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Super-exponential distinguishability of correlated quantum states

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Abstract

In the problem of asymptotic binary i.i.d. state discrimination, the optimal asymptotics of the type I and the type II error probabilities is in general an exponential decrease to zero as a function of the number of samples; the set of achievable exponent pairs is characterized by the quantum Hoeffding bound theorem. A super-exponential decrease for both types of error probabilities is only possible in the trivial case when the two states are orthogonal and hence can be perfectly distinguished using only a single copy of the system. In this paper, we show that a qualitatively different behavior can occur when there is correlation between the samples. Namely, we use gauge-invariant and translation-invariant quasi-free states on the algebra of the canonical anti-commutation relations to exhibit pairs of states on an infinite spin chain with the properties that (a) all finite-size restrictions of the states have invertible density operators and (b) the type I and the type II error probabilities both decrease to zero at least with the speed $e^{-nc\log n}$ with some positive constant c, i.e., with a super-exponential speed in the sample size n. Particular examples of such states include the ground states of the XX model corresponding to different transverse magnetic fields. In fact, we prove our result in the setting of binary composite hypothesis testing, and hence, it can be applied to prove super-exponential distinguishability of the hypotheses that the

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transverse magnetic field is above a certain threshold vs. that it is below a strictly lower value.

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1 Introduction

In the problem of simple binary state discrimination, an experimenter is presented with a quantum system that is either in some state $\omega^{(0)}$ or in another state $\omega^{(1)}$. The experimenter's task is to guess which one the true state of the system is, based on measurements on the system. It is easy to see that even the most elaborate measurement and classical post-processing scheme cannot outperform single 2-outcome (binary) measurements when the goal is to minimize the probability of an erroneous decision. More precisely, there are two types of error probabilities to consider: erroneously identifying the state as $\omega^{(1)}$ (type I error), or erroneously identifying the state as $\omega^{(0)}$ (type II error), and the goal is to minimize some combination of the two. It is easy to see that (in the finite-dimensional case, at least), perfect discrimination (i.e., when both error probabilities are zero) is possible if and only if the density operators of the two states have orthogonal supports.

The error probabilities can be reduced if the experimenter has access to multiple identical copies of the system, and in the asymptotic analysis of the problem one is interested in the achievable asymptotic behaviors of the two error probabilities along all possible sequences of binary measurements (tests) as the number of copies tends to infinity. In general, the best achievable asymptotics is an exponential decrease to zero for both error probabilities; the set of the achievable exponent pairs is described by the quantum Hoeffding bound theorem [4, 12, 26]. Faster (super-exponential) decrease is possible if and only if the supports of the states are different. For instance, if $\operatorname{supp}\omega^{(1)} \not\subseteq \operatorname{supp}\omega^{(0)}$, then there exists a test sequence along which the type I error is constant zero (hence its exponent is $+\infty$), while the type II error decreases exponentially fast (with the exponent being the Rényi zero-divergence of $\omega^{(0)}$ and $\omega^{(1)}$). A faster than exponential decrease for both error probabilities is possible if and only if the supports are orthogonal, in which case both errors can be made zero trivially for any finite number of copies.

The above are well-known in the i.i.d. (independent and identically distributed) case, i.e., when all the samples are prepared in the same state, and there is no correlation between the different samples. Correlated scenarios can be conveniently described using the concept of the C^*-algebra of an infinite spin chain, $\mathcal{C}_{\mathbb{Z}} = \otimes_{k\in\mathbb{Z}}\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional Hilbert space describing a single system. In this case, the candidate states $\omega^{(0)}$ and $\omega^{(1)}$ can be described by positive linear functionals on $\mathcal{C}_{\mathbb{Z}}$ that take 1 on the identity; their restrictions to any subalgebra $\otimes_{k\in\Lambda}\mathcal{B}(\mathcal{H})$ corresponding to a finite subset Λ of samples (equivalently, a finite part of the chain) can be described by density operators in the usual way. A state on the infinite chain is translation-invariant if the density operator of any finite subsystem Λ is the same as that of any of its translates; in particular, the single-site density operators are all the same (i.e., the outcomes of the same measurement performed at different sites are identically distributed). In this picture, a measurement on n consecutive samples is described by a measurement on a length n part of the chain, and the asymptotics is studied in the setting where this length is allowed to go to infinity. Obviously, error exponents are more difficult to determine in the correlated scenario, but rather general results are available in the setting of Stein's lemma, where one of the errors is not required to decrease exponentially [8], and in the setting of the Hoeffding bound for thermal states of translation-invariant finite-range Hamiltonians, and more generally, for states that satisfy a certain factorization property [17]. In these cases, however, the entropic quantities (Umegaki- and quantum Rényi relative entropies) characterizing the achievable exponent pairs are given by regularized formulas and cannot be explicitly computed in general.

A particular class of correlated states where explicit formulas are available can be obtained from translation-invariant and gauge-invariant quasi-free states on the algebra of canonical anti-commutation relations (CAR algebra). Such a state is specified by a measurable function on $[0, 2\pi)$ with values in $[0, 1]$, called the symbol of the state; see Sect. 2 for details. The achievable exponent pairs were determined for a pair of such states in [25], with explicit expressions for the relevant entropic quantities, in the case where the symbols of the two states, denoted by $\hat{q}$ and $\hat{r}$, are bounded away from 0 and 1 in the sense that for some $\eta > 0$, $\eta \le \hat{q}(x), \hat{r}(x) \le 1 - \eta$ for all $x \in [0, 2\pi)$. In this case, the regularized quantum Rényi α-divergences of the two states are finite for every $\alpha > 0$, and the best achievable asymptotics is an exponential decay for both error probabilities.

Our main contribution in this paper is showing that for certain pairs of quasi-free states, super-exponential discrimination is possible. More precisely, we show that if the symbols $\hat{q}$ and $\hat{r}$ are such that there exists a non-degenerate interval on which $\hat{q}$ is constant 0 and $\hat{r}$ is constant 1, then there exists a sequence of tests along which both error probabilities decrease at least with the speed $e^{-nc\log n}$, where n is the sample size. In the same time, unless $\hat{q}$ is constant zero and $\hat{r}$ is constant 1 (up to sets of measure zero), then all the local densities of both states are invertible, and hence, it is not only impossible to make both error probabilities vanish for a finite sample size, but if one of the error probabilities is made zero, then the other is necessarily equal to 1. This is very different from what can be seen in the i.i.d. case, and to the best of our knowledge, this is the first time that such a behavior is presented in the literature.

The structure of the paper is as follows: In Sect. 2.1, we review the necessary basics about quasi-free states on the CAR algebra. In Sect. 2.2, we explain the notions of error exponents and super-exponential distinguishability for translation-invariant states on the spin chain and on the CAR algebra. In Sect. 3, we prove our main result described above. In fact, we state and prove a more general result in the framework of composite state discrimination, showing super-exponential distinguishability of two sets of quasi-free states with invertible local density operators. In Sect. 4, we give various characterizations of super-exponential distinguishability of states in terms of regularized divergences.

2 Preliminaries

2.1 Quasi-free states on the CAR algebra

Here we summarize the necessary basics about quasi-free states on the CAR algebra. For more details and proofs, we refer to [1, 9, 10, 30, 33].

For a complex Hilbert space $\mathcal{H}$, we will denote the set of bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ and use the notation $\mathbb{T}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : 0 \leq T \leq I\}$ for the set of *tests* on $\mathcal{H}$.

For vectors $\varphi_1, \ldots, \varphi_k$ in a complex Hilbert space $\mathcal{H}$, let

$$\varphi_1 \wedge \ldots \wedge \varphi_k := \frac{1}{\sqrt{k!}} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma)\varphi_{\sigma(1)} \otimes \ldots \otimes \varphi_{\sigma(k)}$$

denote their anti-symmetrized tensor product, where $\mathfrak{S}_k$ stands for the set of permutations of k elements and $\varepsilon(\sigma)$ for the sign of the permutation σ. For any $k \in \mathbb{N} \setminus \{0\}$, the k-th anti-symmetric tensor power of $\mathcal{H}$ is

$$\mathcal{H}^{\wedge k} := \overline{\operatorname{span}\{\varphi_1 \wedge \ldots \wedge \varphi_k : \varphi_i \in \mathcal{H}, \, i = 1, \ldots, k\}},$$

where the overline denotes the closure, and we define $\mathcal{H}^{\wedge 0} := \mathbb{C}$. The Hilbert space of a fermionic system with single-particle Hilbert space $\mathcal{H}$ is the *anti-symmetric Fock space*

$$\Gamma(\mathcal{H}) := \bigoplus_{k \in \mathbb{N}} \mathcal{H}^{\wedge k},$$

where $\mathcal{H}^{\wedge k} = \{0\}$ for every $k > \dim \mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, let $A^{\wedge k} := A^{\otimes k}\big|_{\mathcal{H}_1^{\wedge k}}$ as an operator in $\mathcal{B}(\mathcal{H}_1^{\wedge k}, \mathcal{H}_2^{\wedge k})$, and

$$A_F := \bigoplus_{k \in \mathbb{N}} A^{\wedge k},$$

with $A^{\wedge 0} := A^{\otimes 0} := 1 \in \mathcal{B}(\mathbb{C})$. Clearly, A_F is bounded if and only if $\min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\} < +\infty$ or $\|A\| \leq 1$. If $V : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometry/unitary, then V_F is an isometry/unitary from $\Gamma(\mathcal{H}_1)$ into/onto $\Gamma(\mathcal{H}_2)$ with the property $V_F \mathcal{H}_1^{\wedge k} \subseteq \mathcal{H}_2^{\wedge k}$.

For each $\varphi \in \mathcal{H}$, the corresponding *creation operator* $c(\varphi)$ is the unique bounded linear extension of the map

$$\varphi_1 \wedge \ldots \wedge \varphi_k \mapsto \varphi \wedge \varphi_1 \wedge \ldots \wedge \varphi_k, \qquad \varphi_1, \ldots, \varphi_k \in \mathcal{H},$$

and the corresponding *annihilation operator* is its adjoint, $a(\varphi) := c(\varphi)^*$. These operators satisfy the *canonical anti-commutation relations (CARs)*,

$$\{a(\varphi), a(\psi)\} = 0, \qquad \left\{a(\varphi), a^*(\psi)\right\} = \langle \varphi, \psi \rangle \, I, \quad \varphi, \psi \in \mathcal{H}. \tag{2.1}$$

The C*-algebra generated by $\{a(\varphi) : \varphi \in \mathcal{H}\}$ is called the *algebra of the canonical anti-commutation relations* (or *CAR-algebra*) corresponding to the single-particle Hilbert space $\mathcal{H}$ and is denoted by CAR $(\mathcal{H})$. Note that $\varphi \mapsto c(\varphi)$ is complex linear and $\varphi \mapsto a(\varphi)$ is complex anti-linear. Thus, if $\mathcal{H}$ is separable and $(e_i)_{i=1}^{\dim \mathcal{H}}$ is an orthonormal basis (ONB) in it, then CAR $(\mathcal{H})$ is the closure of the linear span of the identity and all the multinomials of the form $a(e_{i_1})^* \dots a(e_{i_n})^* a(e_{j_m}) \dots a(e_{j_1})$, $i_1 < \dots < i_n$, $j_1 < \dots < j_m$. For any isometry/unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$, $V_F(\cdot)V_F^*$ is a homomorphism/isomorphism from CAR $(\mathcal{H}_1)$ to CAR $(\mathcal{H}_2)$ with the property $V_F a(\varphi) V_F^* = a(V\varphi)$, $\varphi \in \mathcal{H}_1$. The *even part* CAR $(\mathcal{H})_+$ of CAR $(\mathcal{H})$ is the subalgebra left invariant by the *parity automorphism* $\pi(\cdot) := (-I)_F(\cdot)(-I)_F$. This is exactly the closure of the linear span of all multinomials with an even number of terms (see Appendix 1 for more details).

If $\mathcal{H}$ is finite-dimensional and $e_1, \dots, e_d$ is an orthonormal basis in $\mathcal{H}$, then $\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \le i_1 < \dots < i_k \le d\}$ is an ONB in $\mathcal{H}^{\wedge k}$; in particular, $\dim \mathcal{H}^{\wedge k} = \binom{d}{k}$ and $\dim \mathcal{F}(\mathcal{H}) = 2^{\dim \mathcal{H}}$. Let $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ be the canonical ONB of $\mathbb{C}^2$. Then,

$$U_e : e_{i_1} \wedge \dots \wedge e_{i_k} \mapsto \otimes_{j=1}^d |x_j\rangle, \quad x_j := \begin{cases} 1, & j \in \{i_1, \dots, i_k\}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

is a unitary from $\Gamma(\mathcal{H})$ to $(\mathbb{C}^2)^{\otimes d}$, and it is easy to verify that

$$U_e a(e_j)^* U_e^* = \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{j-1 \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \underbrace{I \otimes \dots \otimes I}_{d-j \text{ times}}, \tag{2.3}$$

where $\sigma_3 := \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the Pauli z operator. The map $U_e(\cdot)U_e^* : \mathcal{B}(\mathcal{F}(\mathcal{H})) = \text{CAR}\,(\mathcal{H}) \to \mathcal{B}((\mathbb{C}^2)^{\otimes d}) = \mathcal{B}(\mathbb{C}^2)^{\otimes d}$ is called the *Jordan–Wigner isomorphism* corresponding to the given ONB. The *particle number operator* is

$$\begin{aligned} N_{\mathcal{H}} &:= \bigoplus_{k=0}^{d} k I_{\mathcal{H}^{\wedge k}} = \sum_{i=1}^{d} a(e_i)^* a(e_i) \\ &= U_e^* \left(\sum_{i=1}^{d} \underbrace{I \otimes \dots \otimes I}_{i-1 \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \underbrace{I \otimes \dots \otimes I}_{d-i \text{ times}} \right) U_e. \end{aligned}$$

The eigen-values of $N_{\mathcal{H}}$ are $0, \dots, d$, with spectral projections

$$P_k^{N_{\mathcal{H}}} = \underbrace{0 \oplus \dots \oplus 0}_{k \text{ times}} \oplus I_{\mathcal{H}^{\wedge k}} \oplus \underbrace{0 \oplus \dots \oplus 0}_{d-k \text{ times}}$$

$$= U_e^* \left(\sum_{\Lambda \subseteq [d],\, |\Lambda|=k} \left(\bigotimes_{i\in\Lambda} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \left(\bigotimes_{i\in[d]\setminus\Lambda} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) U_e, \tag{2.4}$$

where $[d] := \{1, \ldots, d\}$. Note that $N_{\mathcal{H}}$ is defined in a basis-independent way, and the equalities above are valid for any ONB.

A *state* on CAR $(\mathcal{H})$ is a positive linear functional that takes the value 1 on I. For any positive semi-definite (PSD) operator $Q \in \mathcal{B}(\mathcal{H})$ with $Q \le I$, there exists a unique state ω_Q on CAR $(\mathcal{H})$ (called the *gauge-invariant quasi-free state with symbol* Q) with the property

$$\omega_Q \left(a(\varphi_1)^* \ldots a(\varphi_n)^* a(\psi_m) \ldots a(\psi_1) \right) = \delta_{mn} \det \left\{ \langle \psi_i | Q\varphi_j \rangle \right\}_{i,j=1}^n . \tag{2.5}$$

It is easy to verify that when $\mathcal{H}$ is finite-dimensional, the density operator $\widehat{\omega}_Q$ of ω_Q can be explicitly given as:

$$\widehat{\omega}_Q = \prod_{j=1}^d \left(q_j a(e_j)^* a(e_j) + (1-q_j) a(e_j) a(e_j)^* \right) = U_e^* \left(\bigotimes_{j=1}^d \begin{bmatrix} 1-q_j & 0 \\ 0 & q_j \end{bmatrix} \right) U_e, \tag{2.6}$$

where $Q = \sum_{j=1}^d q_j \left| e_j \middle\rangle\middle\langle e_j \right|$ is any eigen-decomposition of Q, and U_e is the unitary corresponding to the ONB $(e_j)_{j=1}^d$ as in (2.2). Note that for all $1 \le i_1 < \ldots < i_k \le d$, $e_{i_1} \wedge \ldots \wedge e_{i_k}$ is an eigen-vector of $\widehat{\omega}_Q$ with eigen-value $\left(\prod_{j\in\{i_1,\ldots,i_k\}} q_j \right) \cdot \left(\prod_{j\in[d]\setminus\{i_1,\ldots,i_k\}} (1-q_j) \right)$. This implies immediately that if 1 is not an eigen-value of Q then $\widehat{\omega}_Q$ can be written as

$$\widehat{\omega}_Q = \det(I - Q) \left(\frac{Q}{I - Q} \right)_F .$$

Quasi-free states emerge as equilibrium states of non-interacting fermionic systems. For instance, if the single-particle Hamiltonian H of a system of non-interacting fermions is such that $e^{-\beta H}$ is trace-class then the Gibbs state of the system at inverse temperature β is the quasi-free state with symbol $Q = \frac{e^{-\beta H}}{I + e^{-\beta H}}$ (see, e.g., [30, Proposition 5.2.23]).

Consider now a fermionic chain with a single mode at each site. The single-particle Hilbert space of this system is $\mathcal{H} = \ell^2(\mathbb{Z})$, the standard basis of which we denote by $\{\mathbf{1}_{\{k\}} : k \in \mathbb{Z}\}$. The *translation operator* is the unitary $U^{\text{trans}} = \sum_{k\in\mathbb{Z}} \left| \mathbf{1}_{\{k+1\}} \middle\rangle\middle\langle \mathbf{1}_{\{k\}} \right|$, and $\tau(\cdot) := U_F^{\text{trans}}(\cdot)(U^{\text{trans}})_F^*$ gives an automorphism of CAR $\left(\ell^2(\mathbb{Z})\right)$ with the property $\tau(a(\varphi)) = a(U^{\text{trans}}\varphi)$, $\varphi \in \mathcal{H}$. A quasi-free state ω_Q is called *translation-invariant* if $\omega_Q \circ \tau = \omega_Q$, which is easily seen to be equivalent to $U^{\text{trans}} Q = Q U^{\text{trans}}$, i.e., the translation-invariance of the symbol Q. For instance, in the above example a translation-invariant single-particle Hamiltonian H yields a translation-invariant quasi-free state as the equilibrium state of the system. Translation-invariant operators on $\ell^2(\mathbb{Z})$ commute with each other, and they are simultaneously diagonalized by the Fourier transformation:

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2([0, 2\pi)), \quad \mathcal{F}\,\mathbf{1}_{\{k\}} := \chi_k, \quad \chi_k(x) := \frac{e^{ikx}}{\sqrt{2\pi}}, \quad x \in [0, 2\pi), \quad k \in \mathbb{Z}.$$

That is, every translation-invariant operator A arises in the form $A = \mathcal{F}^* M_{\hat{a}} \mathcal{F}$, where $M_{\hat{a}}$ denotes the multiplication operator by a bounded measurable function $\hat{a}$ on $[0, 2\pi)$. As a consequence, the matrix entries of translation-invariant operators in the canonical ONB are constants along diagonals; more explicitly, for any translation-invariant operator $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$,

$$A_{k,j} := \left\langle \mathbf{1}_{\{k\}}, A\mathbf{1}_{\{j\}} \right\rangle = \frac{1}{2\pi} \int_{[0,2\pi)} e^{-i(k-j)x} \hat{a}(x)\, \mathrm{d}x, \qquad k, j \in \mathbb{Z}. \tag{2.7}$$

A measurement on a subsystem corresponding to modes at the sites $\langle n \rangle := \{0, \ldots, n-1\}$ has measurement operators in the C^*-subalgebra $\mathcal{A}_n \subseteq \mathrm{CAR}\left(\ell^2(\mathbb{Z})\right)$ generated by $\{a(\varphi) : \varphi \in \mathcal{H}_n\}$,

$$\mathcal{H}_n := \mathrm{span}\{\mathbf{1}_{\{k\}} : k \in \langle n \rangle\} \subseteq \ell^2(\mathbb{Z}).$$

This subalgebra is naturally isomorphic to $\mathrm{CAR}\left(\mathbb{C}^{\langle n \rangle}\right)$. It is easy to see that if the state of the infinite chain is given by a quasi-free state with symbol Q, then the statistics of any such local measurement is given by the quasi-free state ω_{Q_n} on $\mathrm{CAR}\left(\mathbb{C}^{\langle n \rangle}\right)$ with symbol $Q_n := V_n^* Q V_n$, where V_n is the natural embedding of $\mathbb{C}^{\langle n \rangle}$ into $\ell^2(\mathbb{Z})$.

Lemma 2.1 *Let $\hat{a} : [0, 2\pi) \to [0, +\infty)$ be a nonnegative bounded measurable function, and let $A = \mathcal{F}^* M_{\hat{a}} \mathcal{F}$ be the corresponding translation-invariant operator on $\ell^2(\mathbb{Z})$. The following are equivalent:*

(i) 0 is an eigen-value of $V_n^* A V_n$ for some $n \in \mathbb{N}$;
(ii) 0 is an eigen-value of $V_n^* A V_n$ for every $n \in \mathbb{N}$;
(iii) $A = 0$;
(iv) $\hat{a}$ is equal to 0 almost everywhere.

Proof The equivalence (iv) $\iff$ (iii) is obvious, as are the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), and hence, we only need to prove (i) $\Longrightarrow$ (iv). Assume therefore that $V_n^* A V_n \psi = 0$ for some $\psi \in \mathbb{C}^{\langle n \rangle} \setminus \{0\}$. Then

$$\begin{aligned} 0 = \left\langle \psi, V_n^* A V_n \psi \right\rangle &= \|A^{1/2} V_n \psi\|^2 = \|\mathcal{F} A^{1/2} \mathcal{F}^* \mathcal{F} V_n \psi\|^2 = \|M_{\hat{a}^{1/2}} \mathcal{F} V_n \psi\|^2 \\ &= \|\hat{a}^{1/2} \mathcal{F} V_n \psi\|^2, \end{aligned}$$

whence $\hat{a}^{1/2} \mathcal{F} V_n \psi = 0$ almost everywhere. Since $\mathcal{F} V_n \psi$ is a nonzero trigonometric polynomial that can only have finitely many zeros, this implies that $\hat{a}$ is 0 almost everywhere. $\square$

Corollary 2.2 *Let $Q = \mathcal{F}^* M_{\hat{q}} \mathcal{F} \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be the symbol of a translation-invariant quasi-free state. If $\hat{q}$ is neither almost everywhere zero nor almost everywhere* 1*, then for every $n \in \mathbb{N}$, $\widehat{\omega}_{Q_n}$ is an invertible density operator on $\Gamma(\mathbb{C}^{\langle n \rangle})$.*

Proof Applying Lemma 2.1 to $\hat{a} := \hat{q}$ yields that 0 is not an eigen-value of Q_n for any $n \in \mathbb{N}$. Applying Lemma 2.1 to $\hat{a} := 1 - \hat{q}$ yields that 1 is not an eigen-value of Q_n, either, for any $n \in \mathbb{N}$. Thus, the assertion follows from (2.6). □

Finally, a symbol Q on $\mathbb{C}^{\langle n \rangle}$ is translation-invariant (or rotation-invariant), if it commutes with the n-dimensional translation unitary $U_n^{\text{trans}} = \sum_{k=0}^{n-1} \left|\mathbf{1}_{\{k+1\}}\right\rangle\!\left\langle\mathbf{1}_{\{k\}}\right|$, where the addition is modulo n. Such operators are also called *circular*, and are simultaneously diagonalized by the *n-dimensional discrete Fourier transformation*

$$\mathcal{F}_n : \mathbb{C}^{\langle n \rangle} \to \mathbb{C}^{\langle n \rangle}, \qquad \mathcal{F}_n \mathbf{1}_{\{k\}} := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{n} kj} \mathbf{1}_{\{j\}}, \qquad k \in \langle n \rangle. \tag{2.8}$$

That is, $U_n^{\text{trans}} Q = Q U_n^{\text{trans}}$, $0 \le Q \le I$, if and only if $Q = \mathcal{F}_n^* M_{\hat{q}} \mathcal{F}_n$ for some $\hat{q} \in [0, 1]^{\langle n \rangle}$.

2.2 Asymptotic binary state discrimination

For a finite-dimensional Hilbert space $\mathcal{H}$ and every $\Lambda \subseteq \mathbb{Z}$, let $\mathcal{C}_\Lambda(\mathcal{H}) := \otimes_{i \in \Lambda} \mathcal{B}(\mathcal{H})$. Every $A \in \mathcal{C}_\Lambda(\mathcal{H})$ is naturally identified with $A \otimes \left(\otimes_{i \in \Lambda' \setminus \Lambda} I\right) \in \mathcal{C}_{\Lambda'}(\mathcal{H})$ for $\Lambda' \supseteq \Lambda$, which gives an equivalence relation $\sim$ on $\cup_{\Lambda \subseteq \mathbb{Z} \text{ finite}} \mathcal{C}_\Lambda(\mathcal{H})$. Then, the operator norms on the individual $\mathcal{C}_\Lambda(\mathcal{H})$ naturally define a norm on

$$\left(\bigcup_{\Lambda \subseteq \mathbb{Z} \text{ finite}} \mathcal{C}_\Lambda(\mathcal{H}) \right) \Big/ _{\sim} \tag{2.9}$$

which satisfies the C^*-identity $\|X^* X\| = \|X\|^2$. The completion of (2.9) with respect to this norm gives a C^*-algebra called the *infinite spin chain algebra* with single-site finite-dimensional Hilbert space $\mathcal{H}$, which we denote by

$$\mathcal{C}_{\mathbb{Z}}(\mathcal{H}) := \otimes_{k \in \mathbb{Z}} \mathcal{B}(\mathcal{H}).$$

A translation-invariant state ω on the infinite spin chain is specified by density operators ω_Λ in $\mathcal{C}_\Lambda(\mathcal{H})$ such that $\mathrm{Tr}_{\Lambda' \setminus \Lambda}\, \omega_{\Lambda'} = \omega_\Lambda$ and $\omega_{\Lambda + k} = \omega_\Lambda$ for any finite $\Lambda \subseteq \mathbb{Z}$ and $k \in \mathbb{Z}$. Equivalently, ω is a positive linear functional on the C^*-algebra $\mathcal{C}_{\mathbb{Z}}(\mathcal{H})$, with $\omega(I) = 1$, such that for the translation automorphism τ we have $\omega \circ \tau = \omega$, and the ω_Λ are the density operators of its restrictions onto $\mathcal{C}_\Lambda(\mathcal{H})$.

Given two sets of translation-invariant states $\Omega^{(0)} = \{\omega^{(0,i)}\}_{i \in \mathcal{I}}$ and $\Omega^{(1)} = \{\omega^{(1,j)}\}_{j \in \mathcal{J}}$, a state discrimination protocol of sample size n to decide if the true state of the system belongs to $\Omega^{(0)}$ (null-hypothesis H_0) or to $\Omega^{(1)}$ (alternative hypothesis H_1), is specified by a *test* $T_n \in \mathcal{C}_{[1,n]}(\mathcal{H})$ with $0 \le T_n \le I$, representing a measurement with outcomes 0 and 1, with corresponding measurement operators T_n and

$I - T_n$, respectively. If the outcome of the measurement is k, the experimenter accepts hypothesis H_k to be true. The (worst-case) *type I error probability* of incorrectly rejecting H_0, and the *type II error probability* of incorrectly accepting H_0, respectively, are given by:

$$\alpha_n(T_n) := \sup_{i\in\mathcal{I}} \operatorname{Tr} \omega^{(0,i)}_{[1,n]}(I - T_n), \qquad \beta_n(T_n) := \sup_{j\in\mathcal{J}} \operatorname{Tr} \omega^{(1,j)}_{[1,n]} T_n.$$

A test T_n is *projective*, if $T_n^2 = T_n$. Given a sequence of tests $\vec{T} = (T_n)_{n\in\mathbb{N}}$, with $T_n \in \mathcal{C}_{[1,n]}(\mathcal{H})$, $n \in \mathbb{N}$, the corresponding *type I and type II error exponents* are defined, respectively, as

$$\alpha^{\exp}(\vec{T}) := \liminf_{n\to+\infty} -\frac{1}{n}\log \alpha_n(T_n), \qquad \beta^{\exp}(\vec{T}) := \liminf_{n\to+\infty} -\frac{1}{n}\log \beta_n(T_n). \quad (2.10)$$

We say that $\Omega^{(0)}$ and $\Omega^{(1)}$ can be *super-exponentially distinguished*, if there exists a test sequence $\vec{T}$ along which $\alpha^{\exp}(\vec{T}) = +\infty = \beta^{\exp}(\vec{T})$.

As it was shown in [2] (see also [23, Section 5.3] for a detailed exposition), every translation-invariant gauge-invariant quasi-free state ω on CAR $\left(\ell^2(\mathbb{Z})\right)$ can be mapped into a translation-invariant state $\tilde{\omega}$ on the spin chain $\mathcal{C}_{\mathbb{Z}}(\mathbb{C}^2)$ with the preservation of the locality structure. We give a brief exposition of this in Appendix 1. In particular, given two sets $\Omega^{(0)} = \{\omega^{(0,i)}\}_{i\in\mathcal{I}}$ and $\Omega^{(1)} = \{\omega^{(1,j)}\}_{j\in\mathcal{J}}$ of such states on CAR $\left(\ell^2(\mathbb{Z})\right)$, and numbers $\alpha, \beta \in [0, 1]$, there exists a (projective) test $T_n \in \mathcal{C}_{[1,n]}(\mathbb{C}^2)$ such that $\sup_{i\in\mathcal{I}} \operatorname{Tr} \tilde{\omega}^{(0,i)}_{[1,n]}(I - T_n) = \alpha$, $\sup_{j\in\mathcal{J}} \operatorname{Tr} \tilde{\omega}^{(1,j)}_{[1,n]} T_n = \beta$, if and only if there exists a (projective) test $S_n \in \mathrm{CAR}\,(\mathcal{H}_n)$ such that $\sup_{i\in\mathcal{I}} \omega^{(0,i)}(I - S_n) = \alpha$, $\sup_{j\in\mathcal{J}} \omega^{(1,j)}(S_n) = \beta$. Hence, in order to explore the achievable error exponent pairs for the pair $\tilde{\Omega}^{(0)} = \{\tilde{\omega}^{(0,i)}\}_{i\in\mathcal{I}}$, $\tilde{\Omega}^{(1)} = \{\omega^{(1,j)}\}_{j\in\mathcal{J}}$, one can work directly on the CAR algebra with $\Omega^{(0)}$ and $\Omega^{(1)}$. Thus, we introduce the following:

Definition 2.3 Let $\{\hat{q}_i\}_{i\in\mathcal{I}}$ and $\{\hat{r}_j\}_{j\in\mathcal{J}}$ be measurable functions from $[0, 2\pi)$ to $[0, 1]$, defining the translation-invariant quasi-free states $\Omega_Q := \{\omega_{Q^{(i)}}\}_{i\in\mathcal{I}}$, $\Omega_R := \{\omega_{R^{(j)}}\}_{j\in\mathcal{J}}$ on CAR $\left(\ell^2(\mathbb{Z})\right)$. We say that Ω_Q and Ω_R can be super-exponentially distinguished, if there exists a sequence $T_n \in \mathrm{CAR}\,(\mathcal{H}_n)$, $n \in \mathbb{N}$, such that

$$\liminf_{n\to+\infty} -\frac{1}{n}\log \sup_{i\in\mathcal{I}} \omega_{Q^{(i)}}(I - T_n) = +\infty = \liminf_{n\to+\infty} -\frac{1}{n}\log \sup_{j\in\mathcal{J}} \omega_{R^{(j)}}(T_n).$$

By the above, the sets of states Ω_Q and Ω_R on the CAR algebra are super-exponentially distinguishable if and only if so are the sets of states $\tilde{\Omega}_Q$ and $\tilde{\Omega}_R$ on the spin chain.

3 Super-exponential distinguishability

In this section, we prove the main result of the paper:

Theorem 3.1 *Let $\{\hat{q}_i\}_{i\in\mathcal{I}}$ and $\{\hat{r}_j\}_{j\in\mathcal{J}}$ be measurable functions from $[0, 2\pi)$ to $[0, 1]$, defining the translation-invariant quasi-free states $\Omega_Q := \{\omega_{Q^{(i)}}\}_{i\in\mathcal{I}}$, $\Omega_R := \{\omega_{R^{(j)}}\}_{j\in\mathcal{J}}$ on* $\mathrm{CAR}\left(\ell^2(\mathbb{Z})\right)$. *If there exists an interval $[\mu, \nu] \subseteq [0, 2\pi)$ of positive length such that $\hat{q}_i$ is constant 0 and $\hat{r}_j$ is constant 1 on it for every $i \in \mathcal{I}$ and $j \in \mathcal{J}$, then Ω_Q and Ω_R are super-exponentially distinguishable.*

If, moreover, $\hat{q}_i$ is not almost everywhere 0 and $\hat{r}_j$ is not almost everywhere 1 on $[0, 2\pi)$ for every $i \in \mathcal{I}$ and $j \in \mathcal{J}$, then their local densities $\widehat{\omega}_{Q_n^{(i)}}$ and $\widehat{\omega}_{R_n^{(j)}}$, $n \in \mathbb{N}$, are all invertible.

In fact, the above theorem follows immediately from a more detailed statement given in Theorem 3.8, which we prove in several steps.

The main intuition behind the proof is the following. (For simplicity, we take $|\mathcal{I}| = |\mathcal{J}| = 1$, $Q^{(1)} =: Q$, $R^{(1)} =: R$.) Although the symbols $Q, R \in \mathcal{B}(\ell^2(\mathbb{Z}))$ commute with each other, this is not true anymore for their restrictions Q_n and R_n onto $\mathbb{C}^{\langle n\rangle}$, unless Q or R is a constant multiple of the identity. On the other hand, if instead of restrictions of translation-invariant symbols onto finite-dimensional subspaces we considered translation-invariant symbols Q_n, R_n on the single-particle Hilbert space $\mathbb{C}^{\langle n\rangle}$ of a length n finite chain (with periodic boundary conditions, or equivalently, rotation-invariant symbols on a finite ring), then any two such symbols would commute with each other and would be simultaneously diagonalized by the discrete Fourier transformation; see the end of Sect. 2.1. Now, the analogous condition to the one in Theorem 3.1 in the finite-dimensional case would be that the functions $\hat{q}_n, \hat{r}_n \in \mathbb{C}^{\langle n\rangle}$ satisfy $\hat{q}_n(k) = 0 = 1 - \hat{r}_n(k)$, $k = l+1, \ldots, l+m$ for some $l, m \in \mathbb{C}^{\langle n\rangle}$ (modulo n). Hence, for the projection $E_n := \mathcal{F}_n^* \sum_{k=l+1}^{l+m} \left|\mathbf{1}_{\{k\}}\right\rangle\!\left\langle\mathbf{1}_{\{k\}}\right| \mathcal{F}_n$, we would have $\operatorname{Tr} E_n Q_n = 0 = \operatorname{Tr} E_n(I - R_n)$. A key technical ingredient of our proof, given in Lemma 3.2 and Corollary 3.3, is that for any such projection, one can construct a test, using the spectral decomposition of the particle number operator on the subspace $\operatorname{ran} E_n$, such that the type I and type II error probabilities are upper bounded by a simple expression involving only $\operatorname{Tr} E_n$, $\operatorname{Tr} E_n Q_n$ and $\operatorname{Tr} E_n(I - R_n)$; in particular, if the latter two are 0 then so are the error probabilities. When Q_n and R_n are the non-commuting restrictions of $Q, R \in \mathcal{B}(\ell^2(\mathbb{Z}))$, we can still follow the above strategy, where instead of making the upper bounds exactly zero, we can make them sufficiently small, as shown in Lemmas 3.5, 3.6 and 3.7.

Lemma 3.2 *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and*

$$S_{\mathcal{H}} := \sum_{k=0}^{\lfloor \dim \mathcal{H}/2 \rfloor} P_k^{N_{\mathcal{H}}}. \tag{3.1}$$

For any $A \in \mathcal{B}(\mathcal{H})$ with $0 \le A \le I$,

$$\omega_A(I - S_{\mathcal{H}}) \le \left(\frac{8 \operatorname{Tr} A}{\dim \mathcal{H}}\right)^{\frac{\dim \mathcal{H}}{2}}, \qquad \omega_A(S_{\mathcal{H}}) \le \left(\frac{8 \operatorname{Tr}(I - A)}{\dim \mathcal{H}}\right)^{\frac{\dim \mathcal{H}}{2}}.$$

Proof Let $d := \dim \mathcal{H}$, $S := S_{\mathcal{H}}$, and

$$A = \sum_{i=1}^{d} a_i \, |e_i\rangle\langle e_i| \, ,$$

be an eigen-decomposition of A. By (2.4),

$$S = \sum_{k=0}^{\lfloor d/2 \rfloor} U_e^* \left(\sum_{\Lambda \subseteq [d],\, |\Lambda| = k} \left(\bigotimes_{j \in \Lambda} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \left(\bigotimes_{j \in [d] \setminus \Lambda} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) U_e. \tag{3.2}$$

By (2.6),

$$\widehat{\omega}_A = U_e^* \left(\bigotimes_{j=1}^{d} \begin{bmatrix} 1 - a_j & 0 \\ 0 & a_j \end{bmatrix} \right) U_e \leq \begin{cases} U_e^* \left(\bigotimes_{j=1}^{d} \begin{bmatrix} 1 & 0 \\ 0 & a_j \end{bmatrix} \right) U_e, \\ U_e^* \left(\bigotimes_{j=1}^{d} \begin{bmatrix} 1 - a_j & 0 \\ 0 & 1 \end{bmatrix} \right) U_e. \end{cases} \tag{3.3}$$

Using (3.2) and the first bound in (3.3), we get

$$\begin{aligned} \omega_A(I - S) &= \operatorname{Tr} \widehat{\omega}_A (I - S) \\ &\leq \operatorname{Tr} \sum_{k = \lfloor d/2 \rfloor + 1}^{d} \sum_{\Lambda \subseteq [d],\, |\Lambda| = k} \left(\bigotimes_{j \in \Lambda} \begin{bmatrix} 0 & 0 \\ 0 & a_j \end{bmatrix} \right) \otimes \left(\bigotimes_{j \in [d] \setminus \Lambda} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= \sum_{k = \lfloor d/2 \rfloor + 1}^{d} \sum_{\Lambda \subseteq [d],\, |\Lambda| = k} \prod_{j \in \Lambda} a_j \leq \sum_{\lfloor d/2 \rfloor + 1}^{d} \sum_{\Lambda \subseteq [d],\, |\Lambda| = k} \left(\frac{\sum_{j \in \Lambda} a_j}{k} \right)^k, \end{aligned}$$

where the second inequality follows from the geometric–arithmetic mean inequality. Using that

$$\left(\frac{\sum_{j \in \Lambda} a_j}{k} \right)^k \leq \left(\frac{\sum_{j \in \Lambda} a_j}{k} \right)^{d/2} \leq \left(\frac{\sum_{j \in \Lambda} a_j}{d/2} \right)^{d/2} \leq \left(\frac{\operatorname{Tr} A}{d/2} \right)^{d/2}$$

for every $k \geq \lfloor d/2 \rfloor + 1$, we get

$$\omega_A(I - S) \leq \left(\frac{\operatorname{Tr} A}{d/2} \right)^{d/2} \sum_{k = \lfloor d/2 \rfloor + 1}^{d} \underbrace{\sum_{\Lambda \subseteq [d],\, |\Lambda| = k} 1}_{= \binom{d}{k}} \leq 2^d \left(\frac{\operatorname{Tr} A}{d/2} \right)^{d/2} = \left(\frac{8 \operatorname{Tr} A}{d} \right)^{d/2}.$$

Similarly, using (3.2) and the second bound in (3.3) yields

$$\begin{aligned}\omega_A(S) &= \operatorname{Tr}\widehat{\omega}_A S\\ &\leq \operatorname{Tr}\sum_{k=0}^{\lfloor d/2\rfloor}\sum_{\Lambda\subseteq[d],\,|\Lambda|=k}\left(\bigotimes_{j\in\Lambda}\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)\otimes\left(\bigotimes_{j\in[d]\setminus\Lambda}\begin{bmatrix}1-a_j&0\\0&0\end{bmatrix}\right)\\ &= \sum_{k=0}^{\lfloor d/2\rfloor}\sum_{\Lambda\subseteq[d],\,|\Lambda|=k}\prod_{j\in[d]\setminus\Lambda}(1-a_j) \leq \sum_{k=0}^{\lfloor d/2\rfloor}\sum_{\Lambda\subseteq[d],\,|\Lambda|=k}\left(\frac{\sum_{j\in[d]\setminus\Lambda}(1-a_j)}{d-k}\right)^{d-k}.\end{aligned}$$

Using that

$$\begin{aligned}\left(\frac{\sum_{j\in[d]\setminus\Lambda}(1-a_j)}{d-k}\right)^{d-k} &\leq \left(\frac{\sum_{j\in[d]\setminus\Lambda}(1-a_j)}{d-k}\right)^{d-d/2} \leq \left(\frac{\sum_{j\in[d]\setminus\Lambda}(1-a_j)}{d-d/2}\right)^{d/2}\\ &\leq \left(\frac{\operatorname{Tr}(I-A)}{d-d/2}\right)^{d/2}\end{aligned}$$

for all $k \leq \lfloor d/2\rfloor$, we get

$$\omega_A(S) \leq \left(\frac{\operatorname{Tr}(I-A)}{d/2}\right)^{d/2}\sum_{k=0}^{\lfloor d/2\rfloor}\underbrace{\sum_{\Lambda\subseteq[d],\,|\Lambda|=k} 1}_{=\binom{d}{k}} \leq 2^d\left(\frac{\operatorname{Tr}(I-A)}{d/2}\right)^{d/2} = \left(\frac{8\operatorname{Tr}(I-A)}{d}\right)^{d/2}.$$

□

Corollary 3.3 *Let $\mathcal{H}$ be a Hilbert space. For every nonzero finite-rank projection E on $\mathcal{H}$, there exists an even projection $T \in \operatorname{span}\{a(\varphi) : \varphi \in \operatorname{ran} E\}$ such that for every $A \in \mathcal{B}(\mathcal{H})$, $0 \leq A \leq I$,*

$$\omega_A(I-T) \leq \left(\frac{8\operatorname{Tr} EA}{\operatorname{Tr} E}\right)^{\frac{\operatorname{Tr} E}{2}}, \qquad \omega_A(T) \leq \left(\frac{8\operatorname{Tr} E(I-A)}{\operatorname{Tr} E}\right)^{\frac{\operatorname{Tr} E}{2}}. \tag{3.4}$$

Proof Let V_E be the identical embedding of $\operatorname{ran} E$ into $\mathcal{H}$. For any even projection $S \in \operatorname{CAR}(\operatorname{ran} E)$, $T := (V_E)_F S (V_E)_F^*$ is an even projection in $\operatorname{span}\{a(\varphi) : \varphi \in \operatorname{ran} E\}$, and $\omega_A(I-T) = \omega_{A_E}(I-S)$, $\omega_A(T) = \omega_{A_E}(S)$, where $A_E := V_E^* A V_E$. Noting that $\operatorname{Tr} A_E = \operatorname{Tr} AE$, $\operatorname{Tr}(I_{\operatorname{ran} E} - A_E) = \operatorname{Tr} E(I-A)$, the assertion follows from Lemma 3.2 by choosing $S := S_{\operatorname{ran} E}$ as in (3.1). □

Corollary 3.4 *Let $\{Q^{(i)}\}_{i\in\mathcal{I}}, \{R^{(j)}\}_{j\in\mathcal{J}} \subseteq \mathcal{B}\left(\ell^2(\mathbb{Z})\right)$ be symbols of quasi-free states on $\operatorname{CAR}\left(\ell^2(\mathbb{Z})\right)$. Assume that there exists a sequence of nonzero projections E_n on $\mathbb{C}^{\langle n\rangle}$, $n \in \mathbb{N}$, such that*

$$\liminf_{n\to+\infty}\frac{\operatorname{Tr} E_n}{n}\log\frac{\operatorname{Tr} E_n}{\sup_{i\in\mathcal{I}}\operatorname{Tr} E_n Q_n^{(i)}} = +\infty,$$

$$\liminf_{n\to+\infty} \frac{\mathrm{Tr}\, E_n}{n} \log \frac{\mathrm{Tr}\, E_n}{\sup_{j\in\mathcal{J}} \mathrm{Tr}\, E_n(I - R_n^{(j)})} = +\infty.$$

Then Ω_Q and Ω_R can be super-exponentially distinguished by even projective tests.

Proof Let T_n be the projection corresponding to $V_n E_n V_n^*$ as in Corollary 3.3, where V_n is the canonical embedding of $\mathbb{C}^{\langle n\rangle}$ into $\ell^2(\mathbb{Z})$. Since $\mathrm{Tr}\, E_n = \mathrm{Tr}\, V_n E_n V_n^*$, $\mathrm{Tr}\, E_n Q_n^{(i)} = \mathrm{Tr}\, V_n E_n V_n^* Q^{(i)}$, $\mathrm{Tr}\, E_n(I - R_n^{(j)}) = \mathrm{Tr}\, V_n E_n V_n^*(I - R^{(j)})$, (3.4) yields

$$-\frac{1}{n} \log \omega_{Q^{(i)}}(I - T_n) \geq \frac{\mathrm{Tr}\, E_n}{2n} \log \frac{\mathrm{Tr}\, E_n}{8\, \mathrm{Tr}\, E_n Q_n^{(i)}}, \tag{3.5}$$

$$-\frac{1}{n} \log \omega_{R^{(j)}}(T_n) \geq \frac{\mathrm{Tr}\, E_n}{2n} \log \frac{\mathrm{Tr}\, E_n}{8\, \mathrm{Tr}\, E_n(I - R_n^{(j)})}. \tag{3.6}$$

The statement follows by taking the infima over the respective index sets, and then, the liminf in n in the above inequalities, and noting that $0 \leq \frac{\mathrm{Tr}\, E_n}{2n} \log 8 \leq \frac{1}{2} \log 8$. □

Hence, in order to complete the proof of Theorem 3.1, it is sufficient to show that if Ω_Q and Ω_R are as in Theorem 3.1, then a sequence of projections as in Corollary 3.4 exists. For this, we will need some simple facts about Fourier transforms, see, e.g., [32] for details.

In particular, recall that the n-th partial sum of the Fourier series of an integrable function on $[0, 2\pi)$ is given by

$$(S_n f)(x) := \sum_{k=-n}^{n} e^{ikx} \frac{1}{2\pi} \int_{[0,2\pi)} e^{-ikt} f(t)\, \mathrm{d}t = (f \star D_n)(x),$$

where $D_n(x) := \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi} \frac{\sin((n+1/2)x)}{\sin(x/2)}$ is the *Dirichlet kernel*, and $\star$ stands for the convolution. The n-th Césaro mean of the partial sums is

$$(\widehat{S}_n f)(x) := \frac{1}{n} \sum_{k=0}^{n-1} S_n f(x) = \sum_{k=-n+1}^{n-1} \frac{n - |k|}{n} e^{ikx} \frac{1}{2\pi} \int_{[0,2\pi)} e^{-ikt} f(t)\, \mathrm{d}t = (f \star F_n)(x),$$

where $F_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_n(x) = \frac{1}{2\pi n} \frac{\sin^2(nx/2)}{\sin^2(x/2)}$ is the *Fejér kernel*.

The following may be known; however, as we have not found a reference in the literature, we provide a detailed proof. Recall that V_n is the canonical embedding of $\mathbb{C}^{\langle n\rangle}$ into $\ell^2(\mathbb{Z})$.

Lemma 3.5 *Let $\hat{a}$ be a bounded measurable complex-valued function on $[0, 2\pi)$ and $A = \mathcal{F}^* M_{\hat{a}} \mathcal{F}$. Then, the diagonal matrix entries of $\mathcal{F}_n V_n^* A V_n \mathcal{F}_n^*$ are given by:*

$$(\mathcal{F}_n V_n^* A V_n \mathcal{F}_n^*)_{k,k} = (\widehat{S}_n \hat{a}) \left(\frac{2\pi k}{n} \right), \qquad k \in \langle n\rangle.$$

Proof Let $A_n := V_n^* A V_n$. By (2.7) and (2.8),

$$\begin{aligned}(\mathcal{F}_n\, V_n^* A V_n\, \mathcal{F}_n^*)_{k,k} &= \sum_{j,l=0}^{n-1} (\mathcal{F}_n)_{k,j} (V_n^* A V_n)_{j,l} (\mathcal{F}_n^*)_{l,k} \\ &= \frac{1}{n} \sum_{j,l=0}^{n-1} e^{i\frac{2\pi}{n}kj} e^{-i\frac{2\pi}{n}kl} \frac{1}{2\pi} \int_{[0,2\pi)} e^{-i(j-l)x} \hat{a}(x)\,\mathrm{d}x \\ &= \sum_{m=-n+1}^{n-1} \frac{n-|m|}{n} e^{i\frac{2\pi}{n}km} \frac{1}{2\pi} \int_{[0,2\pi)} e^{-imx} \hat{a}(x)\,\mathrm{d}x \\ &= (\widehat{S}_n \hat{a}) \left(\frac{2\pi k}{n}\right),\end{aligned}$$

where in the third equality above we replaced the summation over j, l with a single summation over $m = j - l$. □

Lemma 3.6 *Let $\hat{a} : [0, 2\pi) \to [0, 1]$ be a measurable function. Assume that $\hat{a}$ is constant c on some interval $[\mu, \nu] \subseteq [0, 2\pi)$. Then for every $0 < \delta < (\nu - \mu)/2$, and every $x \in [\mu + \delta, \nu - \delta]$,*

$$\left|(\widehat{S}_n \hat{a})(x) - c\right| \le \frac{\gamma_\delta}{n},$$

where $\gamma_\delta := \frac{1}{\sin^2 \frac{\delta}{2}}$.

Proof We may extend $\hat{a}$ periodically to $\mathbb{R}$. Then

$$(\widehat{S}_n \hat{a})(x) - c = (\hat{a} \star F_n)(x) - c = \int_{-\pi}^{\pi} F_n(y) \left[\hat{a}(x-y) - c\right] \mathrm{d}y$$

for every x. If $x \in [\mu + \delta, \nu - \delta]$, then

$$\begin{aligned}\left|(\hat{a} \star F_n)(x) - c\right| &\le \int_{|y|<\delta} F_n(y) \underbrace{|\hat{a}(x-y) - c|}_{=0} \mathrm{d}y + \int_{\delta \le |y| \le \pi} F_n(y) \underbrace{|\hat{a}(x-y) - c|}_{\le 1} \mathrm{d}y \\ &\le \frac{1}{2\pi n} \int_{\delta \le |y| \le \pi} \frac{\sin^2 \frac{ny}{2}}{\sin^2 \frac{y}{2}} \mathrm{d}y \le \frac{1}{2\pi n} \int_{\delta \le |y| \le \pi} \frac{1}{\sin^2 \frac{\delta}{2}} \mathrm{d}y \\ &= \frac{\pi - \delta}{\pi n \sin^2 \frac{\delta}{2}} \le \frac{1}{n \sin^2 \frac{\delta}{2}}.\end{aligned}$$

□

Lemma 3.7 *Let $[\mu,\nu]\subseteq[0,2\pi)$ be an interval, let $0<\delta<(\nu-\mu)/2$, and for every $n\in\mathbb{N}$, let*

$$E_{n,\delta}:=\sum_{k:\,\frac{2\pi k}{n}\in[\mu+\delta,\nu-\delta]}\left|\mathcal{F}_n^*\mathbf{1}_{\{k\}}\right\rangle\!\left\langle\mathcal{F}_n^*\mathbf{1}_{\{k\}}\right|.$$

Then $E_{n,\delta}$ is a projection on $\mathbb{C}^{\langle n\rangle}$ such that

$$\operatorname{Tr}E_{n,\delta}\geq\left\lfloor\frac{\nu-\mu-2\delta}{2\pi}n\right\rfloor,\tag{3.7}$$

and for every measurable function $\hat{a}:\,[0,2\pi)\to[0,1]$ that is constant 0 on $[\mu,\nu]$,

$$\operatorname{Tr}E_{n,\delta}A_n\leq(\operatorname{Tr}E_{n,\delta})\frac{\gamma_\delta}{n},\qquad n\in\mathbb{N}.\tag{3.8}$$

Proof Since $\mathcal{F}_n$ is a unitary, $E_{n,\delta}$ is indeed a projection, and the lower bound in (3.7) is obvious. For any k such that $\frac{2\pi k}{n}\in[\mu+\delta,\nu-\delta]$,

$$\operatorname{Tr}\left|\mathcal{F}_n^*\mathbf{1}_{\{k\}}\right\rangle\!\left\langle\mathcal{F}_n^*\mathbf{1}_{\{k\}}\right|A_n=\left\langle\mathbf{1}_{\{k\}},\mathcal{F}_n\,V_n^*AV_n\,\mathcal{F}_n^*\mathbf{1}_{\{k\}}\right\rangle=(\widehat{S}_n\hat{a})\left(\frac{2\pi k}{n}\right)\leq\frac{\gamma_\delta}{n},$$

where the second equality is due to Lemma 3.5, and the inequality follows from Lemma 3.6. This immediately yields (3.8). □

Theorem 3.8 *Let Ω_Q and Ω_R be as in Theorem 3.1. Then, there exists a positive constant c and a sequence of even projections $T_n\in\operatorname{span}\{a(\varphi):\,\varphi\in\mathcal{H}_n\}$ such that*

$$-\frac{1}{n}\log\sup_{i\in\mathcal{I}}\omega_{Q^{(i)}}(I-T_n)\geq c\log n,\qquad-\frac{1}{n}\log\sup_{j\in\mathcal{J}}\omega_{R^{(j)}}(T_n)\geq c\log n,\qquad n\in\mathbb{N}.\tag{3.9}$$

In particular, Ω_Q and Ω_R can be super-exponentially distinguished by even projective tests.

If, moreover, $\hat{q}_i$ is not almost everywhere 0 and $\hat{r}_j$ is not almost everywhere 1 on $[0,2\pi)$ for every $i\in\mathcal{I}$ and $j\in\mathcal{J}$, then their local densities $\widehat{\omega}_{Q_n^{(i)}}$ and $\widehat{\omega}_{R_n^{(j)}}$, $n\in\mathbb{N}$, are all invertible.

Proof Let T_n be as in the proof of Corollary 3.4 with $E_n:=E_{n,\delta}$, $n\in\mathbb{N}$, for some δ as in Lemma 3.7. The inequalities in (3.5)–(3.6) combined with (3.7) and (3.8) yield (3.9).

The assertion about the invertibility of the density operators follows immediately from Corollary 2.2. □

Example 3.9 Consider the XX model with local Hamiltonian on $\mathcal{B}(\mathbb{C}^2)_{[1,n]}$ given by

$$H_n := \frac{1}{2} \sum_{k=-n}^{n-1} \left(\sigma_{x,k}\sigma_{x,k+1} + \sigma_{y,k}\sigma_{y,k+1} \right) + h \sum_{k=-n}^{n} \sigma_{z,k},$$

where $\sigma_{x,k}$ is the Pauli x operator $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ at site k, etc. It is well known that the ground state of this model in the thermodynamic limit is $\tilde{\omega}_{Q^{(h)}}$, where $\omega_{Q^{(h)}}$ is the translation-invariant quasi-free state corresponding to $\hat{q}_h := \hat{r}_h := \mathbf{1}_{[\arccos f(h), 2\pi - \arccos f(h))}$, where $f(h) := \max\{-1, \min\{h, 1\}\}$. (Here $\mathbf{1}_B$ stands for the indicator function of the set B.) See, e.g., [23, Appendix C] for a detailed exposition.

Let $h_0 < h_1$ be such that $h_1 > -1$ or $h_0 < 1$, and consider $\Omega_Q := \{\hat{q}_h : h \le h_0\}$, $\Omega_R := \{\hat{r}_h : h_1 \le h\}$. That is, the experimenter's task is to test whether the transverse magnetic field is below h_0 or above h_1, by making measurements on a finite part of the chain. It is straightforward to verify that $\hat{q}_h$ is constant zero on $[\mu := \arccos f(h_1), \nu := \arccos f(h_0)]$ for every $h \le h_0$, while $\hat{r}_h$ is constant one on $[\mu, \nu]$ for every $h \ge h_1$, and hence, by Theorem 3.1, the two hypotheses can be tested with super-exponentially decreasing error probabilities. By Corollary 2.2, the local densities $\widehat{\omega}_{Q_n^{(h)}}$ are invertible for every $h_1 \le h < 1$, and the local densities $\widehat{\omega}_{R_n^{(h)}}$ are invertible for every $-1 < h \le h_0$.

A variant of the above problem is when the experimenter's task is to test whether the transverse magnetic field is between h_0 and h_0' or between h_1 and h_1', where $-1 < h_0 < h_0' < h_1 < h_1' < 1$. In this case, $\Omega_Q := \{\hat{q}_h : h_0 \le h \le h_0'\}$, $\Omega_R := \{\hat{r}_h : h_1 \le h \le h_1'\}$. It is straightforward to verify that this problem satisfies the conditions in Theorem 3.1 with $\mu = \arccos h_1$, $\nu = \arccos h_0'$, and therefore, the two hypotheses can be tested with super-exponentially decreasing error probabilities, and moreover, all local densities are invertible for every size n.

4 Comments on orthogonality

In this section, we discuss some relations between three concepts: a) the orthogonality of a pair of states, b) their super-exponential distinguishability, and c) certain distinguishability measures taking infinite value on the given pair. We start with an overview of the well-known relations between these for density operators on a finite-dimensional Hilbert space and then discuss a possible extension to pairs of translation-invariant states on an infinite spin chain.

Let $\mathbb{T}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : 0 \le T \le I\}$ denote the set of *tests* on a finite-dimensional Hilbert space $\mathcal{H}$. It is well known [13, 18] and easy to see that for any two density operators $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$,

$$\min_{T \in \mathbb{T}(\mathcal{H})} \{ \underbrace{\operatorname{Tr} \varrho (I - T)}_{=:\alpha(T)} + \underbrace{\operatorname{Tr} \sigma T}_{=:\beta(T)} \} = 1 - \frac{1}{2} \left\| \varrho - \sigma \right\|_1 , \tag{4.1}$$

where $\|X\|_1 := \operatorname{Tr}|X|$, $X \in \mathcal{B}(\mathcal{H})$, is the trace-norm. In particular, we have

$$\begin{aligned}
&\exists\, T \in \mathbb{T}(\mathcal{H}) :\ \alpha(T) = 0 = \beta(T) \\
&\iff \chi(\varrho\|\sigma) := -\log\left(1 - \frac{1}{2}\|\varrho - \sigma\|_1\right) = +\infty \\
&\iff \varrho \perp \sigma, \qquad (4.2)
\end{aligned}$$

where the first condition means perfect distinguishability of ϱ and σ, in the second condition we use the convention $\log 0 := -\infty$, and the orthogonality in the last condition might be formulated in a number of different ways, e.g., as the orthogonality of the supports.

Orthogonality may be equivalently captured by various quantum Rényi divergences. For instance, for any $\alpha \in (0, 1)$ and $z \in (0, +\infty)$, let

$$D_{\alpha,z}(\varrho\|\sigma) := \frac{1}{\alpha - 1}\log \operatorname{Tr}\left(\varrho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{z}}\varrho^{\frac{\alpha}{z}}\right)^z$$

be the Rényi (α, z)-divergence of ϱ and σ [7, 20]. Then,

$$\varrho \perp \sigma \iff D_{\alpha,z}(\varrho\|\sigma) = +\infty \ \text{ for some/all } (\alpha, z) \text{ pairs as above.} \qquad (4.3)$$

Note that the case $\alpha = 1/2$, $z = 1$, expresses the orthogonality of the unit vectors $\varrho^{1/2}, \sigma^{1/2}$ in the Hilbert–Schmidt inner product. Furthermore, for a test $T \in \mathbb{T}(\mathcal{H})$, let

$$\mathcal{T}(X) := (\operatorname{Tr} XT)\,|0\rangle\langle 0| + (\operatorname{Tr} X(I - T))\,|1\rangle\langle 1|\,, \qquad (4.4)$$

where $\{|0\rangle, |1\rangle\}$ is any orthonormal system in some Hilbert space, and let

$$D_\alpha^{\text{test}}(\varrho\|\sigma) := \max_{T\in\mathbb{T}(\mathcal{H})} D_\alpha(\mathcal{T}(\varrho)\|\mathcal{T}(\sigma))$$

be the *test-measured Rényi α-divergence* of ϱ and σ [24]. In the above, for any $z \in (0, +\infty)$.

$$\begin{aligned}
&D_\alpha(\mathcal{T}(\varrho)\|\mathcal{T}(\sigma)) \\
&= D_{\alpha,z}(\mathcal{T}(\varrho)\|\mathcal{T}(\sigma)) \\
&= \frac{1}{\alpha - 1}\log\Big((\operatorname{Tr}\varrho T)^\alpha(\operatorname{Tr}\sigma T)^{1-\alpha} + (\operatorname{Tr}\varrho(I - T))^\alpha(\operatorname{Tr}\sigma(I - T))^{1-\alpha}\Big),
\end{aligned}$$

is the classical Rényi divergence [29] of the commuting pair $\mathcal{T}(\varrho), \mathcal{T}(\sigma)$. Then,

$$\varrho \perp \sigma \iff D_\alpha^{\text{test}}(\varrho\|\sigma) = +\infty \ \text{ for some/all } \alpha \in (0, 1), \qquad (4.5)$$

which is just a reformulation of the equivalence of the first and the last conditions in (4.2).

Consider now translation-invariant states $\omega^{(0)}$, $\omega^{(1)}$ on the infinite spin chain algebra $\mathcal{B}(\mathcal{H})_{\mathbb{Z}}$; see Sect. 2.2. One might define many of the above quantities directly for the states $\omega^{(0)}, \omega^{(1)}$. This is obvious for $\chi(\omega^{(0)}\|\omega^{(1)})$ and $D_\alpha^{\mathrm{test}}(\omega^{(0)}\|\omega^{(1)})$; for the Petz-type Rényi divergences $D_{\alpha,1}\left(\omega^{(0)}\|\omega^{(1)}\right)$ and the sandwiched Rényi divergences $D_{\alpha,\alpha}\left(\omega^{(0)}\|\omega^{(1)}\right)$ with $\alpha \in [1/2, 1)$, see, e.g., [15, 16, 27]. Most of these quantities, however, behave in a singular way for translation-invariant product states. Indeed, let ω_k denote the single-site density operator of $\omega^{(k)}$. Additivity and monotonicity under restriction to subalgebras then give

$$D_{\alpha,z}\left(\omega^{(0)}\|\omega^{(1)}\right) \geq D_{\alpha,z}\left(\omega_0^{\otimes n}\|\omega_1^{\otimes n}\right) = n D_{\alpha,z}\left(\omega_0\|\omega_1\right) \xrightarrow[n\to+\infty]{} +\infty$$

for every $\alpha \in (0, 1)$ and $z = 1$, or $z = \alpha \in [1/2, 1)$, whenever $\omega_0 \neq \omega_1$. Using the Fuchs–van de Graaf inequality [11] in the form $\chi(\varrho\|\sigma) \geq \frac{1}{2} D_{1/2,1/2}(\varrho\|\sigma)$ then yields

$$\chi(\omega^{(0)}\|\omega^{(1)}) \geq \chi(\omega_0^{\otimes n}\|\omega_1^{\otimes n}) \geq \frac{1}{2} D_{1/2,1/2}(\omega_0^{\otimes n}\|\omega_1^{\otimes n}) \xrightarrow[n\to+\infty]{} +\infty.$$

In fact, it is also known that $\chi(\omega^{(0)}\|\omega^{(1)}) = +\infty$ whenever $\omega^{(0)}$ and $\omega^{(1)}$ are different ergodic states [19, Corollary IV.4.2], and it is not too difficult to see that translation-invariant quasi-free states are ergodic (for a hint, see, e.g., [1, Example 7.6]).

In view of the above, the above considered distinguishability measures defined directly for the infinite spin chain states reveal very little about the relation of $\omega^{(0)}$ and $\omega^{(1)}$ from the point of view of state discrimination. One might consider instead the regularized versions of the above quantities, defined for two translation-invariant states $\omega^{(0)}$ and $\omega^{(1)}$ as

$$\overline{\Delta}(\omega^{(0)}\|\omega^{(1)}) := \liminf_{n\to+\infty} \frac{1}{n} \Delta\left(\omega_{[1,n]}^{(0)}\middle\|\omega_{[1,n]}^{(1)}\right), \tag{4.6}$$

where Δ may stand for any distinguishability measure on pairs of states, like χ, $D_{\alpha,z}$, D_α^{test}, etc.

Theorem 4.1 *Let $\omega^{(0)}$ and $\omega^{(1)}$ be translation-invariant states on the infinite spin-chain algebra $\mathcal{B}(\mathcal{H})_{\mathbb{Z}}$. The following are equivalent:*

(i) $\omega^{(0)}$ *and* $\omega^{(1)}$ *can be super-exponentially distinguished.*
(ii) $\overline{\chi}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$.
(iii) $\overline{D}_\alpha^{\mathrm{test}}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ *for every* $\alpha \in (0, 1)$.
(iv) $\overline{D}_\alpha^{\mathrm{test}}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ *for some* $\alpha \in (0, 1)$.
(v) $\overline{D}_{\alpha,z}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ *for every* $\alpha \in (0, 1)$ *and every* $z \geq \max\{\alpha, 1-\alpha\}$.
(vi) $\overline{D}_{\alpha,z}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ *for some* $\alpha \in (0, 1)$ *and some* $z \geq \max\{\alpha, 1-\alpha\}$.

Proof The equivalence (i) $\Longleftrightarrow$(ii) is clear from (4.1). It is straightforward to verify that (i) yields $\overline{D}_\alpha^{\mathrm{test}}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ for every $\alpha \in (0, 1)$, proving (i) $\Longrightarrow$(iii). The implication (iii)$\Longrightarrow$(iv) is obvious.

Assume (iv), i.e., that $\overline{D}_\alpha^{\text{test}}\left(\omega^{(0)}\|\omega^{(1)}\right) = +\infty$ for some $\alpha \in (0, 1)$. Then, there exists a test sequence $T_n \in \mathcal{B}(\mathcal{H})_{[1,n]}$, $n \in \mathbb{N}$, and a sequence $c_n \in [0, +\infty)$, $n \in \mathbb{N}$, with $\lim_n c_n = +\infty$, such that

$$\begin{aligned} e^{-(1-\alpha)nc_n} &= (\operatorname{Tr}\omega_n^{(0)}T_n)^\alpha(\operatorname{Tr}\omega_n^{(1)}T_n)^{1-\alpha} + (\operatorname{Tr}\omega_n^{(0)}(I-T_n))^\alpha(\operatorname{Tr}\omega_n^{(1)}(I-T_n))^{1-\alpha} \\ &\geq \min\left\{(\operatorname{Tr}\omega_n^{(1)}T_n)^{1-\alpha}, (\operatorname{Tr}\omega_n^{(1)}(I-T_n))^{1-\alpha}\right\} \\ &\quad \underbrace{\left((\operatorname{Tr}\omega_n^{(0)}T_n)^\alpha + (\operatorname{Tr}\omega_n^{(0)}(I-T_n))^\alpha\right)}_{\geq 1}, \end{aligned}$$

where $\omega_n^{(k)} := \omega_{[1,n]}^{(k)}$. Let us define a new test sequence $\tilde{T}_n := T_n$ if $\operatorname{Tr}\omega_n^{(1)}T_n \leq 1/2$, and $\tilde{T}_n := I - T_n$ otherwise. Then, the above yields $\operatorname{Tr}\omega_n^{(1)}\tilde{T}_n \leq e^{-nc_n}$, which goes to zero super-exponentially, and

$$\begin{aligned} e^{-(1-\alpha)nc_n} &\geq (\operatorname{Tr}\omega_n^{(0)}(I-\tilde{T}_n))^\alpha(\operatorname{Tr}\omega_n^{(1)}(I-\tilde{T}_n))^{1-\alpha} \\ &\geq (1-e^{-nc_n})^{1-\alpha}\operatorname{Tr}\omega_n^{(0)}(I-\tilde{T}_n))^\alpha, \end{aligned}$$

whence $\operatorname{Tr}\omega_n^{(0)}(I-\tilde{T}_n)$ also goes to zero super-exponentially in n. Thus, we obtain (i).

According to [14, Theorem 1.1] and a standard argument deriving monotonicity under CPTP maps from joint convexity, the Rényi (α, z)-divergences are monotone non-increasing under the joint action of a CPTP map on both of their arguments when $\alpha \in (0, 1)$ and $\max\{\alpha, 1-\alpha\} \leq z$. This immediately implies $\overline{D}_{\alpha,z}\left(\omega^{(0)}\|\omega^{(1)}\right) \geq \overline{D}_\alpha^{\text{test}}\left(\omega^{(0)}\|\omega^{(1)}\right)$ for any such α, z, and thus, the implication (iii)$\Longrightarrow$(v) follows, and (v)$\Longrightarrow$(vi) is trivial.

Finally, (vi)$\Longrightarrow$(ii) follows immediately from Corollary A.2. □

Remark 4.2 It is clear from the proof of Theorem 4.1 that the following also holds. If $n_1 < n_2 < \ldots$, and $\Delta = \chi$, $\Delta = D_{\alpha,z}$ with $\alpha \in (0, 1)$ and $z \geq \max\{\alpha, 1-\alpha\}$, or $\Delta = D_\alpha^{\text{test}}$ with $\alpha \in (0, 1)$, then the following are equivalent:

(i) There exists a sequence of tests $T_{n_k} \in \mathcal{B}(\mathcal{H})_{[1,n_k]}$, $k \in \mathbb{N}$, such that

$$\lim_{k\to+\infty} -\frac{1}{n_k}\log\operatorname{Tr}\omega_{[1,n_k]}^{(0)}(I-T_{n_k}) = +\infty = \lim_{k\to+\infty} -\frac{1}{n_k}\log\operatorname{Tr}\omega_{[1,n_k]}^{(1)}T_{n_k}.$$

(ii) $\lim_{k\to+\infty}\frac{1}{n_k}\Delta\left(\omega_{[1,n_k]}^{(0)}\|\omega_{[1,n_k]}^{(1)}\right) = +\infty.$

Indeed, (i) is equivalent to (ii) with $\Delta = \chi$ due to (4.1), which implies (ii) with $\Delta = D_\alpha^{\text{test}}$ for any given $\alpha \in (0, 1)$; this implies (ii) with $\Delta = D_{\alpha,z}$ for the same α and any $z \geq \max\{\alpha, 1-\alpha\}$, due to monotonicity under CPTP maps, and finally, this implies (ii) with $\Delta = \chi$ due to Corollary A.2.

In particular, Theorem 4.1 remains valid if we replace the lim inf with lim sup in the definition of the error exponents in (2.10), and define super-exponential distinguishability accordingly, and we also replace the lim inf with lim sup in the definition of the regularized distinguishability measures in (4.6).

Remark 4.3 Two states (positive linear normalized functionals) $\omega^{(0)}$ and $\omega^{(1)}$ on a C^*-algebra are defined to be orthogonal in [31, Definition 1.14.1] if $\|\omega^{(0)} - \omega^{(1)}\| = 2$, where the norm is the usual functional norm; this is equivalent to $\chi(\omega^{(0)} \| \omega^{(1)}) = +\infty$ in our notation. The above arguments show that this notion of orthogonality may not be the best suited for the study of asymptotic state discrimination on an infinite spin chain; in particular, any two translation-invariant product states are orthogonal according to this definition, irrespective of whether the density operators of their local restrictions are orthogonal or not.

In contrast, if we define $\omega^{(0)}$ and $\omega^{(1)}$ on an infinite spin chain to be orthogonal if $\overline{\chi}(\omega^{(0)} \| \omega^{(1)}) = +\infty$, then for translation-invariant product states this becomes equivalent to the usual orthogonality of their single-site restrictions $\omega^{(0)}_{[1]}$ and $\omega^{(1)}_{[1]}$. Another appealing feature of this notion of orthogonality of states is that it is equivalent to various regularized distinguishability measures being $+\infty$, according to Theorem 4.1, which gives a nice generalization of the analogous single-site characterizations of orthogonality given in (4.3) and (4.5). Of course, this notion of orthogonality is limited to pairs of translation-invariant states on an infinite spin chain and does not make sense in general for pairs of states on an abstract C^*-algebra.

Remark 4.4 Clearly, if any (and hence all) of (i)–(vi) in Theorem 4.1 holds then we have $\overline{\mathbb{D}}_\alpha(\omega^{(0)} \| \omega^{(1)}) = +\infty$ for any quantum Rényi α-divergence with $\alpha \in (0, 1)$ that is monotone non-increasing under 2-outcome measurements, i.e., under the type of CPTP maps given in (4.4). Here, we say that $\mathbb{D}_\alpha$ is a quantum Rényi α-divergence if it is defined on all pairs of density operators on any finite-dimensional Hilbert space, and for commuting states it reduces to the classical Rényi α-divergence of the diagonal elements of the two density operators in a common eigen-basis. One such example is Matsumoto's maximal α-divergence [22] for every $\alpha \in (0, 1)$; however, at the moment we do not know if the regularized maximal α-divergence being $+\infty$ implies the other properties listed in Theorem 4.1.

5 Conclusion

We have shown that translation-invariant quasi-free states with defining functions $\hat{q}$ and $\hat{r}$ are super-exponentially distinguishable if there is an interval $[\mu, \nu]$ of nonzero length such that one of the functions is constant 0 and the other one is constant 1 on this interval. We have shown that in this case both errors decrease at least as fast as $e^{-nc \log n}$ in the sample size n; it is, however, an open question whether this is in fact the optimal asymptotics, or a faster decrease, e.g., $e^{-cn^{1+\delta}}$ with some $\delta > 0$ can be attained. This can be asked for the class of functions that we considered, but it is also natural to ask if there is any upper bound on the speed of convergence to zero for general pairs of translation-invariant states on a spin chain.

It is known that a translation-invariant quasi-free state ω_Q is pure (i.e., an extremal point of the convex set of states) if and only if the corresponding function $\hat{q}$ is an indicator function, i.e., $\hat{q} = \mathbf{1}_{B_Q}$ for some measurable subset B_Q of $[0, 2\pi)$ (see, e.g., [10]). Two such pure states ω_Q and ω_R are different if and only if B_Q and B_R are different in the measure-theoretic sense, i.e., the Lebesgue measure of $(B_Q \setminus B_R) \cup$

$(B_R \setminus B_Q)$ is positive. This motivates to ask whether the following extension of our result is true: If $\hat{q}$ and $\hat{r}$ are measurable functions from $[0, 2\pi)$ to $[0, 1]$ such that there exists a measurable set $B \subseteq [0, 2\pi)$ of positive Lebesgue measure on which $\hat{q}$ is constant 0 and $\hat{r}$ is constant 1, then ω_Q and ω_R can be super-exponentially distinguished. In particular, this would imply the super-exponential distinguishability of any two different pure translation-invariant quasi-free states.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A: A variant of Audenaert's inequality

It was shown in [6, Theorem 1] that for any two density operators ϱ, σ on a finite-dimensional Hilbert space $\mathcal{H}$,

$$1 - \frac{1}{2}\|\varrho - \sigma\|_1 \leq \operatorname{Tr} \varrho^{\alpha}\sigma^{1-\alpha}, \qquad \alpha \in (0, 1),$$

or equivalently,

$$\chi(\varrho\|\sigma) := -\log\left(1 - \frac{1}{2}\|\varrho - \sigma\|_1\right) \geq (1-\alpha)D_{\alpha,1}(\varrho\|\sigma). \tag{A.1}$$

(see Sect. 4 for the definition of the Rényi (α, z)-divergences.) We will need a simple extension of the above, given in Corollary A.2, which follows by a combination of (A.1), the Araki–Lieb–Thirring (ALT) inequality [3, 21], and its converse given in [5]. Recall that the ALT inequality states that for any two positive semi-definite operators $A, B \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{Tr}\left(A^r B^r A^r\right)^q \leq \operatorname{Tr}(ABA)^{rq}, \qquad q \in [0, +\infty), \quad r \in [0, 1]. \tag{A.2}$$

The converse given in [5, Theorem 2] states that

$$\operatorname{Tr}(ABA)^{rq} \leq \left(\operatorname{Tr}\left(A^r B^r A^r\right)^q\right)^r \left(\|A\|^{2rq} \operatorname{Tr} B^{rq}\right)^{1-r}, \qquad q \in [0, +\infty), \quad r \in [0, 1]. \tag{A.3}$$

Lemma A.1 *Let ϱ, σ be density operators on a finite-dimensional Hilbert space $\mathcal{H}$. For any $\alpha \in (0, 1)$,*

$$D_{\alpha,1}(\varrho\|\sigma) \geq \begin{cases} D_{\alpha,z}(\varrho\|\sigma), & z \in (0, 1], \\ \frac{1}{z} D_{\alpha,z}(\varrho\|\sigma) - \frac{\alpha}{(1-\alpha)^2} \frac{z-1}{z} \log \dim \mathcal{H}, & z > 1. \end{cases} \tag{A.4}$$

Proof Let $z' \leq z''$, $A := \varrho^{\frac{\alpha}{2z'}}$, $B := \sigma^{\frac{1-\alpha}{z'}}$, $q := z''$, $r := z'/z''$. Then, (A.2) gives

$$\operatorname{Tr}\left(\varrho^{\frac{\alpha}{2z''}} \sigma^{\frac{1-\alpha}{z''}} \varrho^{\frac{\alpha}{2z''}}\right)^{z''} \leq \operatorname{Tr}\left(\varrho^{\frac{\alpha}{2z'}} \sigma^{\frac{1-\alpha}{z'}} \varrho^{\frac{\alpha}{2z'}}\right)^{z'},$$

or equivalently, $D_{\alpha,z''}(\varrho\|\sigma) \geq D_{\alpha,z'}(\varrho\|\sigma)$. In particular, the choice $z'' := 1$, $z' := z$ yields the first inequality in (A.4).

Now, let $z' := 1$, $z'' := z > 1$. Then, (A.3) gives

$$\operatorname{Tr} \varrho^\alpha \sigma^{1-\alpha} \leq \left(\operatorname{Tr}\left(\varrho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \varrho^{\frac{\alpha}{2z}}\right)^z\right)^{1/z} \underbrace{\left(\|\varrho\|^\alpha\right)^{1-1/z}}_{\leq 1} \left(\operatorname{Tr} \sigma^{1-\alpha}\right)^{1-1/z}. \tag{A.5}$$

Let $\lambda_1, \ldots, \lambda_d$ be the eigen-values of σ, where $d := \dim \mathcal{H}$. Since $1/(1-\alpha) > 1$, we get

$$(\operatorname{Tr} \sigma^{1-\alpha})^{\frac{1}{1-\alpha}} = \left(\sum_{i=1}^d \lambda_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}} = \left(d \sum_{i=1}^d \frac{1}{d} \lambda_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \leq d^{\frac{1}{1-\alpha}} \sum_{i=1}^d \frac{1}{d} \lambda_i = d^{\frac{\alpha}{1-\alpha}}.$$

Writing this back into (A.5), we get

$$\underbrace{\frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho^\alpha \sigma^{1-\alpha}}_{= D_{\alpha,1}(\varrho\|\sigma)} \geq \frac{1}{z} \underbrace{\frac{1}{\alpha - 1} \log \operatorname{Tr}\left(\varrho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \varrho^{\frac{\alpha}{2z}}\right)^z}_{= D_{\alpha,z}(\varrho\|\sigma)} - \frac{\alpha}{(1-\alpha)^2} \frac{z-1}{z} \log d,$$

which is exactly the second inequality in (A.4). □

Corollary A.2 *Let ϱ, σ be density operators on a finite-dimensional Hilbert space $\mathcal{H}$. For every $\alpha \in (0, 1)$,*

$$\chi(\varrho\|\sigma) \geq \begin{cases} (1-\alpha) D_{\alpha,z}(\varrho\|\sigma), & z \in (0, 1], \\ \frac{1}{z}(1-\alpha) D_{\alpha,z}(\varrho\|\sigma) - \frac{\alpha}{1-\alpha} \frac{z-1}{z} \log \dim \mathcal{H}, & z > 1. \end{cases}$$

Proof Immediate from (A.1) and (A.4). □

Appendix B: Quasi-free states on the spin chain

For simplicity of notation, let $\mathcal{A} := \mathrm{CAR}\left(\ell^2(\mathbb{Z})\right)$, and $a_k := a(\mathbf{1}_{\{k\}})$, where $\mathbf{1}_{\{k\}}$, $k \in \mathbb{Z}$, is the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. As explained in Sect. 2.1, the even part of $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{A}_+ &:= \{x \in \mathcal{A} : \pi(x) = x\}\\ &= \overline{\mathrm{span}\{I,\ a_{i_1}^* \dots a_{i_n}^* a_{j_m} \dots a_{j_1} : i_1 < \dots < i_n,\ j_1 < \dots < j_m,\ n+m \text{ even}\}}.\end{aligned}$$

Similarly, the *odd part* $\mathcal{A}_-$ is defined as

$$\begin{aligned}\mathcal{A}_- &:= \{x \in \mathcal{A} : \pi(x) = -x\}\\ &= \overline{\mathrm{span}\{I,\ a_{i_1}^* \dots a_{i_n}^* a_{j_m} \dots a_{j_1} : i_1 < \dots < i_n,\ j_1 < \dots < j_m,\ n+m \text{ odd}\}}.\end{aligned}$$

Clearly, any $x \in \mathcal{A}$ can be uniquely decomposed as the sum of an even and an odd element as

$$x = \underbrace{\frac{1}{2}(x + \pi(x))}_{\in \mathcal{A}_+} + \underbrace{\frac{1}{2}(x - \pi(x))}_{\in \mathcal{A}_-}.$$

It is obvious from (2.5) that any quasi-free state ω_Q is *even* in the sense that $\omega_Q \circ \pi = \omega_Q$, or equivalently, $\omega_Q(x) = 0$ for every odd element $x \in \mathcal{A}_-$.

Likewise, let $\mathcal{C} := \otimes_{k\in\mathbb{Z}} \mathcal{B}(\mathbb{C}^2)$ be the infinite spin chain algebra with single-site Hilbert space $\mathbb{C}^2$. It is generated by the local Pauli operators $\sigma_k^{(n)}$, $k = 0, 1, 2, 3$, $n \in \mathbb{Z}$, where, e.g.,

$$\sigma_1^{(n)} = \dots \otimes I \otimes \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{site } n} \otimes I \otimes \dots$$

The map

$$\hat{\pi} : \sigma_k^{(n)} \mapsto \begin{cases} -\sigma_k^{(n)}, & k = 1, 2, \\ \sigma_k^{(n)}, & k = 0, 3, \end{cases} \qquad n \in \mathbb{Z},$$

is easily seen to extend to an automorphism of $\mathcal{C}$, which is called the *parity automorphism* of the infinite spin chain, and we can define the even part $\mathcal{C}_+$ and the odd part $\mathcal{C}_-$ of the spin chain analogously to the case of $\mathcal{A}$. Again, any $x \in \mathcal{C}$ can be uniquely decomposed as the sum of an even and an odd element as

$$x = \underbrace{\frac{1}{2}(x + \hat{\pi}(x))}_{\in \mathcal{C}_+} + \underbrace{\frac{1}{2}(x - \hat{\pi}(x))}_{\in \mathcal{C}_-}.$$

The translation automorphism $\hat{\tau}$ of the spin chain is the unique extension of the map $\hat{\tau} : \sigma_k^{(n)} \mapsto \sigma_k^{(n+1)}, k = 0, 1, 2, 3, n \in \mathbb{Z}$.

A formal extension of the Jordan–Wigner automorphism (2.3) would map

$$a_j \quad \text{into} \quad \left(\otimes_{k<j}\sigma_3\right) \otimes \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\text{site } j} \otimes I \otimes \dots , \tag{B.1}$$

which, however, does not correspond to any element of the spin chain. On the other hand, for any even multinomial $a_{i_1}^* \dots a_{i_n}^* a_{j_m} \dots a_{j_1}$, the corresponding product using the formal expression in (B.1) gives an (even) element of $\mathcal{C}$, and the resulting map extends to an isomorphism α between $\mathcal{A}_+$ and $\mathcal{C}_+$. This can be verified by a straightforward computation; for details, see, e.g., [23, Section 5.3]. It is also clear that if $k \le i_1, \dots, i_n, j_1, \dots, j_m \le l$ then $\alpha(a_{i_1}^* \dots a_{i_n}^* a_{j_m} \dots a_{j_1}) \in \mathcal{C}_{[k,l]}$, i.e., the automorphism is compatible with the locality structure of the two algebras, at least for intervals. Moreover, α is compatible with the translations in the sense that $\alpha \circ \tau \circ \alpha^{-1} = \hat{\tau}$. Hence, for any translation-invariant quasi-free state ω_Q, $\hat{\omega}_Q := \omega_Q \circ \alpha^{-1}$ is a translation-invariant state on $\mathcal{C}_+$, and the local restrictions of ω_Q and $\hat{\omega}_Q$ on intervals are mapped into each other by α. Finally, $\hat{\omega}_Q$ has a unique extension to a linear functional $\tilde{\omega}_Q$ on $\mathcal{C}$ by

$$\tilde{\omega}_Q(x) := \tilde{\omega}_Q\left(\frac{1}{2}(x + \hat{\pi}(x))\right), \qquad x \in \mathcal{C}.$$

Since

$$|\tilde{\omega}_Q(x)| = \left|\tilde{\omega}_Q\left(\frac{1}{2}(x + \hat{\pi}(x))\right)\right| \le \frac{1}{2}\left(\|x\| + \left\|\hat{\pi}(x)\right\|\right) = \|x\|, \qquad x \in \mathcal{C},$$

we get that $\left\|\tilde{\omega}_Q\right\| = 1 = \left\|\tilde{\omega}_Q(I)\right\|$, whence the extended $\tilde{\omega}_Q$ is a state on $\mathcal{C}$ (see, e.g., [28, Proposition 2.11]). Clearly, $\tilde{\omega}_Q$ is translation-invariant, it is even, and the density operators of its local restrictions to $\mathcal{C}_{[k,l]}$ coincide with the densities of the corresponding local restrictions of $\hat{\omega}_Q$.

Note that, more generally, the above construction gives a one-to-one correspondence between even translation-invariant states on $\mathcal{A}$ and even translation-invariant states on $\mathcal{C}$.

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