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Spectral curves are transcendental

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Abstract
Some arithmetic properties of spectral curves are discussed: the spectral curve, for example, of a charge $n \geq 2$ Euclidean BPS monopole is not defined over $\overline{\mathbb{Q}}$ if smooth.

Keywords Spectral curve · Periods · Monopole · Transcendental

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1 Introduction

A fundamental ingredient of the modern theory of integrable systems is a curve, the *spectral curve*, and the function theory of this curve enables (via the Baker–Akhiezer function, for example) the solution of the system. Typically, analytic properties of this curve are in the fore: here, we will focus on a less well-developed aspect, its arithmetic properties. We will show that for an integrable system of interest the associated spectral curves are not defined over $\overline{\mathbb{Q}}$, the *transcendental* of the title. This aspect is a manifestation of why it is so difficult to construct specific examples of some systems. The result proven here depends on a number of deep results across several mathematical disciplines and what is novel is bringing them together. For a number theorist, the transcendence of periods is familiar: this paper provides a number of new examples where this is relevant. For an algebraic geometer, defining a curve by properties of lines bundles over it is not new: we see here the arithmetic consequences of this. To be concrete, we will focus on a particular integrable system and remark on other examples. Neither a detailed knowledge of this particular physical system nor the arcane lore of integrable systems will be needed to understand this paper.

The integrable system in focus here is that associated with Nahm's equations and BPS monopoles on $\mathbb{R}^3$, a reduction in the anti-self-dual Yang–Mills equations [1]; for simplicity, we will focus only on the case where the gauge group is $SU(2)$. Some

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years ago, Hitchin [10] gave a description of the regular solutions to this system in terms of a spectral curve $\mathcal{C} \subset T\mathbb{P}^1$ subject to constraints. (These constraints will be reviewed later in the paper.) Although the mathematics associated with these equations has proven remarkably rich, for example, the moduli space of solutions may be given a hyperkähler structure [1], the number of spectral curves that can be explicitly written down are few. Table 1 gives the list of those constructed over a period of some 35 years (see [14] [Ch. 8] for references). Here, η and ζ are the fibre coordinate and affine base coordinate of $T\mathbb{P}^1$ and the degree of η is the "charge"of the monopole. For these introductory comments, let us focus on the charge 2 BPS monopole and return to the others later in the text. Here, we have a one parameter family of solutions

$$0 = \eta^2 + \frac{\boldsymbol{K}(k)^2}{4}\left(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1\right), \tag{1.1}$$

where $K(k)$ is the complete elliptic integral with elliptic modulus k. The scalings of η and ζ here are fixed by the constraints we have mentioned. With these normalisations, this curve is not expressible over $\overline{\mathbb{Q}}$: for if $k \notin \overline{\mathbb{Q}}$ then at least one of $k\, K(k)$ or $K(k)$ must be transcendental; finally, a theorem of Schneider says that if k is algebraic, then $K(k)$ is transcendental. We say the curve is transcendental. Our goal is to establish the following theorem:

Theorem 1.1 *Let $\mathcal{C}$ be a smooth spectral curve of a charge $n \geq 2$ Euclidean BPS monopole. Then, $\mathcal{C}$ is not defined over $\overline{\mathbb{Q}}$.*

We may for the purposes of this introduction understand a curve $\mathcal{C}$ to be defined over a number field as one that can be described by the completion of a curve in $\mathbb{C}^2$ defined by a polynomial with algebraic coefficients. We will return to this point later but note here that the transformation $\tilde{\eta} = 2\eta/\boldsymbol{K}(k)$ of (1.1) (a $\mathbb{C}$-isomorphism that preserves the period matrix of the curve) yields (for $k \in \mathbb{Q}$) a curve definable over $\overline{\mathbb{Q}}$ yet that is not the spectral curve of a monopole. The theorem is a consequence of work of Wüstholz on the vanishing or transcendence of certain periods and the work of a number of authors in developing Hitchin's constraints. Simply put, the integrable system requires certain periods to be integral, but Wüstholz says this cannot be so. We first review the spectral curve and Hitchin's constraints sufficient to indicate their implications for certain periods and then prove the theorem. We conclude with some examples. We remark that Hitchin's construction of harmonic maps from the torus into the three sphere also embodies transcendental constraints on a spectral curve [11] (here, two-third kind differentials are required to have integral periods).

2 The monopole spectral curve and Hitchin's constraints

As already noted, BPS magnetic monopoles describe a class of finite energy solutions to a reduction in the anti-self-dual Yang–Mills equations [1,10]. Assuming a static solution (where the connection is independent of the 'time' coordinate), these partial

Table 1 The known spectral curves

	Curve		Symmetry
1	$0=\eta\prod_{l=1}^{m}(\eta^2+l^2\pi^2\zeta^2)$		Rotational
2	$0=\prod_{l=0}^{m}(\eta^2+[l+\frac{1}{2}]^2\pi^2\zeta^2)$		Rotational
3	$0=\eta^2+\frac{\boldsymbol{K}(k)^2}{4}\left(\zeta^4+2(k^2-k'^2)\zeta^2+1\right)$		–
4	$0=\eta^3+a_3(\zeta^6+5\sqrt{2}\zeta^3-1)$	$a_3=\pm\frac{1}{48\sqrt{3}\pi^{3/2}}\,\Gamma\left(\frac{1}{6}\right)^3\Gamma\left(\frac{1}{3}\right)^3$	Tetrahedral
		$=\pm\frac{1}{48\sqrt{3}}\,B\left(\frac{1}{6},\frac{1}{3}\right)^3$	
5	$0=\eta^4+a_4(\zeta^8+14\zeta^4+1)$	$a_4=\frac{3}{1024\pi^2}\,\Gamma\left(\frac{1}{4}\right)^8=\frac{3}{256}\,B\left(\frac{1}{4},\frac{1}{2}\right)^4$	Octahedral
6	$0=\eta\left(\eta^4-4a_4(\zeta^8+14\zeta^4+1)\right)$		Octahedral
7	$0=\eta\left(\eta^6+a_7\zeta(\zeta^{10}+11\zeta^5-1)\right)$	$a_7=\frac{1}{64\pi^3}\,\Gamma\left(\frac{1}{6}\right)^6\Gamma\left(\frac{1}{3}\right)^6=\frac{1}{64}\,B\left(\frac{1}{6},\frac{1}{3}\right)^6$	Icosahedral
8	$0=\eta^4+36ia\kappa^3\eta\zeta(\zeta^4-1)+3\kappa^4(\zeta^8+14\zeta^4+1)$	$a\in\left(-\sqrt{2}/3^{5/4},\sqrt{2}/3^{5/4}\right)$, κ real half period of	Tetrahedral $a\neq 0$l
		$\wp'(z)^2=4\wp(z)^3-4\wp(z)+12a^2$	Octahedral $a=0$
9	$0=\eta^3-6(a^2+4\epsilon)^{1/3}\kappa^2\eta\zeta^2+2i\kappa^3a(\zeta^5-\zeta)$	$a\in\mathbb{R}$, $\epsilon=\pm1$, κ real half period of	$(x_1,x_2,x_3)\to(x_2,-x_1,-x_3)$
		$\wp'(z)^2=4\wp(z)^3-3(a^2+4\epsilon)^{2/3}\wp(z)+4\epsilon$	$a=\pm2$, $\epsilon=-1$,
			Tetrahedral
10	$0=\eta^3+\alpha\eta\zeta^2+\beta\zeta^6+\gamma\zeta^3-\beta$	With $(a,g)=(\alpha/\beta^{2/3},\gamma/\beta)$ we obtain	C_3
		$g=g(a)$ by requiring a particular period	$\alpha=0$, Tetrahedral
		vanish on the quotient genus 2 curve	

differential equations take the form

$$\star F = D\Phi,$$

where F is the curvature of the connection A for gauge group G with Lie algebra $\mathfrak{g}$, Φ is a Higgs field, $\star$ is the Hodge-$\star$ operator for $\mathbb{R}^3$ (though other 3-manifolds may also be considered). Suitable boundary conditions need to be specified so as to ensure finiteness of the energy; these boundary conditions allow one to define the Higgs field over the 2-sphere "at infinity" and the "charge" of the monopole is the first Chern class of this bundle. Two approaches exist to the problem of constructing these solutions. Just as the self-duality equations may be understood in terms of twistor theory, a reduction in this exists describing monopoles, where mini-twistor space $T\mathbb{P}^1$, the space of lines in $\mathbb{R}^3$, plays the corresponding role. The zero-curvature equation arising from the anti-self-dual Yang–Mills equations leads to $[D_3 - i\Phi, D_{\bar{z}}] = 0$ and considering the operator $D_3 - i\Phi$ (which depends holomorphically on z). The collection of lines in $\mathbb{R}^3$ for which this operator has square integrable solutions forms a curve $\mathcal{C} \subset T\mathbb{P}^1$. A second approach was discovered by Nahm in which the solutions to the partial differential equations were constructed in terms of solutions to a set of matrix ODE's ("Nahm's Equations") and an associated (ordinary) differential operator built from these; this is the Nahm correspondence. Nahm's equations may be viewed as an integrable system and have a Lax pair formulation and corresponding spectral curve given by the vanishing of a characteristic polynomial $P(\eta, \zeta) = \det(\eta - L(\zeta)) = 0$. This spectral curve is precisely the curve $\mathcal{C}$ arising from the mini-twistor viewpoint, and the spectral parameter ζ and η in this approach are identified with coordinates of $T\mathbb{P}^1$. Constructing regular solutions from both approaches becomes one of specifying $\mathcal{C}$, and it was Hitchin [10] who gave necessary and sufficient algebro-geometric constraints on the spectral curve of this integrable system to yield BPS monopoles. The work [4] has shown how one may reconstruct the gauge field data in terms of the function theory of $\mathcal{C}$.

The physical interpretation of the surface $T\mathbb{P}^1$ embues a significance to these coordinates of the spectral curve. Let ζ be a coordinate on $\mathbb{P}^1$ (the direction of the line above) and $(\eta, \zeta) \to \eta \frac{d}{d\zeta} \in T\mathbb{P}^1$ be coordinates for $T\mathbb{P}^1$. The fact the tangent bundle is of degree 2 means that a section may be expressed in terms of a quadratic polynomial; for example,

$$\eta = (x_2 - x_1) - 2x_3\,\zeta - (x_2 + x_1)\,\zeta^2.$$

One can then relate spatial symmetries with fractional linear symmetries of (η, ζ). The spectral curve $\mathcal{C}$ is then specified by the vanishing of the polynomial $P(\eta, \zeta)$ where

$$P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r.$$

This curve, which we will assume smooth, has genus $(n-1)^2$. We note that $T\mathbb{P}^1$ has the antiholomorphic involution $\iota : (\eta, \zeta) \to (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$ which reverses the orientation of lines. We may cover $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ by the two patches $\widehat{\mathcal{U}}_{0,1}$ corresponding to the pre-images of the standard cover $\mathcal{U}_{0,1}$ of $\mathbb{P}^1$. Let $\mathcal{L}^\lambda(m)$ the holomorphic line bundle on $T\mathbb{P}^1$ with transition function $g_{01} = \zeta^m \exp(-\lambda\eta/\zeta)$; setting $\mathcal{L}^\lambda := \mathcal{L}^\lambda(0)$, then $\mathcal{L}^\lambda(m) \equiv \mathcal{L}^\lambda \otimes \pi^*\mathcal{O}(m)$. Hitchin's constraints are then:

H1: $\mathcal{C}$ is real with respect to ι,
H2: $\mathcal{L}^2$ is trivial on $\mathcal{C}$, and $\mathcal{L}^1(n-1)$ is real,
H3: $H^0(\mathcal{C}, \mathcal{L}^s(n-2)) = 0$ for $s \in (0, 2)$.

Here, the parameter s describing the linear flow of Hitchin's line bundles corresponds to the 'time' of the integrable systems evolution, this linear evolution being described by a straight line in $\mathrm{Jac}(\mathcal{C})$. The third condition says that this *real* straight line does not intersect the theta divisor for $s \in (0, 2)$, while it does at $s = 0, 2$. Only the first of these constraints is easily implemented. The reality conditions **H1** mean $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar\zeta})}$, and as a consequence $a_r(\zeta)$ is given by $2r+1$ (real) parameters. It is the difficulty of making effective **H2, 3** that makes the construction of monopoles so difficult.

Ercolani and Sinha [9] made the initial study of **H2**. The triviality of $\mathcal{L}^2$ means that there exists a nowhere-vanishing holomorphic section; in terms of our cover and transition functions, we have $f_0(\eta, \zeta) = \exp\{-2\eta/\zeta\} f_1(\eta, \zeta)$ with f_i holomorphic in $\widehat{\mathcal{U}}_i$. The logarithmic differential of f_0 thus yields a meromorphic differential for which $\exp \oint_\gamma \mathrm{dlog}\, f_0 = 1$ for all $\gamma \in H_1(\mathbb{Z}, \mathcal{C})$, and the flow in the Jacobian is governed by the meromorphic differential

$$\gamma_\infty(P) = \frac{1}{2}\,\mathrm{dlog}\, f_0(P) + \iota\pi \sum_{j=1}^{g} m_j\, \omega_j(P).$$

Here, the ω_i are canonically $\mathfrak{a}$-normalized holomorphic differentials ($\oint_{\mathfrak{a}_k} \omega_j = \delta_{jk}$) and we add an appropriate linear combination so that $\oint_{\mathfrak{a}_k} \gamma_\infty = 0$. These observations, together with the Riemann bilinear relations, yield

Theorem 2.1 (Ercolani–Sinha Constraints [8,9,13]) *The following are equivalent:*

(1) *$\mathcal{L}^2$ is trivial on $\mathcal{C}$.*
(2) $2\boldsymbol{U} \in \Lambda \iff \boldsymbol{U} = \frac{1}{2\pi\iota}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau\boldsymbol{m}$, *where Λ is the period lattice.*
(3) *There exists a 1-cycle $\mathfrak{es} = \boldsymbol{n}\cdot\mathfrak{a} + \boldsymbol{m}\cdot\mathfrak{b}$ such that every holomorphic differential*

$$\Omega = \left[\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)\right] d\zeta/\partial_\eta P$$

has period $\oint_{\mathfrak{es}} \Omega = -2\beta_0$. This 1-cycle satisfies $\iota_*\mathfrak{es} = -\mathfrak{es}$.

A number of remarks are perhaps in order.

(1) Hitchin's constraints do not require $\mathcal{C}$ to be irreducible, and a number of the examples of Table 1 are in fact reducible. These examples show that $\mathcal{C}$ is not defined over $\overline{\mathbb{Q}}$ here as well.

(2) One can say more about $2\boldsymbol{U}$: it is in fact a primitive vector in the period lattice. By tensoring with a section of $\pi^*\mathcal{O}(n-2)|_{\mathcal{C}}$, we obtain a map $\mathcal{O}(\mathcal{L}^s) \hookrightarrow \mathcal{O}(\mathcal{L}^s(n-2))$ and so the vanishing of $H^0(\mathcal{C}, \mathcal{O}(\mathcal{L}^s(n-2)))$ also entails that $H^0(\mathcal{C}, \mathcal{O}(\mathcal{L}^s)) = 0$ for $s \in (0, 2)$; this means that $2\boldsymbol{U}$ is in a primitive vector.

(3) If $\mathcal{A}$ (respectively, $\mathcal{B}$) denotes the matrix of $\mathfrak{a}$- periods (respectively, $\mathfrak{b}$-periods) for a basis of holomorphic differentials, this may be chosen so that (with $\omega = \left(\eta^{n-2}/\partial_\eta P\right) d\zeta$ the final basis element) $(\mathbf{n}, \mathbf{m}) \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = -2(0, \ldots, 0, 1)$. That is the Ercolani–Sinha constraints reflect rational relations between the periods.

(4) It is possible for a curve to satisfy **H2** and yet fail **H3** as seen with

Theorem 2.2 (Braden–Enolski [8]) *To each pair of relatively prime integers $(n, m) = 1$ for which $(m+n)(m-2n) < 0$, we obtain a solution to the Ercolani–Sinha constraints for the curve*

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \qquad b, \chi \in \mathbb{R},$$

as follows. First, we solve for t, where

$$\frac{2n-m}{m+n} = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, 1-t)}. \tag{2.1}$$

Then, $b = \dfrac{1-2t}{\sqrt{t(1-t)}}$, $t = \dfrac{-b+\sqrt{b^2+4}}{2\sqrt{b^2+4}}$. With $\alpha^6 = t/(1-t)$, then

$$\chi^{\frac{1}{3}} = -(n+m)\,\frac{2\pi}{3\sqrt{3}}\,\frac{\alpha}{(1+\alpha^6)^{\frac{1}{3}}}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right).$$

Provided we can solve (2.1), we then have a countable number of curves satisfying **H1,2**. Now, while proving (amongst others) the formula of Ramanujan,

$$\frac{27}{4\pi} = \sum_{m=0}^{\infty} \frac{(2+15m)(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3\,\left(\frac{27}{2}\right)^m},$$

Berndt, Bhargava, and Garvan [3] introduced the following extension of a modular equation of degree n: a *modular equation of degree n and signature r* ($r = 2, 3, 4, 6$) is defined to be a relation between α, β of the form

$$n\,\frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)} = \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)}.$$

This theory enables one to solve (2.1). (The resulting t and b are algebraic and χ transcendental.) Apart from the case $(n, m) = (1, 0), (1, 1)$ (with $t = 1/2 \pm 5\sqrt{3}/18$ and $b = \pm 5\sqrt{2}$) when the curve exhibits tetrahedral symmetry, it is believed no member of this family satisfies **H3** and a conjecture exists [7] for the number of sections the family of line bundles has for $s \in (0, 2)$.

3 Proof and discussion of the theorem

Theorem 1.1 follows from Theorem 2.1 and a deep theorem of Wüstholz.

Theorem 3.1 (Wüstholz [15]) *Let X be a smooth quasi-projective variety over a number field $\mathbb{K}$ possessing a $\mathbb{K}$-rational point and $\omega \in H^0(X, \Omega^1_{X/\mathbb{K}})$ a closed holomorphic differential on X Then, $\int_\gamma \omega$ $(\gamma \in H_1(X, \mathbb{Z}))$ are either zero or transcendental.*

An exposition of this theorem may be found in [2]. This theorem yields many of the classical transcendence results (see [2][§6.3]) including the theorem of Schneider, noted in the introduction, that the periods of an elliptic integral with rational elliptic modulus are transcendental.

To prove the theorem, let us first recall that given a number field $\mathbb{K}$ and $\mathbb{K} \hookrightarrow \mathbb{C}$ we have for a variety $\mathcal{X}$ over $\mathbb{K}$ the schemes

$$\begin{array}{ccc} \mathcal{X} \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{K}) \end{array}$$

The morphism $\mathcal{X} \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ is called the base change of the morphism $\mathcal{X} \to \mathrm{Spec}(\mathbb{K})$, and the fibre product $\mathcal{X} \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ always exists. We say a variety $\mathcal{C}$ over $\mathbb{C}$ is defined over a subfield $\mathbb{K} \subset \mathbb{C}$ if there exists a variety $\mathcal{X}$ over $\mathbb{K}$ such that

$$\mathcal{C} \cong \mathcal{X} \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{C}).$$

Here, we have an isomorphism or birational equivalence over $\mathbb{C}$. Though birational transformations change periods and differentials, the period matrix of the curve is (modulo integral symplectic transformations) fixed in the Siegel upper half plane. In the present setting, periods are being specified and so arbitrary birational transformations are not allowed. The birational transformation of (1.1) noted in the introduction, whilst resulting in a curve defined over $\overline{\mathbb{Q}}$, destroys the integrality of the period required for the monopoles regularity. At root is that Hitchin's conditions specify more than the curve, they also describe a family of line bundles on the curve. (These line bundles enable one to reconstruct the gauge field via the Atiyah–Ward ansatz used by Hitchin.) This has been encoded by the very concrete transition functions in the choice of coordinates describing the curve. With these preliminary remarks, we may now prove the theorem.

Proof Specialising to the case when Wüstholz's variety X is our spectral curve suppose $\mathcal{C}$, and so the polynomial $P(\eta, \zeta)$ is defined over $\overline{\mathbb{Q}}$. We may let $\mathbb{K}$ be the a number field that contains the coefficients of P and the roots of $P(0, \zeta) = a_n(\zeta)$; thus, $\mathcal{C}$ contains a $\mathbb{K}$-rational point. Consider the holomorphic differential $\omega = \left(\eta^{n-2}/\frac{\partial P}{\partial \eta}\right) d\zeta$ (recall $n \geq 2$ in the theorem). We are assuming $\mathcal{C}$ smooth, and so the conditions of Theorem 3.1 are satisfied; thus, the periods of ω are either zero or transcendental. But this contradicts Theorem 2.1 and so $\mathcal{C}$ cannot be defined over $\overline{\mathbb{Q}}$. Thus, Theorem 1.1 is established. $\square$

4 Examples

The known spectral curves in Table 1 all exhibit symmetries; these simplify the problem. Reference [5] shows how questions about the Ercolani–Sinha vector reduce to questions for the quotient curve; the flows of the integrable system are also shown there to simplify using a theorem of Fay and Accola. Examples 4-9 of Table 1 all exhibit a Platonic symmetry group [12], which evidences itself in the Klein polynomials of the appropriate spectral curves; these curves all quotient to an elliptic curve. The elliptic curves for the discrete monopole configurations of examples 4-8 each yield a Beta function of rational arguments, the transcendence of which is also a result Schneider. The transcendence of the one-parameter families 8, 9 both follow by a similar argument to that of the introduction using Schneider's result on the transcendence of the periods of the Weierstrass $\wp$-function for algebraic $g_{2,3}$. Although the examples 1, 2 (for $n \geq 3$), 6, 7 are for reducible curves and so outwith the theorem, they too are transcendental. The final curve has C_3 symmetry and quotients over a genus 2 curve [6]. The transcendence of the periods here requires Theorem (3.1); a genus 2-variant of the AGM due to Richelot may be used for their computation.

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