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On the Strong Convergence of Subgradients of Convex Functions

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Abstract In this paper, results on the strong convergence of subgradients of convex functions along a given direction are presented; that is, the relative compactness (with respect to the norm) of the union of subdifferentials of a convex function along a given direction is investigated.

Keywords Convexity · Subdifferentials · Strong convergence of subgradients · Gâteux derivative

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1 Introduction

There are several results on the strong convergence of subgradients of a sequence of convex functions defined on a Banach space. The most celebrated result is the Attouch theorem; see, for example, [1], where the equivalence of Mosco convergence of lower semicontinuous convex functions to the Painleve–Kuratowski graph convergence of their subdifferentials is established on reflexive Banach space. There are also results extending the Attouch theorem to general Banach spaces; see, for example, [2–7] and references therein. To the best of our knowledge all known results are of the form: there are sequences of points and subgradients such that the strong limits of sequences

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of subgradients exist (limits with respect to the norm of the space); see, for example, Theorem 3.1 in [7]. This is inconvenient. Simply we want to have subgradients with a desired property. The postulate of existence ("there are") does not allow one to guarantee that the subgradients are as good as it is needed. This disadvantage can be observed also, when the directional derivative is calculated. Namely, the difference quotients form sequences of functions with respect to directions, whenever we consider the limit over a discrete subset. It is natural to ask about the convergence of subgradients of this functions; see, for example, Giannessi's questions, which are recalled in (5). The question, about the existence of a convergent subsequence (at least) for this sequence of function, is the question on the existence of a convergent sequence of subgradients along a direction. In a finite-dimensional case, the existence of convergent subsequences is guaranteed by the continuity of the convex function under investigation. However, there can be subsequences with different limits; see [8]; see also [9–11]. It turns out that the set of "wrong directions" (there in no unique limit) has the Lebesgue measure equal zero; see Lemma 3.1. In infinite-dimensional setting, it is hard to expect the convergence. Thus, the basic question in this case, concerning directional convergence of subgradients, is: *when does the union of subdifferentials along a given direction form a relatively compact set (with respect to the norm topology)*? We should also ask about the uniqueness of the limit, which is the essence of Giannessi's questions in the finite-dimensional setting. In the infinite-dimensional case, results of this type are rather unknown, but it would be convenient to have such results at hand. For instance, when the limit exists, then the limiting subgradients inherit properties of a convergent sequence, like: size of norm, being in a specified closed set, a good behaving with respect to the weak convergence of arguments, and so on. In Sect. 3, we present a result which guarantees the relative compactness for some special classes of convex functions; see Theorem 3.1. In Lemmas 2.2 (in the Hilbert space setting) and 3.2 (in the reflexive Banach space setting) examples of functions from the class are provided too.

2 Preliminaries

In this section, some basic notions and their properties are gathered.

In the sequel, $(X, \|\cdot\|)$ stands for a real normed space, X^* for its dual space and $\mathbb{H}$ for a real Hilbert space (with a real inner product). The weak convergence is denoted by $\xrightarrow{weak}$, and the limit from the right is denoted by $t \downarrow a$, which means that $t > a$ and $t \longrightarrow a$.

For every real $r > 0$ and every $x \in X$ we denote by $\mathbb{B}_X(x, r)$ (resp. $\mathbb{B}_X[x, r]$) the open (resp. closed) ball centered at x and of radius r, the sphere is denoted by $\mathbb{S}_X[x, r] := \{y \in X \ : \ \|y - x\| = r\}$ and $\mathbb{S}_X := \mathbb{S}_X[0, 1]$, "cl" stands for the topological closure. A point x is an interior point of D if there exists an open ball centered at x which is completely contained in D, all interior points of D are denoted by int D. For given $x, y \in X$ we put

$$[x, y] := \{tx + (1 - t)y \ : \ t \in [0, 1]\}$$

and

$$]x, y[:= \{tx + (1-t)y \ : \ t \in]0, 1[\}.$$

Following [12, page 2] span M stands for the linear hull of M and it is the smallest (in the sense of inclusion) linear subspace of X containing M. We call the set

$$D[x, x + \mu_0 \mathbb{B}_X[w_0, \beta_0]] := \operatorname{conv}(x \cup (x + \mu_0 \mathbb{B}_X[w_0, \beta_0]))$$

drop, where $\mu_0 > 0$, $\beta_0 > 0$, $x \in X$ and $w_0 \in X \setminus \{0\}$ are given; we refer to [13] and references therein for information on drops. For a given function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ the domain of f is defined by

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

We say that $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is M_0-Lipschitz continuous on a subset $D \subset \operatorname{dom} f$, whenever $|f(u) - f(v)| \leq M_0 \|u - v\|$ for all $u, v \in D$.

Let $p :]0, \infty[\times \{1, 2, 3, \ldots\} \longrightarrow \mathbb{R}$ be a function. Observe that

$$\liminf_{\theta \downarrow 0} \liminf_{i \longrightarrow \infty} p(\theta, i) \geq \liminf_{i \longrightarrow \infty, \theta \downarrow 0} p(\theta, i), \tag{1}$$

where

$$\liminf_{i \longrightarrow \infty, \theta \downarrow 0} p(\theta, i) := \sup_{\epsilon > 0, k \in \mathbb{N}} \inf_{0 < \theta < \epsilon, k < i} p(\theta, i).$$

For every nonempty set $S \subset \mathbb{H}$ the *distance function* from the set S is denoted by $d_S(\cdot)$, that is,

$$d_S(x) = \inf_{u \in S} \|u - x\|, \quad \forall x \in X.$$

Let us assume that $x \in X$ and $\{s_i\}_{i \in \mathbb{N}}$ is a sequence of points from a subset $S \subset X$ such that $\lim_{i \longrightarrow \infty} \|x - s_i\| = d_S(x)$. It is valuable to have results preserving a relative compactness of the sequence, see, for example, Proposition 3.1 in [14]. Below a result of this kind is presented in the Hilbert space setting.

Lemma 2.1 *Let $\mathbb{H}$ be a Hilbert space with a dimension greater than one, W be a closed subspace of $\mathbb{H}$ (thus W is a Hilbert space too), $S \subset \mathbb{H}$ be a nonempty subset and $x \notin \operatorname{cl} S$. Suppose that $y \notin \operatorname{cl} S$, $\|y - x\| = d_S(x)$, $\langle w, x - y \rangle = 0$ for all $w \in W$ and that for some $t \in]0, 1]$ and all $u \in \mathbb{H}$ such that*

$$u - (1-t)x - ty \in \mathbb{B}_{W + \operatorname{span}\{x - y\}} [0, t\|x - y\|]$$

we have

$$d_S(u) \geq \|u - y\|.$$

Then, if $\{s_i\}_{i\in\mathbb{N}}$ *is a sequence of points from a subset* $S \subset X$ *such that*

$$\lim_{i\longrightarrow\infty} \|x - s_i\| = d_S(x),$$

then

$$\lim_{i\longrightarrow\infty} \sup_{w\in\mathbb{S}_W} \langle w, x - s_i\rangle = 0. \tag{2}$$

Proof Let assume the contrary, that is for some $\epsilon > 0$ and for all $i \in \mathbb{N}$ there are $w_i \in \mathbb{S}_W$ such that

$$\limsup_{i\longrightarrow\infty} \langle w_i, x - s_i\rangle > 0.$$

Choose $i_n, k_n \in \mathbb{N}$ such that

$$\lim_{n\longrightarrow\infty} \min\{i_n, k_n\} = \infty \text{ and } \lim_{n\longrightarrow\infty} \langle w_{i_n}, x - s_{i_n}\rangle > 0,$$

and $\|x - s_{i_n}\|^2 \leq k_n^{-2} + \|x - y\|^2$ for all $n \in \mathbb{N}$. Let us define

$$u_n := x - tk_n{}^{-1}\|x - y\|w_{i_n} + tk_n{}^{-2}(y - x)$$

and observe that

$$\|u_n - y\|^2 = \left(1 - 2tk_n{}^{-2} + t^2k_n{}^{-4} + t^2k_n{}^{-2}\right)\|x - y\|^2$$

and

$$\|u_n - (1-t)\,x - ty\|^2 = \left(t^2 - t^2k_n{}^{-2} + t^2k_n{}^{-4}\right)\|x - y\|^2.$$

Moreover,

$$\begin{aligned} d_S^2(u_n) &\leq \|u_n - s_{i_n}\|^2 = \|x - tk_n{}^{-1}\|x - y\|w_{i_n} + tk_n{}^{-2}(y - x) - s_{i_n}\|^2 \\ &\leq k_n^{-2} + \left(1 + t^2k_n{}^{-2} + t^2k_n{}^{-4}\right)\|x - y\|^2 - 2tk_n{}^{-1}\|x - y\|\langle w_{i_n}, x - s_{i_n}\rangle \\ &\quad + 2tk_n{}^{-2}\langle y - x, x - s_{i_n}\rangle. \end{aligned}$$

Hence, for $n \in \mathbb{N}$ large enough we get

$$d_S(u_n) < \|u_n - y\| \text{ and } u_n - (1-t)x - ty \in \mathbb{B}_{W+\text{span}\,\{x-y\}}\left[0, t\|x - y\|\right],$$

which contradicts the assumptions of the lemma. □

The Asplund function, see [15, (1) page 234], is a good tool to investigate the distance function since its convexity, that is the function

$$f_S : \mathbb{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

defined as follows

$$\forall x \in \mathbb{H},\ f_S(x) := 2^{-1}\left(\|x\|^2 - d_S^2(x)\right) = \sup_{s\in S}\langle s, x\rangle - 2^{-1}\|s\|^2, \tag{3}$$

where $S \subset \mathbb{H}$ is a given subset. Below a directional behaving of this function is characterized.

Lemma 2.2 *Let a sequence $\{t_i\}_{i\in\mathbb{N}}$ be such that $t_i > 0$ for all $i \in \mathbb{N}$ and $t_i \downarrow 0$, and let $\mathbb{H}$ be a Hilbert space with a dimension greater than two, W be a closed subspace of $\mathbb{H}$, $S \subset \mathbb{H}$ be a nonempty subset and $x \notin \operatorname{cl} S$, $h \in \mathbb{S}_{\mathbb{H}}$, $S_h \subset S$, $\mu > 0$ be such that $d_S(x+t_ih) = d_{S_h}(x+t_ih)$ for all $t_i \in [0, \mu]$. Suppose that $y \notin \operatorname{cl} S$, $\|y-x\| = d_S(x)$, $\langle w, x-y\rangle = 0$ for all $w \in W$ and that for some $t \in]0, 1]$ and all $u \in \mathbb{H}$ such that*

$$u - (1-t)x - ty \in \mathbb{B}_{W+\operatorname{span}\{x-y\}}[0, t\|x-y\|]$$

we have

$$d_{S_h}(u) \geq \|u-y\|.$$

Then, $f_S'(x; h) = f_{S_h}'(x; h)$ and

$$\begin{aligned} 0 \ &\geq \limsup_{\theta\downarrow 0}\limsup_{i\longrightarrow\infty} \frac{\sup_{z\in\mathbb{B}_Y[0,\theta]} p(t_i, z)}{\theta} \\ &\geq \liminf_{i\longrightarrow\infty,\,\theta\downarrow 0} \frac{\sup_{z\in\mathbb{B}_Y[0,\theta]} p(t_i, z)}{\theta}, \end{aligned} \tag{4}$$

where $Y := \{w \in W : \langle w, h\rangle = 0 \text{ and } \langle w, x\rangle = 0\}$ and

$$p(t_i, z) := \frac{f_{S_h}(x + t_i(h+z)) - f_{S_h}(x + t_ih)}{t_i},$$

and f_{S_h} is the Asplund function for the set S_h, that is, for all $y \in \mathbb{H}$ we put $f_{S_h}(y) := 2^{-1}(\|y\|^2 - d_{S_h}^2(y))$; see (3).

Proof Fix $\theta > 0$ and sequences $\{s_i\}_{i\in\mathbb{N}}$, $\{z_i\}_{i\in\mathbb{N}}$ such that $s_i \in S_h$,

$$z_i \in \mathbb{B}_Y[0,\theta]\setminus\{0\},\ \|x + t_i(h+z_i) - s_i\|^2 \leq d_{S_h}^2(x + t_i(h+z_i)) + t_i^2\|z_i^2\|$$

for all $i \in \mathbb{N}$, and $t_i \downarrow 0$. We have

$$\begin{aligned} p(t_i, z_i) = & \frac{\|x + t_i(h + z_i)\|^2 - \|x + t_i h\|^2}{2t_i} \\ & + \frac{d_{S_h}^2(x + t_i h) - d_{S_h}^2(x + t_i(h + z))}{2t_i} \leq t_i \|z_i\|^2 \\ & + \frac{\|x + t_i h - s_i\|^2 - \|x + t_i(h + z_i) - s_i\|^2}{2t_i} \\ = & \, 2^{-1} t_i \|z_i\|^2 - \langle z_i, x - s_i \rangle. \end{aligned}$$

It follows from (2) that (4) holds true, whenever the inequality in (1) is taken into account. □

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function, which is finite at $x \in X$. The subdifferential of f at $x \in X$ is defined by

$$\partial f(x) := \{x^* \in X^* : \forall h \in X, \langle x^*, h \rangle \leq f(x + h) - f(x)\}.$$

3 Relative Compactness of Sets of Subgradients

Let us recall Giannessi's questions; see [8], see also [9–11] for examples of convex functions in two-dimensional spaces, for which the limit in (5) does not exist:

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, with $n \geq 2$, be a convex function, and set

$$x(t) := (t, 0, \ldots, 0) \in \mathbb{R}^n,$$

with $t \in \mathbb{R}$. Assume that $\nabla f(x(t))$ exists for every $t > 0$, and consider the following limit:

$$\lim_{t \downarrow 0} \nabla f(x(t)). \tag{5}$$

We conjecture that the above limit may not exist. Hence, however, the question is still open. The above question can be generalized in several ways. For instance, $x(t)$ may represent a curve having the origin as endpoint instead of a ray; $\mathbb{R}^n$ may be replaced with an infinite-dimensional space.

Below directions along which we have the weak* convergence of subgradients are indicated.

Lemma 3.1 *Let $(X, \|\cdot\|)$ be a reflexive Banach space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, $x \in \operatorname{dom} f$ and $M \geq 0$, $w_0 \in X$, $x^* \in X^*$ be given. If $f'(x; w_0)$ is a finite real number (the directional derivative at x along w_0 is finite) such that*

$$\forall h \in \mathbb{S}_X, \ \lim_{t \downarrow 0} \frac{f'(x; w_0 + th) - f'(x; w_0) - \langle x^*, th \rangle}{t} = 0, \tag{6}$$

then for all sequences $\{t_i\}_{i\in\mathbb{N}}$ *such that* $t_i > 0$, $\mathbb{B}_{X^*}[0, M] \cap \partial f(x + t_i w_0) \neq \emptyset$ *for all* $i \in \mathbb{N}$*, and* $t_i \downarrow 0$ *we have*

$$w - \lim_{i\longrightarrow\infty} \mathbb{B}_{X^*}[0, M] \cap \partial f(x + t_i w_0) = \{x^*\}, \tag{7}$$

and consequently

$$w - \lim_{t\downarrow 0} \mathbb{B}_{X^*}[0, M] \cap \partial f(x + t w_0) = \{x^*\}, \tag{8}$$

whenever $\mathbb{B}_{X^*}[0, M] \cap \partial f(x + t w_0) \neq \emptyset$ *for all* $t \in]0, \delta[$ *for some* $\delta > 0$*, where*

$$w - \lim_{i\longrightarrow\infty} \mathbb{B}_{X^*}[0, M] \cap \partial f(x + t_i w_0) := \{y^* \in X^* : x_i^* \overset{weak}{\longrightarrow} y^*,$$
$$\textit{with } x_i^* \in \mathbb{B}_{X^*}[0, M] \cap \partial f(x + t_i w_0) \textit{ for all } i \in \mathbb{N}\}.$$

Moreover,

$$f'(x; w_0) = \langle x^*, w_0\rangle. \tag{9}$$

Proof Let us assume that (6) holds true. The equality

$$f'(x; w_0) = \langle x^*, w_0\rangle$$

is easy to verify by a simple algebra. In fact for all $t > 0$ we have

$$f'(x; w_0 + th) - f'(x; w_0) - \langle x^*, th\rangle \leq f'(x; w_0) + f'(x; th) - f'(x; w_0) - \langle x^*, th\rangle.$$

Thus, it follows from (6) that for all $h \in X$ we have

$$0 \leq f'(x; h) - \langle x^*, h\rangle.$$

Again using (6) we get $0 \leq -f'(x; w_0) + \langle x^*, w_0\rangle$, which implies (9).

If $w_0 = 0$, then $\partial f(x) = \{x^*\}$ and we are done. Assume that $w_0 \neq 0$. Take sequences $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i > 0$ for all $i \in \mathbb{N}$ and $\{x_i^*\}_{i\in N}$ such that $x_i^* \in \mathbb{B}_{X^*}[0, M] \cap \partial f(x + t_i w_0)$ for all $i \in \mathbb{N}$. Using the Eberlein–Šhmulyan theorem; see Appendix to Chapter V, Section, page 141 in [16], we may assume that the sequence $\{x_i^*\}_{i\in\mathbb{N}}$ is weakly convergent to some $y^* \in \partial f(x)$, otherwise we choose a proper subsequence. Fix $\epsilon > 0$, $h \in \mathbb{S}_X$ and take $t > 0$ such that

$$\frac{f'(x; w_0 + th) - f'(x; w_0) - \langle x^*, th\rangle}{t} \leq \epsilon.$$

Notice that

$$\begin{aligned}\langle y^*, h\rangle &= \lim_{i\longrightarrow\infty}(\langle y^*, t^{-1}w_0+h\rangle - \langle x_i^*, t^{-1}w_0\rangle)\\ &\le \lim_{i\longrightarrow\infty}\frac{f(x+t_i(w_0+th))-f(x)-(f(x+t_iw_0)-f(x))}{t_it}\\ &\le \frac{f'(x;w_0+th)-f'(x;w_0)}{t}\le\langle x^*, h\rangle+\epsilon,\end{aligned}\tag{10}$$

thus, since $h\in\mathbb{S}_X$ is arbitrary, we get $y^*=x^*$, which translates (7) and the equality

$$w-\lim_{t\downarrow 0}\mathbb{B}_{X^*}[0,M]\cap\partial f(x+tw_0)=\{x^*\}.$$

□

Notice that the weak convergence implies the strong one, whenever $X=\mathbb{R}^n$, thus using Lemma 3.1 we can pick directions along which the strong convergence is preserved and consequently the limit in (5) exists. Below we provide a property which ensures the strong convergence of subgradients. Roughly speaking the union of subdifferentials along the "good" direction forms a relatively compact set with respect to the norm topology. This property is unexpected in the infinite-dimensional setting. Let us distinguish a family of auxiliary functions. Namely, for a given sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i>0$ for all $i\in\mathbb{N}$, and $t_i\downarrow 0$, a given closed subspace $Y\subset X$ and a given positive number $\rho>0$ let us put

$$\begin{aligned}\mathbb{F}(\{t_i\}_{i\in\mathbb{N}},\rho,Y) := \{p:\{0,t_1,t_2,t_3,\ldots\}\times\mathbb{B}_Y[0,\rho]\longrightarrow[0,\infty[\ :\ \text{such that}\\ \forall s\in\mathbb{B}_Y[0,\rho],\ p(0,s)=0,\ \liminf_{i\longrightarrow\infty,\,\theta\downarrow 0}\frac{\sup_{s\in\mathbb{B}_Y[0,\theta]}p(t_i,s)}{\theta}\le 0\\ \text{and } p(t_i,\cdot)\text{ is upper semicontinuous on }\mathbb{B}_Y[0,\rho]\text{ for all }i\in\mathbb{N}\}.\end{aligned}\tag{11}$$

Our first example of function from the class define above is the function p defined in Lemma 2.2, see (4). Observe that the continuity of the Asplund function ensures that (11) is fulfilled for all $\rho>0$ and all sequences $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i>0$ for all $i\in\mathbb{N}$, and $t_i\downarrow 0$, whenever we put $p(0,\cdot)=0$. Below it is shown that weak continuous convex functions can be also used to construct functions from the class.

Lemma 3.2 *Let $(X,\|\cdot\|)$ be a reflexive Banach space and $f:X\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a convex lower semicontinuous function. Assume that f is M_0-Lipschitz continuous on the set $D[x,x+\mu_0\mathbb{B}_X[h,\beta_0]]$, where $x\in\operatorname{dom} f$, $h\in\mathbb{S}_X$, $M_0\ge 0$, $\mu_0>0$, $\beta_0>0$, $x\in\operatorname{dom} f$ are given. If $Y\subset X$ is a subspace such that f is weak continuous on the set $D[x,x+\mu_0\mathbb{B}_Y[h,\beta_0]]$ and*

$$\inf_{\mu>0}\sup_{z\in\mathbb{B}_Y[0,\mu]}\frac{f'(x;h+z)-f'(x;h)}{\mu}=0,\tag{12}$$

then the function p defined on $[0, \infty[\times\mathbb{B}_Y[0,\rho]$, with $\rho \in]0, \beta_0]$, as follows

$$p(t,s) := \begin{cases} \frac{f(x+t(h+s))-f(x)}{t} - f'(x;h), & \text{whenever } t > 0, s \in \mathbb{B}_Y[0,\rho]; \\ 0, & \text{whenever } t = 0, s \in \mathbb{B}_Y[0,\rho], \end{cases} \tag{13}$$

satisfies

$$0 \geq \liminf_{\theta\downarrow 0} \limsup_{i\longrightarrow\infty} \frac{\sup_{s\in\mathbb{B}_Y[0,\theta]} p(t_i,s)}{\theta} \geq \liminf_{i\longrightarrow\infty,\,\theta\downarrow 0} \frac{\sup_{s\in\mathbb{B}_Y[0,\theta]} p(t_i,s)}{\theta}$$

for all sequences $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i > 0$ for all $i \in \mathbb{N}$, and $t_i \downarrow 0$; thus, $p \in \mathbb{F}(\{t_i\}_{i\in\mathbb{N}}, \rho, Y)$ for all $\rho \in]0, \beta_0[$.

Moreover, for all $t \in]0, \mu_0[$ we have

$$\sup_{s\in\mathbb{B}_Y[0,\beta_0]} \frac{f(x+t(h+s))-f(x+th)}{t} - p(t,s) \leq 0.$$

Proof In order to establish the inequality

$$\sup_{s\in\mathbb{B}_Y[0,\beta_0]} \frac{f(x+t(h+s))-f(x+th)}{t} - p(t,s) \leq 0,$$

let us observe that for all $t \in]0, \mu_0[$ by the convexity we have

$$\begin{aligned} \frac{f(x+t(h+s))-f(x+th)}{t} - p(t,s) &= \frac{f(x+t(h+s))-f(x+th)}{t} \\ &\quad - \frac{f(x+t(h+s))-f(x)}{t} + f'(x;h) \\ &= \frac{f(x)-f(x+th)}{t} + f'(x;h) \leq 0. \end{aligned}$$

Let us fix $\rho \in]0, \beta_0[$ and a sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i > 0$ for all $i \in \mathbb{N}$, and $t_i \downarrow 0$. We have

$$\begin{aligned} &\liminf_{\theta\downarrow 0} \limsup_{i\longrightarrow\infty} \frac{\sup_{s\in\mathbb{B}_Y[0,\theta]} p(t_i,s)}{\theta} \\ &= \liminf_{\theta\downarrow 0} \limsup_{i\longrightarrow\infty} \frac{\sup_{s\in\mathbb{B}_Y[0,\theta]} \frac{f(x+t_i(h+s))-f(x)}{t_i} - f'(x;h)}{\theta}. \end{aligned}$$

If

$$\epsilon < \liminf_{\theta\downarrow 0} \limsup_{i\longrightarrow\infty} \frac{\sup_{s\in\mathbb{B}_Y[0,\theta]} p(t_i,s)}{\theta}$$

for some $\epsilon > 0$, then there is $\bar{\theta} \in]0, \rho]$ such that for all $\theta \in]0, \bar{\theta}]$ we have

$$\epsilon < \limsup_{i \longrightarrow \infty} \frac{\sup_{s \in \mathbb{B}_Y[0,\theta]} p(t_i, s)}{\theta}.$$

Fix $\theta \in]0, \bar{\theta}]$ and choose $i_n < i_{n+1}$, $s_n \in \mathbb{B}_Y[0, \theta]$ such that $t_{i_n} > t_{i_{n+1}}$, and

$$\epsilon < \frac{\frac{f(x+t_{i_n}(h+s_n))-f(x)}{t_{i_n}} - f'(x; h)}{\theta}$$

for all $n \in \mathbb{N}$ and $s_n \overset{weak}{\longrightarrow} \bar{s} \in \mathbb{B}_Y[0, \theta]$ (keep in mind the reflexivity of the space). Take $m > n$ and observe that by the convexity and the weak continuity of f we have

$$\begin{aligned} \epsilon &< \frac{\frac{f(x+t_{i_m}(h+s_m))-f(x)}{t_{i_m}} - f'(x; h)}{\theta} \\ &\leq \frac{\frac{f(x+t_{i_n}(h+s_m))-f(x)}{t_{i_n}} - f'(x; h)}{\theta} \longrightarrow \frac{\frac{f(x+t_{i_n}(h+\bar{s}))-f(x)}{t_{i_n}} - f'(x; h)}{\theta}. \end{aligned}$$

Hence,

$$\epsilon \leq \frac{f'(x; h + \bar{s}) - f'(x; h)}{\theta},$$

which contradicts (12). Hence,

$$0 \geq \liminf_{\theta \downarrow 0} \limsup_{i \longrightarrow \infty} \frac{\sup_{s \in \mathbb{B}_Y[0,\theta]} p(t_i, s)}{\theta}$$

and we are done, whenever (1) is used. □

Let us recall the notion of the direct sum of two closed subspaces of a Banach space $(X, \|\cdot\|)$; see Definition 4.20 in [17], where it was given for a vector topological spaces. Suppose Y is a closed subspace of X. If there exists a closed subspace $Z \subset X$ such that $X = Y + Z$ and $Y \cap Z = \{0\}$, then Y is said to be complemented in X. In this case, X is said to be the direct sum of Y and Z, and the notation

$$X = Y \oplus Z$$

is used. It is known that, if Y has a finite codimension, then Y is complemented; see, for example, Lemma 4.21 in [17] or Definition 4.1 and Theorem 5.5 in [12].

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a Banach space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, $M_0 \geq 0$, $\mu_0 > 0$, $\beta_0 > 0$, $x \in \operatorname{dom} f$ and $w_0 \in X \backslash \{0\}$ be given such that f is M_0-Lipschitz continuous on the set $D[x, x + \mu_0 \mathbb{B}_X[w_0, \beta_0]]$. Suppose that for a given sequence $\{t_i\}_{i \in \mathbb{N}}$ such that $t_i \in]0, \mu_0[$ for all $i \in \mathbb{N}$, and*

$t_i \downarrow 0$, there exist positive numbers $\alpha_i^n \in]0,1]$, where $i, n \in \mathbb{N}$ and a closed subspace $Y \subset X$ with a finite codimension, that is $X = Y \oplus Z$, where Z is a vector space with a finite dimension, and a G_δ (a countable intersection of open sets) dense subset $B \subset \mathbb{B}_Y[0, \beta_0]$ for which there are functions $p_n \in \mathbb{F}(\{t_i\}_{i\in\mathbb{N}}, \beta_0, Y)$, for all $n \in \mathbb{N}$, such that

$$\begin{aligned} &\forall s \in B,\ \exists t(s) \in]0, \mu_0[:\ \exists n \in \mathbb{N} :\ \forall t_i \in]0, t(s)], \forall \alpha \in [0,1],\\ &\max\left\{ \frac{f(x+t_i(w_0+\alpha_i^n \alpha s)) - f(x+t_i w_0)}{t_i \alpha_i^n} - p_n(t_i, \alpha s),\right.\\ &\left.\frac{f(x+t_i(w_0-\alpha_i^n \alpha s)) - f(x+t_i w_0)}{t_i \alpha_i^n} - p_n(t_i, -\alpha s)\right\} \le 0. \end{aligned} \tag{14}$$

If $x_i^* \in \partial f(x + t_i w_0)$ for all $i \in \mathbb{N}$, then there is a subsequence $\{i_k\}_{k\in\mathbb{N}}$ such that the subsequence $\{x_{i_k}^*\}_{k\in\mathbb{N}}$ is strongly convergent to some $x^* \in X^*$ on X and to 0 on Y.

Proof There are sequences $\{t_i\}_{i\in\mathbb{N}}$ such that $\mu_0 > t_i > 0$ for all $i \in \mathbb{N}$, $t_i \downarrow 0$, and $\{x_i^*\}_{i\in\mathbb{N}}$ such that $x_i^* \in \partial f(x+t_i w_0)$ for all $i \in \mathbb{N}$, and (14) is fulfilled for a sequence of functions $\{p_n\}_{\in\mathbb{N}}$ from $\mathbb{F}(\{t_i\}_{i\in\mathbb{N}}, \beta_0, Y)$ and positive numbers $\{\alpha_i^n\}_{i,n\in\mathbb{N}}$ from $]0,1]$.

For all $n, i \in \mathbb{N}$ define closed sets (keep in mind the Lipschitz continuity of f and the upper semicontinuity of p_n)

$$\begin{aligned} &D_i^n := \{s \in \mathbb{B}_Y[0,\beta_0] :\\ &\forall \alpha \in [0,1],\ \max\left\{ \frac{f(x+t_i(w_0+\alpha_i^n \alpha s)) - f(x+t_i w_0)}{t_i \alpha_i^n} - p_n(t_i, \alpha s),\right.\\ &\left.\frac{f(x+t_i(w_0-\alpha_i^n \alpha s)) - f(x+t_i w_0)}{t_i \alpha_i^n} - p_n(t_i, -\alpha s)\right\} \le 0\}. \end{aligned}$$

Let us notice that by (14) we have

$$B \subset \bigcup_{n,k\in\mathbb{N}} \bigcap_{i\ge k} D_i^n$$

and that $F_k^n := \bigcap_{i\ge k} D_i^n$ are closed symmetric sets. If $\operatorname{int} F_k^n = \emptyset$ for all $n, k \in N$, then $G_k^n := \mathbb{B}_Y(0,\beta_0) \setminus F_k^n$ are open and dense, so $\bigcap_{n,k\in\mathbb{N}} G_k^n$ is a dense G_δ subset of $\mathbb{B}_Y[0,\beta_0]$ and $B \cap \bigcap_{n,k\in\mathbb{N}} G_k^n = \emptyset$, which is impossible by the Baire category theorem. There are $k_0, n_0, \delta_0 > 0$, $s_0 \in \mathbb{B}_Y[0,\beta_0]$ such that

$$\mathbb{B}_Y[s_0,\delta_0] \cup \mathbb{B}_Y[-s_0,\delta_0] \subset \bigcap_{i\ge k_0} D_i^{n_0},$$

which implies

$$\langle x_i^*, \alpha s\rangle \le \sup_{z\in\mathbb{B}_Y[0,\alpha\beta_0]} p_{n_0}(t_i, z),$$

for all $s \in \mathbb{B}_Y[s_0, \delta_0] \cup \mathbb{B}_Y[-s_0, \delta_0]$, $\alpha \in]0, 1]$ and $i \geq k_0$. Thus,

$$\forall s \in \mathbb{B}_Y[0, \delta_0], \forall i \geq k_0, \forall \theta \in]0, \beta_0], \langle x_i^*, s\rangle \leq \beta_0 \frac{\sup_{z \in \mathbb{B}_Y[0,\theta]} p_{n_0}(t_i, z)}{\theta}.$$

Hence, by (11) we get

$$\liminf_{i \longrightarrow \infty} \sup_{s \in B_Y[0,\delta_0]} \langle x_i^*, s\rangle \leq \beta_0 \liminf_{i \longrightarrow \infty, \theta \downarrow 0} \frac{\sup_{s \in \mathbb{B}_Y[0,\theta]} p_{n_0}(t_i, s)}{\theta} \leq 0,$$

which implies that the sequence $\{x_i\}_{i \in \mathbb{N}}$ has a strongly convergent subsequence to 0 on Y, say that $\{x_{i_k}^*\}_{k \in \mathbb{N}}$ is the subsequence. We recall that $X = Y \oplus Z$, Y is closed subspace of X and Z is a finite-dimensional subspace. In order to get the strong convergence of some subsequence of $\{x_{i_k}^*\}_{k \in \mathbb{N}}$ on X it is enough to observe that

$$\forall i \in \mathbb{N}, \|x_i^*\| \leq M_0 < \infty$$

and we have possibility to choose a strongly convergent subsequence on Z, since the dimension of Z is finite. Having the strong convergence on Y and Z, we have the convergence on X, since $X = Y \oplus Z$. □

4 Conclusions

1. It is shown that, for some convex functions and some direction, it is possible to find a convergent sequence of subgradients along a direction, namely they belong to subdifferentials at points of some segment (a convergent sequence of points) and they form a convergent sequence of functionals, see Theorem 3.1.
2. Examples of convex functions and directions for which Theorem 3.1 can be applied are delivered, see Lemmas 2.2 and 3.2.
3. In Lemma 3.1 an answer to question: *we ask for conditions under which the limit* (5) *exists* is provided. In fact, under assumptions of Lemma 3.1 we have not only the existence of the limit, even for subgradients, but it says, due to the Rademacher Theorem, that the set of directions with the property that the limit in (5) exists is a set of full measure in the finite-dimensional setting, and it is a dense G_δ subset in the weak Asplund space, whenever the weak convergence is postulated instead of the strong one. Thus, Theorem 3.1, together with Lemma 3.1, gives an answer to Giannessi's question in the infinite-dimensional setting.

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