



Tolerance and asymptotic modelling of dynamic thermoelasticity problems for thin micro-periodic cylindrical shells

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Received: 18 September 2019 / Accepted: 22 May 2020 / Published online: 16 June 2020
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Abstract The problem of linear dynamic thermoelasticity in Kirchhoff–Love-type circular cylindrical shells having properties periodically varying in circumferential direction (uniperiodic shells) is considered. In order to describe thermoelastic behaviour of such shells, two mathematical averaged models are proposed—the *non-asymptotic tolerance* and the *consistent asymptotic models*. Considerations are based on the known Kirchhoff–Love theory of elasticity combined with Duhamel–Neumann thermoelastic constitutive relations and on Fourier’s theory of heat conduction. The non-asymptotic tolerance model equations depending on a cell size are derived applying the *tolerance averaging technique* and a *certain extension of the known stationary action principle*. The consistent asymptotic model equations being independent on a microstructure size are obtained by means of the *consistent asymptotic approach*. Governing equations of both the models have constant coefficients, contrary to starting shell

equations with periodic, non-continuous and oscillating coefficients. As examples, two special length-scale problems will be analysed in the framework of the proposed models. The first of them deals with investigation of the effect of a cell size on the shape of initial distributions of temperature micro-fluctuations. The second one deals with study of the effect of a microstructure size on the distribution of total temperature field approximated by sum of an averaged temperature and temperature fluctuations.

Keywords Microheterogeneous cylindrical shells · Dynamic thermoelasticity problems · Tolerance and asymptotic modelling · Length-scale effect

1 Introduction

Thin linearly elastic Kirchhoff–Love-type circular cylindrical shells with a periodically micro-inhomogeneous structure in circumferential direction are objects of consideration. Shells of this kind are termed *uniperiodic*. At the same time, the shells have constant structure in axial direction. By periodic inhomogeneity we shall mean periodically varying thickness and/or periodically varying inertial, elastic and thermal properties of the shell material. We restrict our consideration to those uniperiodic cylindrical shells, which are composed of a large number of identical

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elements. Moreover, every such element, called a *periodicity cell*, can be treated as a thin shell. Typical examples of such shells are presented in Fig. 1 (stiffened shell) and Fig. 2 (a shell composed of two kinds of periodically distributed materials).

Thermoelastic problems of periodic structures (shells, plates, beams) are described by partial differential equations with periodic, highly oscillating and discontinuous coefficients. Thus, these equations are too complicated to constitute the basis for investigations of most of the engineering problems. To obtain averaged equations with constant coefficients, many different approximate modelling methods for structures of this kind have been formulated. Periodic cylindrical shells (plates) are usually described using *homogenized models* derived by applying *asymptotic methods*. These asymptotic models represent certain equivalent structures with constant or slowly varying rigidities and averaged mass densities. Unfortunately, the asymptotic procedures are usually restricted to the first approximation, which leads to homogenized models neglecting *the effect of a periodicity cell size* (called *the length-scale effect*) on the overall shell behaviour. The mathematical foundations of this modelling technique can be found in Bensoussan et al. [1], Jikov et al. [2]. Applications of the asymptotic homogenization procedure to modelling of stationary and non-stationary phenomena for microheterogeneous shells (plates) are presented in a large number of contributions. From the extensive list on this subject we can mention paper by Lutoborski [3] and monograph by Lewiński and Telega [4].

The length-scale effect can be taken into account using *the non-asymptotic tolerance averaging technique*. This technique is based on the concept of *the tolerance and in-discernibility relations* related to the accuracy of the performed measurements and calculations. The mathematical foundations of this modelling technique can be found in Woźniak and Wierzbicki [5], Woźniak et al. [6, 7], Ostrowski [8]. For periodic structures, *governing equations of the tolerance models have constant coefficients dependent also on a cell size*. Some applications of this averaging method to the modelling of mechanical and thermo-mechanical problems for various periodic structures are shown in many works. We can mention here monograph by Tomczyk [9] and papers by Baron [10], Marczak and Jędrzyak [11], Marczak [12], Tomczyk and Litawska [13, 14], where dynamic problems are

investigated and papers by Łaciński and Woźniak [15], Rychlewska et al. [16], where problems of heat conduction are analysed. The extended list of references on this subject can be found in [5–8].

The tolerance averaging technique was also adopted to formulate mathematical models for analysis of various mechanical and thermomechanical problems for functionally graded solids, e.g. for heat conduction in longitudinally graded hollow cylinder by Ostrowski and Michalak [17, 18], for thermoelasticity of transversally graded laminates by Pazera and Jędrzyak [19], Pazera et al. [20], for dynamics of transversally and longitudinally graded thin cylindrical shells by Tomczyk and Szczerba [21–23].

The main aim of this contribution is to formulate and discuss *a new mathematical averaged tolerance model for the analysis of selected dynamic thermoelasticity problems for the uniperiodic cylindrical shells under consideration*. Contrary to the starting exact equations of the shell thermoelasticity with periodic, highly oscillating and discontinuous coefficients, *governing equations of the proposed averaged model have constant coefficients depending also on a cell length dimension λ* . Hence, this model allows to describe the influence of a length scale on the thermoelastic shell behaviour. In order to derive this model we shall apply a certain new approach to *the tolerance modelling* of microheterogeneous media given by Woźniak [7]. This approach is based on the tolerance averaging of integral functionals describing behaviour of the micro-inhomogeneous structures. Then, by using *a certain extension of the known stationary action principle* to the resulting averaged functionals we arrive at the governing equations of tolerance model.

The second aim is to derive *a new averaged consistent asymptotic model for the analysis of selected dynamic thermoelasticity problems for the uniperiodic cylindrical shells under consideration*. The model will be derived applying a certain new approach to *the asymptotic modelling* of microheterogeneous media proposed by Woźniak [7]. This approach is based on the consistent asymptotic averaging of integral functionals describing behaviour of the micro-heterogeneous structures and on using *the extended stationary action principle*. The governing equations of asymptotic model have constant coefficients being independent of a period length. The main advantage of this asymptotic approach is that the

effective elastic and thermal moduli of the structure can be obtained without specification of the periodic cell problem.

As examples, two special length-scale problems will be analysed in the framework of the proposed models. The first of them deals with investigation of the effect of a cell size on the shape of initial distributions of temperature micro-fluctuations. This problem can be studied in the framework of neither the asymptotic model proposed here nor the known asymptotic models commonly used for investigations of vibrations and heat conduction in the micro-periodic shells under consideration. The second one deals with study of the effect of a microstructure size on the distribution of total temperature field approximated by the sum of an averaged temperature and temperature fluctuations.

2 Formulation of the problem: starting equations

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0, L_1)$ and $\zeta \equiv x^2 \in \Xi \equiv (0, L_2)$, where L_1, L_2 are length dimensions of M , cf. Figures 1 and 2. Let $O\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space R^3 and denote $\bar{x} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. Let us introduce the orthonormal parametric representation of the underformed cylindrical shell midsurface M by means of $M \equiv \{ \bar{x} \in R^3 : \bar{x} = \bar{r}(x^1, x^2), (x^1, x^2) \in \Omega \times \Xi \}$, where $\bar{r}(\cdot)$ is the smooth invertible function such that $\partial \bar{r} / \partial x^1 \cdot \partial \bar{r} / \partial x^2 = 0, \partial \bar{r} / \partial x^1 \cdot \partial \bar{r} / \partial x^1 = 1, \partial \bar{r} / \partial x^2 \cdot \partial \bar{r} / \partial x^2 = 1$. Note, that derivative $\partial \bar{r} / \partial x^i, i = 1, 2$, should be understood as differentiation of each component of \bar{r} , i.e. $\partial \bar{r} / \partial x^\alpha = [\partial \bar{r}^1 / \partial x^\alpha, \partial \bar{r}^2 / \partial x^\alpha, \partial \bar{r}^3 / \partial x^\alpha]$ for $\bar{r} = [\bar{r}^1, \bar{r}^2, \bar{r}^3], \alpha = 1, 2$.

Throughout the paper, indices α, β, \dots run over 1,2 and are related to midsurface parameters x^1, x^2 , summation convention holds. Partial differentiation related to x^α is represented by $\partial_\alpha, \partial_\alpha = \partial / \partial x_\alpha$. Moreover, it is denoted $\partial_{\alpha\dots\delta} \equiv \partial_\alpha \dots \partial_\delta$. Let $a_{\alpha\beta}$ and $a^{\alpha\beta}$ stand for the covariant and contravariant midsurface first metric tensors, respectively. Denote by $b_{\alpha\beta}$ the covariant midsurface second metric tensor. Under

orthonormal parametrization introduced on $M, a_{\alpha\beta} = a^{\alpha\beta}$ are unit tensors and components of tensor $b_{\alpha\beta}$ are: $b_{22} = b_{12} = b_{21} = 0, b_{11} = -r^{-1}$. We denote $t \in I = [t_0, t_1]$ as the time coordinate.

Let $d(x), r$ stand for the shell thickness and the midsurface curvature radius, respectively.

The basic cell Δ and an arbitrary cell $\Delta(x)$ with the centre at point $x \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda/2, \lambda/2], \Delta(x) \equiv x + \Delta, x \in \Omega_\Delta, \Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega\}$, where $\lambda \equiv \lambda_1$ is a cell length dimension in $x \equiv x^1$ -direction. The period λ , called the microstructure length parameter, satisfies conditions: $\lambda/d_{\max} > 1, \lambda/r \ll 1$ and $\lambda/L_1 \ll 1$.

It is assumed that the cell Δ has a symmetry axis for $z = 0$, where $z \equiv z^1 \in [-\lambda/2, \lambda/2]$. It is also assumed that inside the cell the geometrical, elastic, inertial and thermal properties of the shell are described by even functions of argument z .

Denote by $u_\alpha = u_\alpha(x, \zeta, t), w = w(x, \zeta, t), (x, \zeta, t) \in \Omega \times \Xi \times I$, the shell displacements in directions tangent and normal to M , respectively. The shell stiffness tensors describing elastic properties of the shell are defined by $D^{\alpha\beta\gamma\delta}(x), B^{\alpha\beta\gamma\delta}(x)$. Let $\mu(x)$ stand for a shell mass density per midsurface unit area. Let $f^\alpha(x, \zeta, t), f(x, \zeta, t)$ be external forces per midsurface unit area, respectively tangent and normal to M . Denote by $\theta(x, \zeta, t)$ the temperature field treated as the temperature increment from a certain constant reference temperature T_0 (by reference temperature we shall mean the zero stress temperature). It is assumed that $\theta/T_0 \ll 1$. Let $\bar{d}^{\alpha\beta}(x)$ stand for the membrane thermal stiffness tensor (tensor of thermo-elastic moduli: $\bar{d}^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\alpha_{\gamma\delta}$, where $\alpha_{\gamma\delta}$ are coefficients of thermal expansion). Denote by $K^{\alpha\beta}(x)$ and $c(x)$ the tensors of heat conductivity and the specific heat, respectively. The heat sources will be neglected. For uniperiodic shells, $D^{\alpha\beta\gamma\delta}(x), B^{\alpha\beta\gamma\delta}(x), \mu(x), \bar{d}^{\alpha\beta}(x), K^{\alpha\beta}(x), c(x)$ are periodic, highly oscillating and non-continuous functions with respect to argument x .

It is assumed that the temperature along the shell thickness is constant. From this restriction it follows that only the coupling between temperature field θ and membrane stresses occurs (describing by tensor $\bar{d}^{\alpha\beta}(x)$) while the coupling of temperature and bending stresses is absent.

The starting equations are the well known governing equations of linear Kirchhoff–Love theory of thin

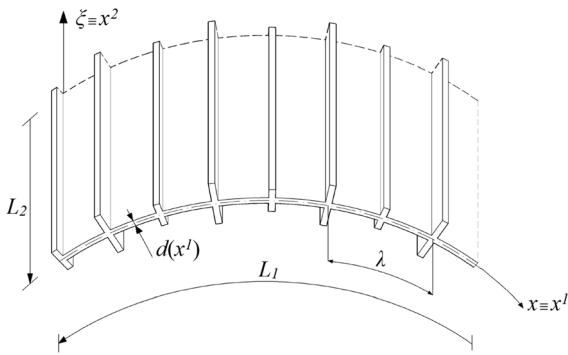


Fig. 1 Fragment of the shell reinforced by two families of uniperiodically spaced ribs

elastic cylindrical shells combined with Duhamel-Neumann thermo-elastic constitutive relations and coupled with the known linearized Fourier heat conduction equation in which the heat sources are neglected [24–28]. Thus, the starting equations consist of

- (a) the Duhamel-Neumann stress–strain–temperature relations

$$\begin{aligned} n^{\alpha\beta}(x, \zeta, t) &= D^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}(x, \zeta, t) - \bar{d}^{\alpha\beta} \theta(x, \zeta, t), \\ m^{\alpha\beta}(x, \zeta, t) &= B^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}(x, \zeta, t), \quad (x, \zeta, t) \in \Omega \times \Xi \times I \end{aligned} \tag{1}$$

where

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta) - b_{\alpha\beta} w, \quad \kappa_{\alpha\beta} = -\partial_\alpha \beta w, \tag{2}$$

- (b) the dynamic equilibrium equations

$$\begin{aligned} \partial_\beta n^{\alpha\beta} - \mu a^{\alpha\beta} \ddot{u}_\beta + f^\alpha &= 0, \\ \partial_{\alpha\beta} m^{\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta} - \mu \ddot{w} + f &= 0, \end{aligned} \tag{3}$$

which after combining with (1) and (2) are expressed in displacement fields u_α , w and temperature field θ

$$\begin{aligned} \partial_\beta (D^{\alpha\beta\gamma\delta} \partial_\delta u_\gamma) + r^{-1} \partial_\beta (D^{\alpha\beta 11} w) - \partial_\beta (\bar{d}^{\alpha\beta} \theta) \\ - \mu a^{\alpha\beta} \ddot{u}_\beta + f^\alpha = 0, \quad r^{-1} D^{\alpha\beta 11} \partial_\beta u_\alpha + \partial_{\alpha\beta} (B^{\alpha\beta\gamma\delta} \partial_\gamma \delta w) \\ - r^{-1} \bar{d}^{11} \theta + r^{-2} D^{1111} w + \mu \ddot{w} - f = 0, \end{aligned} \tag{4}$$

- (c) the linearized heat conduction equation based on the Fourier law coupled with (4)

$$\partial_x (K^{\alpha\beta} \partial_\beta \theta) - c \dot{\theta} = T_0 (\bar{d}^{\alpha\beta} \partial_x \dot{u}_\beta + r^{-1} \bar{d}^{11} \dot{w}). \tag{5}$$

Equations (4) and (5) describe the dynamic thermoelastic problems for the shells under consideration. Coefficients of these equations are periodic, highly oscillating and non-continuous functions in x .

Now, we are to show that Eqs. (4) and (5) can be also derived from the extended principle of stationary action proposed in [7]. These equations cannot be obtained from the principle of stationary action in its classical form because heat conduction Eq. (5) involves the odd derivatives of unknown functions $\theta(x, \zeta, t)$, $u_\alpha(x, \zeta, t)$, $w(x, \zeta, t)$, $(x, \zeta, t) \in \Omega \times \Xi \times I$, with respect to argument t .

We assume that the thermoelastic problems of thin shells considered here are described by the following action functional

$$\begin{aligned} A(u_\alpha, w, p^{\alpha\beta}, \hat{r}) \\ = \int_0^{L_1} \int_0^{L_2} \int_{t_0}^{t_1} L(x, \zeta, t, \partial_\beta u_\alpha, u_\alpha, \dot{u}_\alpha, \partial_{\alpha\beta} w, w, \dot{w}, p^{\alpha\beta}, \hat{r}) dt d\zeta dx, \end{aligned} \tag{6}$$

where Lagrangian L , being a highly oscillating function with respect to x , is determined by

$$\begin{aligned} L = & -\frac{1}{2} (D^{\alpha\beta\gamma\delta} \partial_\beta u_\alpha \partial_\delta u_\gamma + \frac{2}{r} D^{\alpha\beta 11} w \partial_\beta u_\alpha + \frac{1}{r^2} D^{1111} w w \\ & + B^{\alpha\beta\gamma\delta} \partial_{\alpha\beta} w \partial_\gamma \delta w - K^{\alpha\beta} \partial_x \theta \partial_\beta \theta - \mu a^{\alpha\beta} \dot{u}_\alpha \dot{u}_\beta - \mu \dot{w}^2) + f^\alpha u_\alpha + f w \\ & + p^{\alpha\beta} \partial_\beta u_\alpha + \frac{1}{r} p^{11} w + \hat{r} \theta, \end{aligned} \tag{7}$$

and where functions $p^{\alpha\beta}(x, \zeta, t)$, $\hat{r}(x, \zeta, t)$ are highly oscillating with respect to x and determined by independent equations

$$\begin{aligned} p^{\alpha\beta} &= \bar{d}^{\alpha\beta} \theta, \\ \hat{r} &= c \dot{\theta} + T_0 (\bar{d}^{\alpha\beta} \partial_x \dot{u}_\beta + r^{-1} \bar{d}^{11} \dot{w}). \end{aligned} \tag{8}$$

Equations (8) are called the constitutive equations for functions $p^{\alpha\beta}(x, \zeta, t)$, $\hat{r}(x, \zeta, t)$. It has to be emphasized that functions $p^{\alpha\beta}$, \hat{r} are not arguments of Lagrangian (7); they play the role of non-variational parameters.

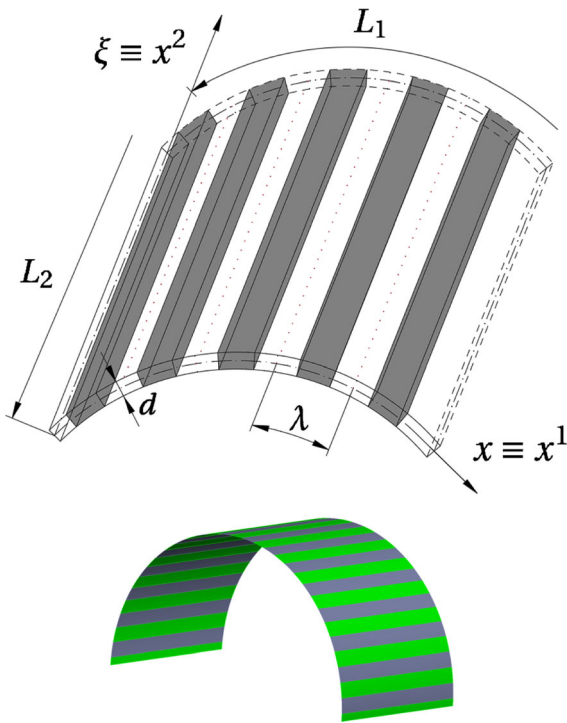


Fig. 2 Fragment of the shell composed of two different materials periodically and densely distributed in circumferential direction

Under assumption that $\partial L/\partial(\partial_\beta u_\alpha)$, $\partial L/\partial(\partial_{\alpha\beta} w)$ and $\partial L/\partial(\partial_\beta \theta)$ are continuous, from the extended principle of stationary action applied to $A(u_\alpha, w, p^{\alpha\beta}, \hat{r})$, we obtain the following system of Euler–Lagrange equations

$$\begin{aligned} \partial_\beta \frac{\partial L}{\partial(\partial_\beta u_\alpha)} - \frac{\partial L}{\partial u_\alpha} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_\alpha} &= 0, \\ -\partial_{\alpha\beta} \frac{\partial L}{\partial(\partial_{\alpha\beta} w)} - \frac{\partial L}{\partial w} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{w}} &= 0, \\ \partial_\beta \frac{\partial L}{\partial(\partial_\beta \theta)} - \frac{\partial L}{\partial \theta} &= 0. \end{aligned} \tag{9}$$

Combining (9) with (7) and (8) we arrive finally at the explicit form of the fundamental equations of the thermoelasticity shell theory under consideration. These equations coincide with the well-known Eqs. (4), (5).

The passage from action functional (6) to Euler–Lagrange equations (9) in which $p^{\alpha\beta}$, \hat{r} are given by constitutive Eq. (8) represents the extended principle of stationary action or the principle of stationary action extended by constitutive equations.

Applying the tolerance averaging technique [7] to Lagrange function (7) and independently to constitutive Eq. (8) and then using the extended stationary action principle we obtain the tolerance model equations describing thermoelastic phenomena in the periodic shells being object of considerations in this paper. All coefficients in the governing equations of the tolerance model are constant in contrast to those from the direct description (4), (5), and some of them depend on a microstructure size.

Moreover, applying the consistent asymptotic procedure [7] to Lagrange function (7) and independently to constitutive Eq. (8) and then using the extended stationary action principle we derive the asymptotic model equations describing thermoelastic behaviour of the periodic shells under consideration. The governing equations of the asymptotic model have constant coefficients but independent on a period length.

To make this paper self-consistent, in the subsequent section we shall outline the main concepts and the fundamental assumptions of the tolerance modelling procedure and of the consistent asymptotic approach, which in the general form are given in the monograph by Woźniak [7].

3 Concepts and assumptions of tolerance and asymptotic modelling techniques

3.1 Main concepts of the tolerance modelling procedure

The fundamental concepts of the tolerance modelling procedure under consideration are those of two tolerance relations between points and real numbers determined by tolerance parameters, slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation.

Below, the mentioned above concepts and assumptions will be specified with respect to one-dimensional region $\Omega = (0, L_1)$ defined in this paper.

3.1.1 Tolerance between points

Let λ be a positive real number. Points x, y belonging to $\Omega = (0, L_1)$ are said to be in tolerance determined by λ , if and only if the distance between points x, y does not exceed λ , i.e. $\|x - y\| \leq \lambda$.

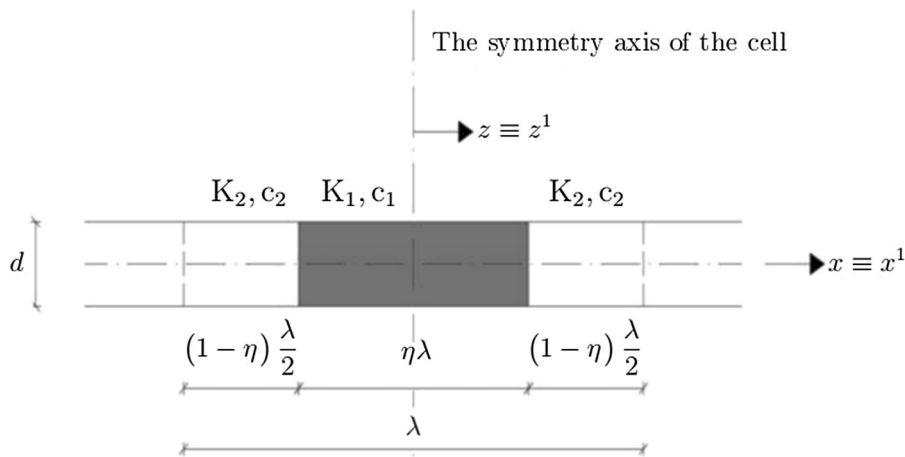


Fig. 3 Basic cell $\Delta \equiv [-\lambda/2, \lambda/2] \subset \Omega$ of the uniperiodic shell

3.1.2 Tolerance between real numbers

Let $\tilde{\delta}$ be a positive real number. Real numbers μ, ν are said to be in tolerance determined by $\tilde{\delta}$, if and only if $|\mu - \nu| \leq \tilde{\delta}$.

The above relations are denoted by: $x \overset{\tilde{\delta}}{\approx} y, \mu \overset{\tilde{\delta}}{\approx} \nu$. Positive parameters $\lambda, \tilde{\delta}$ are called *tolerance parameters*.

3.1.3 Slowly-varying functions

Let $F(x)$ be a function defined in $\bar{\Omega} = [0, L_1]$, which is continuous, bounded and differentiable in $\bar{\Omega}$ together with their derivatives up to the R -th order. Note, that function F can also depend on $\xi \in \bar{\Xi} = [0, L_2]$ and time coordinate t as parameters. Let $\delta \equiv (\lambda, \delta_0, \delta_1, \dots, \delta_R)$ be the set of tolerance parameters. The first of them represents the distances between points in $\bar{\Omega}$. The second one and the k -th one, $k = 1, \dots, R$, are related to the differences between the values of function $F(\cdot)$ and its derivative $\partial_1^k F(x)$ in points x, y belonging to $\bar{\Omega}$ such that $\|x - y\| \leq \lambda$. A function $F(\cdot)$ is said to be *slowly-varying of the R -th kind* with respect to cell Δ and tolerance parameters $\delta, F \in SV_\delta^R(\Omega, \Delta)$, if and only if the following two conditions are fulfilled

$$\begin{aligned}
 &(\forall (x, y) \in \Omega^2)[(x \overset{\lambda}{\approx} y) \\
 &\Rightarrow F(x) \overset{\delta_0}{\approx} F(y) \text{ and } \partial_1^k F(x) \overset{\delta_k}{\approx} \partial_1^k F(y), \quad k \\
 &= 1, 2, \dots, R],
 \end{aligned}
 \tag{10}$$

$$(\forall x \in \Omega)[\lambda |\partial_1^k F(x)| \overset{\delta_k}{\approx} 0, \quad k = 1, 2, \dots, R]. \tag{11}$$

It is worth to know that tolerance parameter λ in every problem under consideration is known a priori as a characteristic cell length dimension, whereas values of tolerance parameters $\delta_0, \delta_1, \dots, \delta_R$ can be determined only a posteriori, i.e. after obtaining unique solution to the considered initial-boundary value problem.

3.1.4 Tolerance-periodic functions

An integrable and essentially bounded function $f(x)$ defined in $\bar{\Omega} = [0, L_1]$, which can also depend on $\xi \in \bar{\Xi}$ and time coordinate t as parameters, is called *tolerance-periodic* in reference to cell Δ and tolerance parameters $\delta \equiv (\lambda, \delta_0)$, if for every $x \in \Omega_\Delta$ there exist Δ -periodic function $\tilde{f}(\cdot)$ such that $f|_{\Delta(x) \cap \text{Dom} f}$ and $\tilde{f}|_{\Delta(x) \cap \text{Dom} \tilde{f}}$ are indiscernible in tolerance determined by $\delta \equiv (\lambda, \delta_0)$. Function \tilde{f} is a Δ -periodic approximation of f in $\Delta(x)$. For function $f(\cdot)$ being tolerance-periodic together with its derivatives up to the R -th order, we shall write $f \in TP_\delta^R(\Omega, \Delta)$, $\delta \equiv (\lambda, \delta_0, \delta_1, \dots, \delta_R)$. It should be noted that for periodic structures being objects of considerations in

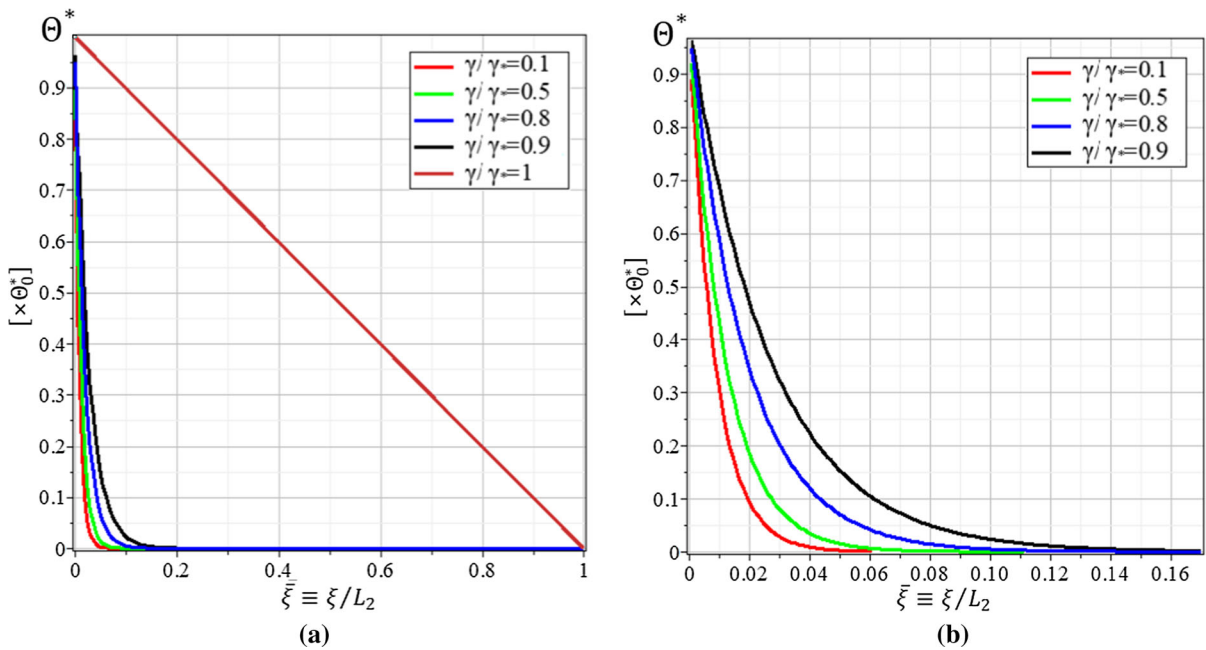


Fig. 4 Diagrams of dimensionless initial temperature microfluctuations $\Theta^*(\tilde{\xi})$ being **a** exponentially or linearly decaying solutions (46), (47) to Eq. (44) versus dimensionless coordinate

this work, function $\tilde{f}(x, z)$ has the same analytical form in every cell $\Delta(x)$ with a centre at $x \in \Omega$. Hence, $\tilde{f}(x, z)$ is independent of x and we have $\tilde{f} = \tilde{f}(z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$. In the general case, i.e. for tolerance periodic structures (i.e. structures which in small neighbourhoods of $\Delta(x)$ can be approximately regarded as periodic), $\tilde{f}(x, z)$ depends on x and hence we have $\tilde{f} = \tilde{f}(x, z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$.

3.1.5 Fluctuation shape functions

Let $h(x)$ be a continuous, highly oscillating, λ -periodic function defined in $\bar{\Omega} = [0, L_1]$, which has continuous derivatives $\partial_1^k h$, $k = 1, \dots, R - 1$, and either continuous or piecewise continuous bounded derivative $\partial_1^R h$. Function $h(\cdot)$ will be called *the fluctuation shape function of the R-th kind*, $h(\cdot) \in FS^R(\Omega, \Delta)$, if it satisfies conditions

$$\begin{aligned}
 h \in O(\lambda^R), \quad \partial_1^k h \in O(\lambda^{R-k}), \quad k = 1, 2, \dots, R, \\
 \int_{\Delta(x)} \mu(z) h(z) dz = 0, \quad z \in \Delta(x), \quad x \in \Omega_\Delta,
 \end{aligned}
 \tag{12}$$

$\tilde{\xi} \in [0, 1]$, **b** exponentially decaying solutions (46) to Eq. (44) versus $\tilde{\xi} \in [0, 0.2]$; $\lambda/L_1 = 0.1$, $K_2/K_1 = 0.5$

where $\mu(\cdot)$ is a certain positive valued λ -periodic function defined in $\bar{\Omega}$. Nonnegative integer R is specified in every discussed problem.

Moreover, for every $F(x) \in SV_\delta^R(\Omega, \Delta)$ and $h(x) \in FS^R(\Omega, \Delta)$ function $\vartheta(x) \equiv h(x)F(x) \in TP_\delta^R(\Omega, \Delta)$ satisfies condition

$$\begin{aligned}
 \int_{\Delta(x)} \partial^k \vartheta(z) dz = F(x) \int_{\Delta(x)} \partial^k h(z) dz, \quad z \in \Delta(x), \quad x \in \Omega_\Delta, \\
 k = 0, 1, 2, \dots, R, \quad \partial^0 \vartheta \equiv \vartheta, \quad \partial^0 h \equiv h.
 \end{aligned}
 \tag{13}$$

3.1.6 Averaging operation

Let $f(x)$ be a function defined in $\bar{\Omega} \equiv [0, L_1]$, which is integrable and bounded in every cell $\Delta(x)$, $x \in \Omega_\Delta$. The averaging operation of $f(\cdot)$ is defined by

$$\langle f \rangle (x) \equiv \frac{1}{|\Delta|} \int_{\Delta(x)} f(z) dz, \quad z \in \Delta(x), \quad x \in \Omega_\Delta.
 \tag{14}$$

It can be observed that if $f(\cdot)$ is Δ -periodic then $\langle f \rangle$ is constant.

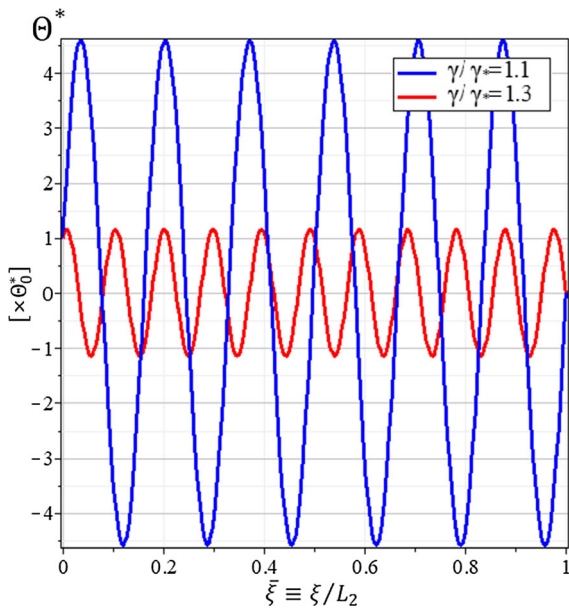


Fig. 5 Diagrams of dimensionless initial temperature micro-fluctuations $\Theta^*(\bar{\xi})$ being oscillating solutions (48) to Eq. (44) versus dimensionless coordinate $\bar{\xi} \in [0, 1]$ made for $\gamma/\gamma_* = 1.1$ and $\gamma/\gamma_* = 1.3$; $\lambda/L_1 = 0.1, K_2/K_1 = 0.5$

3.2 Basic assumptions of the tolerance modelling procedure

The tolerance modelling is based on two assumptions. The first of them is termed *the tolerance averaging approximation*. The second one is called *the micro-macro decomposition*.

3.2.1 Tolerance averaging approximation

For an integrable periodic function $\varphi(x)$ defined in $\bar{\Omega} = [0, L_1]$ and for $H(x) \in SV_\delta^1(\Omega, \Delta), S(x) \in SV_\delta^2(\Omega, \Delta), h(x) \in FS^1(\Omega, \Delta), s(x) \in FS^2(\Omega, \Delta), x \in \bar{\Omega}$, the following tolerance relations, called *the tolerance averaging approximation*, hold for every $x \in \Omega_\Delta$

$$\begin{aligned} \langle \varphi \partial_1^R H \rangle (x) &= \langle \varphi \rangle \partial_1^R H(x) + O(\delta), \quad R = 0, 1, \quad \partial_1^0 H \equiv H, \\ \langle \varphi \partial_1^R S \rangle (x) &= \langle \varphi \rangle \partial_1^R S(x) + O(\delta), \quad R = 0, 1, 2, \quad \partial_1^0 S \equiv S, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \langle \varphi \partial_1(hH) \rangle (x) &= \langle \varphi \partial_1 h \rangle H(x) + O(\delta), \\ \langle \varphi \partial_1(sS) \rangle (x) &= \langle \varphi \partial_1 s \rangle S(x) + O(\delta), \\ \langle \varphi \partial_1^2(sS) \rangle (x) &= \langle \varphi \partial_1^2 s \rangle S(x) + O(\delta). \end{aligned} \tag{16}$$

In the course of modelling, terms $O(\delta)$ in (15) and (16) are neglected. Approximations (15), (16) follow directly from conditions (10), (11) which hold for the slowly-varying functions and from condition (13) satisfied by the fluctuation shape functions.

3.2.2 Micro-macro decomposition assumption

The second fundamental assumption, called *the micro-macro decomposition*, states that the displacement and temperature fields occurring in the Lagrangian under consideration can be decomposed into *macroscopic and microscopic parts*. The macroscopic part is represented by *unknown averaged displacements and temperature* being slowly-varying functions in periodicity direction. The microscopic part is described by *the known highly oscillating periodic fluctuation shape functions multiplied by unknown temperature fluctuation amplitudes and displacement fluctuation amplitudes* being slowly-varying in x .

Micro-macro decomposition introduced in the thermoelastic problem discussed in this paper is presented in Sect. 4.1.

3.3 Basic concepts and assumptions of the asymptotic modelling procedure

The basic notions of the consistent asymptotic procedure [7] are those of *the fluctuation shape functions and the averaging operation*. These concepts have been explained in Sect. 3.1. The consistent asymptotic modelling does not require notions of tolerance-periodic and slowly-varying functions.

The consistent asymptotic decomposition is the basic assumption imposed on the starting Lagrangian under consideration. It states that the displacement and temperature fields occurring in the Lagrangian must be replaced by families of fields depending on parameter $0 < \varepsilon \ll 1$ and defined in an arbitrary cell. These families of displacements and temperature are decomposed into averaged part independent of ε and highly-oscillating part depending on ε .

Consistent asymptotic decomposition introduced in the thermoelastic problem discussed in this paper is presented in Sect. 5.1.

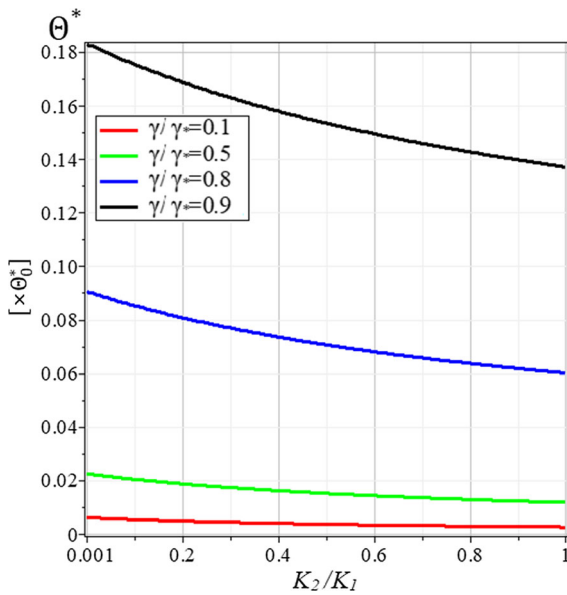


Fig. 6 Diagrams of dimensionless initial temperature micro-fluctuations $\Theta^*(\bar{\xi})$ being exponentially decaying solutions (46) to Eq. (44) versus ratio K_2/K_1 ; $\lambda/L_1 = 0.1$, $\bar{\xi} = 0.05$

4 Tolerance modelling

4.1 Governing equations of the tolerance model

The tolerance modelling procedure for Euler–Lagrange equations (9) is realized in two steps.

The first step is based on the tolerance averaging of Lagrangian (7) under micro-macro decomposition of displacements $u_\alpha(x, \xi, t) \in TP_\delta^1(\Omega, \Delta)$, $w(x, \xi, t) \in TP_\delta^2(\Omega, \Delta)$ and temperature field $\theta(x, \xi, t) \in TP_\delta^1(\Omega, \Delta)$ $x \in \Omega$, $(\xi, t) \in \Xi \times I$, which in the problem discussed here is assumed in the form

$$\begin{aligned} u_\alpha(x, \xi, t) &= u_\alpha^0(x, \xi, t) + h(x)U_\alpha(x, \xi, t), \\ w(x, \xi, t) &= w^0(x, \xi, t) + g(x)W(x, \xi, t), \\ \theta(x, \xi, t) &= \theta^0(x, \xi, t) + b(x)\Theta(x, \xi, t), \end{aligned} \tag{17}$$

where

$$\begin{aligned} u_\alpha^0(x, \xi, t), U_\alpha(x, \xi, t), \theta^0(x, \xi, t), \Theta(x, \xi, t) &\in SV_\delta^1(\Omega, \Delta), \\ \delta &\equiv (\lambda, \delta_0, \delta_1), w^0(x, \xi, t), W(x, \xi, t) \in SV_\delta^2(\Omega, \Delta), \\ \delta &\equiv (\lambda, \delta_0, \delta_1, \delta_2). \end{aligned} \tag{18}$$

Macrodisplacements u_α^0, w^0 and macrotemperature θ^0 as well as displacement fluctuation amplitudes U_α, W and temperature fluctuation amplitude Θ are the new unknowns.

Fluctuation shape functions for displacements $h(x) \in FS^1(\Omega, \Delta)$, $g(x) \in FS^2(\Omega, \Delta)$ and fluctuation shape function for temperature $b(x) \in FS^1(\Omega, \Delta)$ are the known, λ -periodic, continuous and highly-oscillating functions. Agree with (12) they have to satisfy conditions: $h \in O(\lambda)$, $\lambda \partial_1 h \in O(\lambda)$, $g \in O(\lambda^2)$, $\lambda \partial_1 g \in O(\lambda^2)$, $\lambda^2 \partial_{11} g \in O(\lambda^2)$, $b \in O(\lambda)$, $\lambda \partial_1 b \in O(\lambda)$, $\langle \mu h \rangle = \langle \mu g \rangle = \langle cb \rangle = 0$.

We substitute the right-hand sides of (17) into starting Lagrangian (7) and then we average the result over the cell applying formula (14) and tolerance averaging approximation (15), (16). As a result we obtain function $\langle L_{hg} \rangle$ being the averaged form of Lagrangian (7) in $\Delta(x)$. Under the additional approximation $1 + \lambda/r \approx 1$ (i.e. after neglecting terms of an order of λ/r) the final result has the form

$$\begin{aligned} \langle L_{hg} \rangle &(\partial_\beta u_\alpha^0, u_\alpha^0, \partial_2 U_\alpha, U_\alpha, u_\alpha^0, \dot{U}_\alpha, \partial_{\alpha\beta} w^0, w^0, \partial_{22} W, \\ &\partial_2 W, W, w^0, \dot{W}, \partial_x \theta^0, \theta^0, \partial_2 \Theta, \Theta, p^{2\beta}, \bar{r}) \\ &= -\frac{1}{2} [\langle D^{2\beta\gamma\delta} \rangle \partial_\beta u_\alpha^0 \partial_\gamma u_\alpha^0 + 2 \langle D^{2\beta\gamma1} \partial_1 h \rangle \partial_\beta u_\alpha^0 U_\gamma + \\ &+ \langle D^{211\gamma} (\partial_1 h)^2 \rangle U_\gamma U_\alpha + \langle D^{222\delta} (h)^2 \rangle \partial_2 U_\gamma \partial_2 U_\alpha + \\ &+ 2r^{-1} (\langle D^{2\beta11} \rangle \partial_\beta u_\alpha^0 w^0 + \langle D^{2111} \partial_1 h \rangle w^0 U_\alpha) + \\ &+ r^{-2} \langle D^{1111} \rangle w^0 w^0 + \langle B^{2\beta\gamma\delta} \rangle \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 + \\ &+ 2 (\langle B^{2\beta11} \partial_{11} g \rangle \partial_{\alpha\beta} w^0 W + \langle B^{2\beta22} g \rangle \partial_{\alpha\beta} w^0 \partial_{22} W + \\ &+ \langle B^{1122} g \partial_{11} g \rangle \partial_{22} WW) + 4 \langle B^{1212} (\partial_1 g)^2 \rangle (\partial_2 W)^2 + \\ &+ \langle B^{1111} (\partial_{11} g)^2 \rangle W^2 + \langle B^{2222} (g)^2 \rangle (\partial_{22} W)^2 + \\ &- \langle K^{2\beta} \rangle \partial_x \theta^0 \partial_\beta \theta^0 - 2 \langle K^{1\beta} \partial_1 b \rangle \partial_\beta \theta^0 \Theta + \\ &- 2 \langle K^{2\beta} b \rangle \partial_\beta \theta^0 \partial_2 \Theta - \langle K^{11} (\partial_1 b)^2 \rangle \Theta^2 \\ &+ \langle K^{22} (b)^2 \rangle (\partial_2 \Theta)^2 + - \langle \mu \rangle a^{\alpha\beta} u_\alpha^0 u_\beta^0 - \langle \mu \rangle (w^0)^2 \\ &- \langle \mu (h)^2 \rangle a^{\alpha\beta} \dot{U}_\alpha \dot{U}_\beta - \langle \mu (g)^2 \rangle (\dot{W})^2 \\ &+ \langle f^\alpha \rangle u_\alpha^0 + \langle f^2 h \rangle U_\alpha + \langle f \rangle w^0 + \langle fg \rangle W + \\ &+ \langle p^{2\beta} \rangle \partial_\beta u_\alpha^0 + \langle p^{\alpha 1} \partial_1 h \rangle U_\alpha + \langle p^{22} h \rangle \partial_2 U_\alpha \\ &+ r^{-1} \langle p^{11} \rangle w^0 + \langle \bar{r} \rangle \theta^0 + \langle \bar{r} b \rangle \Theta, \end{aligned} \tag{19}$$

with averaged constitutive equations given by

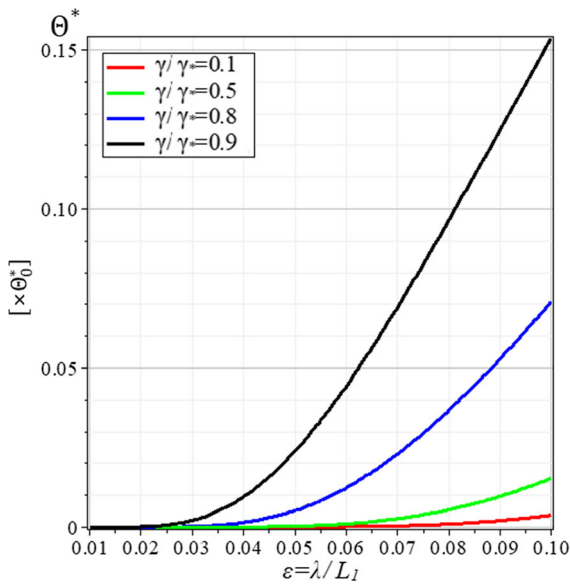


Fig. 7 Diagrams of dimensionless initial temperature microfluctuations $\Theta^*(\bar{\xi})$ being exponentially decaying solutions (46) to Eq. (44) versus dimensionless microstructure length parameter $\varepsilon \equiv \lambda/L_1$; $K_2/K_1 = 0.5$, $\bar{\xi} = 0.05$

$$\begin{aligned}
 \langle p^{\alpha\beta} \rangle &= \langle \bar{d}^{\alpha\beta} \rangle \theta^0 + \langle \bar{d}^{\alpha\beta} b \rangle \Theta, \\
 \langle p^{\alpha 1} \partial_1 h \rangle &= \langle \bar{d}^{\alpha 1} \partial_1 h \rangle \theta^0 + \langle \bar{d}^{\alpha 1} b \rangle \Theta, \\
 \langle p^{\alpha 2} h \rangle &= \langle \bar{d}^{\alpha 2} h \rangle \theta^0 + \langle \bar{d}^{\alpha 2} h b \rangle \Theta, \\
 \langle p^{11} h \rangle &= \langle \bar{d}^{11} g \rangle \theta^0 + \langle \bar{d}^{11} g b \rangle \Theta, \\
 \langle \hat{r} \rangle &= \langle c \rangle \hat{\theta}^0 + T_0 [\langle \bar{d}^{\alpha\beta} \rangle \partial_\alpha u_\beta^0 + \langle \bar{d}^{1\beta} \partial_1 h \rangle \dot{U}_\beta \\
 &\quad + \langle \bar{d}^{2\beta} h \rangle \partial_2 \dot{U}_\beta + r^{-1} \langle \bar{d}^{11} \rangle w^0], \\
 \langle \hat{r} b \rangle &= \langle c b^2 \rangle \hat{\theta}^0 + T_0 [\langle b \bar{d}^{\alpha\beta} \rangle \partial_\alpha u_\beta^0 \\
 &\quad + \langle b \bar{d}^{1\beta} \partial_1 h \rangle \dot{U}_\beta \\
 &\quad + \langle b \bar{d}^{2\beta} h \rangle \partial_2 \dot{U}_\beta].
 \end{aligned}
 \tag{20}$$

The underlined terms in (19), (20) depend on a period length λ .

Action functional

$$A_{hg}(u_\alpha^0, U_\alpha, w^0, W, p^{\alpha\beta}, \hat{r}) = \int_0^{L_1} \int_0^{L_2} \int_{t_0}^{t_1} \langle L_{hg} \rangle dt d\xi dx,
 \tag{21}$$

with $\langle L_{hg} \rangle$ given by (19) and with averaged constitutive equations for functions $p^{\alpha\beta}$, \hat{r} expressed by (20), is called *the tolerance averaging of starting action functional* $A(u_\alpha, w, p^{\alpha\beta}, \hat{r})$, cf. formula (6), under decomposition (17).

In the second step, we apply the extended principle of stationary action to averaged Lagrange function (19). In this step, expressions (20) are treated as non-variational parameters.

Under assumption that $\partial \langle L_{hg} \rangle / \partial (\partial_\beta u_\alpha^0)$, $\partial \langle L_{hg} \rangle / \partial (\partial_\alpha \beta w^0)$, $\partial \langle L_{hg} \rangle / \partial (\partial_2 U_\alpha)$, $\partial \langle L_{hg} \rangle / \partial (\partial_2 W)$, $\partial \langle L_{hg} \rangle / \partial (\partial_{22} W)$, $\partial \langle L_{hg} \rangle / \partial (\partial_\beta \theta^0)$, $\partial \langle L_{hg} \rangle / \partial (\partial_2 \Theta)$ are continuous, from the extended principle of stationary action applied to (21) we obtain the following system of Euler–Lagrange equations for $u_\alpha^0, w^0, U_\alpha, W, \theta^0, \Theta$ as the basic unknowns

$$\begin{aligned}
 \partial_\beta \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_\beta u_\alpha^0)} - \frac{\partial \langle L_{hg} \rangle}{\partial u_\alpha^0} + \frac{\partial \partial \langle L_{hg} \rangle}{\partial t} \frac{\partial u_\alpha^0}{\partial t} &= 0, \\
 - \partial_\alpha \beta \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_\alpha \beta w^0)} - \frac{\partial \langle L_{hg} \rangle}{\partial w^0} + \frac{\partial \partial \langle L_{hg} \rangle}{\partial t} \frac{\partial w^0}{\partial t} &= 0, \\
 \frac{\partial \partial \langle L_{hg} \rangle}{\partial t} \frac{\partial U_\alpha}{\partial U_\alpha} - \frac{\partial \langle L_{hg} \rangle}{\partial U_\alpha} \\
 + \partial_2 \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_2 U_\alpha)} &= 0, \frac{\partial \partial \langle L_{hg} \rangle}{\partial t} \frac{\partial W}{\partial W} - \frac{\partial \langle L_{hg} \rangle}{\partial W} \\
 + \partial_2 \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_2 W)} - \partial_{22} \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_{22} W)} &= 0, \\
 \partial_\beta \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_\beta \theta^0)} - \frac{\partial \langle L_{hg} \rangle}{\partial \theta^0} &= 0, \\
 - \frac{\partial \langle L_{hg} \rangle}{\partial \Theta} - \partial_2 \frac{\partial \langle L_{hg} \rangle}{\partial (\partial_2 \Theta)} &= 0.
 \end{aligned}
 \tag{22}$$

Combining (22) with (19) and (20) we obtain finally the explicit form of *the tolerance model equations*. These equations can be written in the form of

- the stress–strain–temperature relations

$$\begin{aligned}
 N^{\alpha\beta} &= \langle D^{\alpha\beta\gamma\delta} \rangle \partial_\delta u_\gamma^0 + r^{-1} \langle D^{\alpha\beta 11} \rangle w^0 + \langle D^{\alpha\beta\gamma 1} \partial_1 h \rangle U_\gamma \\
 &\quad + \langle D^{\alpha\beta\gamma 2} h \rangle \partial_2 U_\gamma - \langle \bar{d}^{\alpha\beta} \rangle \theta^0 - \langle \bar{d}^{\alpha\beta} b \rangle \Theta, \\
 M^{\alpha\beta} &= \langle B^{\alpha\beta\gamma\delta} \rangle \partial_\gamma \delta w^0 + \langle B^{\alpha\beta 11} \partial_{11} g \rangle W + \\
 &\quad + 2 \langle B^{\alpha\beta 12} \partial_{12} g \rangle \partial_2 W + \langle B^{\alpha\beta 22} g \rangle \partial_{22} W,
 \end{aligned}
 \tag{23}$$

$$\begin{aligned}
 H^\beta &= \langle \partial_1 h D^{\beta 1\gamma\delta} \rangle \partial_\delta u_\gamma^0 - \langle h D^{\beta 2\gamma\delta} \rangle \partial_{2\delta} u_\gamma^0 + \\
 &\quad + \langle D^{\beta 11\gamma} (\partial_1 h)^2 \rangle U_\gamma - \langle D^{\beta 22\gamma} (h)^2 \rangle \partial_{2\gamma} U_\gamma + \\
 &\quad + r^{-1} \langle \partial_1 h D^{\beta 111} \rangle w^0 - \langle \partial_1 h \bar{d}^{\beta 11} \rangle \theta^0 + \langle \bar{d}^{\beta 2} h \rangle \partial_2 \theta^0 + \\
 &\quad - \langle \bar{d}^{\beta 1} \partial_1 h b \rangle \Theta + \langle \bar{d}^{\beta 2} h b \rangle \partial_2 \Theta, \\
 G &= \langle \partial_{11} g B^{11\alpha\beta} \rangle \partial_\alpha \beta w^0 - 2 \langle \partial_{12} g B^{\alpha\beta 12} \rangle \partial_\alpha \beta w^0 + \\
 &\quad + \langle g B^{\alpha\beta 22} \rangle \partial_\alpha \beta w^0 + \langle (\partial_{11} g)^2 B^{1111} \rangle W + (2 \langle \partial_{11} g B^{1122} g \rangle + \\
 &\quad - 4 \langle (\partial_{11} g)^2 B^{1212} \rangle) \partial_{22} W + \langle (g)^2 B^{2222} \rangle \partial_{2222} W,
 \end{aligned}
 \tag{24}$$

- the dynamic equilibrium equations

$$\begin{aligned}
 \partial_z N^{\alpha\beta} - \langle \mu \rangle a^{\alpha\beta} \dot{u}_\alpha^0 + \langle f^\beta \rangle &= 0, \\
 \partial_{z\beta} M^{\alpha\beta} + r^{-1} N^{11} + \langle \mu \rangle \dot{w}^0 - \langle f \rangle &= 0, \\
 \underline{\langle \mu (h)^2 \rangle} + a^{\alpha\beta} \dot{U}_\alpha + H^\beta - \underline{\langle f^\beta h \rangle} &= 0, \\
 \underline{\langle \mu (g)^2 \rangle} + \dot{W} + G - \underline{\langle fg \rangle} &= 0,
 \end{aligned}
 \tag{25}$$

- the heat conduction equations

$$\begin{aligned}
 \langle K^{\alpha\beta} \rangle \partial_{z\beta} \theta^0 + \langle K^{1\beta} \partial_1 b \rangle \partial_\beta \Theta + \underline{\langle K^{2\beta} b \rangle} \partial_{z\beta} \Theta - \langle c \rangle \dot{\theta}^0 &= \\
 = T_0 [\langle \bar{d}^{\alpha\beta} \rangle \partial_{z\alpha} \dot{u}_\beta^0 + \langle \bar{d}^{1\beta} \partial_1 h \rangle \dot{U}_\beta + \underline{\langle \bar{d}^{2\beta} h \rangle} \partial_2 \dot{U}_\beta + r^{-1} \langle \bar{d}^{11} \rangle \dot{w}^0], \\
 \underline{\langle K^{2\beta} b \rangle} \partial_{z\beta} \theta^0 - \langle K^{1\beta} \partial_1 b \rangle \partial_\beta \theta^0 + \underline{\langle K^{22} (b)^2 \rangle} \partial_{z2} \Theta - \langle K^{11} (\partial_1 b)^2 \rangle \Theta + \\
 - \underline{\langle c (b)^2 \rangle} \dot{\Theta} = T_0 [\underline{\langle b \bar{d}^{\alpha\beta} \rangle} \partial_{z\alpha} \dot{u}_\beta^0 + \underline{\langle \bar{d}^{1\beta} b \partial_1 h \rangle} \dot{U}_\beta + \underline{\langle \bar{d}^{2\beta} b h \rangle} \partial_2 \dot{U}_\beta].
 \end{aligned}
 \tag{26}$$

Equations (23)–(26) together with the *micro-macro decomposition* (17) and *physical reliability conditions* (18) constitute the *tolerance model for the analysis of selected dynamic thermoelasticity problems for uniperiodic shells under consideration*. Coefficients of the derived model equations are *constant* and some of them *involve microstructure length parameter λ* (underlined terms).

4.2 Discussion of results

The important features of the *tolerance model* proposed here are listed below.

- In contrast to exact thermoelastic shell Eqs. (4), (5) with periodic, discontinuous and highly oscillating coefficients, the tolerance model Eqs. (23)–(26) proposed here *have constant coefficients*. Moreover, *some of them depend on a period length λ* (underlined terms). Hence, the tolerance model makes it possible to *describe the effect of a microstructure size on the global thermoelastic shell behaviour*. Moreover, we can analyse the length-scale effect not only in non-stationary but also in stationary problems for the uniperiodic shells considered here.
- The number and form of boundary conditions for macrodisplacements u_α^0, w^0 and macrotemperature θ^0 are the same as in the classical shell theory governed by thermoelasticity Eqs. (4), (5). The boundary conditions for kinematic fluctuation amplitudes U_α, W and for thermal fluctuation

amplitudes Θ should be defined only on boundaries $\zeta = 0, \zeta = L_2$.

- Decomposition (17) and hence also governing Eqs. (23)–(26) of the tolerance model are uniquely determined by the given a priori highly oscillating periodic *fluctuations shape functions for displacements* $h(x) \in FS^1(\Omega, \Delta), h \in O(\lambda), g(x) \in FS^2(\Omega, \Delta), g \in O(\lambda^2)$, and *fluctuations shape function for temperature* $b(x) \in FS^1(\Omega, \Delta), b \in O(\lambda)$, which represent oscillations of displacement and temperature fields inside a cell. These functions can be derived as solutions to periodic eigenvalue cell problems. For the most cases of those problems an approximate forms of these solutions are taken into account. The choice of these functions can be also based on the experience or intuition of the researcher.
- It has to be emphasized that solutions to selected initial/boundary value problems formulated in the framework of the tolerance model have a physical sense only if conditions (18) hold for the pertinent tolerance parameters δ , i.e. if unknowns $u_\alpha^0, w^0, \theta^0, U_\alpha, W, \Theta$ of the tolerance model equations are *slowly-varying functions* in periodicity direction. These conditions can be also used for the a posteriori evaluation of tolerance parameters δ and hence, for the verification of the physical reliability of the obtained solutions.
- For a homogeneous shell with a constant thickness $D^{\alpha\beta\gamma\delta}(x), B^{\alpha\beta\gamma\delta}(x), \mu(x), \bar{d}^{\alpha\beta}(x), K^{\alpha\beta}(x), c(x)$ are constant and because $\langle \mu h \rangle = \langle \mu g \rangle = \langle cb \rangle = 0$, we obtain $\langle h \rangle = \langle g \rangle = \langle b \rangle = 0$, and hence $\langle \partial_1 h^a \rangle = \langle \partial_1 g^A \rangle = \langle \partial_{11} g^A \rangle = \langle \partial_1 b \rangle = 0$. In this case equations (25)_{1,2} and (26)₁ reduced to the well known shell equations of motion for averaged displacements $u_\alpha^0(x, \zeta, t), w^0(x, \zeta, t)$ and to the heat conduction equation for averaged temperature $\theta^0(x, \zeta, t)$. Independently, for fluctuation amplitudes $U_\alpha(x, \zeta, t), W(x, \zeta, t), \Theta(x, \zeta, t)$ we arrive at the system of equations, which under condition $\langle f^\beta h \rangle = \langle fg \rangle = 0$ and under homogeneous initial conditions for U_α, W and Θ has only trivial solution $U_\alpha = W = \Theta = 0$. Hence, from

decomposition (17) it follows that $u_x = u_x^0$, $w = w^0$, $\theta = \theta^0$. It means that Eqs. (23)–(26) generated by tolerance averaged Lagrange function (19) reduce to the starting Eqs. (4), (5) generated by Lagrange function (7).

5 Consistent asymptotic modelling

The asymptotic model equations can be obtained directly from the tolerance model Eqs. (23)–(26) by the formal limit passage $\lambda \rightarrow 0$. However, the same results can be obtained independently of the tolerance model by applying the consistent asymptotic procedure (variational approach) proposed in [7]. In this approach the concepts of tolerance-periodic and slowly-varying functions are not introduced. On passing from the tolerance to asymptotic modelling, we retain only the concepts of fluctuation shape function and averaged operation. Below, asymptotic model equations will be derived by applying the consistent asymptotic modelling.

5.1 Asymptotic model equations

Asymptotic modelling procedure for Euler–Lagrange equations (9) is realized in two steps.

The first step is based on the consistent asymptotic averaging of Lagrangian (7) and independently on the consistent asymptotic averaging of constitutive Eq. (8) for functions $p^{\alpha\beta}(x, \zeta, t)$, $\hat{r}(x, \zeta, t)$ being the non-variational parameters of Lagrange function (7).

In order to do it, we shall restrict considerations to displacement fields $u_x = u_x(z, \zeta, t)$, $w = w(z, \zeta, t)$ and temperature field $\theta(z, \zeta, t)$ defined in $\Delta(x) \times \Xi \times I$, $z \equiv z^1 \in \Delta(x)$, $x \in \Omega_\Delta$, $(\zeta, t) \in \Xi \times I$. Then, we replace $u_x(z, \zeta, t)$, $w(z, \zeta, t)$ and $\theta(z, \zeta, t)$ by families of displacements $u_{ex}(z, \zeta, t) \equiv u_x(z/\varepsilon, \zeta, t)$, $w_\varepsilon(z, \zeta, t) \equiv w(z/\varepsilon, \zeta, t)$ and family of temperature field $\theta_\varepsilon(z, \zeta, t) \equiv \theta(z/\varepsilon, \zeta, t)$ where $0 < \varepsilon < \ll 1$, $z \in \Delta_\varepsilon(x)$, $\Delta_\varepsilon \equiv (-\varepsilon\lambda/2, \varepsilon\lambda/2)$ (scaled cell), $\Delta_\varepsilon(x) \equiv x + \Delta_\varepsilon$, $x \in \Omega_{\Delta_\varepsilon}$ (scaled cell with a centre at $x \in \Omega_{\Delta_\varepsilon}$).

We introduce the consistent asymptotic decomposition of displacement and temperature families $u_{ex}(z, \zeta, t)$, $w_\varepsilon(z, \zeta, t)$, $\theta_\varepsilon(z, \zeta, t)$, $(z, \zeta, t) \in \Delta_\varepsilon \times \Xi \times I$ in the area of every ε -scaled cell

$$\begin{aligned} u_{ex}(z, \zeta, t) &\equiv u_x(z/\varepsilon, \zeta, t) = u_x^0(z, \zeta, t) + \varepsilon h_\varepsilon(z)U_x(z, \zeta, t), \\ w_\varepsilon(z, \zeta, t) &\equiv w(z/\varepsilon, \zeta, t) = w^0(z, \zeta, t) + \varepsilon^2 g_\varepsilon(z)W(z, \zeta, t), \\ \theta_\varepsilon(z, \zeta, t) &\equiv \theta(z/\varepsilon, \zeta, t) = \theta^0(z, \zeta, t) + \varepsilon b_\varepsilon(z)\Theta(z, \zeta, t). \end{aligned} \tag{27}$$

Functions u_x^0, w^0 and U_x, W are termed *macrodisplacements* and *displacement fluctuation amplitudes*, respectively. Functions θ^0, Θ are called *macrotemperature* and *temperature fluctuation amplitudes*. All unknowns mentioned above are assumed to be continuous and bounded in $\bar{\Omega}$. It has to be emphasized that these unknown functions are independent of ε .

Fluctuation shape functions for displacements $h_\varepsilon(z) \equiv h(z/\varepsilon) \in FS^1(\Omega, \Delta)$, $g_\varepsilon(z) \equiv g(z/\varepsilon) \in FS^2(\Omega, \Delta)$ and fluctuation shape function for temperature $b_\varepsilon(z) \equiv b(z/\varepsilon) \in FS^1(\Omega, \Delta)$ in (27) are highly oscillating and Δ -periodic. They have to be known in every problem under consideration. They depend on λ as a parameter and have to satisfy conditions: $h_\varepsilon \in O(\varepsilon\lambda)$, $\lambda\partial_1 h_\varepsilon \in O(\varepsilon\lambda)$, $g_\varepsilon \in O((\varepsilon\lambda)^2)$, $\lambda\partial_1 g_\varepsilon \in O((\varepsilon\lambda)^2)$, $\lambda^2\partial_{11} g_\varepsilon \in O((\varepsilon\lambda)^2)$, $b_\varepsilon \in O(\varepsilon\lambda)$, $\lambda\partial_1 b_\varepsilon \in O(\varepsilon\lambda)$, $\langle \mu h_\varepsilon \rangle = \langle \mu g_\varepsilon \rangle = \langle cb_\varepsilon \rangle = 0$. Bearing in mind that $h_\varepsilon(z) \equiv h(z/\varepsilon)$, $g_\varepsilon(z) \equiv g(z/\varepsilon)$, $b_\varepsilon(z) \equiv b(z/\varepsilon)$ and setting $\partial_1 h_\varepsilon(z) \equiv \frac{1}{\varepsilon}\partial_1 h(z/\varepsilon)$, $\partial_1 g_\varepsilon(z) \equiv \frac{1}{\varepsilon}\partial_1 g(z/\varepsilon)$, $\partial_{11} g_\varepsilon(z) \equiv \frac{1}{\varepsilon^2}\partial_{11} g(z/\varepsilon)$, $\partial_1 b_\varepsilon(z) \equiv \frac{1}{\varepsilon}\partial_1 b(z/\varepsilon)$, from (27) we obtain

$$\begin{aligned} \partial_1 u_{ex}(z, \zeta, t) &= \partial_1 u_x^0(z, \zeta, t) + \partial_1 h(z/\varepsilon)U_x(z, \zeta, t) + O(\varepsilon), \\ \partial_2 u_{ex}(z, \zeta, t) &= \partial_2 u_x^0(z, \zeta, t) + O(\varepsilon), \\ \dot{u}_{ex}(z, \zeta, t) &= \dot{u}_x^0(z, \zeta, t) + O(\varepsilon), \\ \partial_1 \dot{u}_{ex}(z, \zeta, t) &= \partial_1 \dot{u}_x^0(z, \zeta, t) + \partial_1 h(z/\varepsilon)\dot{U}_x(z, \zeta, t) + O(\varepsilon), \\ \partial_2 \dot{u}_{ex}(z, \zeta, t) &= \partial_2 \dot{u}_x^0(z, \zeta, t) + O(\varepsilon) \\ \partial_1 w_\varepsilon(z, \zeta, t) &= \partial_1 w^0(z, \zeta, t) + O(\varepsilon), \\ \partial_{11} w_\varepsilon(z, \zeta, t) &= \partial_{11} w^0(z, \zeta, t) + \partial_{11} b_\varepsilon W(z, \zeta, t) + O(\varepsilon), \\ \partial_{12} w_\varepsilon(z, \zeta, t) &= \partial_{12} w^0(z, \zeta, t) + O(\varepsilon) = \partial_{21} w_\varepsilon(z, \zeta, t), \\ \partial_2 w_\varepsilon(z, \zeta, t) &= \partial_2 w^0(z, \zeta, t) + O(\varepsilon), \\ \partial_{22} w_\varepsilon(z, \zeta, t) &= \partial_{22} w^0(z, \zeta, t) + O(\varepsilon), \\ \dot{w}_\varepsilon(z, \zeta, t) &= \dot{w}^0(z, \zeta, t) + O(\varepsilon), \end{aligned} \tag{28}$$

and

$$\begin{aligned} \partial_1 \theta_\varepsilon(z, \zeta, t) &= \partial_1 \theta^0(z, \zeta, t) + \partial_1 b(z/\varepsilon)\Theta(z, \zeta, t) + O(\varepsilon), \\ \partial_2 \theta_\varepsilon(z, \zeta, t) &= \partial_2 \theta^0(z, \zeta, t) + O(\varepsilon), \\ \dot{\theta}_\varepsilon(z, \zeta, t) &= \dot{\theta}^0(z, \zeta, t) + O(\varepsilon), \end{aligned} \tag{29}$$

where $z \in \Delta_\varepsilon(x), x \in \Omega_{\Delta_\varepsilon}$.

Because of Lagrangian L defined by (7) is highly oscillating with respect to x and essentially bounded in its domain, then there exists Lagrangian $\tilde{L}(z, \xi, t, \partial_\beta u_x, u_\alpha, \dot{u}_\alpha, \partial_{\alpha\beta} w, w, \dot{w}, p^{\alpha\beta}, \hat{r})$ being the periodic approximation of Lagrangian L in $\Delta(x), z \in \Delta(x), x \in \Omega_\Delta$. Let \tilde{L}_ε be a family of functions given by

$$\begin{aligned} \tilde{L}_\varepsilon &= \tilde{L}(z/\varepsilon, \xi, t, \partial_\beta u_{\varepsilon x}, u_{\varepsilon x}, \dot{u}_{\varepsilon x}, \partial_{\alpha\beta} w_\varepsilon, w_\varepsilon, \dot{w}_\varepsilon, \partial_\beta \theta_\varepsilon, \theta_\varepsilon, p_\varepsilon^{\alpha\beta}, \hat{r}_\varepsilon) = \\ &= -\frac{1}{2}[D^{2\beta\gamma\delta}\partial_\beta u_{\varepsilon x}\partial_\delta u_{\varepsilon\gamma} + 2r^{-1}D^{2\beta 11}w_\varepsilon\partial_\beta u_{\varepsilon x} + \\ &\quad + r^{-2}D^{1111}w_\varepsilon w_\varepsilon + B^{2\beta\gamma\delta}\partial_{\alpha\beta} w_\varepsilon\partial_\gamma w_\varepsilon + \\ &\quad - K^{2\beta}\partial_x\theta_\varepsilon\partial_\beta\theta_\varepsilon - \mu\alpha^{2\beta}\dot{u}_{\varepsilon x}\dot{u}_{\varepsilon\beta} - \mu(\dot{w}_\varepsilon)^2] + \\ &\quad + f^\alpha u_{\varepsilon x} + f w_\varepsilon + p_\varepsilon^{\alpha\beta}\partial_\beta u_\varepsilon^\alpha + r^{-1}p_\varepsilon^{11}w_\varepsilon + \hat{r}_\varepsilon\theta_\varepsilon, \end{aligned} \tag{30}$$

where $p_\varepsilon^{\alpha\beta}, \hat{r}_\varepsilon$ play the role of parameters and are given by independent equations

$$\begin{aligned} p_\varepsilon^{\alpha\beta} &= \bar{d}^{\alpha\beta}\theta_\varepsilon, \\ \hat{r}_\varepsilon &= c\dot{\theta}_\varepsilon + T_0(\bar{d}^{\alpha\beta}\partial_\alpha\dot{u}_{\varepsilon\beta} + r^{-1}\bar{d}^{11}\dot{w}_\varepsilon). \end{aligned} \tag{31}$$

We substitute the right-hand sides of (27)–(29) into (30) and independently to (31). Then, we take into account that under limit passage $\varepsilon \rightarrow 0$, terms $O(\varepsilon), O(\varepsilon^2)$ can be neglected and every continuous and bounded function $f(z, t)$ of argument $z \in \Delta_\varepsilon(x)$ tends to function $f(x, t)$ of argument $x \in \bar{\Omega}$. Moreover, if $\varepsilon \rightarrow 0$ then, by means of a property of the mean value, cf. Jikov et al. [2], the obtained result tends weakly to function L^0 being the averaged form of starting Lagrangian (7) under consistent asymptotic decomposition (27)

$$\begin{aligned} L_0(\partial_\beta u_x^0, u_x^0, U_x, u_\alpha^0, \partial_{\alpha\beta} w^0, w^0, W, \dot{w}^0, \partial_\beta \theta^0, \theta^0, \Theta, p^{\alpha\beta}, \hat{r}) &= \\ &= -\frac{1}{2}[\langle D^{2\beta\gamma\delta}(z) \rangle \partial_\beta u_\alpha^0 \partial_\delta u_\alpha^0 + 2\langle D^{2\beta\gamma 1}(z) \rangle \partial_1 h(z) \rangle \partial_\beta u_\alpha^0 U_\gamma + \\ &\quad + \langle D^{2111}(z) \rangle (\partial_1 h(z))^2 \rangle U_\gamma U_x + 2r^{-1}(\langle D^{2\beta 11}(z) \rangle \partial_\beta u_\alpha^0 w^0 + \\ &\quad + \langle D^{2111}(z) \rangle \partial_1 h(z) \rangle w^0 U_x) + r^{-2} \langle D^{1111}(z) \rangle (w^0)^2 + \\ &\quad + \langle B^{2\beta\gamma\delta}(z) \rangle \partial_{\alpha\beta} w^0 \partial_\gamma w^0 + 2\langle B^{2\beta 11}(z) \rangle \partial_1 h(z) \rangle \partial_{\alpha\beta} w^0 W + \\ &\quad + \langle B^{1111}(z) \rangle (\partial_1 h(z))^2 \rangle (W)^2 + \\ &\quad - \langle K^{2\beta}(z) \rangle \partial_x \theta^0 \partial_\beta \theta^0 - 2\langle K^{1\beta}(z) \rangle \partial_1 b(z) \rangle \partial_\beta \theta^0 \Theta \\ &\quad - \langle K^{11}(z) \rangle (\partial_1 b(z))^2 \rangle (\Theta)^2 + \\ &\quad - \langle \mu(z) \rangle \alpha^{2\beta} \dot{u}_\alpha^0 \dot{u}_\alpha^0 - \langle \mu(z) \rangle (w^0)^2 + \langle f^\alpha \rangle u_\alpha^0 \\ &\quad + \langle f \rangle w^0 + \langle p^{\alpha\beta} \rangle \partial_\beta u_\alpha^0 + \langle p^{21} \rangle \partial_1 h \rangle U_x \\ &\quad + r^{-1} \langle p^{11} \rangle w^0 + \langle \hat{r} \rangle \Theta, \end{aligned} \tag{32}$$

where averaged constitutive equations for functions $\langle p^{\alpha\beta} \rangle, \langle \hat{r} \rangle$ have the form

$$\begin{aligned} \langle p^{\alpha\beta} \rangle &= \langle \bar{d}^{\alpha\beta} \rangle \theta^0, \quad \langle p^{21} \rangle \partial_1 h \rangle = \langle \bar{d}^{21} \rangle \partial_1 h \rangle \theta^0, \\ \langle \hat{r} \rangle &= \langle c \rangle \theta^0 + T_0[\langle \bar{d}^{2\beta} \rangle \partial_\alpha \dot{u}_\alpha^0 + \langle \bar{d}^{1\beta} \rangle \partial_1 h \rangle U_\beta + r^{-1} \langle \bar{d}^{11} \rangle w^0]. \end{aligned} \tag{33}$$

Averages $\langle \cdot \rangle$ occurring in (32), (33) are constant and calculated by means of (14).

In the framework of consistent asymptotic procedure we introduce the consistent asymptotic action functional

$$A_{hg}^0(u_\alpha^0, U_x, w^0, W, p^{\alpha\beta}, \hat{r}) = \int_0^{L_1} \int_0^{L_2} \int_{t_0}^{t_1} L_0 dt d\xi dx, \tag{34}$$

where L_0 is given by (32).

The second step in the asymptotic modelling of Euler–Lagrange equations (9) is to apply the extended principle of stationary action to averaged Lagrange function (32). In this step, expressions (33) are treated as non-variational parameters.

Under assumption that $\partial L_0 / \partial(\partial_\beta u_\alpha^0), \partial L_0 / \partial(\partial_{\alpha\beta} w^0), \partial L_0 / \partial(\partial_\beta \theta^0)$ are continuous, from the extended principle of stationary action applied to (34) we obtain the following system of Euler–Lagrange equations for $u_\alpha^0, w^0, U_x, W, \theta^0, \Theta$ as the basic unknowns

$$\begin{aligned} \partial_\beta \frac{\partial L_0}{\partial(\partial_\beta u_\alpha^0)} - \frac{\partial L_0}{\partial u_\alpha^0} + \frac{\partial}{\partial t} \frac{\partial L_0}{\partial \dot{u}_\alpha^0} &= 0, \\ -\partial_{\alpha\beta} \frac{\partial L_0}{\partial(\partial_{\alpha\beta} w^0)} - \frac{\partial L_0}{\partial w^0} + \frac{\partial}{\partial t} \frac{\partial L_0}{\partial \dot{w}^0} &= 0, \\ \frac{\partial L_0}{\partial U_x} &= 0, \quad \frac{\partial L_0}{\partial W} = 0, \\ \partial_\beta \frac{\partial L_0}{\partial(\partial_\beta \theta^0)} - \frac{\partial L_0}{\partial \theta^0} &= 0, \quad \frac{\partial L_0}{\partial \Theta} = 0. \end{aligned} \tag{35}$$

Combining (35) with (32) and (33) we arrive at the explicit form of the consistent asymptotic model equations for $u_\alpha^0(x, \xi, t), w^0(x, \xi, t), U_x(x, \xi, t), W(x, \xi, t), \theta^0(x, \xi, t), \Theta(x, \xi, t), x \in \Omega, (\xi, t) \in \Xi \times I$

$$\begin{aligned} \langle D^{2\beta\gamma\delta} \rangle \partial_\beta u_\alpha^0 + r^{-1} \langle D^{2\beta 11} \rangle \partial_\beta w^0 + \langle D^{2\beta\gamma 1} \rangle \partial_1 h \rangle \partial_\beta U_\gamma - \langle \bar{d}^{\alpha\beta} \rangle \partial_\beta \theta^0 + \\ - \langle \mu \rangle \alpha^{2\beta} \dot{u}_\alpha^0 + \langle f^\alpha \rangle = 0, \\ \langle B^{2\beta\gamma\delta} \rangle \partial_{\alpha\beta} w^0 + \langle D^{2\beta 11} \rangle \partial_1 h \rangle \partial_{\alpha\beta} W + r^{-1} \langle D^{11\gamma\delta} \rangle \partial_\gamma w^0 + \\ + r^{-1} \langle D^{1111} \rangle w^0 + r^{-1} \langle D^{11\gamma\delta} \rangle \partial_1 h \rangle U_\delta - \langle \bar{d}^{11} \rangle \theta^0 + \\ + \langle \mu \rangle w^0 - \langle f \rangle = 0, \\ \langle (\partial_1 h)^2 \rangle D^{2\beta 11} \rangle U_\gamma = - \langle \partial_1 h D^{2\beta\gamma\delta} \rangle \partial_\beta u_\alpha^0 - r^{-1} \langle \partial_1 h D^{2\beta 11} \rangle w^0 + \langle \partial_1 h \bar{d}^{11} \rangle \theta^0, \\ \langle (\partial_1 h)^2 \rangle B^{1111} \rangle W = - \langle \partial_1 h B^{11\gamma\delta} \rangle \partial_\gamma w^0, \\ \langle K^{2\beta} \rangle \partial_{\alpha\beta} \theta^0 + \langle K^{1\beta} \rangle \partial_1 b \rangle \partial_\beta \Theta - \langle c \rangle \theta^0 \\ = T_0[\langle \bar{d}^{\alpha\beta} \rangle \partial_\alpha \dot{u}_\alpha^0 + \langle \bar{d}^{1\beta} \rangle \partial_1 h \rangle U_\beta + r^{-1} \langle \bar{d}^{11} \rangle w^0], \\ \langle K^{11} \rangle (\partial_1 b)^2 \rangle \Theta = - \langle K^{1\beta} \rangle \partial_1 b \rangle \partial_\beta \theta^0. \end{aligned} \tag{36}$$

Equation (36) consist of partial differential equations for macrodisplacements u_α^0, w^0 and macrotemperature θ^0 coupled with linear algebraic equations for kinematic fluctuation amplitudes U_α, W and thermal fluctuation amplitude Θ . After eliminating fluctuation amplitudes from the governing equations by means of

$$\begin{aligned} U_\gamma &= -G_{\gamma\eta}^{-1} [\langle \partial_1 h D^{1\eta\mu\theta} \rangle \partial_\mu u_\mu^0 + r^{-1} \langle \partial_1 h D^{1\eta 11} \rangle \\ &\quad \times w^0 - \langle \partial_1 h \bar{d}^{\eta 1} \rangle \theta^0], \\ W &= -E^{-1} \langle \partial_{11} g B^{11\gamma\delta} \rangle \partial_{\gamma\delta} w^0, \\ \Theta &= -C^{-1} \langle K^{1\beta} \partial_1 b \rangle \partial_\beta \theta^0, \end{aligned} \tag{37}$$

where $G_{\alpha\gamma} = \langle D^{\alpha\gamma 11} (\partial_1 h)^2 \rangle$, $E = \langle B^{1111} (\partial_{11} g)^2 \rangle$, $C = \langle K^{11} (\partial_1 b)^2 \rangle$, $G_{\alpha\gamma} G_{\gamma\eta}^{-1} = \delta_{\alpha\eta}$ ($\delta_{\alpha\eta}$ is an unit tensor) we arrive finally at the asymptotic model equations expressed only in macrodisplacements u_α^0, w^0 and macrotemperature θ^0

$$\begin{aligned} D_h^{\alpha\beta\gamma\delta} \partial_{\beta\delta} u_\gamma^0 + r^{-1} D_h^{\alpha\beta 11} \partial_\beta w^0 - \bar{D}_b^{\alpha\beta} \partial_\beta \theta^0 \\ - \langle \mu \rangle \alpha^{\alpha\beta} \dot{u}_\beta^0 + \langle f^\alpha \rangle = 0, \\ B_g^{\alpha\beta\gamma\delta} \partial_{\alpha\beta\gamma\delta} w^0 + r^{-1} D_h^{11\gamma\delta} \partial_\delta u_\gamma^0 + r^{-2} D_h^{1111} w^0 \\ - r^{-1} \bar{D}_h^{11} \theta^0 + \langle \mu \rangle \dot{w}^0 - \langle f \rangle = 0, \tag{38} \\ \bar{K}_b^{\alpha\beta} \partial_{\alpha\beta} \theta^0 - [\langle c \rangle + T_0 \langle \bar{d}^{1\beta} \partial_1 h \rangle G_{\beta\eta}^{-1} \\ \langle \partial_1 h \bar{d}^{\eta 1} \rangle] \dot{\theta}^0 = \\ = T_0 [\bar{D}_h^{\alpha\beta} \partial_\alpha \dot{u}_\beta^0 + r^{-1} \bar{D}_h^{11} \dot{w}^0], \end{aligned}$$

where

$$\begin{aligned} D_h^{\alpha\beta\gamma\delta} &\equiv \langle D^{\alpha\beta\gamma\delta} \rangle - \langle D^{\alpha\beta\eta 1} \partial_1 h \rangle G_{\eta\zeta}^{-1} \langle \partial_1 h D^{1\zeta\gamma\delta} \rangle, \\ B_g^{\alpha\beta\gamma\delta} &\equiv \langle B^{\alpha\beta\gamma\delta} \rangle - \langle B^{\alpha\beta 11} \partial_{11} g \rangle E^{-1} \langle \partial_{11} g B^{11\gamma\delta} \rangle, \\ \bar{D}_h^{\alpha\beta} &\equiv \langle \bar{d}^{\alpha\beta} \rangle - \langle D^{\alpha\beta\gamma 1} \partial_1 h \rangle G_{\gamma\eta}^{-1} \langle \partial_1 h \bar{d}^{\eta 1} \rangle, \\ \bar{K}_b^{\alpha\beta} &\equiv \langle K^{\alpha\beta} \rangle - \langle K^{1\alpha} \partial_1 b \rangle C^{-1} \langle K^{1\beta} \partial_1 b \rangle. \end{aligned} \tag{39}$$

Tensors $D_h^{\alpha\beta\gamma\delta}, B_g^{\alpha\beta\gamma\delta}$ are tensors of effective elastic moduli for uniperiodic shells considered here.

Tensor $\bar{D}_h^{\alpha\beta}$ is a tensor of effective elastic-thermal moduli.

Tensor $\bar{K}_b^{\alpha\beta}$ is a tensor of effective thermal moduli.

Because of functions $u_\alpha(x, \zeta, t), w(x, \zeta, t), \theta(x, \zeta, t)$ have to be uniquely defined in $\Omega \times \Xi \times I$, we conclude that $u_\alpha(x, \zeta, t), w(x, \zeta, t), \theta(x, \zeta, t)$ must take the form of (17) with U_α, W, Θ given by (37). However, now

unknowns $u_\alpha^0, w^0, U_\alpha, W, \theta^0, \Theta$ in (17) are not assumed to be slowly-varying in the sense given by (10), (11).

Equations (38) together with decomposition (17) represent the consistent asymptotic model of selected dynamic thermoelasticity problems for the thin uniperiodic cylindrical shells under consideration.

5.2 Discussion of results

The important features of the derived consistent asymptotic model are listed below.

- Contrary to starting Eqs. (4), (5) with periodic, highly oscillating and discontinuous coefficients, the asymptotic model Eq. (38) formulated here have constant coefficients but independent of a period length. It means that this model is not able to describe the influence of a cell size on the global shell thermoelasticity.
- Unknown functions $u_\alpha^0, U_\alpha, w^0, W$ and θ^0, Θ in (38) are demanded to be bounded and continuous in $\bar{\Omega}$ together with their appropriate derivatives.
- Within the asymptotic model we formulate boundary conditions only for the macrodisplacements u_α^0, w^0 and macrotemperature θ^0 . The number and form of these conditions are the same as in the classical shell theory governed by starting Eqs. (4), (5).
- The extra unknown functions U_α, W, Θ called fluctuation amplitudes are governed by a system of linear algebraic equations and can be always eliminated from the governing equations. Hence, the unknowns of final asymptotic model Eq. (38) are only macrodisplacements u_α^0, w^0 and macrotemperature θ^0 .
- The resulting asymptotic model Eq. (38) are uniquely determined by the postulated a priori periodic fluctuations shape functions, $h(x) \in FS^1(\Omega, \Delta)$, $h \in O(\lambda)$, $g(x) \in FS^2(\Omega, \Delta)$, $g \in O(\lambda^2)$, $b(x) \in FS^1(\Omega, \Delta)$, $b \in O(\lambda)$ representing oscillations of displacement and temperature fields inside a cell.
- Taking into account that for a homogeneous shell with a constant thickness $D^{\alpha\beta\gamma\delta}(x), B^{\alpha\beta\gamma\delta}(x), \mu(x), \bar{d}^{\alpha\beta}(x), K^{\alpha\beta}(x), c(x)$ are constant and bearing in mind that $\langle \partial_1 h \rangle = \langle \partial_{11} g \rangle = \langle \partial_1 b \rangle > 0$ we obtained from (37) that $U_\alpha = W = \theta = 0$ and from

(39) that $D_h^{\alpha\beta\gamma\delta} \equiv D^{\alpha\beta\gamma\delta}$, $B_g^{\alpha\beta\gamma\delta} \equiv B^{\alpha\beta\gamma\delta}$, $\bar{D}_h^{\alpha\beta} = \bar{d}^{\alpha\beta}$, $\bar{K}_b^{\alpha\beta} = K^{\alpha\beta}$. Hence, from decomposition (17) it follows that $u_x = u_x^0$, $w = w^0$, $\theta = \theta^0$. It means that Eq. (38) generated by asymptotically averaged Lagrange function (32) together with asymptotically averaged constitutive Eq. (33) reduce to the starting Eqs. (4), (5) generated by Lagrange function (7) together with constitutive Eq. (8) for invariational parameters occurring in (7).

6 Examples of applications

The biggest advantage of the tolerance model derived in this paper is that it makes it possible to describe phenomena depending on the cell size. For that reason, in this section we shall study two special length-scale problems applying governing Eqs. (23)–(26) of the tolerance model. The first of them deals with investigation of the effect of a cell size λ on the shape of initial distributions of temperature micro-fluctuations $\Theta(\cdot, t)$ in the closed uniperiodic shell. This special problem can be studied in the framework of neither the asymptotic models nor the known commercial numerical models for the periodic shells under consideration. The second one deals with study of the effect of a microstructure size λ on the distribution of total temperature field $\theta(\cdot, t) = \theta^0(\cdot, t) + b(\cdot)\Theta(\cdot, t)$, $t \in I = [t_0, t_1]$, cf. Eq. (17)₃, in the open shell with infinite axial length dimension (shell strip). In the second example, the results obtained from the tolerance model will be compared with those derived from the asymptotic model proposed in this contribution.

6.1 Example 1: The effect of a cell size on the initial distributions of temperature micro-fluctuations

6.1.1 Introduction

The object of considerations is a thin cylindrical circular closed shell with r , $L_1 = 2\pi r$, L_2 , d as its midsurface curvature radius, circumferential length, axial length and constant thickness, respectively. It is assumed that $L_2 \geq L_1$. The shell has a periodically inhomogeneous structure in circumferential direction and constant structure in the axial direction. It is

assumed that the shell is made of two homogeneous thermoelastic isotropic materials, which are perfectly bonded on interfaces. Fragment of such a shell is shown in Fig. 2.

We recall that the basic cell Δ is defined by: $\Delta \equiv [-\lambda/2, \lambda/2]$, where λ is a cell length dimension in $x \equiv x^1$ -direction, cf. Figs. 2 and 3. We also recall that for *period length* λ the following conditions hold: $\lambda/d \gg 1$, $\lambda/r \ll 1$ and $\lambda/L_1 \ll 1$. Setting $z \equiv z^1 \in [-\lambda/2, \lambda/2]$, we assume that the cell has a symmetry axis for $z = 0$, cf. Fig. 3. Inside the cell, the geometrical, elastic, inertial and thermal properties of the shell are described by symmetric (i.e. even) functions of argument z .

The influence of a period length λ on the shape of the initial distributions of temperature micro-fluctuations $\Theta(\cdot, t)$ will be studied by applying tolerance model Eqs. (23)–(26).

In order to analyse this problem, we assume that the external forces f^β , f are equal to zero. We neglect the forces of inertia $\langle \mu \rangle a^{\alpha\beta} \ddot{u}_\alpha^0$, $\langle \mu(h)^2 \rangle a^{\alpha\beta} \ddot{U}_\alpha$ in directions tangential to the shell midsurface. At the same time we also neglect terms containing the first time derivatives of macrodisplacements $u_\alpha^0(\cdot, t)$ and of displacement fluctuation amplitudes $U_\alpha(\cdot, t)$ as sufficiently small when compared to terms containing the first time derivatives of kinematical unknowns $w^0(\cdot, t)$, $W(\cdot, t)$.

The investigated problem is rotationally symmetric with a period λ/r ; hence u_1^0 , $U_1 = 0$ and the remaining unknowns of the tolerance model u_2^0 , U_2 , w^0 , W , θ^0 , Θ are independent of x -midsurface parameter. It has to be emphasized that only unknowns mentioned above are independent of x . The total displacements u_2 , w and total temperature field θ in decomposition (17) depend on x because the fluctuation shape functions depend on this argument.

Bearing in mind the symmetric form of a cell $\Delta \equiv [-\lambda/2, \lambda/2]$, we assume that fluctuation shape functions for displacements $h(\cdot) \in FS^1(\Omega, \Delta)$ and for temperature $b(\cdot) \in FS^1(\Omega, \Delta)$ are odd with respect to $z \in [-\lambda/2, \lambda/2]$ whereas fluctuation shape function for displacements $g(\cdot) \in FS^2(\Omega, \Delta)$ is even with respect to z .

We recall that considerations are restricted to uniperiodic shells composed of homogeneous, isotropic constituents. In this case the components of

membrane thermal stiffness tensor $\bar{d}^{2\beta}(x)$ and of heat conduction tensor $K^{2\beta}(x)$, $x \in \Omega$, are: $\bar{d}^{12} = \bar{d}^{21} = 0$, $\bar{d}^{11} = \bar{d}^{22}$ and $K^{12} = K^{21} = 0$, $K^{11} = K^{22}$.

6.1.2 Analytical results

Under assumptions given in Sect. 6.1.1, the system of tolerance model Eqs. (25)–(26) separates into the following system of five equations for $u_2^0(\xi, t)$, $w^0(\xi, t)$, $U_2(\xi, t)$, $W(\xi, t)$, $\theta^0(\xi, t)$, $(\xi, t) \in \Xi \times I$

$$\begin{aligned} &<D^{2222}> \partial_{22}u_2^0 + r^{-1} <D^{2211}> \partial_2 w^0 - <\bar{d}^{22}> \partial_2 \theta^0 = 0, \\ &<B^{2222}> \partial_{2222}w^0 + <B^{2211}\partial_{11}g> \partial_{22}W + <\underline{B^{2222}g}> \partial_{2222}W \\ &\quad + <\mu> \dot{w}^0 = 0, \\ &<(h)^2 D^{2222}> \partial_{22}U_2 - <(\partial_1 h)^2 D^{2212}> U_2 - <\bar{d}^{22}bh> \partial_2 \Theta = 0, \\ &<\partial_{11}gB^{1122}> \partial_{22}w^0 + <gB^{2222}> \partial_{2222}w^0 + <(\partial_{11}g)^2 B^{1111}> W \\ &\quad + (2 <\partial_{11}gB^{1122}g> - 4 <(\partial_{11}g)^2 B^{1212}>) \partial_{22}W \\ &\quad + <(g)^2 B^{2222}> \partial_{2222}W + <\mu(g)^2> \ddot{W} = 0, \\ &<K^{22}> \partial_{22}\theta^0 - <c> \dot{\theta}^0 = r^{-1}T_0 <\bar{d}^{11}> \dot{w}^0, \end{aligned} \tag{40}$$

and independent equation for temperature fluctuation amplitude $\Theta(\xi, t)$, $(\xi, t) \in \Xi \times I$

$$\begin{aligned} &<K^{22}(b)^2> \partial_{22}\Theta - <K^{11}(\partial_1 b)^2> \Theta \\ &\quad - \underline{<c(b)^2>} \dot{\Theta} \\ &= 0. \end{aligned} \tag{41}$$

The underlined averages in (40) and (41) depend on microstructure length parameter λ .

The subsequent analysis will be restricted to Eq. (41) describing micro-fluctuations of temperature field in axial direction caused by periodic structure of the shells under consideration.

Setting $\bar{\Theta}(\bar{\xi}, t) \equiv \Theta(L_2\bar{\xi}, t)$, where $\bar{\xi} \equiv \xi/L_2$, $\bar{\xi} \in [0, 1]$, and denote $K(x) \equiv K^{11}(x) = K^{22}(x)$, $x \in \Omega$, we transform Eq. (41) to the following dimensionless form with respect to dimensionless argument $\bar{\xi}$

$$\begin{aligned} &(L_2)^{-2} <K(b)^2> \partial_{22}\bar{\Theta} - <K(\partial_1 b)^2> \bar{\Theta} \\ &\quad - \underline{<c(b)^2>} \dot{\bar{\Theta}} \\ &= 0. \end{aligned} \tag{42}$$

We shall investigate the problem of time decaying of the temperature fluctuation amplitude $\bar{\Theta}(\bar{\xi}, t)$ setting

$$\bar{\Theta}(\bar{\xi}, t) = \Theta^*(\bar{\xi}) \exp(-\gamma t), \quad t \geq 0, \tag{43}$$

with $\gamma > 0$ as a time decaying coefficient. Function $\Theta^*(\bar{\xi})$ represents an initial distribution of temperature micro-fluctuations, i.e. $\bar{\Theta}(\bar{\xi}, t = 0) = \Theta^*(\bar{\xi})$.

Hence, under denotations

$$\tilde{k}^2 \equiv \frac{(L_2)^2 <K(\partial_1 b)^2>}{\lambda^2 <K(\bar{b})^2>}, \quad \gamma_* \equiv \frac{<K(\partial_1 b)^2>}{\lambda^2 <c(\bar{b})^2>},$$

where $\bar{b}(\cdot) = \lambda^{-1}b(\cdot)$, Eq. (42) yields

$$\partial_{22}\Theta^*(\bar{\xi}) - \tilde{k}^2[1 - (\gamma/\gamma_*)]\Theta^*(\bar{\xi}) = 0, \tag{44}$$

where $\gamma_* > 0$ is a certain new time decaying coefficient depending on microstructure length parameter λ . It can be shown that averages $<K(\partial_1 b)^2>$, $<K(\bar{b})^2>$, $<c(\bar{b})^2>$ are greater than zero; hence $\tilde{k}^2 > 0$ and $\gamma_* > 0$. The boundary conditions for $\Theta^*(\bar{\xi})$ are assumed in the form

$$\Theta^*(\bar{\xi} = 0) = \Theta_0^*, \quad \Theta^*(\bar{\xi} = 1) = 0, \tag{45}$$

where Θ_0^* is the known constant.

The solution to Eq. (44) depends on relations between time decaying coefficients γ and γ_* . The following special cases can be taken into account.

1⁰) If $0 < \gamma < \gamma_*$ and setting $k_\gamma^2 \equiv \tilde{k}^2[1 - (\gamma/\gamma_*)]$ then

$$\begin{aligned} \Theta^*(\bar{\xi}) &= \Theta_0^*[\exp(-\tilde{k}_\gamma \bar{\xi})(1 - \exp(-2\tilde{k}_\gamma))^{-1} \\ &\quad + \exp(\tilde{k}_\gamma \bar{\xi})(1 - \exp(2\tilde{k}_\gamma))^{-1}]; \end{aligned} \tag{46}$$

in this case the initial temperature micro-fluctuations decay exponentially.

2⁰) If $\gamma = \gamma_*$ then

$$\Theta^*(\bar{\xi}) = \Theta_0^*(1 - \bar{\xi}); \tag{47}$$

we deal with a linear decaying of the initial temperature micro-fluctuation amplitude.

4⁰) If $\gamma > \gamma_*$ and setting $\kappa^2 \equiv \tilde{k}^2[(\gamma/\gamma_*) - 1] \neq (n\pi)^2(L_2)^{-2}$ then

$$\Theta^*(\bar{\xi}) = \Theta_0^* \sin(\kappa(1 - \bar{\xi}))(\sin(\kappa))^{-1}; \tag{48}$$

the temperature micro-fluctuations oscillate.

5⁰) If $\gamma > \gamma_*$ and $\kappa^2 \equiv \tilde{k}^2[(\gamma/\gamma_*) - 1] = (n\pi)^2(L_2)^{-2}$ then the solution doesn't exist.

6.1.3 Numerical calculations

The numerical analysis is based on solutions (46)–(48) to Eq. (44).

We recall that thermal properties $K(x) \equiv K^{11}(x) = K^{22}(x)$, $c(x)$, $x \in \Omega$, of the composite shell under consideration are λ -periodic functions in $x \in \Omega$. We assume that thermal properties of the component materials are described by constant heat conduction coefficients K_1, K_2 and constant specific heats c_1, c_2 , cf. Fig. 3. Inside the cell, functions $K(z), c(z)$, $z \in \Delta$, have the form

$$K(z), c(z) = \begin{cases} K_1, c_1 & \text{for } z \in (-\eta\lambda/2, \eta\lambda/2), \\ K_2, c_2 & \text{for } z = [-\lambda/2, -\eta\lambda/2] \cup [\eta\lambda/2, \lambda/2], \end{cases} \tag{49}$$

where $\eta \in [0, 1]$ is a parameter describing distribution of material properties in the cell, cf. Fig. 3.

The fluctuation shape function $b(x) \in FS^1(\Omega, \Delta)$, $x \in \Omega$ describes the expected form of temperature disturbances caused by a periodic structure of the shell. This function has to be λ -periodic in x and must satisfy condition $\langle b(z)c(z) \rangle = 0$, $z \in \Delta$. Note, that for every function being λ -periodic in $x \in \Omega$, we can restrict its domain Ω to cell Δ . On the basis of knowledge of the physically reasonable temperature fluctuations in periodic structures being under boundary conditions similar to those introduced in this paper, cf. [7], in the problem under consideration the fluctuation shape function can be taken as: $b(z) = \lambda \sin(2\pi z/\lambda)$, $z \in \Delta \equiv [-\lambda/2, \lambda/2] \subset \Omega$.

Calculational results based on solutions (46)–(48) to Eq. (44) are shown in Figs. 4, 5, 6 and 7.

The calculations are made for $\eta = 0.4$, for fixed ratio $L_2/L_1 = 2$ and for various ratios $\varepsilon \equiv \lambda/L_1 \in [0.01, 0.1]$, $K_2/K_1 \in [0.001, 1]$. It can be observed that under assumption $L_2/L_1 = 2$, values of ratio $\varepsilon \equiv \lambda/L_1$ imply the following values of ratio λ/L_2 : $\lambda/L_2 = \lambda/(2L_1) = 0.5\varepsilon$. We recall that the problem discussed here deals with a closed circular shell and hence $L_1 = 2\pi r$.

Plots of the exponentially and linearly decaying solutions $\Theta^*(\bar{\xi})$ ($(\gamma/\gamma_*)^2 \leq 1$: $(\gamma/\gamma_*)^2 = 0.1, 0.5, 0.8, 0.9, 1$) versus dimensionless coordinate $\bar{\xi} \equiv \xi/L_2 \in [0, 1]$ are presented in Fig. 4a. Plots for the exponentially decaying solutions $\Theta^*(\bar{\xi})$ ($(\gamma/\gamma_*)^2 < 1$: $(\gamma/\gamma_*)^2 = 0.1, 0.5, 0.8, 0.9$) versus

dimensionless coordinate $\bar{\xi} \equiv \xi/L_2 \in [0, 0.2]$ are shown in Fig. 4b. All diagrams mentioned above are performed for ratios $\lambda/L_1 = 0.1$, $K_2/K_1 = 0.5$.

In Fig. 5 there are diagrams of the oscillating solutions $\Theta^*(\bar{\xi})$ made for $(\gamma/\gamma_*)^2 > 1$: $(\gamma/\gamma_*)^2 = 1.1$ and for $(\gamma/\gamma_*)^2 > 1$: $(\gamma/\gamma_*)^2 = 1.3$ versus dimensionless coordinate $\bar{\xi} \equiv \xi/L_2 \in [0, 1]$. These diagrams are performed for ratios $\lambda/L_1 = 0.1$, $K_2/K_1 = 0.5$.

In Fig. 6 there are diagrams of exponentially decaying solutions $\Theta^*(\bar{\xi})$ ($(\gamma/\gamma_*)^2 < 1$: $(\gamma/\gamma_*)^2 = 0.1, 0.5, 0.8, 0.9$) versus ratio $K_2/K_1 \in [0.001, 1]$ made for $\lambda/L_1 = 0.1$ and $\bar{\xi} = 0.05$.

Plots of exponentially decaying solutions $\Theta^*(\bar{\xi})$ ($(\gamma/\gamma_*)^2 < 1$: $(\gamma/\gamma_*)^2 = 0.1, 0.5, 0.8, 0.9$) versus ratio $\lambda/L_1 \in [0.01, 0.1]$ performed for $K_2/K_1 = 0.5$ and $\bar{\xi} = 0.05$ are presented in Fig. 7.

6.1.4 Discussion of analytical and computational results

On the basis of analytical results (46)–(48) and computational results shown in Figs. 4, 5, 6 and 7, the following conclusions can be formulated:

1. The shape of initial temperature micro-fluctuations $\Theta^*(\bar{\xi})$ depends on relations between the given time decaying coefficient $\gamma > 0$ and a certain time decaying coefficient $\gamma_* > 0$ depending on microstructure length parameter λ . The initial temperature micro-fluctuations decay exponentially for $0 < \gamma < \gamma_*$, cf. Eq. (46) and Fig. 4. They decay linearly for $\gamma = \gamma_*$, cf. Eq. (47) and Fig. 4a. If $\gamma > \gamma_*$ then the temperature micro-fluctuations have non-decayed form; they oscillate, cf. Eq. (48) and Fig. 5.
2. From results shown in Fig. 4a, b, it can be observed that the strongest decaying of function $\Theta^*(\bar{\xi})$ occurs for a very small value of ratio γ/γ_* , i.e. for $\gamma/\gamma_* = 0.1$. In this case, values of $\Theta^*(\bar{\xi})$ can be treated as equal to zero for $\bar{\xi} > 0.06$. It means that for $0 < \gamma < \gamma_*$ the initial temperature micro-fluctuations can be treated as equal to zero outside a very narrow layer near boundary $\xi = 0$. Thus, the tolerance model proposed here enables analysis of the boundary-layer phenomena.
3. Analysing results presented in Fig. 5, we can observe that with increasing values of ratio γ/γ_*

for $\gamma > \gamma_*$, the initial micro-fluctuations $\Theta^*(\bar{\xi})$ stronger oscillate. At the same time, the maximum values of function $\Theta^*(\bar{\xi})$ decrease strongly as the ratio γ/γ_* increases.

4. From results shown in Fig. 6 it follows that values of dimensionless initial temperature micro-fluctuations $\Theta^*(\bar{\xi})$ decrease with the increasing of ratio $K_2/K_1 \in [0.001, 1]$, i.e. with the decreasing of differences between *thermal properties* of the component materials described by constant heat conduction coefficients K_1, K_2 . Because the value of K_1 for the thermally stronger material is fixed then these differences decrease if values of K_2 tend to value of K_1 .
5. Analysing results presented in Fig. 7, it can be seen that values of dimensionless initial temperature micro-fluctuations $\Theta^*(\bar{\xi})$ (exponentially decaying solutions) increase with the increasing of ratio λ/L , i.e. with the decrease of differences between cell size λ and the length dimension L of the shell midsurface in periodicity direction.
6. *The length-scale special problem discussed here can be analysed in the framework of neither the asymptotic model (38) formulated in this contribution nor the known asymptotic models commonly used for investigations of thermoelastic problems for micro-periodically shells under consideration.* It can be observed that within the asymptotic model, after neglecting the length-scale terms, Eq. (42) reduces to equation $\langle K(\partial_1 b)^2 \rangle \bar{\Theta} = 0$, which has only trivial solution $\bar{\Theta} = 0$.
7. Notice that from Eq. (43) it follows that for an arbitrary but fixed time argument t the shape of temperature micro-fluctuation amplitude $\bar{\Theta}(\bar{\xi}, t)$ is the same as the form of initial temperature micro-fluctuation amplitude $\Theta^*(\bar{\xi})$.

6.2 Example 2: The effect of a cell size on the distribution of the total temperature field

The effect of a microstructure size λ on the distribution of total temperature field $\theta(\cdot, t)$ approximated by micro-macro decomposition (17)₃, i.e. $\theta(\cdot, t) = \theta^0(\cdot, t) + b(\cdot)\Theta(\cdot, t)$, $t \in I = [t_0, t_1]$, will be investigated for an open shell with infinite axial length

dimension (shell strip). The shell strip has a micro-periodic structure along circumferential direction and constant structure in the axial direction. Examples of shells with an uniperiodic structure are shown in Figs. 1 and 2, where now axial dimension L_2 is treated as infinite.

Considerations will be restricted to the heat conduction problem independent of ξ -coordinate. We assume that the uniperiodic shell strip is composed of homogeneous, isotropic constituents. We also assume that temperature fluctuation shape function $b(\cdot) \in FS^1(\Omega, \Delta)$ is odd with respect to $z \in \Delta \equiv [-\lambda/2, \lambda/2]$ (the cell has a symmetry axis for $z = 0$) and that $b(x = 0) = b(x = L) = 0$, where $L \equiv L_1$.

Now, the heat conduction Eq. (26) reduce to the form

$$\begin{aligned} \langle K^{11} \rangle \partial_{11} \theta^0 + \langle K^{11} \partial_1 b \rangle \partial_1 \Theta - \langle c \rangle \dot{\theta}^0 &= 0, \\ \langle K^{11} \partial_1 b \rangle \partial_1 \theta^0 + \langle K^{11} (\partial_1 b)^2 \rangle \Theta + \langle c(b)^2 \rangle \dot{\Theta} &= 0. \end{aligned} \tag{50}$$

The micro-macro decomposition (17)₃ has now the form

$$\theta(x, t) = \theta^0(x, t) + b(x)\Theta(x, t), \quad (x, t) \in \Omega \times I, \tag{51}$$

where θ^0 and Θ are governed by Eq. (50).

Let the shell be subjected to the initial temperature distribution given by $\theta(x, t = 0) = \tilde{a}x + \tilde{b}$ and the constant temperature distribution given by $\theta(x = 0, t) = \alpha_0$, $\theta(x = L, t) = \alpha_L$. Under assumption $b(x = 0) = b(x = L) = 0$, from (51) it follows that the boundary conditions for averaged temperature θ^0 are: $\theta^0(x = 0, t) = \alpha_0$, $\theta^0(x = L, t) = \alpha_L$. The initial conditions have the form: $\theta^0(x, t = 0) = \tilde{a}x + \tilde{b}$, $\Theta(x, t = 0) = 0$. The solution to Eq. (50), satisfying the aforementioned boundary and initial conditions, exists provided that $\tilde{a} = (\alpha_L - \alpha_0)/L$ and $\tilde{b} = \alpha_0$. Under denotations

$$\tau \equiv \frac{\langle c(b)^2 \rangle}{\langle K^{11} (\partial_1 b)^2 \rangle}, \quad \kappa \equiv \frac{\langle K^{11} \partial_1 b \rangle}{\langle K^{11} (\partial_1 b)^2 \rangle}, \tag{52}$$

where $\tau > 0$ is a certain time parameter depending on a cell size λ , this solution is given by

$$\begin{aligned} \theta^0(x, t) &= \tilde{a}x + \tilde{b}, \\ \Theta(x, t) &= \kappa \tilde{a} \left(\exp\left(-\frac{t}{\tau}\right) - 1 \right). \end{aligned} \tag{53}$$

It follows that the distribution of temperature in the shell strip considered here can be approximated by

$$\theta(x, t) = \tilde{a}x + \tilde{b} - \kappa \tilde{a} \left(\exp\left(-\frac{t}{\tau}\right) - 1 \right) b(x). \tag{54}$$

Let us compare result (54) with the result from the asymptotic model. This model can be directly derived from Eq. (50) by neglecting underlined term involving the microstructure length parameter λ

$$\begin{aligned} \langle K^{11} \rangle &> \partial_{11}\theta^0 + \langle K^{11}\partial_1 b \rangle > \partial_1\Theta - \langle c \rangle \dot{\theta}^0 = 0, \\ \langle K^{11}\partial_1 b \rangle &> \partial_1\theta^0 + \langle K^{11}(\partial_1 b)^2 \rangle > \Theta = 0. \end{aligned} \tag{55}$$

For asymptotic model Eq. (55) we obtain

$$\theta(x, t) = \tilde{a}x + \tilde{b} + \kappa \tilde{a} b(x). \tag{56}$$

It is easy to see that if $t \rightarrow \infty$ then the solution (54) obtained in the framework of tolerance model (50) tends to the solution (56) derived from asymptotic model (55). Thus the effect of a period length λ on the distribution of temperature is significant only for small values of t and can be neglected for $t \gg \tau$. This effect is also proportional to $\langle K^{11}\partial_1 b \rangle \tilde{a}$, being caused simultaneously by the heterogeneity of shell material described by $K^{11}(x)$ and the gradient $\tilde{a} = (\alpha_L - \alpha_0)/L$ of the averaged temperature field θ^0 .

7 Remarks and conclusions

The objects of analysis are thin linearly thermoelastic Kirchhoff–Love-type circular cylindrical shells having a periodically micro-heterogeneous structure in circumferential direction (uniperiodic shells), cf. Figs. 1 and 2.

Considerations are based on the known Kirchhoff–Love theory of elasticity combined with Duhamel–Neumann thermoelastic constitutive relations and on Fourier’s theory of heat conduction. The exact shell Eqs. (4) and (5) describing the dynamic thermoelastic problems for the shells considered in this contribution have highly oscillating, non-continuous and periodic coefficients.

The main aim of this paper is to formulate and discuss a new mathematical non-asymptotic averaged model of thermoelastic problems for the periodic shells under consideration. In order to do it, the tolerance modelling technique [5–8] and a certain extension of the known stationary action principle [7] are applied. The tolerance model derived here is represented by the stress–strain–temperature relations (23), (24) and the dynamic equilibrium Eq. (25) coupled with the heat conduction Eq. (26). The tolerance model equations have constant coefficients depending also on a cell size. Hence, this model makes it possible to analyse the effect of a period length on the global thermodynamic shell behaviour (the length-scale effect). Solutions to the initial-boundary value problems have the physical sense only if the basic kinematic and thermal unknowns of the tolerance model are slowly-varying functions in periodicity direction. This requirement can be verified only a posteriori and it determines the range of the physical applicability of the model.

The second aim is to formulate a certain asymptotic model of dynamic thermoelasticity problems for the shells under consideration. As a tool of modelling we shall apply the consistent asymptotic approach [7, 8] and extended stationary action principle. Governing Eq. (38) of the asymptotic model have constant coefficients being independent on a microstructure size.

Both the tolerance and asymptotic models are uniquely determined by the periodic, highly oscillating fluctuation shape functions representing disturbances of temperature and displacement fields inside a cell. These functions must be known in every considered problem.

As illustrative examples, certain special length-scale problems were discussed. The first of them dealt with time decaying of initial fluctuations of temperature field in the uniperiodic closed shell. It was analysed in the framework of the proposed tolerance model (23)–(26). It was shown that in the uniperiodic shells under consideration the form of initial temperature micro-fluctuations depends on relations between the given time decaying coefficient $\gamma > 0$ and a certain time decaying coefficient $\gamma_* > 0$ depending on microstructure length parameter λ . The initial temperature micro-fluctuations can decay exponentially. They can decay linearly. For a certain relation between γ and γ_* , the temperature micro-fluctuations

have *non-decayed form*; they *oscillate*. Moreover, if $0 < \gamma < \gamma_*$ then the micro-fluctuations are strongly decaying near the boundary $\xi = 0$ and can be treated as equal to zero outside a certain narrow layer near this boundary. Thus, it has been shown that the tolerance model proposed here enables analysis of the *boundary-layer phenomena*. All the effects mentioned above cannot be investigated in the framework of the asymptotic models.

The second length-scale problem dealt with the effect of a microstructure size λ on the distribution of total temperature field $\theta(\cdot, t)$ in an open uniperiodic shell with infinite axial length dimension (shell strip). The heat conduction Eq. (26) of the tolerance model were applied. The result obtained from the tolerance model was compared with that derived from the asymptotic one. It was shown that the length-scale effect is significant only for small values of argument t and can be neglected for $t > \tau$, where τ is a certain time parameter depending on a cell size λ .

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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