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Fixed points of Lyapunov integral operators and Gibbs measures

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Abstract In this paper we shall consider the connections between Lyapunov integral operators and Gibbs measures for models with four competing interactions and uncountable (i.e. [0, 1]) set of spin values on a Cayley tree. We prove the existence of fixed points of Lyapunov integral operators and give a condition of uniqueness of a fixed point.

Keywords Cayley tree · Gibbs measures · Lyapunov integral operator · Fixed point

Mathematics Subject Classification Primary 82B05 · 82B20; Secondary 60K35

1 Preliminaries

A Cayley tree $\Gamma^k = (V, L)$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k + 1$ edges incident to each vertices. Here V is the set of vertices and L that of edges (arcs). Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y\rangle$. The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d \mid x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \textit{ such that the pairs } \langle x_0, x_1\rangle, ..., \langle x_{d-1}, x_d\rangle \textit{ are neighboring vertices}\}.$$

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Let $x^0 \in V$ be a fixed and we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\},$$
$$L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\},$$

The set of the direct successors of x is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1} \mid d(x, y) = 1\}, \ x \in W_n.$$

We observe that for any vertex $x \neq x^0$, x has k direct successors and x^0 has $k + 1$. The vertices x and y are called second neighbor which is denoted by $\rangle x, y\langle$, if there exist a vertex $z \in V$ such that x, z and y, z are nearest neighbors. We will consider only second neighbors $\rangle x, y\langle$, for which there exist n such that $x, y \in W_n$. Three vertices x, y and z are called a triple of neighbors and they are denoted by $\langle x, y, z\rangle$, if $\langle x, y\rangle$, $\langle y, z\rangle$ are nearest neighbors and x, $z \in W_n$, $y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Now we consider models with four competing interactions where the spin takes values in the set $[0, 1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \to [0, 1]$ is called a configuration and the set of all configurations on A we denote by $\Omega_A = [0, 1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi_1 : (t, u, v) \in [0, 1]^3 \to \xi_1(t, u, v) \in R$, $\xi_i : (u, v) \in [0, 1]^2 \to \xi_i(u, v) \in R$, $i \in \{2, 3\}$ are given bounded, measurable functions. Then we consider the model with four competing interactions on the Cayley tree which is defined by following Hamiltonian

$$\begin{aligned} H(\sigma) = &- J_3 \sum_{\langle x,y,z\rangle} \xi_1\left(\sigma(x), \sigma(y), \sigma(z)\right) - J \sum_{\rangle x,y\langle} \xi_2\left(\sigma(x), \sigma(z)\right) \\ &- J_1 \sum_{\langle x,y\rangle} \xi_3\left(\sigma(x), \sigma(y)\right) - \alpha \sum_{x\in V} \sigma(x), \end{aligned} \tag{1.1}$$

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and $J, J_1, J_3, \alpha \in R\setminus\{0\}$. Let $h : (t, x) \in [0, 1] \times V\setminus\{x^0\} \to h_{t,x} \in \mathbb{R}$ and $|h_{t,x}| < C$ where x^0 is a root of Cayley tree and C is a constant which does not depend on t. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right), \tag{1.2}$$

$$Z_n = \int\!\!\ldots\!\!\int_{\Omega^{(p)}_{V_{n-1}}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda^{(p)}_{V_{n-1}}(d\widetilde{\sigma}_n), \tag{1.3}$$

where for a set $A \subset V$ we denoted

$$\underbrace{\Omega_A \times \Omega_A \times ... \times \Omega_A}_{3\cdot 2^{p-1}} = \Omega_A^{(p)}, \quad \underbrace{\lambda_A \times \lambda_A \times ... \times \lambda_A}_{3\cdot 2^{p-1}} = \lambda_A^{(p)}, \ n, p \in \mathbb{N},$$

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$\iint\limits_{\Omega_{W_n} \times \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)(\lambda_{W_n} \times \lambda_{W_n})(d\omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{1.4}$$

By Kolmogorov's extension theorem there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$. The measure μ is called *splitting Gibbs measure* corresponding to Hamiltonian (1.1) and function $x \mapsto h_x = \{h_{x,t}\}$, $x \neq x^0$ (see [1,2,5,7]).

Denote

$$\begin{aligned} K(t,u,v) = \exp\{&J_3\beta\xi_1(t,u,v) + J\beta\xi_2(u,v) + J_1\beta(\xi_3(t,u) + \xi_3(t,v)) \\ &+ \alpha\beta(u+v)\}, \end{aligned} \tag{1.5}$$

and

$$f(t,x) = \exp(h_{t,x} - h_{0,x}), \quad (t,u,v) \in [0,1]^3, \ x \in V\backslash\{x^0\}.$$

The following statement describes conditions on h_x guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

Proposition 1.1 [6] *Let $k = 2$. The measure $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfies the consistency condition (1.4) iff for any $x \in V\backslash\{x^0\}$ the following equation holds:*

$$f(t,x) = \frac{\int_0^1\int_0^1 K(t,u,v)f(u,y)f(v,z)dudv}{\int_0^1\int_0^1 K(0,u,v)f(u,y)f(v,z)dudv}, \tag{1.6}$$

where $S(x) = \{y, z\}$.

2 Existence of a fixed point of the operator $\mathcal{L}$

Now we prove that there exist at least one fixed point of Lyapunov integral equation, namely there is a splitting Gibbs measure corresponding to Hamiltonian (1.1).

Proposition 2.1 *Let $k = 2$, $J_3 = J = \alpha = 0$ and $J_1 \neq 0$. Then (1.6) is equivalent to*

$$f(t,x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp\{J_1\beta\xi_3(t,u)\}f(u,y)du}{\int_0^1 \exp\{J_1\beta\xi_3(0,u)\}f(u,y)du}, \tag{2.1}$$

where $f(t,x)=\exp(h_{t,x}-h_{0,x}),\ t\in[0,1],\ x\in V.$

Proof For $J_3=J=\alpha=0$ and $J_1\neq 0$ one gets $K(t,u,v)=\exp\{J_1\beta(\xi_3(u,t)+\xi_3(v,t))\}$. Then (1.6) can be written as

$$f(t,x)=\frac{\int_0^1\int_0^1\exp\{J_1\beta(\xi_3(t,u)+\xi_3(t,v))\}f(u,y)f(v,z)dudv}{\int_0^1\int_0^1\exp\{J_1\beta(\xi_3(0,u)+\xi_3(0,v))\}f(u,y)f(v,z)dudv}$$
$$=\frac{\int_0^1\exp\{J_1\beta\xi_3(t,u)\}f(u,y)du\cdot\int_0^1\exp\{J_1\beta\xi_3(t,v)\}f(v,z)dv}{\int_0^1\exp\{J_1\beta\xi_3(0,u)\}f(u,y)du\cdot\int_0^1\exp\{J_1\beta\xi_3(0,v)\}f(v,z)dv}. \tag{2.2}$$

Since $\rangle y,z\langle = S(x)$ Eq. (2.2) is equivalent to (2.1). □

Now we consider the model (1.1) in the class of translational-invariant functions $f(t,x)$ i.e $f(t,x)=f(t)$, for any $x\in V$. For such functions Eq. (1.1) can be written as

$$f(t)=\frac{\int_0^1\int_0^1 K(t,u,v)f(u)f(v)dudv}{\int_0^1\int_0^1 K(0,u,v)f(u)f(v)dudv}, \tag{2.3}$$

where $K(t,u,v)=\exp\{J_3\beta\xi_1(t,u,v)+J\beta\xi_2(u,v)+J_1\beta(\xi_3(t,u)+\xi_3(t,v))+\alpha\beta(u+v)\}$, $f(t)>0,\ t,u\in[0,1]$.

We shall find positive continuous solutions to (2.3) i.e. such that $f\in C^+[0,1]=\{f\in C[0,1]: f(x)>0\}$.

Define a nonlinear operator H on the cone of positive continuous functions on $[0,1]$:

$$(Hf)(t)=\frac{\int_0^1\int_0^1 K(t,s,u)f(s)f(u)dsdu}{\int_0^1\int_0^1 K(0,s,u)f(s)f(u)dsdu}.$$

We'll study the existence of positive fixed points for the nonlinear operator H (i.e., solutions of the Eq. (2.3)).

We define the Lyapunov integral operator $\mathcal{L}$ on $C[0,1]$ by the equality (see [3])

$$\mathcal{L}f(t)=\int_0^1 K(t,s,u)f(s)f(u)dsdu.$$

Put

$$\mathcal{M}_0=\left\{f\in C^+[0,1]: f(0)=1\right\}.$$

Lemma 2.2 *The equation $Hf=f$ has a nontrivial positive solution iff the Lyapunov equation $\mathcal{L}g=g$ has a nontrivial positive solution.*

Proof At first we shall prove that the equation

$$Hf=f,\ \ f\in C_0^+[0,1] \tag{2.4}$$

has a positive solution iff the Lyapunov equation

$$\mathcal{L}g = \lambda g, \quad g \in C^{+}[0, 1] \tag{2.5}$$

has a positive solution in $\mathcal{M}_0$ for some $\lambda > 0$.

Let λ_0 be a positive eigenvalue of the Lyapunov operator $\mathcal{L}$. Then there exists $f_0 \in C_0^{+}[0, 1]$ such that $\mathcal{L} f_0 = \lambda_0 f_0$. Take $\lambda \in (0, +\infty)$, $\lambda \neq \lambda_0$. Define the function $h_0(t) \in C_0^{+}[0, 1]$ by $h_0(t) = \frac{\lambda}{\lambda_0} f_0(t), \quad t \in [0, 1]$. Then $\mathcal{L}h_0 = \lambda h_0$, i.e., the number λ is an eigenvalue of Lyapunov operator $\mathcal{L}$ corresponding the eigenfunction $h_0(t)$. It's easy to check that if the number $\lambda_0 > 0$ is an eigenvalue of the operator $\mathcal{L}$, then an arbitrary positive number is eigenvalue of the operator $\mathcal{L}$. Now we shall prove the lemma. Let Eq. (2.4) holds then the function $\frac{1}{\lambda} g(t)$ be a fixed point of the operator $\mathcal{L}$. Analogously, since H is non-linear operator we can correspond to the fixed point if there exist any eigenvector. □

Proposition 2.3 *The equation*

$$\mathcal{L} f = \lambda f, \quad \lambda > 0 \tag{2.6}$$

has at least one solution in $C_0^{+}[0, 1]$.

Proof Clearly, that the Lyapunov operator $\mathcal{L}$ is a compact on the cone $C^{+}[0, 1]$. By the other hand we have

$$\mathcal{L} f(t) \geq m \left(\int_0^1 f(s) ds \right)^2,$$

for all $f \in C^{+}[0, 1]$, where $m = \min K(t, s, u) > 0$.

Put $\Gamma = \{f : \ \|f\| = r, \ \ f \in C[0, 1]\}$. We define the set Γ_+ by

$$\Gamma_+ = \Gamma \cap C^{+}[0, 1].$$

Then we obtain

$$\inf_{f \in \Gamma_+} \|\mathcal{L} f\| > 0.$$

Then by Schauder's theorem (see [4], p.20) there exists a number $\lambda_0 > 0$ and a function $f_0 \in \Gamma_+$ such that, $\mathcal{L} f_0 = \lambda_0 f_0$. □

Denote by $N_{fix.p}(H)$ and $N_{fix.p}(\mathcal{L})$ the set of positive numbers of nontrivial positive fixed points of the operators H and L, respectively. By Lemma 2.2 and Proposition 2.3 we can conclude that:

Proposition 2.4 *(a) The Eq.* (2.4) *has at least one solution in* $C_0^{+}[0, 1]$.

(b) The equality $N_{fix.p}(H) = N_{fix.p}(\mathcal{L})$ *is hold.*

From Propositions 1.1 and 2.4 we get the following theorem.

Theorem 2.5 *The set of splitting Gibbs measures corresponding to Hamiltonian* (1.1) *is non-empty.*

3 The uniqueness of fixed point of the operator $\mathcal{L}$

In this section we shall give a condition of the uniqueness of fixed point of the operator $\mathcal{L}$.

Theorem 3.1 *Let the kernel $K(t,u,v)$ satisfies the condition*

$$\max_{(t,u,v)\in[0,1]^3} K(t,u,v) < c \min_{(t,u,v)\in[0,1]^3} K(t,u,v), \quad c \in \left(1, \frac{1}{2}\sqrt{\sqrt{17}+1}\right). \tag{3.1}$$

Then the operator $\mathcal{L}$ has the unique fixed point in $C_0^+[0,1]$.

Proof Let $\max_{(t,u,v)\in[0,1]^3} K(t,u,v) = \mathcal{K}$ and $\min_{(t,u,v)\in[0,1]^3} K(t,u,v) = k$. At first we shall prove that if $g \in C_0^+[0,1]$ is a solution of the equation $\mathcal{L}f = f$ then $g \in \mathcal{G}$ where

$$\mathcal{G} = \left\{ f \in C[0,1] : \frac{k}{\mathcal{K}^2} \le f(t) \le \frac{\mathcal{K}}{k^2} \right\}.$$

Let $s \in \mathcal{L}(C^+[0,1])$ be an arbitrary function. Then there exists a function $h \in C^+[0,1]$ such that $s = \mathcal{L}h$. Since s is continuous on $[0,1]$, there exists $t_1, t_2 \in [0,1]$ such that

$$s_{\min} = \min_{t\in[0,1]} s(t) = s(t_1) = (\mathcal{L}h)(t_1), \quad s_{\max} = \max_{t\in[0,1]} s(t) = s(t_2) = (\mathcal{L}h)(t_2).$$

Consequently we get

$$s_{\min} \ge k\int_0^1\int_0^1 h(u)h(v)dudv \ge k\int_0^1\int_0^1 \frac{K(t_2,u,v)}{\mathcal{K}} h(u)h(v)dudv = \frac{k}{\mathcal{K}} s_{max}. \tag{3.2}$$

Since g is a fixed point of the operator $\mathcal{L}$ we have $\|g\| \le \mathcal{K}\|g\|^2 \Rightarrow \|g\| \ge \frac{1}{\mathcal{K}}$.

From (3.2)

$$g(t) \ge g_{\min} = \min_{t\in[0,1]} g(t) \ge \frac{k}{\mathcal{K}}\|g\| \Rightarrow g(t) \ge \frac{k}{\mathcal{K}^2}.$$

Similarly,

$$g(t) = (\mathcal{L}g)(t) \ge k\int_0^1\int_0^1 g(u)g(v)dudv \ge k g_{\min}^2 \Rightarrow g_{\min} \le \frac{1}{k}.$$

Hence

$$g(t) \le g_{\max} \le \frac{\mathcal{K}}{k} g_{\min} \le \frac{\mathcal{K}}{k^2}.$$

Thus we have $g \in \mathcal{G}$.

Now we show that $\mathcal{L}$ has the unique fixed point. By Proposition 2.4, $\mathcal{L}g = g$ has at least one solution. Assume that there are two solutions $g_1 \in C_0^+[0,1]$ and $g_2 \in C_0^+[0,1]$, i.e $\mathcal{L}g_i = g_i$, $i = 1, 2$.

Let a function $f \in C[0,1]$ changes its sign on $[0, 1]$. Then it is easy to check that for every $a \in \mathbb{R}$ the following inequality holds: $\|f(t) - a\| \geq \frac{1}{2}\|f\|$.

Put $\xi(t) = g_1(t) - g_2(t)$. Since $\xi(t)$ changes its sign on [0, 1], we get

$$\max_{t\in[0,1]}\left|\xi(t) - \left(\frac{k^2}{\mathcal{K}^2} + \frac{\mathcal{K}^2}{k^2}\right)\int_0^1 \xi(s)ds\right| \geq \frac{1}{2}\|\xi\|,$$

$$\xi(t) = 2\int_0^1\int_0^1 K(t,u,v)\,(g_1(u)g_1(v) - g_2(u)g_2(v))\,dudv.$$

The last equation can be written as

$$\xi(t) = \int_0^1\int_0^1 K(t,u,v)\eta(u,v)\,(|\xi(u) - \xi(v)| + \xi(u) + \xi(v))\,dudv,$$

where

$$\min\{g_1(t), g_2(t)\} \leq \eta(u,v) \leq \max\{g_1(t), g_2(t)\},\ t \in [0,1].$$

Since $g_i(t) \in \mathcal{G},\ i = \overline{1,2}$ we get $\frac{k}{\mathcal{K}^2} \leq \eta(u,v) \leq \frac{\mathcal{K}}{k^2},\ (u,v) \in [0,1]^2$. Hence

$$\left|2\cdot K(t,u,v)\eta(u,v) - \left(\frac{\mathcal{K}^2}{k^2} + \frac{k^2}{\mathcal{K}^2}\right)\right| \leq \frac{\mathcal{K}^2}{k^2} - \frac{k^2}{\mathcal{K}^2}.$$

Then

$$\left|\xi(t) - \left(\frac{\mathcal{K}^2}{k^2} + \frac{k^2}{\mathcal{K}^2}\right)\int_0^1\int_0^1 (|\xi(u) - \xi(v)| + \xi(u) + \xi(v))\,dudv\right| \leq \left(\frac{\mathcal{K}^2}{k^2} - \frac{k^2}{\mathcal{K}^2}\right)\|\xi\|. \tag{3.3}$$

Assume the kernel $K(t,u,v)$ satisfies the condition (3.1). Then $\mathcal{K}^4 - k^4 < (\mathcal{K}k)^2 \Rightarrow \mathcal{K} < ck$ but it's contradict to the following: if $\xi \in C[0,1]$ changes its sign on [0, 1] then for every $a \in \mathbb{R}$ the following inequality holds $\|\xi - a\| \geq \frac{1}{2}\|\xi\|$. This completes the proof. □

Theorem 3.2 *Let* $k \geq 2$. *If the function* $K(t,u,v)$ *which defined in* (1.5) *satisfies the condition* (3.1)*, then the model* (1.1) *has the unique-translational invariant Gibbs measure.*

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