

# On property of injectivity for real $W^*$ -algebras and JW-algebras

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**Abstract** In this paper injective real  $W^*$ -algebras are investigated. It is shown that injectivity is equivalent to the property of E (extension property). It is proven that a real  $W^*$ -algebra is injective iff its hermitian part is injective, and it is equivalent to, that the enveloping  $W^*$ -algebra is also injective. Moreover, it is shown that if the second dual space of a real  $C^*$ -algebra is injective, then the real  $C^*$ -algebra is nuclear.

**Keywords** A real  $W^*$ -algebra · A JW-algebra · Injectivity

**Mathematics Subject Classification** 46L10, 46L37

## 1 Introduction

The theory of operator algebras, acting on a Hilbert space was initiated in thirties by papers of Murray and von Neumann. In the papers they have studied the structure of algebras which were later called von Neumann algebras or  $W^*$ -algebras. They are weakly closed complex  $*$ -algebras of operators on a Hilbert space. At present the theory of von Neumann algebras is a deeply developed theory with various applications.

In the middle of sixtieth D.Topping and E.Stormer have initiated the study of Jordan (non associative and real) analogues of von Neumann algebras—so called JW-algebras, i.e. real linear spaces of self-adjoint operators on a complex Hilbert space, which contain the identity operator  $\mathbb{1}$ , closed with respect to the Jordan (i.e. symmetrised) product  $x \circ y = (xy + yx)/2$  and closed in the weak operator topology. The structure of

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these algebras has happened to be close to the structure of von Neumann algebras and it was possible to apply ideas and methods similar to von Neumann algebras theory in the study of JW-algebras. Thus D. Topping has classified JW-algebras into those of type I,  $\text{II}_1$ ,  $\text{II}_\infty$ , III, later E. Stormer and Sh. Ayupov considered the problem on connections between the type of a JW-algebra and the type of its enveloping  $W^*$ -algebra. Moreover, E. Stormer gave a complete study of type I JW-algebras and has also proved that any reversible JW-algebra  $A$  (in particular of type II and III) is isomorphic to the direct sum  $A_c \oplus A_r$ , where the JW-algebra  $A_c$  is the self-adjoint part  $\mathcal{U}(A_c)_s$  of its enveloping  $W^*$ -algebra  $\mathcal{U}(A_c)$ , whence the JW-algebra  $A_r$  coincides with the self-adjoint part  $R(A_r)_s$  of the real enveloping  $W^*$ -algebra  $R(A_r)$  such that  $R(A_r) \cap iR(A_r) = \{0\}$  (so called real  $W^*$ -algebra). In this connection the study of real  $W^*$ -algebras was carried out parallel to the theory of JW-algebras.

Thus the structure theory of real  $W^*$ -algebras is relatively new, though in many aspects it is similar to the classical case of complex  $W^*$ -algebras (von Neumann algebras). In this paper we introduce the notions of injectivity and nuclearity for real  $W^*$ -algebras (in general for real  $C^*$ -algebras) and study their relations with property of E (the extension property).

## 2 Preliminaries

Let  $B(H)$  be the algebra of all bounded linear operators, acting on a complex Hilbert space  $H$ . Recall that a weakly closed  $*$ -subalgebra  $A \subset B(H)$  with the identity  $\mathbb{1}$  is called a  $W^*$ -algebra. A real  $*$ -subalgebra  $R \subset B(H)$  with  $\mathbb{1}$  is called a *real  $W^*$ -algebra*, if it is weakly closed and  $R \cap iR = \{0\}$ . A *JW-algebra* is a real subspace of the space of self-adjoint operators on a real or complex Hilbert space, closed under the operator Jordan product  $a \circ b = (ab + ba)/2$  and closed in the weak operator topology.

Let  $A$  be a Banach  $*$ -algebra over the field  $\mathbb{C}$ . The algebra  $A$  is called a  $C^*$ -algebra, if  $\|aa^*\| = \|a\|^2$  for any  $a \in A$ . A real Banach  $*$ -algebra  $R$  is called a *real  $C^*$ -algebra*, if  $\|aa^*\| = \|a\|^2$  and an element  $\mathbb{1} + aa^*$  is invertible for any  $a \in R$ . It is easy to see that  $R$  is a real  $C^*$ -algebra if and only if a norm on  $R$  can be extended onto the complexification  $A = R + iR$  of the algebra  $R$  so that algebra  $A$  is a  $C^*$ -algebra (see [9, 5.1.1]). Denote by  $M_n(A)$  algebra of all  $n \times n$  matrices over  $A$  which is also a  $C^*$ -algebra. Recall that a continuous linear map  $\varphi$  between two  $C^*$ -algebras  $A$  and  $B$  is called *completely positive*, if for any  $n \geq 1$ , the natural map  $\varphi_n$  from the  $C^*$ -algebra  $A \otimes M_n$  to the  $C^*$ -algebra  $B \otimes M_n$ , defining by

$$\varphi_n((a_{i,j})_{i,j=1}^n) = (\varphi(a_{i,j}))_{i,j=1}^n$$

is positive, where  $M_n$  is the  $C^*$ -algebra of  $n \times n$  matrices over  $\mathbb{C}$ . We say that a  $W^*$ -algebra  $A$  is called *injective* if the following condition is held: for every  $C^*$ -algebra  $B$ , for every self-adjoint linear subspace  $S$  of  $B$ , containing the identity  $\mathbb{1}$ , and for every completely positive linear map  $\varphi : S \rightarrow A$ , there is a completely positive linear map  $\bar{\varphi} : B \rightarrow A$  such that  $\bar{\varphi}|_S = \varphi$  [1, 10]. All notions above are defined similarly for real  $C^*$ - and  $W^*$ -algebras. Recall [10], that a  $W^*$ -algebra  $A \subset B(H)$  is injective if and

only if it has the *property of  $E$*  (the extension property), i.e. there exists a projection  $P : B(H) \rightarrow A$  such that  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ . In this case the map  $P$  is completely positive (see [10]).

### 3 Injective real $W^*$ -algebras

In the following lemma we will show that the map  $P$  between two real  $W^*$ -algebras is completely positive.

**Lemma 1** *Let  $R \subset B(H)$  be a real  $W^*$ -algebra. If  $P : B(H) \rightarrow R$  is a projection such that  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ , then  $P$  is completely positive.*

*Proof* By [6, Theorem 3.2] there exists a projection  $\bar{P}$  from  $B(H)$  onto  $R + iR$  such that  $\|\bar{P}\| = 1, \bar{P}(\mathbb{1}) = \mathbb{1}$  and  $P = P_1 \circ \bar{P}$ , where  $P_1(x + iy) = x$ . By [10, Lemma] the map  $\bar{P}$  is completely positive. Let  $a \in M_n(B(H)), a \geq 0, n \geq 1$ . Then  $\bar{P}_n(a) \geq 0$ . Let  $\bar{P}_n(a) = c + id$ , where  $c, d \in M_n(R)$ . By [2, Lemma 1.3.4] we have  $c \geq 0$ . Then

$$P_n(a) = (P_1 \circ \bar{P}_n)(a) = P_1(c + id) = c \geq 0,$$

therefore,  $P_n$  is positive. Hence, the map  $P$  is completely positive. □

**Theorem 1** *Let  $R \subset B(H)$  be a real  $W^*$ -algebra. If the algebra  $R$  is injective, then it has the property of  $E$ , i.e. there exists a projection  $P : B(H) \rightarrow R$  such that  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ .*

*Proof* By [9, Proposition 6.1.2]  $R$  is isometrically  $*$ -isomorphic to a real  $W^*$ -algebra on some real Hilbert space  $H_r$ . Therefore, we may suppose without loss of generality that  $B(H) = B(H_r) + iB(H_r)$  and  $R \subset B(H_r)$ . Since the identity map  $id : R \rightarrow R$  is completely positive, by the definition of injectivity the map  $id$  has a completely positive extension  $\varphi : B(H_r) \rightarrow R$ . Let  $P : B(H) \rightarrow R$  be a map, is defined as  $P(x + iy) = \varphi(x)$ , where  $x, y \in B(H_r)$ . It is clear that  $\|P\| = 1$  and  $P(\mathbb{1}) = \mathbb{1}$ . □

In order to show the converse of Theorem 1 we will similarly prove the real analogues of Stinespring’s theorem and some lemmas about completely positive maps in [1].

**Theorem 2** (The real analogy of Stinespring’s theorem.) *Let  $R$  be a real  $C^*$ -algebra with identity and  $H_r$  be a real Hilbert space. Then every completely positive linear map  $\varphi : R \rightarrow B(H_r)$  has the form  $\varphi(x) = V^*\pi(x)V$ , where  $\pi$  is a representation of  $R$  on some real Hilbert space  $K_r$  and  $V$  is a bounded operator from  $H_r$  to  $K_r$ .*

*Proof* Consider the vector space  $R \otimes H_r$  as tensor product of two the vector spaces  $R$  and  $H_r$ . We define a bilinear form  $\langle \cdot, \cdot \rangle$  on  $R \otimes H_r$  as follows; if  $u = x_1 \otimes \xi_1 + x_2 \otimes \xi_2 + \dots + x_m \otimes \xi_m$  and  $v = y_1 \otimes \eta_1 + y_2 \otimes \eta_2 + \dots + y_n \otimes \eta_n$ , put  $\langle u, v \rangle = \sum_{i,j} (\varphi(y_i^* x_j) \xi_j, \eta_i)$ . The fact that  $\varphi$  is completely positive guarantees that  $\langle \cdot, \cdot \rangle$  is positive semi-definite. For each  $x \in R$ , define a linear transformation  $\pi_0(x)$  on  $R \otimes H_r$  by  $\pi_0(x) : \sum x_j \otimes \xi_j \rightarrow \sum xx_j \otimes \xi_j$ .  $\pi_0$  is an algebraic homomorphism for

which  $\langle u, \pi_0(x)v \rangle = \langle \pi_0(x^*)u, v \rangle$  for all  $u, v \in R \otimes H_r$ . It follows that, for fixed  $u, \varrho(x) = \langle \pi_0(x)u, u \rangle$  defines positive a linear functional on  $R$ , i.e.  $\varrho(x^*x) \geq 0$ , hence  $\langle \pi_0(x)u, \pi_0(x)u \rangle = \langle \pi_0(x^*)\pi_0(x)u, u \rangle = \langle \pi_0(x^*x)u, u \rangle = \varrho(x^*x) \leq \|x^*x\|_{\varrho(\mathbb{1})} = \|x\|^2 \langle u, u \rangle$ .

Now let  $S = \{u \in R \otimes H_r : \langle u, u \rangle = 0\}$ .  $S$  is a linear subspace of  $R \otimes H_r$ , invariant under  $\pi_0(x)$  for every  $x \in R$  and  $\langle \cdot, \cdot \rangle$  determines a positive definite inner product on the quotient  $(R \otimes H_r)/S$  in the usual way:  $\langle u + S, v + S \rangle = \langle u, v \rangle$ . Letting  $K_r$  be the real Hilbert space completion of the quotient, the preceding paragraph implies that there is a unique representation  $\pi$  of  $R$  on  $K_r$  such that  $\pi(x)(u + S) = \pi_0(x)u + S, x \in R, u \in R \otimes H_r$ .

Finally, define a linear map  $V$  of  $H_r$  into  $K_r$ , by  $V\xi = \mathbb{1} \otimes \xi + S$ . It follows that  $\|V\xi\|^2 = \langle \varphi(\mathbb{1})\xi, \xi \rangle \leq \|\varphi(\mathbb{1})\|\|\xi\|^2$ , so that  $V$  is bounded, and the required formula  $\varphi(x) = V^*\pi(x)V$  follows from the definition of  $V$  by a routine computation.  $\square$

Let  $S$  be a subspace of a real  $C^*$ -algebra  $R$ , and  $H_r$  be a real Hilbert space.  $B(S, H_r)$  (resp.  $B(R, H_r)$ ) will denote the vector space of all bounded linear maps of  $S$  (resp.  $R$ ) into  $B(H_r)$ . Note that  $B(S, H_r)$  is a real Banach space in the obvious norm. We shall endow  $B(S, H_r)$  with a certain topology, relative to which it becomes the dual space of another real Banach space.

For  $l > 0$ , let  $B_l(S, H_r)$  denote a closed ball of radius  $l$ :  $B_l(S, H_r) = \{\varphi \in B(S, H_r) : \|\varphi(a)\| \leq l\|a\| \text{ for all } a \in S\}$ . Firstly, topologize  $B_l$  as follows: by definition, a net  $\varphi_\nu \in B_l(S, H_r)$  converges to  $\varphi \in B_l(S, H_r)$  if  $\varphi_\nu(a) \rightarrow \varphi(a)$  in the weak operator topology for every  $a \in S$ . A convex subset  $U$  of  $B(S, H_r)$  is open, if  $U \cap B_l(S, H_r)$  is an open subset of  $B_l(S, H_r)$  for every  $l > 0$ . The convex open sets form a base for a locally convex Hausdorff topology on  $B(S, H_r)$ , which we shall call the *BW-topology*. We can show that  $B(S+iS, H_r+iH_r) = B(S, H_r) + iB(S, H_r)$  and the BW-topology on  $B(S, H_r)$  is the restriction of BW-topology of  $B(S+iS, H_r+iH_r)$ . Equivalently, the BW-topology is the strongest locally convex topology on  $B(S, H_r)$  which relativize to the prescribed topology on each ball  $B_l(S, H_r), l > 0$ . It is clear that a linear functional  $f$  on  $B(S, H_r)$  is BW-continuous iff the restriction of  $f$  to every  $B_l(S, H_r)$  is continuous. By linearity of  $f$ , we conclude that  $f$  is BW-continuous iff the restriction of  $f$  to  $B_1(S, H_r)$  is continuous.

*Remark 1* 1)  $B_l(S, H_r)$  is compact in the relative BW-topology.  
 2) The restriction map  $\varphi \rightarrow \varphi|_S$  of  $B(R, H_r)$  into  $B(S, H_r)$  is BW-continuous.

Let  $CP(S, H_r)$  (resp.  $CP(R, H_r)$ ) denote the set of all completely positive linear maps of  $S$  (resp.  $R$ ) into  $B(H_r)$ . Each is a subset of  $B(S, H_r)$  and  $B(R, H_r)$ , respectively, and thus inherits a BW-topology from the larger space as above. In addition, it is apparent that both  $CP(S, H_r)$  and  $CP(R, H_r)$  are convex cones, and the set  $CP(R, H_r)|_S$  of all restrictions of maps in  $CP(R, H_r)$  to  $S$  is a subcone of  $CP(S, H_r)$ .

**Lemma 2**  $CP(R, H_r)|_S$  is a closed cone in  $B(S, H_r)$ , relative to the BW-topology.

*Proof* We claim first that  $\|\varphi\| = \|\varphi|_S\|$ , for every  $\varphi \in CP(R, H_r)$ . Choose  $\pi$  and  $V$ , as in Theorem 2, such that  $\varphi(x) = V^*\pi(x)V, x \in R$ . Then  $\|\varphi\| \leq \|V^*\| \cdot \|V\| = \|V^*V\| = \|\varphi(\mathbb{1})\|$ ; since  $\mathbb{1} \in S$  it follows that  $\|\varphi\| \leq \|\varphi|_S\|$ . The opposite inequality is trivial.

Nextly, observe that  $CP(R, H_r)$  is a BW-closed subset of  $B(R, H_r)$ ; indeed, since  $CP(R, H_r)$  is convex, then by definition, it is closed iff  $CP(R, H_r) \cap B_l(R, H_r)$  is (relatively) closed for every  $l > 0$ . But, if  $\varphi_\nu$  is a bounded net in  $CP(R, H_r)$  such that  $\varphi_\nu \rightarrow \varphi \in B(R, H_r)$  (BW), then  $\varphi_\nu(x) \rightarrow \varphi(x)$  in the weak operator topology for every  $x \in R$ , and this makes it plain that  $\varphi$  must also be completely positive. By Remark 1 1), it follows that for every  $l > 0$ ,  $CP(R, H_r) \cap B_l(R, H_r)$  is BW-compact. The first paragraph of the proof shows that restriction map  $\varphi \rightarrow \varphi|_S$  carries  $CP(R, H_r) \cap B_l(R, H_r)$  onto  $CP(R, H_r)|_S \cap B_l(S, H_r)$ , and by Remark 1 2), the restriction is BW-continuous; we conclude that  $CP(R, H_r)|_S \cap B_l(S, H_r)$  is compact and closed. Since  $CP(R, H_r)|_S$  is convex, it follows from the definition of the BW-topology that this set is closed.  $\square$

**Lemma 3** *If  $f$  is an arbitrary BW-continuous linear functional with*

$$f(CP(R, H_r)|_S) \geq 0,$$

*then  $f(\varphi) \geq 0$  for every  $\varphi \in CP(S, H_r)$ .*

The lemma is proven analogically to Lemmas 1.2.5 and 1.2.6 in [1].

**Proposition 1** *The set  $CP(S, H_r)$  is a subcone of  $CP(R, H_r)|_S$ .*

The proof is completed by Lemmas 2 and 3 and a standard separation theorem. Hence, we obtain:

**Corollary 1** *The set  $CP(S, H_r)$  coincides with  $CP(R, H_r)|_S$ , i.e.*

$$CP(R, H_r)|_S = CP(S, H_r).$$

**Theorem 3**  *$B(H_r)$  is an injective real  $W^*$ -algebra.*

*Proof* Let  $S$  be a self-adjoint linear subspace of a real  $C^*$ -algebra  $R$ , which contains the identity  $\mathbb{1}$  of  $R$ , and let  $\varphi : S \rightarrow B(H_r)$  be an arbitrary completely positive linear map. Then  $\varphi \in CP(S, H_r)$ . By Corollary 1,  $\varphi \in CP(R, H_r)|_S$ , i.e. the map  $\varphi$  has a completely positive extension on  $R$ . Hence, we have that  $B(H_r)$  is injective.  $\square$

And now, we will show the converse of Theorem 1.

**Theorem 4** *Let  $R \subset B(H)$  be a real  $W^*$ -algebra. If the algebra  $R$  has the property of  $E$ , then it is injective.*

*Proof* Let there be a projection  $P : B(H) \rightarrow R$  such that  $\|P\| = 1$ ,  $P(\mathbb{1}) = \mathbb{1}$  and let  $\varphi : S \rightarrow R$  be a completely positive map, where  $S$  is a self-adjoint subspace of a real  $C^*$ -algebra  $B$ , which contains the identity  $\mathbb{1}$  of  $B$ . Since  $B(H_r)$  is injective, the map  $\varphi : S \rightarrow R \subset B(H_r)$  can be extended to a completely positive map  $\bar{\varphi}$  on  $B$ , i.e. there is a map  $\bar{\varphi} : B \rightarrow B(H_r)$  such that  $\bar{\varphi}|_S = \varphi$ . Consider a map  $\varphi_1 = P \circ \bar{\varphi} : B \xrightarrow{\bar{\varphi}} B(H_r) \xrightarrow{P} R$ . It is easy to see the map  $\varphi_1$  is completely positive and  $\varphi_1|_S = \varphi$ . Hence,  $R$  is injective.  $\square$

From Theorems 1 and 4 we have the following result.

**Corollary 2** *Let  $R \subset B(H)$  be a real  $W^*$ -algebra. The algebra  $R$  is injective, if and only if it has the property of  $E$ .*

Recall that if  $R$  is a real  $W^*$ -algebra, then  $R_s = \{x \in R : x = x^*\}$  is a JW-algebra by Jordan product  $x \circ y = (xy + yx)/2$ . A JW-algebra (or a JC-algebra)  $R_s$  is called injective if for any  $C^*$ -algebras  $B \subset C$  and any morphism  $\varphi : B_s \rightarrow R_s$  there is a morphism  $\bar{\varphi} : C_s \rightarrow R_s$  such that  $\bar{\varphi}|_{B_s} = \varphi$ .

The following result has been proven in [8, Theorem 1.2]. But, there is a gap in the second part of the proof of Theorem 1.2 in [8]. In the following theorem, using Theorem 3 we will complete the second part of the proof of Theorem 1.2 in [8].

**Theorem 5** *If the JW-algebra  $R_s$  has the property of  $E$ , i.e. there is a projection  $P : B(H) \rightarrow R_s$  such that  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ , then  $R_s$  is injective.*

*Proof* Let  $B \subset C$  be  $C^*$ -algebras and  $\varphi : B_s \rightarrow R_s$  be a morphism. Let  $i_1 : R_s \hookrightarrow B(H)_s \subset B(H_r)$  be an identity map, which is the natural injection. Since  $i_1$  is completely positive, the map  $\psi = i_1 \circ \varphi : B_s \rightarrow B(H_r)$  is also completely positive. By injectivity of  $B(H_r)$  the map  $\psi$  has a completely positive extension  $\bar{\psi} : C \rightarrow B(H_r)$  because we can consider  $C$  as a real  $C^*$ -algebra with  $B \subset C$ . Then the map  $\bar{\varphi} = P \circ \bar{\psi} : C_s \xrightarrow{\bar{\psi}} B(H_r) \xrightarrow{P} R_s$  is completely positive and  $\bar{\varphi}|_{B_s} = \varphi$ .  $\square$

Then, using the first part of Theorem 1.2 in [8] and Theorem 5 we can formulate the following corollary.

**Corollary 3** *The JW-algebra  $R_s$  is injective, if and only if it has the property of  $E$ .*

**Theorem 6** *Let  $R \subset B(H)$  be a real  $W^*$ -algebra. The algebra  $R$  has the property of  $E$  if and only if the JW-algebra  $R_s$  has the property of  $E$ .*

*Proof* Suppose, that  $R$  has the property of  $E$ , i.e. there exists a projection  $P : B(H) \rightarrow R$  with  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ . We define a map  $P_1 : B(H) \rightarrow R_s$ , as

$$P_1 = E_1 \circ P : B(H) \xrightarrow{P} R \xrightarrow{E_1} R_s,$$

where  $E_1(x) = \frac{1}{2}(x + x^*)$ . It is easy to see that the map  $P_1$  is a projection with  $\|P_1\| = 1, P_1(\mathbb{1}) = \mathbb{1}$ . Hence,  $R_s$  has the property of  $E$ .

Conversely, let  $P : B(H) \rightarrow R_s$  is a projection with  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ . Then by [6, Theorem 3.2] there exists a projection  $\bar{P}$  from  $B(H)$  onto  $R + iR$  such that  $\|\bar{P}\| = 1, \bar{P}(\mathbb{1}) = \mathbb{1}$ . And now, we consider the map

$$P_1 = E_2 \circ \bar{P} : B(H) \xrightarrow{\bar{P}} R + iR \xrightarrow{E_2} R,$$

where  $E_2(x + iy) = x$ . It is clear that  $P_1$  is a projection such that  $\|P_1\| = 1, P_1(\mathbb{1}) = \mathbb{1}$ .  $\square$

**Theorem 7** *A real  $W^*$ -algebra  $R$  has the property of E if and only if the enveloping  $W^*$ -algebra  $R + iR$  of  $R$  has the property of E.*

*Proof* Let  $R$  has the property of E, i.e. there exists a projection  $P : B(H) \rightarrow R$  with  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ . Then by [6, Theorem 3.2] there exists a projection  $\overline{P}$  from  $B(H)$  onto  $R + iR$  such that  $\|\overline{P}\| = 1, \overline{P}(\mathbb{1}) = \mathbb{1}$ ; therefore,  $R + iR$  has the property of E.

Conversely, let  $\overline{P}$  is a projection  $\overline{P}$  from  $B(H)$  onto  $R + iR$  such that  $\|\overline{P}\| = 1, \overline{P}(\mathbb{1}) = \mathbb{1}$ . Consider the map  $E(x) = \frac{1}{2}(x + \alpha_R(x^*))$ , where  $x \in R + iR$  and  $\alpha_R$  is involutive  $*$ -antiautomorphism of  $R + iR$  with  $\alpha_R(a + ib) = a^* + ib^*$  for  $a, b \in R$  (see [3]). It is easy to see that  $E$  is a projection from  $R + iR$  onto  $R$ . Indeed, let  $y = E(x)$ , then we have

$$\alpha(y) = \frac{1}{2}\alpha(x + \alpha(x^*)) = \frac{1}{2}(\alpha(x) + \alpha^2(x^*)) = \frac{1}{2}(\alpha(x) + x^*) = y^*,$$

i.e.  $y \in R$ . It is clear, that  $\|E\| = 1, E(\mathbb{1}) = \mathbb{1}$ . Then, the map  $P = E \circ \overline{P}$  is a projection from  $B(H)$  onto  $R$  such that  $\|P\| = 1, P(\mathbb{1}) = \mathbb{1}$ , i.e.  $R$  has the property of E. □

This result is true for injectivity, too.

**Theorem 8** *A real  $W^*$ -algebra  $R$  is injective if and only if the enveloping  $W^*$ -algebra  $R + iR$  of  $R$  is injective.*

*Proof* Let  $R$  be injective. Then, by Theorem 1,  $R$  has the property of E and by Theorem 7,  $R + iR$  has the property of E. By [10, Theorem],  $R + iR$  is injective. Conversely, let  $R + iR$  be injective. By [10, Theorem],  $R + iR$  has the property of E. From Theorem 7, it follows that  $R$  has the property of E. Then, by Theorem 4,  $R$  is injective. □

From theorems above we have the following result.

**Corollary 4** 1) *A JW-algebra  $R_s$  has the property of E if and only if the enveloping  $W^*$ -algebra  $R + iR$  of  $R$  has the property of E.*  
 2) *JW-algebra  $R_s$  is injective if and only if the enveloping  $W^*$ -algebra  $R + iR$  of  $R$  is injective.*

*Remark 2* Thus, for the  $*$ -algebras  $R_s, R, R + iR$  the notion of injectivity and the property of E coincide.

Now, we will consider the dual space.

**Theorem 9** *If  $R$  is a real  $C^*$ -algebra, then the second dual  $R^{**}$  of  $R$  is a real  $W^*$ -algebra.*

*Proof* It is known that the complexification  $A = R + iR$  of  $R$  is a  $C^*$ -algebra. By [9, Proposition 1.1.4] we have  $A^{**} = R^{**} + iR^{**}$ . By [12, Theorem 1.17.2]  $A^{**}$  is a  $W^*$ -algebra, therefore  $R^{**}$  is a real  $W^*$ -algebra. □

Let  $A$  and  $B$  be  $C^*$ -algebras,  $A \otimes B$  be their tensor product over the field  $\mathbb{C}$ . A norm  $\|\cdot\|_\gamma$  on the  $*$ -algebra  $A \otimes B$  is called a  $C^*$ -norm, if  $\|xx^*\|_\gamma = \|x\|_\gamma^2$  for all  $x \in A \otimes B$ . Every  $C^*$ -norm on  $A \otimes B$  is *crossnorm*, i.e. it satisfies the condition  $\|a \otimes b\|_\gamma = \|a\|\|b\|$  for all  $a \in A, b \in B$ . If  $\|\cdot\|_\gamma$  is  $C^*$ -norm, then the closure of the  $*$ -algebra  $A \otimes B$ , by the norm  $\|\cdot\|_\gamma$  is a  $C^*$ -algebra which we denote by  $A \otimes_\gamma B$ . There are two following crossnorms:

$$\begin{aligned} \|u\|_{min} &= \sup\{|(f \otimes g)(u)| : f \in A^*, g \in B^*, \|f\| \leq 1, \|g\| \leq 1\}, \\ \|u\|_{max} &= \inf\{\sum_j \|a_j\| \cdot \|b_j\|\}, \text{ where } u = \sum_j a_j \otimes b_j, (f \otimes g)(u) = \sum_j f(a_j) \cdot g(b_j) \text{ and } a_j \in A, b_j \in B. \end{aligned}$$

It can be easily shown, that for any  $C^*$ -norm  $\|\cdot\|_\gamma$  it is held that:  $\|u\|_{min} \leq \|\cdot\|_\gamma \leq \|u\|_{max}$ . It is known [12] that if  $A$  and  $B$  are  $W^*$ -algebras, then there is a unique central projection  $Z$  in  $(A \otimes_{min} B)^{**}$  such that  $A \otimes_{min} B$  is identified with a weak- $*$  dense  $C^*$ -subalgebra of  $(A \otimes_{min} B)^{**}Z$ . The  $W^*$ -algebra  $(A \otimes_{min} B)^{**}Z$  is called the  $W^*$ -tensor product of  $A$  and  $B$ , and is denoted by  $A \overline{\otimes} B$ .

**Theorem 10** *Let  $R$  and  $Q$  be real  $W^*$ -algebras,  $\mathcal{U}(R) = R + iR$  and  $\mathcal{U}(Q) = Q + iQ$  be their enveloping  $W^*$ -algebras, respectively. Then  $\mathcal{U}(R \overline{\otimes} Q) = \mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q)$ , i.e.  $(R + iR) \overline{\otimes} (Q + iQ) = R \overline{\otimes} Q + i(R \overline{\otimes} Q)$ .*

*Proof* Let  $\alpha_R$  and  $\alpha_Q$  be canonical  $*$ -antiautomorphisms of  $\mathcal{U}(R)$  and  $\mathcal{U}(Q)$ , generating  $R$  and  $Q$ , respectively, i.e.  $R = \{x \in \mathcal{U}(R) : \alpha_R(x) = x^*\}$  and  $Q = \{y \in \mathcal{U}(Q) : \alpha_Q(y) = y^*\}$ . Consider the  $*$ -antiautomorphism

$$\alpha_R \overline{\otimes} \alpha_Q : \mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q) \rightarrow \mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q),$$

defining by  $\alpha_R \overline{\otimes} \alpha_Q(x \otimes y) = \alpha_R(x) \otimes \alpha_Q(y)$ . Since  $(\alpha_R \overline{\otimes} \alpha_Q)^2$  is normal for any

$$z = \sup_n \sum_{i=1}^n (x_i \otimes y_i) \in \mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q)$$

it is held:

$$\begin{aligned} (\alpha_R \overline{\otimes} \alpha_Q)^2(z) &= \sup_n (\alpha_R \overline{\otimes} \alpha_Q)^2\left(\sum_{i=1}^n (x_i \otimes y_i)\right) = \sup_n \sum_{i=1}^n (\alpha_R \overline{\otimes} \alpha_Q)^2(x \otimes y) \\ &= \sup_n \sum_{i=1}^n \alpha_R(x)^2 \otimes \alpha_Q(y)^2 = \sup_n \sum_{i=1}^n (x_i \otimes y_i) = z, \end{aligned}$$

i.e.  $(\alpha_R \overline{\otimes} \alpha_Q)^2$  is an identity map on  $\mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q)$ . Consider a real  $W^*$ -algebra

$$F = \{z \in \mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q) : \alpha_R \overline{\otimes} \alpha_Q(z) = z^*\}.$$

Then it can be easily shown that  $F \cap iF = \{0\}$  and  $\mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q) = F + iF$ . If  $x \in R$  and  $y \in Q$ , then  $\alpha_R \overline{\otimes} \alpha_Q(x \otimes y) = \alpha_R(x) \otimes \alpha_Q(y) = x^* \otimes y^* = (x \otimes y)^*$ , therefore,  $R \overline{\otimes} Q \subset F$ . Since  $F$  is weak  $*$ -closed,  $R \overline{\otimes} Q \subset F$ . Hence,  $\mathcal{U}(R \overline{\otimes} Q) \subset \mathcal{U}(R) \overline{\otimes} \mathcal{U}(Q)$ . Let  $z \in \mathcal{U}(R) \otimes \mathcal{U}(Q)$ . Then,  $z = \sum_{k=1}^n (a_k + ib_k) \otimes (c_k + id_k)$ , where  $a_k, b_k \in R$ ,



$c_k, d_k \in Q$ . Since  $z = \sum_{k=1}^n (a_k \otimes c_k - b_k \otimes d_k) + i \sum_{k=1}^n (b_k \otimes c_k + a_k \otimes d_k)$ , then  $z \in R \otimes Q + i(R \otimes Q) = U(R \otimes Q)$ . Therefore,  $U(R) \otimes U(Q) = U(R \otimes Q)$ , hence  $U(R) \overline{\otimes} U(Q) \subset U(R \overline{\otimes} Q)$ .  $\square$

*Remark 3* Recall that the Jordan analogue of Theorem 10, is proven in [7], i.e. for the JW algebras  $R$  and  $Q$  it is shown that  $U(R \overline{\otimes} Q) = U(R) \overline{\otimes} U(Q)$ .

**Definition 1** A  $C^*$ -algebra  $A$  is called *nuclear*, if for any  $C^*$ -algebra  $B$  the norms  $\| \cdot \|_{min}$  and  $\| \cdot \|_{max}$  on  $A \otimes B$  (all of  $C^*$ -norms on  $A \otimes B$ ) coincide, i.e.  $A$  is called *nuclear*, if there is the unique  $C^*$ -norm on  $A \otimes B$ .

The notion of the nuclearity is defined similarly for real  $C^*$ -algebras.

From Theorem 10 we can have the following result.

**Proposition 2** *A real  $C^*$ -algebra  $R$  is nuclear, if the  $C^*$ -algebra  $R + iR$  is nuclear.*

*Proof* Let  $R$  be a real  $C^*$ -algebra. Then  $Q + iQ$  is a  $C^*$ -algebra and by Theorem 9

$$(R + iR) \overline{\otimes} (Q + iQ) = R \overline{\otimes} Q + i(R \overline{\otimes} Q).$$

Since there is the unique  $C^*$ -norm on  $(R + iR) \otimes (Q + iQ) = R \otimes Q + i(R \otimes Q)$ , by  $(R + iR) \overline{\otimes} (Q + iQ) = R \overline{\otimes} Q + i(R \overline{\otimes} Q)$  there is also the unique  $C^*$ -norm on  $R \otimes Q$ .  $\square$

*Remark 4* In the paper [Lemma 6.2 (2), [4]] using the complexification of the tensor product of two real  $C^*$ -algebras it was stated in passing that a real  $C^*$ -algebra  $R$  is nuclear iff the enveloping  $C^*$ -algebra  $M = R + iR$  is nuclear. However, generally speaking, it is not clear that  $R + iR$  is nuclear when  $R$  is nuclear. Indeed, it is known, that any real  $W^*$ -algebra is generated by the involutive  $*$ -antiautomorphism of enveloping  $W^*$ -algebra (see [3]). Since there is a  $W^*$ -algebra  $N$  which does not have a  $*$ -antiautomorphism algebra  $M \overline{\otimes} N$  can also not have a  $*$ -antiautomorphism. Therefore,  $C^*$ -algebra  $M \overline{\otimes} N$  can not be expressed as complexification of a real  $C^*$ -algebra.

Therefore, we will formulate the following problem.

**Problem 1** Is the  $C^*$ -algebra  $R + iR$  nuclear, whether  $R$  is a nuclear real  $C^*$ -algebra?

Let  $R$  be a real  $C^*$ -algebra. It is known, that the (complex) second dual space  $(R + iR)^{**}$  is a  $W^*$ -algebra. Then, by  $(R + iR)^{**} = R^{**} + iR^{**}$  the (real) second dual space  $R^{**}$  is a real  $W^*$ -algebra. There is the following connection, relative to injectivity and nuclearity between a real  $W^*$ -algebra and its second dual space.

**Theorem 11** *Let  $R$  be a real  $C^*$ -algebra. If the real  $W^*$ -algebra  $R^{**}$  has the property of E, then  $R$  is nuclear.*

*Proof* Since  $A^{**} = R^{**} + iR^{**}$ , by Theorem 7  $A^{**}$  also has the property of E. So, by [10, Theorem]  $A^{**}$  is injective. Then, by [5, Theorem 3],  $A$  is nuclear. By Proposition 2  $R$  is also nuclear.  $\square$

The converse is not obvious, therefore we will formulate the last problem of the paper.

**Problem 2** Does the real  $W^*$ -algebra  $R^{**}$  have the property of E, whether  $R$  is a nuclear real  $C^*$ -algebra?

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