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Khinchine type inequalities with optimal constants via ultra log-concavity

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Abstract We derive Khinchine type inequalities for even moments with optimal constants from the result of Walkup (J Appl Probab 13:76–85, 1976) which states that the class of log-concave sequences is closed under the binomial convolution.

Keywords Log-concavity · Ultra log-concavity · Khinchine inequality · Factorial moments

Mathematics Subject Classification (2000) 60E15 · 26D15

1 Introduction

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and let $r_1, r_2, \ldots, r_n$ be independent symmetric ± 1 random variables. The classical Khinchine inequality [8], states that for any positive $p > q$ there exists a constant $C_{p,q}$ (which does not depend on $n, \alpha_1, \alpha_2, \ldots, \alpha_n$) such that

$$(\mathbb{E}|S|^p)^{1/p} \leq C_{p,q} \cdot (\mathbb{E}|S|^q)^{1/q},$$

where $S = \sum_{i=1}^n \alpha_i r_i$.

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There was a long pursuit for the optimal values of the constants $C_{p,q}$. The best values of $C_{p,2}$ for $p \geq 3$ were established by Whittle [16] while the optimal $C_{2,1}$ constant was proved to be equal to $\sqrt{2}$ by Szarek [14] (see also [12] for a short proof which extends to the normed linear space setting; this approach was later extended in [10, 13]). Finally, Haagerup [6], found best values of $C_{p,2}$ for all $p \in (2,3)$, and of $C_{2,q}$ for all $q \in (0,2)$, thus solving the part of the problem which is most important for applications since $ES^2 = \sum_{i=1}^{n} \alpha_i^2$ is a quantity particularly easy to deal with. However, the general problem of finding optimal values of the constants $C_{p,q}$ is open and probably quite difficult. Its special case when both p and q are even numbers, and p is divisible by q, was settled by Czerwiński in his unpublished Master thesis [4]. His method was based on some algebraic-combinatorial identities and does not seem to generalize to other situations. On the other hand, König and Kwapień [9], and Baernstein and Culverhouse [1], have obtained comparison of moments inequalities with best constants, similar in spirit to Haagerup's result but with the symmetric Bernoulli random variables replaced by some multidimensional rotationally invariant random vectors of special form (for example, uniformly distributed on spheres or balls). Again, as in Haagerup's approach, it was crucial for their main argument to work to have $p = 2$ or $q = 2$. In the present paper we establish the optimal values of $C_{p,q}$ for even $p > q > 0$ (the assumption of $q \mid p$ no longer needed) both in the classical Khinchine inequality and its high-dimensional counterparts.

The main tool in our approach is Walkup's theorem (Theorem 1 of [15]) which states that the binomial convolution of two log-concave sequences is also log-concave:

Definition 1 A sequence $(a_i)_{i=0}^{\infty}$ of non-negative real numbers is called log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i \geq 1$ and the set $\{i \geq 0 \mid a_i > 0\}$ is an interval of integers.

Theorem 1 (Walkup [15]) *Let $(a_i)_{i=0}^{\infty}$ and $(b_i)_{i=0}^{\infty}$ be two log-concave sequences of positive real numbers. Define*

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

Then the sequence $(c_n)_{n=0}^{\infty}$ is log-concave.

Using Liggett's terminology [11] we may also rephrase this to another statement: the class of ultra log-concave sequences is closed under standard convolution operation, where a sequence of positive numbers $(a_i)_{i=0}^{\infty}$ is called *ultra log-concave* if and only if the sequence $(i! \cdot a_i)_{i=0}^{\infty}$ is log-concave.

There are at least three proofs of this theorem in the literature. Walkup's original proof is a bit difficult for non-experts whereas Liggett's proof [11] is very elementary but quite long, as it covers a more general result than just Theorem 1. Recently, Gurvits [5] published a short proof which, however, relies on the powerful Alexander-Fenchel inequalities for mixed volumes of convex bodies. For reader's convenience we provide yet another proof, more similar to Liggett's than to Walkup's, but shorter than Liggett's proof and very direct. We postpone it till Sect. 3.

2 Main results

Now we will present an application of Theorem 1. Let us denote the standard Euclidean norm on $\mathbb{R}^d$ by $\|\cdot\|$. In what follows we consider rotation invariance with respect to the same standard Euclidean structure. Furthermore, let G be a Gaussian random variable with the standard $\mathcal{N}(0,1)$ distribution while by $\mathbf{G}$ we denote an $\mathbb{R}^d$-valued Gaussian vector with the standard $\mathcal{N}(0, Id_d)$ distribution.

Lemma 1 *Let $\Pi : \mathbb{R}^d \to \mathbb{R}$ be the projection to the first coordinate. For $p > 0$ assume that X is a rotation invariant $\mathbb{R}^d$-valued random vector with finite pth moment. Then*

$$\frac{\mathbb{E}|\Pi X|^p}{\mathbb{E}|G|^p} = \frac{\mathbb{E}\|X\|^p}{\mathbb{E}\|\mathbf{G}\|^p}.$$

Proof Let θ be a random vector uniformly distributed on the unit sphere of $(\mathbb{R}^d, \|\cdot\|)$ and independent of X. Since X is rotation invariant it has the same distribution as $\|X\|\cdot\theta$ and thus ΠX has the same distribution as $\|X\|\cdot\Pi\theta$. Therefore $\mathbb{E}|\Pi X|^p = \mathbb{E}\|X\|^p\cdot\mathbb{E}|\Pi\theta|^p$. The same argument used for $\mathbf{G}$ instead of X yields

$$\mathbb{E}|G|^p = \mathbb{E}|\Pi\mathbf{G}|^p = \mathbb{E}\|\mathbf{G}\|^p\cdot\mathbb{E}|\Pi\theta|^p.$$

□

Definition 2 We will say that an $\mathbb{R}^d$-valued random vector X is *ultra sub-Gaussian* if either $X = 0$ a.s., or X is rotation invariant (i.e. symmetric if $d = 1$), has all moments finite, and the sequence $(a_i)_{i=0}^{\infty}$ defined by
$a_i = \mathbb{E}\|X\|^{2i}/\mathbb{E}\|\mathbf{G}\|^{2i}$ for $i \geq 1$, and $a_0 = 1$, is log-concave.

Lemma 2 *If X and Y are independent ultra sub-Gaussian $\mathbb{R}^d$-valued random vectors then $X + Y$ is also ultra sub-Gaussian.*

Proof If X or Y is equal to zero a.s. then the assertion is obvious. Let

$$\begin{aligned} a_i &= \mathbb{E}\|X\|^{2i}/\mathbb{E}\|\mathbf{G}\|^{2i} = \mathbb{E}(\Pi X)^{2i}/\mathbb{E}G^{2i}, \\ b_i &= \mathbb{E}\|Y\|^{2i}/\mathbb{E}\|\mathbf{G}\|^{2i} = \mathbb{E}(\Pi Y)^{2i}/\mathbb{E}G^{2i}, \\ c_i &= \mathbb{E}\|X+Y\|^{2i}/\mathbb{E}\|\mathbf{G}\|^{2i} = \mathbb{E}(\Pi X + \Pi Y)^{2i}/\mathbb{E}G^{2i} \end{aligned}$$

for $i \geq 1$, and let $a_0 = b_0 = c_0 = 1$. It remains to notice that

$$\begin{aligned} c_n &= \frac{\mathbb{E}(\Pi X + \Pi Y)^{2n}}{\mathbb{E}G^{2n}} = \frac{1}{(2n-1)!!}\sum_{i=0}^{n}\binom{2n}{2i}\mathbb{E}(\Pi X)^{2i}\mathbb{E}(\Pi Y)^{2n-2i} \\ &= \sum_{i=0}^{n}\frac{(2n)!!}{(2i)!!(2n-2i)!!}a_i b_{n-i} = \sum_{i=0}^{n}\binom{n}{i}a_i b_{n-i}, \end{aligned}$$

where we have used the fact that ΠX and ΠY are independent and symmetric. The double factorial $N!!$ denotes the product of all positive integers which have the same parity as N and do not exceed N. We adopt the standard convention that $(-1)!! = 0!! = 1$. The assertion immediately follows from Theorem 1. □

Lemma 3 *Assume that an $\mathbb{R}^d$-valued random vector X and a non-negative random variable R are independent, and that $R \cdot X$ has distribution $\mathcal{N}(0, Id_d)$. Then X is ultra sub-Gaussian.*

Proof Clearly, X is rotation invariant. Note that for $p > 0$ we have

$$\mathbb{E}\|X\|^p \cdot \mathbb{E}R^p = \mathbb{E}\|\mathbf{G}\|^p \in (0, \infty),$$

so that X has all moments finite and strictly positive. Let $a_i = \mathbb{E}\|X\|^{2i}/\mathbb{E}\|\mathbf{G}\|^{2i}$. By the Schwarz inequality for $i \geq 1$ we have

$$1/a_i^2 = (\mathbb{E}R^{2i})^2 \leq \mathbb{E}R^{2(i-1)} \cdot \mathbb{E}R^{2(i+1)} = 1/(a_{i-1}a_{i+1})$$

which proves that the sequence $(a_i)_{i=0}^{\infty}$ is log-concave. □

Corollary 1 *Assume that X is a random vector uniformly distributed on*

(i) *the Euclidean sphere $r \cdot S^{d-1}$ (if $d = 1$ this is symmetric $\pm r$ distribution) or*
(ii) *the Euclidean ball $r \cdot B^d$*
for some $r > 0$. Then X is ultra sub-Gaussian.

Proof Distribution of any $\mathbb{R}^d$-valued random vector which is rotation invariant and unimodal (i.e. it has a rotation invariant density which is non-increasing as a function of distance to zero) can be expressed as an integral mean of measures uniformly distributed on balls with center in zero. Since the standard normal distribution $\mathcal{N}(0, Id_d)$ is rotation invariant and unimodal the corollary is established in the case (ii).

For the reader's convenience, however, we provide an explicit description of this factorization. Let us denote the volume of the unit ball B^d by $v_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ and let $\varphi_d(s) = (2\pi)^{-d/2}e^{-s^2/2}$, i.e. $\varphi_d(\|x\|)$ is a density of $\mathcal{N}(0, Id_d)$. Furthermore, for $s > 0$ set $u_s(x) = v_d^{-1}s^{-d}1_{\{x \in \mathbb{R}^d : \|x\| \leq s\}}$, so that u_s is a density of a random vector uniformly distributed on $s \cdot B^d$. Hence

$$\begin{aligned}\varphi_d(\|x\|) &= \int_{\|x\|}^{\infty} (-\varphi_d'(s))\,ds = \int_0^{\infty} 1_{\{x \in \mathbb{R}^d : \|x\| \leq s\}} s\varphi_d(s)\,ds \\ &= \int_0^{\infty} v_d s^{d+1} \varphi_d(s) u_s(x)\,ds.\end{aligned}$$

Thus the product of a random vector uniformly distributed on B^d with an independent positive random variable $\tilde{R}$ with density $v_d s^{d+1}\varphi_d(s)1_{(0,\infty)}(s)$ has distribution $\mathcal{N}(0, Id_d)$. Setting $R = \tilde{R}/r$ ends the construction.

The case (i) is simpler, it just suffices to note that

$$\mathbf{G} = \frac{\|\mathbf{G}\|}{r} \cdot r \frac{\mathbf{G}}{\|\mathbf{G}\|},$$

where the factors are independent, and the second of them is uniformly distributed on $r \cdot S^{d-1}$. □

Corollary 2 *For $\alpha > 0$ let X be an $\mathbb{R}^d$-valued random vector with density*

$$g_X(x) = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{\alpha}+1)} \pi^{-d/2} e^{-\|x\|^\alpha}.$$

If $\alpha > 2$ then X is ultra sub-Gaussian.

Proof Let $\beta \in (0, \alpha)$ and let Y be an $\mathbb{R}^d$-valued random vector with density

$$g_Y(x) = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{\beta}+1)} \pi^{-d/2} e^{-\|x\|^\beta}.$$

It is a well-known fact that Y is a mixture of dilatations of X and thus (for $\beta = 2 < \alpha$) the assertion follows. For the sake of completeness we provide a detailed argument. Let Z be a standard positive β/α-stable random variable, so that $\mathbb{E}e^{-wZ} = e^{-w^{\beta/\alpha}}$ for every $w \geq 0$. Note that for $\mu > 0$ we have

$$\begin{aligned}\mathbb{E}Z^{-\mu} &= \mathbb{E}\frac{1}{\Gamma(\mu)} \int_0^\infty e^{-tZ} t^{\mu-1}\, dt = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} \mathbb{E}e^{-tZ}\, dt \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-t^{\beta/\alpha}}\, dt = \frac{\alpha\Gamma(\alpha\mu/\beta)}{\beta\Gamma(\mu)}.\end{aligned}$$

Let g_Z denote the density of Z and let W be a positive random variable independent of X with density

$$g_W(t) = \frac{\beta\Gamma(d/\alpha)}{\alpha\Gamma(d/\beta)} t^{-d/\alpha} g_Z(t).$$

We will prove that $W^{-1/\alpha}X$ has the same distribution as Y. Since both random vectors are rotation invariant it suffices to prove that $W^{-1/\alpha}\|X\|$ has the same distribution as $\|Y\|$ which immediately follows from the fact that the Laplace transforms of logarithms of these random variables are equal:

$$\mathbb{E}(W^{-1/\alpha}\|X\|)^\lambda = \mathbb{E}W^{-\lambda/\alpha} \cdot \mathbb{E}\|X\|^\lambda = \mathbb{E}\|Y\|^\lambda$$

for every $\lambda \geq 0$. Indeed, by a standard and direct computation we obtain $\mathbb{E}\|X\|^\lambda = \Gamma(\frac{\lambda+d}{\alpha})/\Gamma(d/\alpha)$ and $\mathbb{E}\|Y\|^\lambda = \Gamma(\frac{\lambda+d}{\beta})/\Gamma(d/\beta)$, whereas

$$\mathbb{E}W^{-\lambda/\alpha} = \frac{\beta\Gamma(d/\alpha)}{\alpha\Gamma(d/\beta)}\mathbb{E}Z^{-(\lambda+d)/\alpha} = \frac{\Gamma(d/\alpha)\Gamma(\frac{\lambda+d}{\beta})}{\Gamma(d/\beta)\Gamma(\frac{\lambda+d}{\alpha})}.$$

□

Now we are in position to state and prove our main results:

Theorem 2 *Let n and d be positive integers and let $p > q \geq 2$ be even integers. Let $X_1, X_2, \ldots, X_n$ be independent $\mathbb{R}^d$-valued ultra sub-Gaussian random vectors. Then*

$$(\mathbb{E}\|S\|^p)^{1/p} \leq \frac{(\mathbb{E}\|\mathbf{G}\|^p)^{1/p}}{(\mathbb{E}\|\mathbf{G}\|^q)^{1/q}} \cdot \mathbb{E}(\|S\|^q)^{1/q}$$

where $S = X_1 + X_2 + \cdots + X_n$.

Theorem 3 *Let n and d be positive integers and let $p > q \geq 2$ be even integers. Let $X_1, X_2, \ldots, X_n$ be independent $\mathbb{R}^d$-valued random vectors and assume that each of them is either uniformly distributed on a Euclidean sphere or uniformly distributed on a Euclidean ball. Then*

$$(\mathbb{E}\|S\|^p)^{1/p} \leq \frac{(\mathbb{E}\|\mathbf{G}\|^p)^{1/p}}{(\mathbb{E}\|\mathbf{G}\|^q)^{1/q}} \cdot \mathbb{E}(\|S\|^q)^{1/q}$$

where $S = X_1 + X_2 + \cdots + X_n$.

The constant $(\mathbb{E}\|\mathbf{G}\|^p)^{1/p}/(\mathbb{E}\|\mathbf{G}\|^q)^{1/q} = \left(\Gamma(\frac{p+d}{2})\right)^{\frac{1}{p}} \left(\Gamma(\frac{q+d}{2})\right)^{-\frac{1}{q}} \left(\Gamma(\frac{d}{2})\right)^{\frac{1}{q}-\frac{1}{p}}$ is obviously optimal, as indicated by the example of i.i.d. centered X_i's with $n \to \infty$ (by the Central Limit Theorem). For $d = 1$ this is the classical Khinchine inequality.

Proof of Theorems 2 and 3 Without loss of generality we may and will assume that all the spheres and balls mentioned in the assumptions of Theorem 3 are centered at zero, i.e. X_i's are rotationally invariant. Indeed, it suffices to notice that $S - \mathbb{E}S = \sum_{i=1}^n (X_i - \mathbb{E}X_i)$ is rotation invariant and thus $\|S\|$ has the same distribution as $\|(S - \mathbb{E}S) + \|\mathbb{E}S\| \cdot \theta\|$, where θ is uniformly distributed on the unit sphere and independent of S. Thus by increasing number of variables by one we have reduced the problem to the case of rotationally invariant random vectors. Corollary 1 allows us to deduce Theorem 3 from Theorem 2.

Now it is enough to note that S is ultra log-concave by Lemma 2, so that the sequence $(a_k)_{k=0}^\infty$ given by $a_k = \mathbb{E}\|S\|^{2k}/\mathbb{E}\|\mathbf{G}\|^{2k}$ (with $a_0 = 1$) is log-concave. By multiplying inequalities $a_k^{2k} \geq a_{k-1}^k a_{k+1}^k$ for $k = 1, 2, \ldots, s$ we deduce that the sequence $(a_s^{1/s})_{s=1}^\infty$ is non-increasing. In particular, $a_{p/2}^{2/p} \leq a_{q/2}^{2/q}$ which is equivalent to the assertion of Theorem 2. □

3 Proof of Walkup's theorem

We assume that $\binom{n}{k} = 0$ for $k < 0$ and $k > n$, where $n \geq 0, k, n \in \mathbb{Z}$. Let us also set $a_i = b_i = 0$ for $i < 0, i \in \mathbb{Z}$.

Lemma 4 *Let $n \geq 1$ and $k \leq n$ be non-negative integers. Then*

$$\binom{n+1}{i}\binom{n-1}{k-i} \geq \binom{n+1}{k-i+1}\binom{n-1}{i-1} \tag{1}$$

for $0 \leq i \leq \lfloor k/2 \rfloor$.

Proof For $i = 0$ the inequality (1) is obvious. For $i > 0$ it is equivalent to

$$(n-k+i)(k-i+1) \geq i(n+1-i).$$

Since $n \geq k$ and $i \leq k/2$ we have

$$(n-k+i)(k-i+1) - i(n+1-i) = (n-k)(k-2i+1) \geq 0.$$

□

Lemma 5 *For $n \geq 1, k \leq n$ and $0 \leq i \leq \lfloor k/2 \rfloor$ consider a sequence*

$$s_i = 2\binom{n}{i}\binom{n}{k-i} - \binom{n-1}{i}\binom{n+1}{k-i} - \binom{n+1}{i}\binom{n-1}{k-i}.$$

Then the sequence $(\operatorname{sgn}(s_i))_{i=0}^{\lfloor k/2 \rfloor}$ is non-decreasing.

Proof After some simple reductions we get

$$\operatorname{sgn}(s_i) = \operatorname{sgn}\left[\frac{2n}{n+1} - \frac{n-i}{n-k+i+1} - \frac{n-k+i}{n+1-i}\right].$$

Let $m = n - k \geq 0$. We have

$$\frac{n-i}{n-k+i+1} + \frac{n-k+i}{n+1-i} = \frac{m+n+1}{m+i+1} - 1 + \frac{m+n+1}{n+1-i} - 1 = \frac{(m+n+1)(m+n+2)}{(m+i+1)(n+1-i)} - 2$$

therefore it suffices to notice that the function $i \mapsto (m+i+1)(n+1-i)$ is positive and non-decreasing on $[0, k/2]$. □

Lemma 6 *Let $(s_i)_{i=0}^n, (l_i)_{i=0}^n$ be two sequences of real numbers. Assume that $0 \leq l_1 \leq l_2 \cdots \leq l_n$, $\sum_{i=0}^n s_i = 0$ and the sequence $(\operatorname{sgn}(s_i))_{i=0}^n$ is non-decreasing. Then $\sum_{i=0}^n s_i l_i \geq 0$.*

Proof Let $i_0 = \min\{0 \le i \le n \mid s_i \ge 0\}$. Then

$$\sum_{i=0}^{n} s_i l_i = \sum_{i<i_0} s_i l_i + \sum_{i\ge i_0} s_i l_i \ge l_{i_0} \sum_{i<i_0} s_i + l_{i_0} \sum_{i\ge i_0} s_i = 0.$$

□

Before we give a proof of Theorem 1, let us make some remarks. Let $(a_i)_{i=0}^{\infty}$, $(b_i)_{i=0}^{\infty}$ be log-concave. Fix $1 \le i \le j$. Observe that

$$a_i a_j \ge a_{i-1} a_{j+1}, \quad \text{for } 0 \le i \le j. \tag{2}$$

Indeed, it suffices to consider the case when a_{i-1} and a_{j+1} are positive. Then $\{i-1, i, \dots, j, j+1\} \subset \{k \ge 0 \mid a_k > 0\}$. By multiplying the inequalities $a_k^2 \ge a_{k-1}a_{k+1}$ for $k = i, \dots, j$ and dividing by $a_i a_{i+1}^2 \cdot \ldots a_{j-1}^2 a_j > 0$ we arrive at (2). Note that we have used the fact that $\{k \ge 0 \mid a_k > 0\}$ is an interval of integers.

If $0 \le i \le j$ and $0 \le k \le l$ then from (2) we get

$$(a_i a_j - a_{i-1} a_{j+1})(b_k b_l - b_{k-1} b_{l+1}) \ge 0,$$

therefore

$$a_i a_j b_k b_l + a_{i-1} a_{j+1} b_{k-1} b_{l+1} \ge a_i a_j b_{k-1} b_{l+1} + a_{i-1} a_{j+1} b_k b_l, \quad 0 \le i \le j, 0 \le k \le l. \tag{3}$$

For fixed $k, n, k \le n$ we will use the notation $L_i = a_i a_{k-i} b_{n-i} b_{n-k+i}$ and $R_i = a_i a_{k-i} b_{n-i+1} b_{n-k+i-1}$. Note that $L_i = L_{k-i}$ and $L_{-1} = 0$.

Proof of Theorem 1 It is easy to check that the set $\{i \ge 0 \mid c_i > 0\}$ is an interval of integers. Therefore, for $n \ge 1$ we have to prove the inequality

$$\sum_{\substack{j=0,\dots,n \\ i=0,\dots,n}} \binom{n}{i}\binom{n}{j} a_i a_j b_{n-i} b_{n-j} \ge \sum_{\substack{j=0,\dots,n-1 \\ i=0,\dots,n+1}} \binom{n+1}{i}\binom{n-1}{j} a_i a_j b_{n+1-i} b_{n-1-j}.$$

It suffices to prove that

$$\sum_{i=0}^{k} \binom{n}{i}\binom{n}{k-i} a_i a_{k-i} b_{n-i} b_{n-k+i} \ge \sum_{i=0}^{k} \binom{n+1}{i}\binom{n-1}{k-i} a_i a_{k-i} b_{n+1-i} b_{n-1-k+i} \tag{4}$$

for $k = 0, \dots, 2n$. Inequality (4) for $k \ge n$ and a pair $((a_i)_{i=0}^{\infty}, (b_i)_{i=0}^{\infty})$ is equivalent to (4) for $\tilde{k} = 2n - k \le n$ and a pair $((b_i)_{i=0}^{\infty}, (a_i)_{i=0}^{\infty})$, so we can consider only $k \le n$.

For $i = 0, \dots, \lfloor k/2 \rfloor$ we have $i \le k - i$ and $n - k + i \le n - i$, therefore (3) yields

$$\binom{n+1}{k-i+1}\binom{n-1}{i-1}(a_i a_{k-i} b_{n-i} b_{n-k+i} + a_{i-1} a_{k-i+1} b_{n-i+1} b_{n-k+i-1})$$
$$\ge \binom{n+1}{k-i+1}\binom{n-1}{i-1}(a_i a_{k-i} b_{n-i+1} b_{n-k+i-1} + a_{i-1} a_{k-i+1} b_{n-i} b_{n-k+i}).$$

Moreover, using Lemma 4 and (2) we have

$$\left[\binom{n+1}{i}\binom{n-1}{k-i} - \binom{n+1}{k-i+1}\binom{n-1}{i-1}\right] a_i a_{k-i} b_{n-i+1} b_{n-k+i-1}$$
$$\le \left[\binom{n+1}{i}\binom{n-1}{k-i} - \binom{n+1}{k-i+1}\binom{n-1}{i-1}\right] a_i a_{k-i} b_{n-i} b_{n-k+i}.$$

We can rewrite these inequalities in the form of

$$\binom{n+1}{k-i+1}\binom{n-1}{i-1}(L_i + L_{i-1}) \ge \binom{n+1}{k-i+1}\binom{n-1}{i-1}(R_i + R_{k-i+1}) \tag{5}$$

and

$$\left[\binom{n+1}{i}\binom{n-1}{k-i} - \binom{n+1}{k-i+1}\binom{n-1}{i-1}\right] L_i \ge \left[\binom{n+1}{i}\binom{n-1}{k-i}\right.$$
$$\left. - \binom{n+1}{k-i+1}\binom{n-1}{i-1}\right] R_i \tag{6}$$

for $i = 0, \dots, \lfloor k/2 \rfloor$. Note that if k is odd then $L_{\lfloor k/2 \rfloor} = R_{\lfloor k/2 \rfloor + 1}$. In order to estimate the RHS of (4) we add (5) and (6) for $i = 0, \dots, \lfloor k/2 \rfloor$, and the equality

$$\binom{n-1}{\lfloor k/2 \rfloor}\binom{n+1}{k - \lfloor k/2 \rfloor} L_{\lfloor k/2 \rfloor} = \binom{n-1}{\lfloor k/2 \rfloor}\binom{n+1}{k - \lfloor k/2 \rfloor} R_{\lfloor k/2 \rfloor + 1}$$

in the case of k odd. We arrive at

$$\sum_{i=0}^{k}\binom{n+1}{i}\binom{n-1}{k-i} R_i \le \sum_{i=0}^{\lfloor k/2 \rfloor - 1}\left[\binom{n-1}{i}\binom{n+1}{k-i} + \binom{n+1}{i}\binom{n-1}{k-i}\right] L_i$$
$$+\theta_k\left[\binom{n-1}{\lfloor k/2 \rfloor}\binom{n+1}{k-\lfloor k/2 \rfloor} + \binom{n+1}{\lfloor k/2 \rfloor}\binom{n-1}{k-\lfloor k/2 \rfloor}\right] L_{\lfloor k/2 \rfloor},$$

where $\theta_k = 1/2$ if k is even and $\theta_k = 1$ when k is odd. Since the LHS of (4) is equal to

$$\sum_{i=0}^{\lfloor k/2 \rfloor - 1} 2\binom{n}{i}\binom{n}{k-i} L_i + 2\theta_k \binom{n}{\lfloor k/2 \rfloor}\binom{n}{k - \lfloor k/2 \rfloor} L_{\lfloor k/2 \rfloor},$$

it suffices to prove the inequality

$$\sum_{i=0}^{\lfloor k/2\rfloor-1} s_i L_i + \theta_k s_{\lfloor k/2\rfloor} L_{\lfloor k/2\rfloor} \geq 0.$$

We have

$$\sum_{i=0}^{\lfloor k/2\rfloor-1} s_i \ + \ \theta_k s_{\lfloor k/2\rfloor} \ = \frac{1}{2}\sum_{i=0}^{k} s_i \ = \ 0$$

and $0 \leq L_0 \leq L_1 \leq \cdots \leq L_{\lfloor k/2\rfloor}$. To finish the proof it suffices to use Lemma 5 and Lemma 6. □

Remark 1 Consider sequences $(a_n)_{n=0}^{\infty} = (1, 0, 0, 1, 0, 0, \ldots)$ and $(b_n)_{n=0}^{\infty} = (1, 1, 1, \ldots)$ and note that the binomial convolution of these sequences is not log-concave. Therefore, without the second assumption in the Definition (1) it is impossible to prove Walkup's Theorem.

4 Inequalities for factorial moments

For a positive integer n and any real number x we define the Pochhammer symbol $(x)_n = x(x-1) \cdot \ldots \cdot (x-n+1)$, with $(x)_0 = 1$, and in a standard way we put $\binom{x}{n} = (x)_n/n!$. Let n be a non-negative integer and let X be a nonnegative random variable X with $\mathbb{E}|X|^n < \infty$. Then $\mathbb{E}(X)_n$ is called the nth factorial moment of X.

Definition 3 *Let $\kappa \in \mathbb{R}$. We will say that a nonnegative integer-valued random variable X is κ-*good *if the sequence $\mathbb{E}e^{\kappa X}(X)_n$ for $n \geq 0$ is log-concave (we assume that all the expectations are finite).*

Lemma 7 *If X and Y are independent κ-good random variables then $X+Y$ is also κ-good.*

Proof Multiplying the obvious identity

$$e^{\kappa(X+Y)}\binom{X+Y}{k} = \sum_{i=0}^{k} e^{\kappa X}\binom{X}{i} \cdot e^{\kappa Y}\binom{Y}{k-i}$$

by $k!$ we arrive at

$$e^{\kappa(X+Y)}(X+Y)_k = \sum_{i=0}^{k}\binom{k}{i} e^{\kappa X}(X)_i \cdot e^{\kappa Y}(Y)_{k-i}.$$

To finish the proof it suffices to take expectation of both sides of this equality, use Theorem 1 and independence of X and Y. □

Now we can conclude with the following inequality:

Theorem 4 *Let $\kappa \in \mathbb{R}$ and let $X_1, X_2, \ldots, X_n$ be independent κ-good random variables (e.g. $\{0, 1\}$ Bernoulli random variables) and let p be a positive integer. Then for $S = X_1 + X_2 + \cdots + X_n$ we have*

$$(\mathbb{E}e^{\kappa S}(S)_p)^2 \geq \mathbb{E}e^{\kappa S}(S)_{p-1} \cdot \mathbb{E}e^{\kappa S}(S)_{p+1}.$$

Since $\lim_{\kappa \to -\infty} e^{-\kappa p}\mathbb{E}e^{\kappa X}(X)_p = p! \cdot \mathbb{P}(X = p)$ the case of $\kappa \to -\infty$ refers to the ultra log-concavity of the sequence $\mathbb{P}(X = p)$ which was carefully investigated and successfully applied for example in a recent paper of Johnson [7]. In fact, the log-concavity of the sequence $\mathbb{E}e^{\kappa S}(S)_p$ may be easily deduced from the ultra log-concavity of the sequence $\mathbb{P}(S = p)$, which in particular covers the case of Bernoulli sums. However, sometimes the sequence $\mathbb{E}(X)_p$ or, more generally, $\mathbb{E}(X)_p e^{\kappa X}$ may be log-concave even though the sequence $\mathbb{P}(X = p)$ is not ultra log-concave.

5 Tail to moments log-concavity trick

We finish with a discussion of a result of Borell ([3], formulated there in a slightly different and more general setting) which, in fact, can be traced back to the work of Barlow, Marshall, and Proschan [2, p. 384] although there it appears with a slighlty restricting additional assumption. It is very standard by now and has many different proofs, some of them very simple, but still we think that it may be of some interest to provide yet another argument, especially because it is quite similar in spirit to the one we used in our proof of Walkup's theorem.

Theorem 5 [2,3] *Assume that $\varphi : (0, \infty) \to (0, \infty)$ is a log-concave function (i.e. $\log \varphi$ is concave). Then also $\Psi : (0, \infty) \to (0, \infty)$ defined by*

$$\Psi(q) = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1}\varphi(t)\, dt$$

is log-concave.

Proof It suffices to prove that $\Psi(q)^2 \geq \Psi(q - \varepsilon)\Psi(q + \varepsilon)$ for $q > \varepsilon > 0$, which obviously follows from

$$\frac{1}{\Gamma(q)^2} \int_0^t s^{q-1}(t - s)^{q-1}\varphi(s)\varphi(t - s)\, ds$$
$$\geq \frac{1}{\Gamma(q - \varepsilon)\Gamma(q + \varepsilon)} \int_0^t s^{q-\varepsilon-1}(t - s)^{q+\varepsilon-1}\varphi(s)\varphi(t - s)\, ds$$

by integration over $t > 0$ and using the Fubini theorem.

Since φ is log-concave the function $s \mapsto \varphi(s)\varphi(t-s)$ is non-decreasing on $(0, t/2]$ and non-increasing on $[t/2, t)$. Also, it is obviously symmetric with respect to $t/2$. Hence it suffices to prove that

$$\frac{1}{\Gamma(q)^2}\int_a^{t-a} s^{q-1}(t-s)^{q-1}\,ds \geq \frac{1}{\Gamma(q-\varepsilon)\Gamma(q+\varepsilon)}\int_a^{t-a} s^{q-\varepsilon-1}(t-s)^{q+\varepsilon-1}\,ds$$

for $0 < a < t/2$, which upon using the homogeneity reduces to proving that

$$f(\alpha)=B(q-\varepsilon,q+\varepsilon)\int_\alpha^{1-\alpha} w^{q-1}(1-w)^{q-1}dw - B(q,q)\int_\alpha^{1-\alpha} w^{q-\varepsilon-1}(1-w)^{q+\varepsilon-1}dw$$

is non-negative for $\alpha \in [0, 1/2]$. Obviously, $f(0) = f(1/2) = 0$. Now it is enough to notice that $f'(\alpha) > 0$ if and only if

$$\left(\frac{\alpha}{1-\alpha}\right)^{\varepsilon} + \left(\frac{1-\alpha}{\alpha}\right)^{\varepsilon} > 2B(q-\varepsilon, q+\varepsilon)/B(q,q). \tag{7}$$

The left hand side of (7) is decreasing in α, so that there exists some $\alpha_0 \in (0, 1/2)$ such that f is increasing on $[0, \alpha_0]$ and then decreasing on $[\alpha_0, 1/2]$. Thus $f \geq 0$ on $[0, 1/2]$ and the proof is finished. □

The following well known corollary follows (note that it reveals some intriguing similarity to a way in which we used Walkup's theorem to derive the Khinchine inequalities):

Corollary 3 *Assume that Z is a positive random variable with log-concave tails, i.e. the function $\varphi(t) = \mathbb{P}(Z > t)$ is log-concave on $(0, \infty)$. Let $\mathcal{E}$ be an exponential random variable with parameter 1. Then*

$$(\mathbb{E}Z^p)^{1/p} \leq \frac{(\mathbb{E}\mathcal{E}^p)^{1/p}}{(\mathbb{E}\mathcal{E}^q)^{1/q}} \cdot (\mathbb{E}Z^q)^{1/q}$$

for all $p > q > 0$.

The constant $(\mathbb{E}\mathcal{E}^p)^{1/p}/(\mathbb{E}\mathcal{E}^q)^{1/q} = \Gamma(p+1)^{1/p}/\Gamma(q+1)^{1/q}$ obviously cannot be improved in general since $\mathcal{E}$ has log-concave tails.

Proof From Theorem 5 we infer that $\Psi : [0, \infty) \to (0, \infty)$ defined by

$$\Psi(q) = \frac{1}{\Gamma(q)}\int_0^\infty t^{q-1}\varphi(t)\,dt = \frac{\mathbb{E}Z^q}{\mathbb{E}\mathcal{E}^q}$$

for $q > 0$, and by $\Psi(0) = 1$, is log-concave (it is an easy exercise to check that Ψ is right-continuous at zero). Hence $g(q) = \log\Psi(q)$ is concave with $g(0) = 0$, so that $q \mapsto g(q)/q$ is a non-increasing function on $(0, \infty)$, which is equivalent to the assertion of Corollary 3. □

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