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Bochner representable operators on Banach function spaces

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Abstract Let $(E, \|\cdot\|_E)$ be a Banach function space, E' the Köthe dual of E and $(X, \|\cdot\|_X)$ be a Banach space. It is shown that every Bochner representable operator $T : E \to X$ maps relatively $\sigma(E, E')$-compact sets in E onto relatively norm compact sets in X. If, in particular, the associated norm $\|\cdot\|_{E'}$ on E' is order continuous, then every Bochner representable operator $T : E \to X$ is $(\gamma_E, \|\cdot\|_X)$-compact, where γ_E stands for the natural mixed topology on E. Applications to Bochner representable operators on Orlicz spaces are given.

Keywords Banach function spaces · Orlicz spaces · Mixed topologies · Bochner representable operators · Compact operators

Mathematics Subject Classification 47B38 · 46B40 · 46E30 · 47B05

1 Introduction and preliminaries

We assume that $(X, \|\cdot\|_X)$ is a real Banach space with the Banach dual $(X^*, \|\cdot\|_{X^*})$. Let B_X stand for the closed unit ball in X. For terminology concerning Riesz spaces and function spaces, we refer the reader to [1,2,9,10,20].

We assume that (Ω, Σ, μ) is a complete σ-finite measure space. By Σ_f we denote the δ-ring of all sets $A \in \Sigma$ with $\mu(A) < \infty$. Let L^0 denote the corresponding space of equivalence classes of all Σ-measurable real functions on Ω. Then L^0 is a

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super Dedekind complete Riesz space, equipped with the F-norm topology $\mathcal{T}_0$ of convergence in measure on sets of finite measure.

Let $(E, \|\cdot\|_E)$ be a Banach function space, that is, E is an order ideal of L^0 with $\operatorname{supp} E = \Omega$ and $\|\cdot\|_E$ is a Riesz norm. By $\mathcal{T}_E$ we denote the topology of the norm $\|\cdot\|_E$. For $r > 0$, let $B_E(r) := \{u \in E : \|u\|_E \le r\}$. The Köthe dual E' of E is defined by:

$$E' := \left\{ v \in L^0 : \int_\Omega |u(\omega)v(\omega)|\, d\mu < \infty \text{ for all } u \in E \right\}.$$

The associated norm $\|\cdot\|_{E'}$ on E' is defined for $v \in E'$ by

$$\|v\|_{E'} := \sup \left\{ \left| \int_\Omega u(\omega)v(\omega)\, d\mu \right| : u \in E, \|u\|_E \le 1 \right\}.$$

Then $\operatorname{supp} E' = \Omega$ (see [9, Theorem 6.1.5]). The σ-order continuous dual $E_c^\sim$ of E separates the points of E and $E_c^\sim$ can be identified with E' through the Riesz isomorphism $E' \ni v \mapsto F_v \in E_c^\sim$, where

$$F_v(u) = \int_\Omega u(\omega)v(\omega)\, d\mu \text{ for } u \in E$$

(see [9, Theorem 6.1.1]). The Mackey topology $\tau(E, E')$ is a locally convex-solid Hausdorff topology with the σ-Lebesgue property (see [1, Exercise 18, p. 178]). Note that $\tau(E, E') = \mathcal{T}_E$ if and only if $\|\cdot\|_E$ is σ-order continuous.

In addition to these facts, the following assumptions will be in force without futher mention throughout this paper:

(A) $E \subset L^1_{loc}$, i.e., $A \in \Sigma_f$ implies $u\mathbb{1}_A \in L^1$ for all $u \in E$.
(B) E is perfect, i.e., $E = E''$ (equivalently, $\|\cdot\|_E$ satisfies both the σ-Fatou property and the σ-Levy property (see [1, Definition 3.14], [20, § 110 and Theorem 112.2]).

Note that from (A) it follows that for every $A \in \Sigma_f$, $\mathbb{1}_A \in E'$. Moreover, from (B) it follows that E is $\sigma(E, E')$-sequentially complete (see [13, Proposition 1.2]). Note that if a subset H of E is $\sigma(E, E')$-bounded, then $\sup_{u\in H} \|u\|_E < \infty$ (see [10, Lemma 1, p. 20], [9, Theorem 6.1.6]). By $(E')_a$ we denote the ideal in E' of all elements of order continuous norms, that is,

$$(E')_a := \left\{ v \in E' : \|v_n\|_{E'} \to 0 \text{ if } |v(\omega)| \ge v_n(\omega) \downarrow 0 \ \mu - a.e. \right\}.$$

Then $\|\cdot\|_{E'}$ is order continuous if and only if $(E')_a = E'$.

Let $L^0(X)$ stand for the linear space of μ-equivalence classes of all strongly Σ-measurable functions $g : \Omega \to X$. Let

$$E'(X) = \left\{ g \in L^0(X) : \|g(\cdot)\|_X \in E' \right\}.$$

Definition 1.1 A bounded linear operator $T : E \to X$ is said to be *Bochner representable*, if there exists $g \in E'(X)$ such that

$$T(u) = \int_\Omega u(\omega)g(\omega)\, d\mu \ \text{ for } \ u \in E.$$

2 Bochner representable operators on Banach spaces

It is known that every Bochner representable operator $T : L^1 \to X$ (where $\mu(\Omega) < \infty$) maps relatively $\sigma(L^1, L^\infty)$-compact sets onto relatively norm compact sets in X (see [5, Lemma 11, pp. 74–75]). Now we extend this result to Bochner representable operators $T : E \to X$.

Theorem 2.1 *Assume that $T : E \to X$ is a Bochner representable operator. Then the following statements hold:*

(i) *T is $(\tau(E, E'), \|\cdot\|_X)$-continuous.*
(ii) *T maps relatively $\sigma(E, E')$-compact sets in E onto relatively norm compact sets in X.*

Proof There exists $g \in E'(X)$ such that

$$T(u) = \int_\Omega u(\omega)g(\omega)\, d\mu \ \text{ for } \ u \in E.$$

(i) Assume that $x^* \in X^*$. Then for $u \in E$,

$$\begin{aligned} |x^*(T(u))| &= \left| \int_\Omega u(\omega)x^*(g(\omega))\, d\mu \right| \\ &\le \int_\Omega |u(\omega)| \cdot \|x^*\|_{X^*} \|g(\omega)\|_X\, d\mu = \|x^*\|_{X^*} F_{\|g(\cdot)\|_X}(|u|). \end{aligned}$$

It follows that $x^* \circ T \in E_c^\sim$, so T is $(\sigma(E, E'), \sigma(X, X^*))$-continuous (see [2, Theorem 9.26]), and hence T is $(\tau(E, E'), \|\cdot\|_X)$-continuous (see [8, Theorem 8.6.1]).

(ii) Assume that H is a relatively $\sigma(E, E')$-compact subset of E. It follows that H is relatively $\sigma(E, E')$-sequentially compact (see [10, Lemma 11, p. 31]). Since E is $\sigma(E, E')$-sequentially complete, in view of [13, Proposition 1.1] the set $\{u\|g(\cdot)\|_X : u \in H\}$ in L^1 is uniformly integrable. Hence given $\varepsilon > 0$ there exist $\Omega_0 \in \Sigma_f$ and $\delta > 0$ such that

$$\sup_{u \in H} \int_{\Omega \setminus \Omega_0} |u(\omega)| \cdot \|g(\omega)\|_X\, d\mu \le \frac{\varepsilon}{2}, \tag{2.1}$$

$$\sup_{u \in H} \int_A |u(\omega)| \cdot \|g(\omega)\|_X\, d\mu \le \frac{\varepsilon}{2}, \quad \text{whenever} \quad A \in \Sigma \quad \text{and} \quad \mu(A) \le \delta. \tag{2.2}$$

Choose a sequence (s_n) of X-valued strongly μ-measurable step functions on Ω such that $s_n(\omega) \to g(\omega)$ μ-a.e. and $\|s_n(\omega)\|_X \le \|g(\omega)\|_X$ μ-a.e. for all $n \in \mathbb{N}$ (see

[7, Theorem 6, p. 4]). Hence $s_n \in E'(X)$ for $n \in \mathbb{N}$. By the Egorov theorem (see [7, Theorem 42, p. 18]), there exists $A_0 \in \Sigma$ with $A_0 \subset \Omega_0$, $\mu(\Omega_0 \setminus A_0) \le \delta$ and $\sup_{\omega \in A_0} \|s_n(\omega) - g(\omega)\|_X \to 0$. For every $n \in \mathbb{N}$ define an operator $T_n : E \to X$ by

$$T_n(u) := \int_{A_0} u(\omega) s_n(\omega)\, d\mu \ \text{ for } \ u \in E.$$

Note that for every $n \in \mathbb{N}$, T_n is a bounded operator of finite dimensional range, so that T_n is a compact operator. Define an operator $T_{A_0} : E \to X$ by

$$T_{A_0}(u) := \int_{A_0} u(\omega) g(\omega)\, d\mu \ \ for \ \ u \in E.$$

Then for $u \in B_E(1)$, we have

$$\begin{aligned}\left\| \int_{A_0} u(\omega)(g(\omega) - s_n(\omega))\, d\mu \right\|_X &\le \int_{A_0} |u(\omega)| \cdot \|g(\omega) - s_n(\omega)\|_X \, d\mu \\ &\le \left(\int_{\Omega} |u(\omega)| \mathbb{1}_{A_0}(\omega)\, d\mu \right) \sup_{\omega \in A_0} \|g(\omega) - s_n(\omega)\|_X \\ &\le \left\| \mathbb{1}_{A_0} \right\|_{E'} \sup_{\omega \in A_0} \|g(\omega) - s_n(\omega)\|_X .\end{aligned}$$

Hence $\|T_{A_0} - T_n\| \le \|\mathbb{1}_{A_0}\|_{E'} \sup_{\omega \in A_0} \|g(\omega) - s_n(\omega)\|_X$, and hence $\|T_{A_0} - T_n\| \to 0$. It follows that T_{A_0} is a compact operator. Since $\sup_{u \in H} \|u\|_E < \infty$, we have that $K_\varepsilon := \{T(\mathbb{1}_{A_0} u) : u \in H\}$ is a relatively norm compact set in X.

Moreover, since $\mu(\Omega_0 \setminus A_0) \le \delta$, for every $u \in H$, by (2.2) we get

$$\left\| T\left(\mathbb{1}_{\Omega_0 \setminus A_0} u\right)\right\|_X \le \int_{\Omega \setminus A_0} |u(\omega)| \cdot \|g(\omega)\|_X \, d\mu \le \frac{\varepsilon}{2}$$

i.e., $\{T(\mathbb{1}_{\Omega_0 \setminus A_0} u) : u \in H\} \subset \frac{\varepsilon}{2} B_X$. For every $u \in H$, by (2.1) we have

$$\left\| T\left(\mathbb{1}_{\Omega \setminus \Omega_0} u\right)\right\|_X \le \int_{\Omega \setminus \Omega_0} |u(\omega)| \cdot \|g(\omega)\|_X \, d\mu \le \frac{\varepsilon}{2},$$

i.e., $\{T(\mathbb{1}_{\Omega \setminus \Omega_0} u) : u \in H\} \subset \frac{\varepsilon}{2} B_X$. Hence for every $u \in H$, we have

$$T(u) = T(\mathbb{1}_{\Omega \setminus \Omega_0} u) + T(\mathbb{1}_{\Omega_0 \setminus A_0} u) + T(\mathbb{1}_{A_0} u) \in \varepsilon B_X + K_\varepsilon .$$

In view of the Grothendieck's compactness criterion (see [4, Exercise 4(iii)]) $T(H)$ is a relatively norm compact set in X, as desired. □

By $\gamma[\mathcal{T}_E, \mathcal{T}_0]$ (in brief, γ_E) we denote the *natural mixed topology* on E in the sense of Wiweger (see [3,11,12,19], for more details). Then $\mathcal{T}_0\big|_E \subset \gamma_E \subset \mathcal{T}_E$ and γ_E is the finest linear topology on E that agrees with $\mathcal{T}_0$ on every ball $B_E(r)$, $r > 0$ (see [19, 2.2.2]). $\mathcal{T}_E$ and γ_E have the same bounded sets in E, and for a sequence (u_n)

in E, $u_n \to 0$ in γ_E if and only if $u_n \to 0$ in $\mathcal{T}_0$ and $\sup_n \|u_n\|_E < \infty$ (see [19, Corollary, p. 56 and Theorem 2.6.1]). If, in particular, $\operatorname{supp}(E')_a = \Omega$, then γ_E is a locally convex-solid Hausdorff topology and γ_E coincides with the mixed topology $\gamma[\mathcal{T}_E, |\sigma|(E, (E')_a)]$ (see [11, Theorem 3.3]). Note that, then (E, γ_E) is a generalized DF-space (see [15] for more details). In view of [11, Theorem 3.1], we have:

$$(E, \gamma_E)^* = \{F_v : v \in (E')_a\}. \tag{2.3}$$

Theorem 2.2 *Assume that $(E, \|\cdot\|_E)$ is a Banach function space with the order continuous accociated norm $\|\cdot\|_{E'}$ on E'. Let $T : E \to X$ be a Bochner representable operator. Then the following statements hold:*

(i) *T is $(\gamma_E, \|\cdot\|_X)$-continuous and norm-compact operator.*
(ii) *T is $(\gamma_E, \|\cdot\|_X)$-compact, that is, there exists a γ_E-neighborhood V of 0 in E such that $T(V)$ is a relatively norm compact set in X.*

Proof (i) There exists $g \in (E')_a(X)$ such that

$$T(u) = \int_\Omega u(\omega)g(\omega)\,d\mu \quad \text{for } u \in E.$$

For $u \in E$, we have

$$\|T(u)\|_X \leq \int_\Omega |u(\omega)|\ \|g(\omega)\|_X\,d\mu = F_{\|g(\cdot)\|_X}(|u|),$$

where $\|g(\cdot)\|_X \in (E')_a$. Using (2.3) we obtain that T is $(\gamma_E, \|\cdot\|_X)$-continuous. Choose a sequence (s_n) of X-valued strongly μ-measurable step-functions on Ω such that $\|s_n(\omega) - g(\omega)\|_X \to 0$ μ-a.e. and $\|s_n(\omega)\|_X \leq \|g(\omega)\|_X$ μ-a.e. and all $n \in \mathbb{N}$ (see [7, Theorem 6, p. 4]). Hence $s_n \in (E')_a(X)$ and $\|s_n(\omega) - g(\omega)\|_X \leq 2\|g(\omega)\|_X$ μ-a.e. for all $n \in \mathbb{N}$. Let $v_n(\omega) = \sup_{k \geq n} \|s_n(\omega) - g(\omega)\|_X$ for $\omega \in \Omega$. Then $2\|g(\omega)\|_X \geq v_n(\omega) \downarrow 0$ μ-a.e., so $\|\ \|s_n(\cdot) - g(\cdot)\|_X\ \|_{E'} \leq \|v_n\|_{E'} \to 0$.

For each $n \in \mathbb{N}$, let $T_n : E \to X$ be a linear operator defined by

$$T_n(u) := \int_\Omega u(\omega)s_n(\omega)\,d\mu \quad \text{for } u \in E.$$

Note that the range of each T_n is contained in the span of the finite set of values of s_n. Therefore T_n is compact for each $n \in \mathbb{N}$, and for each $u \in E$,

$$\begin{aligned}\|(T_n - T)(u)\|_X &= \left\| \int_\Omega u(\omega)(s_n(\omega) - g(\omega))\,d\mu \right\|_X \\ &\leq \int_\Omega |u(\omega)| \cdot \|s_n(\omega) - g(\omega)\|_X d\,\mu \\ &\leq \|u\|_E \cdot \left\| \|s_n(\cdot) - g(\cdot)\|_X \right\|_{E'}.\end{aligned}$$

It follows that $\|T - T_n\| \to 0$, so T is a compact operator.

(ii) Since (E, γ_E) is a generalized DF-space, it is quasinormable (see [15, p. 422]). Hence in view of (i) by the Grothendieck's classical results (see [15, p. 429]), we obtain that T is $(\gamma_E, \|\cdot\|)$-compact. □

Remark 2.1 A related result to Theorem 2.2 can be found in [17, Corollary 4.8].

An important class of Banach function spaces are Orlicz spaces (see [10,16] for more details). By a Young function we mean here s convex continuous mapping $\Phi : [0,\infty) \to [0,\infty)$ that vanishes only at 0 and $\Phi(t)/t \to 0$ as $t \to 0$ and $\Phi(t)/t \to \infty$ as $t \to \infty$. By Φ^* we denote the Young function complementary to Φ in the sense of Young.

The Orlicz space

$$L^\Phi := \left\{ u \in L^0 : \int_\Omega \Phi(\lambda|u(\omega)|)\, d\mu < \infty \ \text{ for some } \ \lambda > 0 \right\},$$

equipped with the topology $\mathcal{T}_\Phi$ of the norm:

$$\|u\|_\Phi := \inf\left\{ \lambda > 0 : \int_\Omega \Phi(|u(\omega)|/\lambda)\, d\mu \le 1 \right\}$$

is a perfect Banach function space (see [10,16]). Then $(L^\Phi)' = L^{\Phi^*}$ and $(L^{\Phi^*})_a = E^{\Phi^*} = \{v \in L^{\Phi^*} : \int_\Omega \Phi(\lambda|v(\omega)|)d\mu < \infty$ for all $\lambda > 0\}$. In particular, $E^{\Phi^*} = L^{\Phi^*}$ if Φ^* satisfies the Δ_2-condition, i.e., $\Phi^*(2t) \le d\Phi^*(t)$ for some $d > 1$ and all $t \ge 0$.

We say that a Young function Ψ *increases more rapidly* than Φ (in symbols, $\Phi \prec \Psi$) if for an arbitrary $c > 0$ there exists $d > 0$ such that $c\Phi(t) \le \frac{1}{d}\Psi(dt)$ for all $t \ge 0$.

If $\Phi \prec \Psi$, then $L^\Psi \subset L^\Phi$ and by $i_\Psi : L^\Psi \to L^\Phi$ we denote the inclusion map.

Proposition 2.3 *Let $T : L^\Phi \to X$ be a Bochner representable operator. Then for every Young function Ψ with $\Phi \prec \Psi$ the operator $T \circ i_\Psi : L^\Psi \to X$ is compact.*

Proof Assume that Ψ is a Young function with $\Phi \prec \Psi$. Then by [16, Theorem 5.3.3, p. 171] the closed unit ball $B_\Psi(1)$ in L^Ψ is relatively $\sigma(L^\Phi, L^{\Phi^*})$-compact in L^Φ. Hence according to Theorem 2.1 $T(B_\Psi(1))$ is relatively norm compact, and this means that $T \circ i_\Psi : L^\Psi \to X$ is compact. □

We say that a Young function Φ *increases essentially more rapidly than another* Ψ (in symbols, $\Psi \ll \Phi$) if for an arbitrary $c > 0$, $\Psi(ct)/\Phi(t) \to 0$ as $t \to 0$ and $t \to \infty$.

The following characterization of the mixed topology $\gamma_\Phi (= \gamma[\mathcal{T}_\Phi, \mathcal{T}_0])$ on L^Φ will be useful (see [14, Theorem 2.1]).

Theorem 2.4 *Let Φ be a Young function. Then the mixed topology γ_Φ on L^Φ is generated by the family of norms $\{\|\cdot\|_\Psi\big|_{L^\Phi} : \Psi \ll \Phi\}$.*

As a consequence of Theorems 2.2 and 2.4 we have:

Corollary 2.5 *Assume that a Young function Φ^* satisfies the Δ_2-condition. Let $T : L^\Phi \to X$ be a Bochner representable operator. Then there exists a Young function Ψ with $\Psi \ll \Phi$ such that $T(B_\Psi(1) \cap L^\Phi)$ is a relatively norm compact set in X.*

Remark (i) The result of Corollary 2.5 was established in a different way in [14, Theorem 2.3].

(ii) For a bounded linear operator $T : L^\Phi \to X$, following [6] one can define its functional norm $|||T|||$ by $|||T||| = \sup \Sigma \|\alpha_i T(\mathbb{1}_{A_i})\|_X$, where the supremum is taken over all finite Σ-partition (A_i) of Ω and all $\alpha_i \in \mathbb{R}$ such that $\|\Sigma \alpha_i \mathbb{1}_{A_i}\|_\Phi \leq 1$.

Uhl [18, Theorem 1] showed that if X either is reflexive or is a separable dual Banach space and Φ obeys the Δ_2-condition, then every bounded linear operator $T : L^\Phi \to X$ with $|||T||| < \infty$ is Bochner representable. If, in addition, Φ^* also obeys the Δ_2-condition, then every bounded operator $T : L^\Phi \to X$ with $|||T||| < \infty$ is compact (see [18, Corollary 2]).

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