with any $\gamma<\log\dfrac{1}{1-p_\eta}$ provided the chain $X_n$ satisfies the condition (22) with $N=0$.

The last result may be generalised to the case where there is no common minorant for jumps, but the drift is everywhere positive. In order to produce such a generalisation, we start with the following statement, where

$$\tau_i := \inf\{n\geq 1 : X_n\leq i-1\}.$$

**Proposition 5** Assume that, for any $i\in\mathbb{Z}^+$, there exists a positive decreasing function $g_i(j)$ such that $g_i(X_{n\wedge\tau_i})$ is a supermartingale and

$$p_1 := \sup_{i\geq 1}\frac{g_i(i)}{g_i(i-1)}<1. \tag{27}$$

If, in addition,

$$p_2 := \inf_{i\geq 0}\mathbb{P}_i\{X_1\geq i+1\}>0, \tag{28}$$

then (23) holds.

**Proof** For any $i \geq 1$, applying Doob's inequality to the supermartingale $g_i(X_{n\wedge\tau_i})$ with $X_0 = i$, we obtain that

$$\mathbb{P}_i\Big\{\inf_{n\geq 1} X_n \leq i-1\Big\} = \mathbb{P}_i\Big\{\sup_{n\geq 1} g_i(X_{n\wedge\tau_i}) \geq g_i(i-1)\Big\}$$
$$\leq \frac{\mathbb{E}_i g_i(X_0)}{g_i(i-1)} = \frac{g_i(i)}{g_i(i-1)} \leq p_1.$$

Therefore,

$$\mathbb{P}_i\Big\{\inf_{n\geq 1} X_n \geq i+1\Big\} \geq \sum_{j=i+1}^{\infty} \mathbb{P}_i\{X_1 = j\}\Big(1 - \mathbb{P}_j\Big\{\inf_{n\geq 1} X_n \leq i\Big\}\Big)$$
$$\geq (1-p_1)\sum_{j>i} \mathbb{P}_i\{X_1 = j\}$$
$$\geq (1-p_1)p_2 > 0,$$

uniformly for all $i \in \mathbb{Z}^+$. Then

$$\mathbb{P}\{\ell(i) \geq k+1\} \leq (1-p)^k, \qquad \text{where } p := (1-p_1)p_2 > 0;$$

hence, (23) holds for all $\gamma < \log \dfrac{1}{1-p}$. □

The last proposition helps us to deduce finiteness of exponential moments of local times for a Markov chain with everywhere positive drift.

**Proposition 6** Assume that there exist $\varepsilon > 0$ and $M < \infty$ such that

$$\mathbb{E}_i\{X_1 - i; X_1 - i \leq M\} \geq \varepsilon \quad \text{for all } i \in \mathbb{Z}^+. \tag{29}$$

Assume also that there exists a random variable $\zeta \geq 0$ with $\mathbb{E}\zeta < \infty$ such that

$$\mathbb{P}_i\{X_1 - i \leq -j\} \leq \mathbb{P}\{\zeta \geq j\} \quad \text{for all } i, j \in \mathbb{Z}^+. \tag{30}$$

Then (23) holds.

**Proof** Since $\mathbb{E}\zeta < \infty$, there exists a decreasing integrable function $h_1(x) : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\mathbb{P}\{\zeta > x\} = o(h_1(x))$ as $x \to \infty$. In turn, by [4], there exists a function $h_2(x)$ which is continuous, decreasing, integrable and regularly varying at infinity with index $-1$ such that $h_1(x) \leq h_2(x)$. Put

$$J := \int_0^\infty h_2(y)\mathrm{d}y < \infty.$$

Then there exists a sufficiently small $\delta > 0$ and a sufficiently large $T$ such that

$$J\mathbb{P}\{\zeta > T\} \le \varepsilon h_2(2T+M), \tag{31}$$

and, for all $t \ge T$,

$$h_2(T+(1-\delta)t) - h_2(t+T+M) \le \frac{\varepsilon}{2\mathbb{E}\zeta} h_2(t+T+M), \tag{32}$$

$$J\mathbb{P}\{\zeta \ge \delta t\} \le \frac{\varepsilon}{2} h_2(t+T+M). \tag{33}$$

Now take

$$h(t) := \min(h_2(2T+M), h_2(t)) = \begin{cases} h_2(2T+M) & \text{for } t \in [0, 2T+M], \\ h_2(t) & \text{for } t \ge 2T+M, \end{cases}$$

and consider a positive decreasing function

$$g(x) := \int_{(x+T)^+}^{\infty} h(y)\mathrm{d}y, \quad x \in \mathbb{R},$$

which is linear for $x \in [-T, T+M]$ with slope $g'(0) = -h(T) < 0$ and $h(-T) \le J$. By (32) and (33), the function $h$ satisfies

$$h(T+(1-\delta)t) \le (1+\varepsilon/2\mathbb{E}\zeta)h(t+T+M) \quad \text{for all } t \ge 0, \tag{34}$$

$$J\mathbb{P}\{\zeta \ge \delta t\} \le \frac{\varepsilon}{2} h(t+T+M) \quad \text{for all } t \ge T. \tag{35}$$

Define $g_i(j) := g(j-i)$. By construction,

$$\frac{g_i(i)}{g_i(i-1)} = \frac{g(0)}{g(-1)} = \frac{\int_T^\infty h(y)\mathrm{d}y}{\int_{T-1}^\infty h(y)\mathrm{d}y} < 1,$$

which guarantees fulfilment of (27). The condition (28) follows from (29). Then it remains to show that, for every $i \in \mathbb{Z}^+$, $g_i(X_{n\wedge\tau_i})$ is a supermartingale, that is,

$$\mathbb{E}_j g(X_1 - i) \le g(j-i) \quad \text{for all } j \ge i.$$

Indeed, for all $j$ between $i$ and $i+T$, $g(x)$ is linear on $[-T, j-i+M]$. Therefore, for $i \le j \le i+T$,

$$\begin{aligned}
\mathbb{E}_j g(X_1 - i) - g(j-i) &\le \mathbb{E}_j\{g(X_1 - i) - g(j-i); X_1 - j \le M\} \\
&= g'(0)\mathbb{E}_j\{X_1 - j;\ -T + i - j \le X_1 - j \le M\} \\
&+ (g(-T) - g(j-i))\mathbb{P}_j\{X_1 - j < -T + i - j\} \\
&\le -h(0)\mathbb{E}_j\{X_1 - j; X_1 - j \le M\} + g(-T)\mathbb{P}_j\{X_1 - j < -T\} \\
&\le -h_2(2T+M)\varepsilon + J\mathbb{P}\{\zeta > T\} \ \le\ 0,
\end{aligned} \tag{36}$$

by (29) and (31). Now let us consider the case $j > i + T$. Since $g$ is decreasing,

$$\begin{aligned}
\mathbb{E}_j g(X_1 - i) - g(j-i) &\le \mathbb{E}_j\{g(X_1 - i) - g(j-i); X_1 - j \le -\delta(j-i)\} \\
&+ \mathbb{E}_j\{g(X_1 - i) - g(j-i);\ -\delta(j-i) < X_1 - j \le 0\} \\
&+ \mathbb{E}_j\{g(X_1 - i) - g(j-i); 0 < X_1 - j \le M\} \\
&= E_1 + E_2 + E_3.
\end{aligned} \tag{37}$$

We have, by the upper bound $g(x) \le J$ and (35),

$$E_1 \le J\mathbb{P}\{\zeta \ge \delta(j-i)\} \ \le\ \frac{\varepsilon}{2}h(j - i + T + M). \tag{38}$$

Since $g'(x) = -h(x+T)$ for all $x \ge -T$, the second term possesses the following upper bound:

$$\begin{aligned}
E_2 &\le g'((1-\delta)(j-i))\mathbb{E}_j\{X_1 - j;\ -\delta(j-i) < X_1 - j \le 0\} \\
&= -h(T + (1-\delta)(j-i))\mathbb{E}_j\{X_1 - j;\ -\delta(j-i) < X_1 - j \le 0\} \\
&\le -h(T + (1-\delta)(j-i))\mathbb{E}_j\{X_1 - j; X_1 - j \le 0\}.
\end{aligned}$$

Therefore, due to (34),

$$\begin{aligned}
E_2 &\le -(1 + \varepsilon/2\mathbb{E}\zeta)h(j - i + T + M)\mathbb{E}_j\{X_1 - j; X_1 - j \le 0\} \\
&\le -h(j - i + T + M)\mathbb{E}_j\{X_1 - j; X_1 - j \le 0\} + \frac{\varepsilon}{2}h(j - i + T + M).
\end{aligned} \tag{39}$$

The third term is not greater than

$$\begin{aligned}
E_3 &\le g'(j - i + M)\mathbb{E}_j\{X_1 - j; 0 < X_1 - j \le M\} \\
&= -h(j - i + T + M)\mathbb{E}_j\{X_1 - j; 0 < X_1 - j \le M\}.
\end{aligned} \tag{40}$$

Substituting (38)–(40) into (37), we get the desired inequality $\mathbb{E}_j g(X_1 - i) - g(j-i) \le 0$ for all $j > i + T$. Together with (36), this proves that

$g_i(X_{n\wedge\tau_i})$ constitutes a nonnegative bounded supermartingale and the proof is complete. □

## 4 Random walk with negative drift conditioned to stay nonnegative

Consider the simplest application of our method of construction of harmonic functions. It deals with random walk conditioned to stay nonnegative. Let $S_0 = 0$, $S_n = \sum_{k=1}^{n} \xi_k$ be a random walk with independent and identically distributed jumps, $\mathbb{E}\xi_k < 0$. One of the possible ways to define a random walk conditioned to stay nonnegative—see for instance [1]—consists of performing Doob's $h$-transform over $S_n$ killed at leaving $\mathbb{Z}^+$, that is, a Markov chain on $\mathbb{Z}^+$ with transition probabilities

$$P(i,j) = \frac{f(j)}{f(i)}\mathbb{P}\{i + \xi_1 = j\}, \quad i, j \in \mathbb{Z}^+,$$

where $f$ is a positive harmonic function for the killed random walk, that is,

$$f(i) = \sum_{j=0}^{\infty} \mathbb{P}\{\xi_1 = j - i\}f(j), \quad i \geq 0.$$

According to Theorem 1 of [5], such a function exists if and only if

$$\mathbb{E}e^{\beta\xi_1} = 1 \text{ for some } \beta > 0.$$

This function is unique (up to a constant factor) and is defined in [5] as

$$f(i) = \sum_{j=0}^{i} e^{\beta(i-j)}u(j), \tag{41}$$

where $u(j)$ stands for the mass function of the renewal process of strict descending ladder heights of $S_n$.

Now let us show how our approach provides another representation of the harmonic function $f(i)$. Start with the following transition kernel on $\mathbb{Z}^+$:

$$Q(i,j) := e^{(j-i)\beta}\mathbb{P}\{\xi_1 = j - i\}, \quad i,j \in \mathbb{Z}^+.$$

This kernel represents the transition probabilities for the random walk $S_n^{(\beta)}$ killed at leaving $\mathbb{Z}^+$, where $S_n^{(\beta)}$ is the result of an exponential change of measure with parameter $\beta$, that is,

$$S_0^{(\beta)} = 0, \quad S_n^{(\beta)} = \sum_{k=1}^{n} \xi_k^{(\beta)},$$
$$\mathbb{P}\{\xi_k^{(\beta)} = j\} = e^{\beta j}\mathbb{P}\{\xi_k = j\}, \quad j \in \mathbb{Z}.$$

As we have already mentioned in the introduction, see (4),

$$f^*(i) := \mathbb{P}\{i + S_n^{(\beta)} \ge 0 \text{ for all } n \ge 0\} = \mathbb{P}\Big\{\min_{n\ge 0} S_n^{(\beta)} \ge -i\Big\}$$

is harmonic for the kernel $Q$. Hence, the function

$$f(i) := e^{\beta i} f^*(i) = e^{\beta i}\mathbb{P}\Big\{\min_{n\ge 0} S_n^{(\beta)} \ge -i\Big\} \tag{42}$$

is harmonic for the random walk $S_n$ killed at leaving $\mathbb{Z}^+$. Notice that this harmonic function possesses the following lower and upper bounds:

$$e^{\beta i} - e^{-\beta} \le f(i) \le e^{\beta i}. \tag{43}$$

The upper bound immediately follows from $f^*(i) \le 1$. The lower bound follows by the Cramér–Lundberg estimate

$$\mathbb{P}\Big\{\min_{n\ge 0} S_n^{(\beta)} \le -i-1\Big\} \le e^{-\beta(i+1)}.$$

In addition to (43), notice that, by the Cramér–Lundberg approximation,

$$f(i) - e^{\beta i} \to C \in (-e^{-\beta}, 0) \quad \text{as } i \to \infty.$$

Let us show that the functions defined in (41) and (42) coincide up to a constant factor. Indeed, let $(\tau_k, \chi_k)$ and $(\tau_k^{(\beta)}, \chi_k^{(\beta)})$ denote descending ladder processes for $S_n$ and $S_n^{(\beta)}$, respectively. It follows from the definition of $S_n^{(\beta)}$ that

$$\mathbb{P}\{\chi_k^{(\beta)} = x, \tau_k^{(\beta)} = j\} = e^{-\beta x}\mathbb{P}\{\chi_k = x, \tau_k = j\}$$

for all $x$, $j > 0$. Therefore,

$$\begin{aligned}
\mathbb{P}\Big\{\min_{n\ge 0} S_n^{(\beta)} = -l\Big\} &= \sum_{k=0}^{\infty}\mathbb{P}\Big\{\sum_{j=0}^{k}\chi_j^{(\beta)} = l, \tau_1^{(\beta)} < \infty, \dots, \tau_k^{(\beta)} < \infty, \tau_{k+1}^{(\beta)} = \infty\Big\}\\
&= \mathbb{P}\{\tau_1^{(\beta)} = \infty\}\sum_{k=0}^{\infty}\mathbb{P}\Big\{\sum_{j=0}^{k}\chi_j = l\Big\}e^{-\beta l}\\
&= \mathbb{P}\{\tau_1^{(\beta)} = \infty\}e^{-\beta l}u(l).
\end{aligned}$$

This gives us the desired equivalence with the multiplier $\mathbb{P}\{\tau_1^{(\beta)} = \infty\} = 1 - \mathbb{E}e^{-\beta\chi_1}$.

The random walk conditioned to stay nonnegative is the simplest Markov chain where the general scheme of construction of a harmonic function helps. In the next section, we follow almost the same techniques in our study of tail behaviour for asymptotically homogeneous in space Markov chains with negative drift under Cramér-type assumptions. Although the scheme is the same in the main aspects, some additional arguments are required.

## 5 Positive recurrent Markov chains: asymptotic behaviour of stationary distribution

In this section, we consider an asymptotically homogeneous in space Markov chain $X_n$ with jumps $\xi(i)$, that is, $\xi(i) \Rightarrow \xi$ as $i \to \infty$. Our next result describes the case when the convergence of jumps is so fast that the stationary measure $\pi$ of $X_n$ is asymptotically proportional to that of the random walk $W_n$ delayed at the origin.

**Theorem 2** Suppose that the jumps of $X_n$ possess a stochastic majorant $\Xi$,

$$\xi(i) \le_{st} \Xi,\ i \in \mathbb{Z}^+, \tag{44}$$

such that $\mathbb{E}\Xi e^{\beta\Xi} < \infty$. If

$$\sum_{i=0}^{\infty} |\mathbb{E}e^{\beta\xi(i)} - 1| < \infty, \tag{45}$$

then $\pi(i) \sim ce^{-\beta i}$ as $i \to \infty$, where $c > 0$.

It is worth mentioning that (45) is weaker than conditions we found in the literature. First, using so-called equilibrium identities, Borovkov and Korshunov [2] proved exponential asymptotics for $\pi$ under the condition

$$\sum_{i=0}^{\infty} \int_{-\infty}^{\infty} e^{\beta y} |\mathbb{P}\{\xi(i) < y\} - \mathbb{P}\{\xi < y\}| dy < \infty,$$

which is definitely stronger than (45) and implies, in particular, that the expectations of $\xi^{(\beta)}(i)$ also converge with a summable speed. Further, to show that the constant $c$ in front of $e^{-\beta i}$ is positive they introduced the following condition:

$$\sum_{i=0}^{\infty} \left(\mathbb{E}e^{\beta\xi(i)} - 1\right)^- i \log i < \infty.$$

Second, Foley and McDonald [8] used an assumption which can be rewritten in our notations as follows:

$$\sum_{i=0}^{\infty} \sum_{j \in \mathbb{Z}} e^{\beta j} |\mathbb{P}\{\xi(i) = j\} - \mathbb{P}\{\xi = j\}| < \infty.$$

Furthermore, the condition (45) is quite close to the optimal one. If, for example, $\mathbb{E}e^{\beta\xi(i)} - 1$ are of the same sign and not summable, then $\pi(i)e^{\beta i}$ converges either to zero or to infinity; see Corollary 2 below. Thus, if (45) is violated, then $\pi(i)$ may have exponential asymptotics only in the case when $\mathbb{E}e^{\beta\xi(i)} - 1$ is changing its sign infinitely often.

**Example 3** Consider a Markov chain $X_n$ which jumps to neighbours only:

$$\mathbb{P}\{\xi(i)=1\}=1-\mathbb{P}\{\xi(i)=-1\}=p+\varphi(i).$$

Assume that, as $i\to\infty$,

$$\varphi(i) \;=\; \begin{cases} i^{-\gamma}+O(i^{-\gamma-1}) & i=2k,\\ -i^{-\gamma}+O(i^{-\gamma-1}) & i=2k+1,\end{cases}$$

with some $\gamma\in(1/2,1)$. Clearly, (45) is not satisfied. Let us have a look at the values of $X_n$ at even time moments, i.e. $Y_k=X_{2k}$, $k\ge 0$. Then we have

$$\begin{aligned}\mathbb{P}_i\{Y_1-i=-2\}&=(q-\varphi(i))(q-\varphi(i-1)),\\ \mathbb{P}_i\{Y_1-i=0\}&=(q-\varphi(i))(p+\varphi(i-1))+(p+\varphi(i))(q-\varphi(i+1)),\\ \mathbb{P}_i\{Y_1-i=2\}&=(p+\varphi(i))(p+\varphi(i+1)),\end{aligned}$$

where $q:=1-p$. From these equalities, we obtain

$$\begin{aligned}&\mathbb{E}_i\left[\left(\frac{q}{p}\right)^{Y_1-i}\right]-1=\left(\frac{p^2}{q^2}-1\right)\mathbb{P}_i\{Y_1-i=-2\}+\left(\frac{q^2}{p^2}-1\right)\mathbb{P}_i\{Y_1-i=2\}\\ &=\left(\frac{p^2}{q^2}-1\right)(q-\varphi(i))(q-\varphi(i-1))+\left(\frac{q^2}{p^2}-1\right)(p+\varphi(i))(p+\varphi(i+1))\\ &=-q\left(\frac{p^2}{q^2}-1\right)(\varphi(i)+\varphi(i-1))+p\left(\frac{q^2}{p^2}-1\right)(\varphi(i)+\varphi(i+1))+O(i^{-2\gamma}).\end{aligned}$$

Noting that $\varphi(i)+\varphi(i+1)=O(i^{-\gamma-1})$, we conclude that the sequence $|\mathbb{E}_i(q/p)^{Y_1-i}-1|$ is summable and, consequently, we may apply Theorem 2. Since $\pi$ is stationary for $Y_n$ too, we obtain $\pi(i)\sim c(p/q)^i$ as $i\to\infty$. $\square$

**Proof of Theorem 2** Fix some $N\in\mathbb{Z}^+$. As is well known (see, for example, [12, Theorem 10.4.9]), the invariant measure $\pi$ satisfies the equality

$$\pi(i)=\sum_{j=0}^{N}\pi(j)\sum_{n=0}^{\infty}\mathbb{P}_j\{X_n=i;\tau_N>n\}, \tag{46}$$

where

$$\tau_N:=\inf\{n\ge 1: X_n\le N\}.$$

Let $h(i)$ be a harmonic function for $X_n$ killed at entering $[0,N]$, that is,

$$\mathbb{E}_i\{h(X_1);X_1>N\}=h(i)\quad \text{for all } i>N.$$

Then we can perform Doob's $h$-transform on $X_n$ killed at entering $[0, N]$ and define a new Markov chain $\widehat{X}_n$ on $\mathbb{Z}^+$ with the following transition kernel:

$$\mathbb{P}_i\{\widehat{X}_1 = j\} = \frac{h(j)}{h(i)}\mathbb{P}_i\{X_1 = j; \tau_N > 1\}$$

if $h(i) > 0$, and $\mathbb{P}_i\{\widehat{X}_1 = j\}$ being arbitrarily defined if $h(i) = 0$. Since $h$ is harmonic, then we also have

$$\mathbb{P}_i\{\widehat{X}_n = j\} = \frac{h(j)}{h(i)}\mathbb{P}_i\{X_n = j; \tau_N > n\} \text{ for all } n. \tag{47}$$

Combining (47) and (46), we get

$$\begin{aligned}\pi(i) =& \frac{1}{h(i)} \sum_{j=0}^{N} \pi(j)h(j) \sum_{n=0}^{\infty} \mathbb{P}_j\{\widehat{X}_n = i\} \\ =& \frac{\widehat{U}(i)}{h(i)} \sum_{j=0}^{N} \pi(j)h(j),\end{aligned} \tag{48}$$

where

$$\widehat{U}(i) := \sum_{n=0}^{\infty} \mathbb{P}\{\widehat{X}_n = i\}$$

is the renewal measure generated by the chain $\widehat{X}_n$ with initial distribution

$$\mathbb{P}\{\widehat{X}_0 = j\} = \widehat{c}\pi(j)h(j), \quad j \le N, \quad \text{where } \widehat{c} := \frac{1}{\sum_{j=0}^{N} \pi(j)h(j)}.$$

Suppose that the harmonic function $h(i)$ is such that the jumps $\widehat{\xi}(i)$ of the chain $\widehat{X}_n$ satisfy the following conditions:

$$\widehat{\xi}(i) \Rightarrow \widehat{\xi} \quad \text{as } x \to \infty, \quad \mathbb{E}\widehat{\xi} > 0, \tag{49}$$

the family of random variables $\{|\widehat{\xi}(i)|, i \in \mathbb{Z}^+\}$ admits an integrable majorant $\widehat{\Xi}$, that is,

$$|\widehat{\xi}(i)| \le_{\mathrm{st}} \widehat{\Xi} \text{ for all } i \in \mathbb{Z}^+, \quad \mathbb{E}\widehat{\Xi} < \infty; \tag{50}$$

and

$$\sup_{i\in\mathbb{Z}^+} \widehat{U}(i) <\infty. \tag{51}$$

Then the key renewal theorem for asymptotically homogeneous in space Markov chains from Korshunov [9] states that $\widehat{U}(i) \to 1/\mathbb{E}\widehat{\xi}$ as $i \to \infty$. Substituting this into (48), we deduce the following asymptotics:

$$\pi(i) \sim \frac{\sum_{j=0}^{N} \pi(j)h(j)}{\mathbb{E}\widehat{\xi}} \frac{1}{h(i)} \quad \text{as } i \to \infty. \tag{52}$$

So, now we need to choose a level $N$ and to construct a harmonic function $h(i)$ for $X_n$ killed at entering $[0, N]$ such that $h$ satisfies the conditions (49)–(51). The intuition behind our construction of the function $h$ is simple. Since we consider asymptotically homogeneous Markov chain, the chain behaves similarly to the random walk with jumps like $\xi$. We assume that the limiting jump satisfies Cramér's condition; hence, it should be such that $h(i) \sim e^{\beta i}$ as $i \to \infty$.

Consider the transition kernel

$$Q_N^{(\beta)}(i,j) := \frac{e^{\beta j}}{e^{\beta i}} P(i,j)\mathbb{I}\{j > N\},$$

which is the result of an exponential change of measure. By the theorem conditions, $Q_N^{(\beta)}(i,\mathbb{Z}^+)$ is finite for all $i$. Let us find a level $N$ such that the kernel $Q_N^{(\beta)}$ satisfies the conditions of Theorem 1.

Denote

$$\begin{aligned}
\delta_N(i) &:= \log Q_N^{(\beta)}(i,\mathbb{Z}^+) \\
&= \log \sum_{j=N+1}^{\infty} e^{\beta(j-i)} P(i,j) \\
&= \log \mathbb{E}\{e^{\beta\xi(i)}; i + \xi(i) > N\}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i=0}^{\infty} |\mathbb{E}\{e^{\beta\xi(i)}; i + \xi(i) > N\} - 1| &\le \sum_{i=0}^{\infty} |\mathbb{E}e^{\beta\xi(i)} - 1| + \sum_{i=0}^{\infty} \mathbb{E}\{e^{\beta\xi(i)}; i + \xi(i) \le N\} \\
&\le \sum_{i=0}^{\infty} |\mathbb{E}e^{\beta\xi(i)} - 1| + \sum_{i=0}^{\infty} e^{-\beta(i-N)},
\end{aligned}$$

we conclude by the condition (45) that the condition (7) of Theorem 1 holds for any $N \in \mathbb{Z}^+$.

Further, $\delta_N^+(i) \to 0$ as $N \to \infty$, for every $i \in \mathbb{Z}^+$. Moreover, $\delta_N^+(i) \le \log^+ \mathbb{E}e^{\beta\xi(i)}$, where the series

$$\sum_{i=0}^{\infty} \log^+ \mathbb{E}e^{\beta\xi(i)}$$

is convergent, due to the condition (45). Then the dominated convergence theorem yields that

$$\delta_N := \sum_{i=0}^{\infty} \delta_N^+(i) \to 0 \quad \text{as } N \to \infty.$$

So, if we proved that, for some $\gamma > 0$,

$$\sup_{i\in\mathbb{Z}^+} \mathbb{E}_i e^{\gamma \ell_N^{(\beta)}(i)} < \infty, \tag{53}$$

then we may choose a sufficiently large $N$ such that condition (8) of Theorem 1 holds; here $\ell_N^{(\beta)}(i)$ is the local time at state $i$ of the underlying chain $X_{N,n}^{(\beta)}$, $n \ge 0$, of the kernel $Q_N^{(\beta)}$.

By Proposition 4, the relation (53) would follow if we constructed, for a sufficiently large $N$, a minorant with positive mean for the jumps $\xi_N^{(\beta)}(i)$ of the chain $X_{N,n}^{(\beta)}$. The asymptotic homogeneity of the Markov chain $X_n$ implies that

$$\xi_N^{(\beta)}(i) \Rightarrow \xi^{(\beta)} \quad \text{as } i \to \infty, \tag{54}$$

where the limiting random variable has distribution

$$\mathbb{P}\{\xi^{(\beta)} = j\} = e^{\beta j}\mathbb{P}\{\xi = j\}$$

with positive mean $\mathbb{E}\xi e^{\beta\xi}$. The left tails of $\xi_N^{(\beta)}(i)$ are exponentially bounded. Therefore, there exist a sufficiently large $N$ and a random variable $\eta$ with positive mean, $\mathbb{E}\eta > 0$, such that

$$\xi_N^{(\beta)}(i) \ge_{st} \eta \quad \text{for all } i \ge N,$$

and a minorant is identified.

Finally, (13) also follows from minorisation and convergence: for every fixed $N$,

$$\mathbb{P}\{i + \eta_1 + \cdots + \eta_n \ge N \text{ for all } n \ge 1\} \to 1 \quad \text{as } i \to \infty.$$

So, for a sufficiently large $N$, the kernel $Q_N^{(\beta)}$ satisfies all the conditions of Theorem 1. Therefore, there exists a positive harmonic function $f$ for this kernel such that $f(i) \to 1$ as $i \to \infty$.

Let us consider a function $h(i) := e^{\beta i} f(i)$. For any $i \in \mathbb{Z}^+$, we have the equality

$$\begin{aligned}\sum_{j=N+1}^{\infty} P(i,j)h(j) &= \sum_{j=N+1}^{\infty} P(i,j)e^{\beta j}f(j)\\ &= e^{\beta i}\sum_{j=N+1}^{\infty} Q_N^{(\beta)}(i,j)f(j)\\ &= e^{\beta i}f(i) = h(i),\end{aligned}$$

and, hence, $h(i)$ is a harmonic function for the Markov chain $X_n$ killed at entering $[0, N]$. Let us check that $h$ produces $\widehat{X}_n$ satisfying the conditions (49)–(51). First, the condition (49) holds because, for any $j \geq N - i$,

$$\mathbb{P}\{\widehat{\xi}(i) = j\} = \frac{e^{\beta(i+j)}f(i+j)}{e^{\beta i}f(i)}\mathbb{P}\{\xi(i) = j\} \to e^{\beta j}\mathbb{P}\{\xi = j\} \quad \text{as } i \to \infty.$$

Notice that $\mathbb{E}\widehat{\xi} = \mathbb{E}\xi e^{\beta\xi}$.

Second, let us prove that the condition (50) holds. From the upper bound

$$\begin{aligned}\mathbb{P}\{\widehat{\xi}(i) > j\} &= \sum_{k=j+1}^{\infty} \frac{f(i+k)}{f(i)}e^{\beta k}\mathbb{P}\{\xi(i) = k\}\\ &\leq c_1\mathbb{E}\{e^{\beta\xi(i)};\xi(i) > j\}\\ &\leq c_1\mathbb{E}\{e^{\beta\Xi};\Xi > j\}, \quad c_1 < \infty,\end{aligned}$$

owing to the condition (44). We deduce that $\widehat{\xi}(i) \leq_{st} \Xi_1$, where

$$\mathbb{E}\Xi_1 \leq c_1 \sum_{j=0}^{\infty} \mathbb{E}\{e^{\beta\Xi};\Xi > j\} \leq c_1\mathbb{E}\Xi e^{\beta\Xi} < \infty. \tag{55}$$

On the other hand,

$$\begin{aligned}\mathbb{P}\{\widehat{\xi}(i) < -j\} &= \sum_{k=j+1}^{i} \frac{f(i-k)}{f(i)}e^{-\beta k}\mathbb{P}\{\xi(i) = -k\}\\ &\leq c_2\sum_{k=j+1}^{\infty} e^{-\beta k} \leq \frac{c_2}{\beta}e^{-\beta j}, \quad c_2 < \infty,\end{aligned}$$

so that $\widehat{\xi}(i) \ge -\Xi_2$, where $\Xi_2$ has some positive exponential moment finite. Together with (55), this implies fulfilment of the condition (50) for the function $h(i) = e^{\beta i} f(i)$.

Third, the condition (51) follows from the equalities

$$\begin{aligned}\widehat{U}(i) =& \sum_{n=0}^{\infty} \mathbb{P}\{\widehat{X}_n = i\}\\ =& f(i) \sum_{n=0}^{\infty} \mathbb{P}\{X^{(\beta)}_{N,n} = i\} = f(i)\mathbb{E}\ell^{(\beta)}_N(i)\end{aligned}$$

and from boundedness (53) of an exponential moment of the local times of $X^{(\beta)}_{N,n}$.

Therefore, we may apply (52) and deduce that

$$\pi(i) \sim \frac{\sum_{j=0}^{N} \pi(j)h(j)}{\mathbb{E}\widehat{\xi}} \frac{e^{-\beta i}}{f(i)} \sim \frac{\sum_{j=0}^{N} \pi(j)e^{\beta j} f(j)}{\mathbb{E}\xi e^{\beta\xi}} e^{-\beta i} \quad \text{as } i \to \infty.$$

The proof of Theorem 2 is complete. □

We now turn our attention to the case where $\mathbb{E}e^{\beta\xi(i)}$ converges to 1 in a nonsummable way. The next result describes the behaviour of $\pi$ in terms of a nonuniform exponential change of measure.

**Theorem 3** Suppose that, for some $\varepsilon > 0$,

$$\sup_{i\in\mathbb{Z}^+} \mathbb{E}e^{(\beta+\varepsilon)\xi(i)} < \infty. \tag{56}$$

Assume also that there exists a differentiable function $\beta(x)$ such that

$$\sum_{i=0}^{\infty} |\mathbb{E}e^{\beta(i)\xi(i)} - 1| < \infty, \tag{57}$$

and $|\beta'(x)| \le \gamma(x)$, where $\gamma(x)$ is a decreasing function which is integrable at infinity of order $o(1/x)$. Then, for some $c > 0$,

$$\pi(i) \sim c e^{-\int_0^i \beta(y)\mathrm{d}y} \quad \text{as } i \to \infty.$$

It should be noticed that Theorem 2 can be seen as a special case of Theorem 3 with $\beta(x) \equiv \beta$. We have decided to split these two statements because of the milder moment condition (44) in Theorem 2 and since the proof of Theorem 3 is simpler via a reduction to the case of summable rate of convergence, which has been considered in Theorem 2.

**Proof of Theorem 3** Consider the function $g(x) := e^{\int_0^x \beta(y)\mathrm{d}y}$ and perform the following change of measure:

$$Q^{(g)}(i,j) := \frac{g(j)}{g(i)}P(i,j).$$

First let us estimate

$$\begin{aligned} Q^{(g)}(i,\mathbb{Z}^+) - 1 &= \frac{\mathbb{E}g(i+\xi(i)) - g(i)}{g(i)} \\ &= \mathbb{E}e^{\int_i^{i+\xi(i)} \beta(y)\mathrm{d}y} - 1. \end{aligned}$$

Observe that, with necessity, $\beta(i) \to \beta$ so that, by the condition (56),

$$\mathbb{E}\left\{e^{\int_i^{i+\xi(i)} \beta(y)\mathrm{d}y} - 1; |\xi(i)| > \sqrt{i}\right\} = o\left(e^{-\varepsilon\sqrt{i}/2}\right) \quad \text{as } i \to \infty.$$

Further, the condition on the derivative of $\beta(y)$ implies that

$$\begin{aligned} \left|\int_i^{i+\xi(i)} \beta(y)\mathrm{d}y - \beta(i)\xi(i)\right| &\le \int_i^{i+\xi(i)} |\beta(y) - \beta(i)|\mathrm{d}y \\ &\le \sup_{|y|\le\sqrt{i}} |\beta'(i+y)|\xi^2(i)/2 \\ &\le \gamma\left(i - \sqrt{i}\right)\xi^2(i)/2, \end{aligned}$$

uniformly for $|\xi(i)| \le \sqrt{i}$. We have $\gamma\left(i-\sqrt{i}\right)\xi^2(i) \le \gamma\left(i-\sqrt{i}\right)i \to 0$ as $i \to \infty$. Therefore, again in view of the condition (56),

$$\mathbb{E}\left\{e^{\int_i^{i+\xi(i)} \beta(y)\mathrm{d}y}; |\xi(i)| \le \sqrt{i}\right\} = \mathbb{E}e^{\beta(i)\xi(i)} + O\left(\gamma\left(i-\sqrt{i}\right) + e^{-\varepsilon\sqrt{i}/2}\right) \quad \text{as } i \to \infty.$$

Hence,

$$|Q^{(g)}(i,\mathbb{Z}^+) - 1| \le |\mathbb{E}e^{\beta(i)\xi(i)} - 1| + O\left(\gamma\left(i-\sqrt{i}\right) + e^{-\varepsilon\sqrt{i}/2}\right).$$

Taking into account (57) and summability of the sequence $\gamma\left(i-\sqrt{i}\right)$, we conclude that

$$\sum_{i=0}^{\infty} |Q^{(g)}(i,\mathbb{Z}^+) - 1| < \infty.$$

This allows us to apply Theorem 1 to the kernel $Q^{(g)}$ killed at entering some set $[0, N]$ in the same way as in the proof of Theorem 2 and to deduce that $\pi(i) \sim c/g(i)$ as $i \to \infty$, which completes the proof. $\square$

Since the function $\beta(x)$ is stated implicitly in Theorem 3, it calls for specification of some cases where $\beta(x)$ can be expressed in terms of the difference $\mathbb{E}e^{\beta\xi(i)} - 1$.

**Corollary 1** Assume the condition (56) and that there exists a differentiable function $\alpha(x)$ such that $\alpha'(x)$ is regularly varying at infinity with index $r \in (-2, -3/2)$ and

$$\mathbb{E}e^{\beta\xi(i)} - 1 = \alpha(i) + \gamma(i), \tag{58}$$

where $\sum_{j=0}^{\infty} |\gamma(i)| < \infty$. Suppose also that

$$\sum_{j=0}^{\infty} |\alpha(i)(\mathbb{E}\xi(i)e^{\beta\xi(i)} - m)| < \infty, \tag{59}$$

where $m := \mathbb{E}\xi e^{\beta\xi}$. Then

$$\pi(i) \sim ce^{-\beta i + A(i)/m} \qquad \text{as } i \to \infty, \tag{60}$$

where $c > 0$ and $A(x) := \int_0^x \alpha(y)\mathrm{d}y$.

**Proof** Notice that, since $r \in (-2, -3/2)$, $\alpha(x)$ is regularly varying at infinity with index $r + 1 \in (-1, -1/2)$, $A(x) \to \infty$, $A(x) = o(x)$ as $x \to \infty$ and $\sum_{i=1}^{\infty} \alpha^2(i) < \infty$.

Take $\beta(x) := \beta - \alpha(x)/m$. Since $r < -3/2$, $\alpha(i) = o\left(1/\sqrt{i}\right)$. Hence, by Taylor's theorem, uniformly for $|\xi(i)| \le \sqrt{i}$,

$$e^{-\alpha(i)\xi(i)/m} = 1 - \alpha(i)\xi(i)/m + O(\alpha^2(i)\xi^2(i)),$$

which yields

$$\begin{aligned}
\mathbb{E}e^{\beta(i)\xi(i)} &= \mathbb{E}e^{\beta\xi(i)} - \alpha(i)\mathbb{E}\xi(i)e^{\beta\xi(i)}/m + O(\alpha^2(i)) \\
&= \mathbb{E}e^{\beta\xi(i)} - \alpha(i) + O(|\alpha(i)(\mathbb{E}\xi(i)e^{\beta\xi(i)} - m)| + \alpha^2(i)) \\
&= 1 + \gamma(i) + O(|\alpha(i)(\mathbb{E}\xi(i)e^{\beta\xi(i)} - m)| + \alpha^2(i)).
\end{aligned}$$

Thus, the function $\beta(x)$ satisfies all the conditions of Theorem 3 and the proof is complete. $\square$

Notice that the key condition on the rate of convergence of $\mathbb{E}e^{\beta\xi(i)}$ to 1 that implies the asymptotics (60) in the last corollary is that the sequence $\alpha^2(i)$ is summable. If it is not so, that is, if the index $r + 1$ of regular variation in the function $\alpha(x)$ is between

$-1/2$ and 0, then the asymptotic behaviour of $\pi(i)$ is different from (60). This is specified in the following corollary.

**Corollary 2** Assume the condition (56) and that there exists a differentiable function $\alpha(x)$ such that

$$|\alpha(x)| \leq \frac{c}{(1+x)^{\frac{1}{M+1}+\varepsilon}}$$

for some $c < \infty$, $M \in \mathbb{N}$, and $\varepsilon > 0$,

$$|\alpha'(x)| \leq \gamma_1(x) \tag{61}$$

for some decreasing and integrable at infinity function $\gamma_1(x)$, and

$$\mathbb{E}e^{\beta\xi(i)} - 1 = \alpha(i) + \gamma_2(i), \quad i \geq 0,$$

where $\sum_{i=0}^{\infty} \gamma_2(i) < \infty$. Assume also that, for every $k = 1, 2, ..., M$, there exist constants $D_{k,j}$ such that

$$m_k(i) = m_k + \sum_{j=1}^{M-k} D_{k,j}\alpha^j(i) + O(\alpha^{M-k+1}(i)), \tag{62}$$

where $m_k(i) := \mathbb{E}\xi^k(i)e^{\beta\xi(i)}$ and $m_k := \mathbb{E}\xi^k e^{\beta\xi}$. Then there exist real numbers $c > 0$, $R_1, R_2, ..., R_M \in \mathbb{R}$ such that

$$\pi(i) \sim c\exp\left\{-\beta i - \sum_{k=1}^{M} R_k \int_0^i \alpha^k(x)\mathrm{d}x\right\} \quad \text{as } i \to \infty. \tag{63}$$

**Proof** Define

$$\Delta(x) := \sum_{k=1}^{M} R_k \alpha^k(x).$$

In view of Theorem 3, it suffices to show that there exist $R_1, R_2, ..., R_M \in \mathbb{R}$ such that

$$\sum_{i=1}^{\infty} \left|\mathbb{E}e^{(\beta+\Delta(i))\xi(i)} - 1\right| < \infty. \tag{64}$$

Indeed, $\Delta(x)$ is differentiable and $|\Delta'(x)| \leq C|\alpha'(x)|$. Therefore, we may apply Theorem 3 with $\beta(x) = \beta + \Delta(x)$.

By Taylor's theorem, calculations similar to those in the previous corollary show that, as $i \to \infty$,

$$\begin{aligned}\mathbb{E}e^{(\beta+\Delta(i))\xi(i)} &= \mathbb{E}e^{\beta\xi(i)} + \sum_{k=1}^{M} \frac{m_k(i)}{k!}\Delta^k(i) + O(\Delta^{M+1}(i)) \\ &= 1 + \alpha(i) + \gamma_2(i) + \sum_{k=1}^{M} \frac{m_k(i)}{k!}\Delta^k(i) + O(\alpha^{M+1}(i) + e^{-\varepsilon\sqrt{i}/2}).\end{aligned}$$

From this equality, we infer that we can determine $R_1$, $R_2$,..., $R_M$ by the relation

$$\alpha(i) + \sum_{k=1}^{M} \frac{m_k(i)}{k!}\Delta^k(i) = O(\alpha^{M+1}(i)). \tag{65}$$

It follows from the assumption (62) and the bound $\Delta(x) = O(\alpha(x))$ that (65) is equivalent to

$$z + \sum_{k=1}^{M} \frac{1}{k!}\left(m_k + \sum_{j=1}^{M-k} D_{k,j}z^j\right)\left(\sum_{j=1}^{M} R_j z^j\right)^k = O(z^{M+1}) \quad \text{as } z \to 0.$$

Consequently, the coefficient of $z^k$ should be zero for all $k \le M$, and we can determine all $R_k$ recursively. For example, the coefficient of $z$ equals $1 + m_1 R_1$. Thus, $R_1 = -1/m_1$. Further, the coefficient of $z^2$ is $D_{1,1}R_1 + m_1 R_2 + m_2 R_1^2/2$ and, consequently,

$$R_2 = \frac{D_{1,1}}{m_1^2} - \frac{m_2}{2m_1^3}.$$

All further coefficients can be found in the same way. □

If $\alpha(x)$ from Corollary 2 decreases slower than any power of $x$, but (61) and (62) remain valid, then one has, by the same arguments,

$$\pi(i) = \exp\left\{-\beta i - \sum_{k=1}^{M} R_k \int_0^i \alpha^k(x)\mathrm{d}x + O\left(\int_0^i \alpha^{M+1}(x)\mathrm{d}x\right)\right\},$$

which can be seen as a corrected logarithmic asymptotic for $\pi$. To obtain precise asymptotics, one needs more information on the moments of the jumps $m_k(i)$.

**Corollary 3** Assume the condition (56) and that there exists a differentiable function $\alpha(x)$ such that (61) holds,

$$\mathbb{E}e^{\beta\xi(i)} - 1 = \alpha(i), \quad i \geq 1, \tag{66}$$

and there exist constants $D_{k,j}$ such that

$$m_k(i) = m_k + \sum_{j=1}^{\infty} D_{k,j}\alpha^j(i), \tag{67}$$

for all $k \geq 1$, where $m_k(i) := \mathbb{E}\xi^k(i)e^{\beta\xi(i)}$ and $m_k := \mathbb{E}\xi^k e^{\beta\xi}$. Assume further that

$$\sup_{k\geq 1} \sum_{j=1}^{\infty} D_{k,j} r^j < \infty \quad \text{for some } r > 0.$$

Then there exist real numbers $R_1, R_2, \ldots$, such that

$$\pi(i) \sim c \exp\left\{ -\beta i - \sum_{k=1}^{\infty} R_k \int_0^i \alpha^k(x)\mathrm{d}x \right\} \quad \text{as } i \to \infty.$$

**Proof** For any $i \geq 1$, let $\beta(i)$ denote a positive solution to the equation

$$\mathbb{E}e^{\beta(i)\xi(i)} = 1.$$

Since $\mathbb{E}e^{\gamma\xi(i)}$ is finite for all $\gamma \leq \beta + \varepsilon$, we can rewrite the last equation using Taylor's series:

$$\mathbb{E}e^{\beta\xi(i)} + \sum_{k=1}^{\infty} \frac{\Delta^k(i)}{k!} \mathbb{E}\xi^k(i)e^{\beta\xi(i)} = 1,$$

where $\Delta(i) = \beta(i) - \beta$. Taking into account (66) and (67), we get then

$$\alpha(x) + \sum_{k=1}^{\infty} \frac{\Delta^k(x)}{k!} \sum_{j=0}^{\infty} D_{k,j}\alpha^j(x) = 0, \quad x \geq 0. \tag{68}$$

Set $D_{0,1} = 1$ and define

$$F(z, w) := \sum_{k,j\geq 0} \frac{D_{k,j}}{k!} z^j w^k.$$

Therefore, (68) can be rewritten as $F(\alpha(x), \Delta(x)) = 0$. In other words, we are looking for a function $w(z)$ satisfying $F(z, w(z)) = 0$. Since $F(0, 0) = 0$ and $\frac{\partial}{\partial w}F(0, 0) = m_1 > 0$, we can apply Theorem B.4 from Flajolet and Sedgewick [7] which says that $w(z)$ is analytic in a vicinity of zero, that is, there exists a $\rho > 0$ such that

$$w(z) = \sum_{n=1}^{\infty} R_n z^n, \quad |z| < \rho.$$

Consequently,

$$\Delta(x) = \sum_{n=1}^{\infty} R_n \alpha^n(x)$$

for all $i$ such that $|\alpha(i)| < \rho$.

Applying Theorem 3 with $\beta(x) = \beta + \Delta(x)$, we get

$$\pi(i) \sim ce^{-\beta(i) - \int_0^i \Delta(y)\mathrm{d}y}.$$

Integrating $\Delta(y)$ as a series, we complete the proof. □

We conclude with the following remark. In the proof of Corollary 3, we have adapted the derivation of the Cramér series in large deviations for sums of independent random variables; see, for example, Petrov [13]. There is just one difference: we need analyticity of an implicit function instead of analyticity of an inverse function.

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