J Optim Theory Appl (2011) 149: 494–512
DOI 10.1007/s10957-011-9800-z

Feedback Control Design for Systems with x-Discontinuous Right-Hand Side

V.I. Korobov · Y.V. Korotyayeva

Published online: 28 January 2011

Abstract The problem of the admissible feedback synthesis for nonlinear systems with discontinuous right-hand side is considered. Sufficient conditions for solvability of this problem are proved. The neighborhood of the origin is broken in a finite number of domains $G_1, G_2, \dots, G_k$. In each G_j a control system $\dot{x} = f_j(x, u)$ is given. The problem of the admissible feedback synthesis is completely studied for control systems of the form $\dot{x} = a_j(x) + \gamma_j(x, u)b_j(x)$, where $u \in \Omega_j \subset \mathbb{R}$ for $x \in G_j$. The controllability function method is used to construct the feedback control.

Keywords Control system · x-discontinuous right-hand side · Feedback · Synthesis problem · Pass point

1 Introduction

The optimal synthesis problem is one of central problems of the mathematical control theory. The problem consists in constructing a control of the form $u = u(x)$ satisfying the given constraints, such that the minimum (maximum) of a certain functional be reached on trajectories of the closed system. Many papers and monographs are

Communicated by B.T. Polyak

The work was partially supported by Polish Ministry of Science and High Education grant N N514 238438.

V.I. Korobov (✉)
University of Szczecin, Szczecin, Poland
e-mail: korobow@univ.szczecin.pl

V.I. Korobov · Y.V. Korotyayeva
Karazin National University, Kharkov, Ukraine

Y.V. Korotyayeva
e-mail: liz-korotjaeva@yandex.ru

devoted to optimal synthesis problems, beginning from [1, 2]. In particular, the time optimal control problem consists in constructing of a control which transfers an arbitrary point to the given one in minimal time. The analytic solution of the linear time optimal problem, by using the moment min-problem, is given in [3, 4].

Many works are devoted to the method of dynamic programming, beginning from investigations of R. Bellman, R. Isaacs, G. Leitmann. The investigation of the optimal synthesis problem, based on the dynamic programming method, is given in [5]. A. Kurzhanski considers the feedback control design in systems allowing pulse actions of finite order [6]. The approach of dynamic programming is exploited to solve feedback pulse control problems [7].

We consider a nonlinear system of the form

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R},\ f(0, 0) = 0, \tag{1}$$

with the constraint on a control $u \in \Omega \subset \mathbb{R}$ ($0 \in \operatorname{int} \Omega$). Here $f(x, u)$ is an arbitrary function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. The problem consists in constructing a control in the form $u = u(x)$, satisfying the given constraints, such that the trajectory of the system $\dot{x} = f(x, u(x))$, starting at an arbitrary point x_0 from a certain neighborhood of the origin, ends in zero at the moment of time $T(x_0) < \infty$.

The statement of this problem is justified by the following: in many applied problems, it is sufficient to find the feedback control satisfying the given constraints. This control is required to be smooth, for example. Besides, the control must guarantee the time finiteness of movement. For example, when astronauts return to the Earth, initial conditions can be different. It is important to land in a finite time using a simple enough control. However, it is not necessary to minimize the landing time. Sometimes it is impossible to construct the optimal control in a real-time mode. In some games each gamer should analyze some positions in a limited time and make the admissible move. This move is not necessarily the best one.

The controllability function method (based on the idea of Lyapunov function) was suggested in 1979 for solving the admissible feedback synthesis problem for systems with smooth right-hand sides [8–10].

Several methods for constructing the controllability function and the feedback control have been proposed: for linear systems both in finite and infinite dimensional spaces [11, 12], for nonlinear systems with smooth right-hand sides using the first approximation, for control systems with disturbances [13–16]. The feedback synthesis problem was investigated for the class of nonlinear systems using mapping to linear systems ([17–20] and other works). This problem was considered also under the assumption that there exist constraints on the derivatives of the control [21].

Conditions for the origin (point of rest) to be uniformly, finitely, strongly stable or to be quasi-finitely, weakly, stable in terms of Lyapunov function are given by E. Roxin [22]. It is also noted that "no continuity is required for the Lyapunov function, and this makes it difficult to characterize the based conditions from the theorem by one and the same Lyapunov function". The controllability function method can be extended to the case where the origin is not a stationary point. Then the synthesis is unstable, because, after getting to the origin, the trajectory does not stay at the origin and even leaves the neighborhood and afterwards gets back to the origin in a finite time. The controllability function method allows one to construct analytical (even smooth) feedback control. In the optimal time control problem, the time of movement coincides with the controllability function. In this case, the controllability function design for linear systems is based on the moment min-problem.

The idea of controllability function design is based on the generalization of the nest method [16]. This method consists in constructing an infinite sequence of domains B_k, $k = 1, 2, \dots$, $B_{k+1} \subset B_k$, $\bigcap_k B_k = 0$. The boundary of each B_k is the level surface of $V_k(x)$ ($V_k(x)$ is the Lyapunov function). In $Q_k = B_k/B_{k+1}$ a control $u_k(x)$ is chosen in accordance with $V_k(x)$. The control design is determined by convergence to infinity of stability power of the system $\dot{x} = f(x, u_k(x))$ as $k \to \infty$. Thus, the trajectory gets to the origin in a finite time.

Our paper deals with the feedback synthesis problem in the statement, which has not been investigated before. A neighborhood of the origin is broken into a finite number of domains. In each domain a certain control system is given. This problem can be interpreted as the admissible synthesis problem for control systems with x-discontinuous right-hand sides.

Let the surfaces $\{\Gamma_i\}_{i=1}^m$ break the neighborhood of the origin of $\mathbb{R}^n$ into a finite number of open sets $\{G_j\}_{j=1}^k$. In each set G_j a control system

$$\dot{x} = f_j(x, u), \quad u \in \Omega_j \subset \mathbb{R},\ 0 \in \operatorname{int} \Omega_j,$$

is given. Besides, suppose that $f_j(0, 0) = 0$, if $0 \in \overline{G}_j$.

These systems describe a transfer of a certain object from one medium to another. In the case of pendulum (Example 5.2) it means the presence of a magnetic field in one of the domains G_j or the appearance of a resisting force, an elastic force, a frictional force, etc.

Different classes of dynamical systems, with discontinuous right-hand sides and different statements of problems, were considered beginning from [23]. Modern approaches treat the class of differential equations, where the right-hand side is discontinuous on its variables, as differential inclusions. This class of systems includes as a subclass the so-called variable structure systems [24]. Hybrid systems are defined as dynamical systems, whose state has two components: one that evolves in a continuous set such as $\mathbb{R}^n$ (according to a differential equation), and another one that evolves in a discrete set such as $\mathbb{N}$ (according to some transition logic-based rule) [25, 26]. The book [27] presents theoretical developments in the field of stability analysis and control synthesis of systems, that combine continuous dynamics with switching events (switching systems).

The problem of feedback control synthesis for differential inclusions was considered in [10, 13, 16]. This problem is closely connected with the finite-time stability problem, which is investigated up to date, for example, in [28] (and for the case of uncertain control systems in [29]). In fact, the authors of [28] "rediscovered", independently each other the known result from [30].

In [31] a method of the Lyapunov function design, which allows one to estimate the convergence time, is presented for dynamics given by an ordinary differential equation with a discontinuous right-hand side.

The control systems studied in this paper do not belong to any class described above. We do not use properties of differential inclusions, since we assume that the trajectory of the closed system be unique.

In the present paper, a sufficient condition for solvability of the local feedback synthesis problem for the system

$$\dot{x} = f(x,u), \quad f(x,u) = f_j(x,u),\ x \in G_j,\ j = 1,2,\dots,k,$$

with an initial condition $x(0) = x_0$, $x_0 \in Q$ (Q is a certain neighborhood of the origin) is proved by use of the controllability function method. In addition to the sufficient condition for solvability of admissible feedback synthesis problem a way of control design for the systems with x-discontinuous right-hand side of a special form is offered.

Let emphasize the importance of giving a well-defined concept for the solution of the system $\dot{x} = f(x,u(x))$, $x(0) = x_0$, where $f(x,u(x)) = f_j(x,u_j(x))$, $x \in G_j$, $j = 1,2,\dots,k$, at the boundary of domains G_j. The feedback control is chosen, such that the corresponding trajectory cannot slide along surfaces Γ_i. Moreover, this trajectory intersects Γ_i at the moments of time $\tau_1 < \cdots < \tau_p$, $p < \infty$, outside any closed ball of a sufficiently small radius. For t from intervals (τ_i, τ_{i+1}), $i = 1,\dots,p-1$, the trajectory $x(t)$ belongs to a certain domain G_j. This condition together with the condition of the existence of controllability function, provides getting of the trajectory of the closed system to the origin in a finite time.

Mayer and Lagrange optimization problems for controlled variable structure systems are investigated in [32].

V. Boltyanski considers control systems with discontinuous right-hand sides in [33]. He focuses on the optimal synthesis problem. The maximum principle for Mayer and Lagrange problems is formulated for determining of the program control $u = u(t)$. In this paper we consider a more common case comparing with [33]: points of the boundaries of G_j may belong to the intersection of a finite number of $(n-1)$-dimensional smooth surfaces.

The paper is organized as follows. In Sect. 2, preliminary notations and results are given, and solutions of control systems with x-discontinuous right-hand sides are defined. The main result (a sufficient condition for solvability of the feedback synthesis problem) and a brief description of the essence of the controllability function method are provided in Sect. 3. Further, in Sect. 4, the feedback synthesis problem for control systems of the form $\dot{x} = a_j(x) + \gamma_j(x,u)b_j(x)$, where $u \in \Omega_j \subset \mathbb{R}$ for $x \in G_j$, is completely studied. In Sect. 5, the obtained results are illustrated by examples.

2 Preliminary Notations and Results

Let Q be a closed neighborhood of the origin (a closure of an open set, $0 \in \operatorname{int} Q$). Let ε be an arbitrary positive number. Denote by S_ε a closed ball of radius $\varepsilon > 0$ ($S_\varepsilon = \{x \in R^n : \|x\| \le \varepsilon\}$) such that S_ε belongs to Q.

For the description of $\{G_j\}_{j=1}^k$, let us take m functions $\{\psi_i(x)\}_{i=1}^m$, having continuous partial derivatives of the first order. Each function $\psi_i(x)$ defines two domains

$$\psi_i^+ = \{x \in \mathbb{R}^n : \psi_i(x) > 0\}, \qquad \psi_i^- = \{x \in \mathbb{R}^n : \psi_i(x) < 0\}$$

and a surface $\Gamma_i = \{x \in \mathbb{R}^n : \psi_i(x) = 0\}$. Assume that $\nabla\psi_i(x) \neq 0$ at the points of Γ_i. Denote by $\mathbb{I} = \{1,\dots,m\}$, $\mathbb{J} = \{1,\dots,k\}$.

Call the set $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$ a boundary. Assume that the neighborhood of the origin be broken by Γ in a finite number of disjoint domains $G_1, G_2, \dots, G_k$. Refer to the point $x \in \Gamma$ as a boundary point. Denote by I_x a set of indexes of functions ψ_i : $\psi_i(x) = 0$, and by ν_i, $i \in I_x$, a unit normal at the point $x \in \Gamma$. Any point $x \in \Gamma$ is a common boundary point of G_j, $j \in J_x$. Assume that normals at any point $x \in \Gamma$ be linearly independent, $\nu_i(x) = \pm \frac{\nabla \psi_i(x)}{\|\nabla \psi_i(x)\|}$. The orientation of normals is defined below.

Hence, either any point $x \in Q$ belongs to the boundary $x \in \Gamma$; i.e., there exists i_0 such that $\psi_{i_0}(x) = 0$, or x is an inner point of a certain domain G_j; i.e., for $i \in \mathbb{I}$ inequalities $\psi_i(x) \neq 0$ are true. Each function $\{\varphi_i(x)\}_{i=1}^{m}$ does not change the sign in any domain G_j.

Let K_x be a cone with the apex in zero, formed by normals at the point $x \in \Gamma$, i.e. $K_x = \{\nu \in \mathbb{R}^n : \nu = \sum_{i \in I_x} \lambda_i \nu_i(x), \lambda_i > 0\}$. Denote by K_x^* an adjoint cone to the cone K_x, i.e. $K_x^* = \{h \in \mathbb{R}^n : (h, \nu_i(x)) \geq 0,\ i \in I_x\}$. Then $\overset{\circ}{K}{}_x^* = \{h \in \mathbb{R}^n : (h, \nu_i(x)) > 0,\ i \in I_x\}$ is an interior of the cone K_x^*, $\overset{\circ}{K}{}_x^* \neq \emptyset$.

Consider the control process described by (1), where

$$f(x,u) = f_j(x,u), \quad x \in G_j, \quad f_j(x,u) = (f_{1j}(x,u), \dots, f_{nj}(x,u))^*,$$

$u \in \Omega_j \subset \mathbb{R}$.

Henceforward, we assume that suitable conditions be true.

Condition A Each vector-function $f_j(x,u)$ is defined in the domain $V_j \times \Omega_j$, where V_j is a certain neighborhood of G_j (V_j is an open set, $G_j \subset V_j$). Suppose also that $f_j(x,u)$ satisfies the Lipschitz condition in the domain $\{(x,u) : x \in V_j, 0 < \rho_1 \leq \|x\| \leq \rho_2, u \in \Omega_j\}$, i.e.

$$\|f_j(x',u') - f_j(x'',u'')\| \leq L_1(\rho_1,\rho_2)(\|x'' - x'\| + |u'' - u'|). \tag{2}$$

Condition B Let functions $u_j(x)$, $x \in V_j$, $u_j(x) \in \Omega_j$ exist, such that:

(*B1*) $u_j(x)$ satisfies the Lipschitz condition in the closed domain

$$K_j(\rho_1,\rho_2) = \{x : x \in V_j, 0 < \rho_1 \leq \|x\| \leq \rho_2\};$$

i.e.,

$$|u_j(x') - u_j(x'')| \leq L_2(\rho_1,\rho_2)\|x'' - x'\| \quad \forall x', x'' \in K_j(\rho_1,\rho_2); \tag{3}$$

(*B2*) there exists the orientation of normals ν_i, $i \in I_x$, at the point $x \in \Gamma$ (the common boundary point of G_j, $j \in J_x$), such that

$$f_j(x, u_j(x)) \in \overset{\circ}{K}{}_x^*, \ j \in J_x. \tag{4}$$

Define the function $u(x) = u_j(x)$ in each domain G_j, $j \in \mathbb{J}$. Further we define the solution of system

$$\dot{x} = f(x, u(x)), \tag{5}$$

with an initial condition $x(0)=x_0$, $x_0\in Q/S_\varepsilon$, and prove, that the trajectory cannot slide along surfaces Γ_i due to condition *B2*.

Lemma 2.1 *Consider the control process given by* (1), *where*

$$f(x,u)=f_j(x,u),\quad x\in G_j,\ u\in\Omega_j\subset\mathbb{R}.$$

Let conditions A *and* B *be satisfied. Then*, *there exists a unique trajectory of system* (5) *defined on some time segment* $[0,\tau]$. *The trajectory starts at an arbitrary point* $x_0\in(Q/S_\varepsilon)\bigcap\overline{G}_j$ *for* $j\in\mathbb{J}$, *and* $x(t)\in G_j/S_\varepsilon$ *for* $0<t\le\tau$.

Proof If x_0 does not belong to the boundary, $x_0\in G_j$ for some $j\in\mathbb{J}$, then the inequality

$$\|f_j(x',u_j'(x'))-f_j(x'',u_j''(x''))\|\le L_1(\varepsilon,R)(1+L_2(\varepsilon,R))\|x''-x'\| \tag{6}$$

holds for points x', $x''\in K_j(\varepsilon,R)$, $0<\varepsilon<R$, R being a certain positive number. Therefore, a unique solution of the Cauchy problem

$$\dot{x}=f_j(x,u_j(x)),\quad x(0)=x_0,$$

exists on some time interval $0\le t<\tau$, $\tau>0: x(t)\in G_j$ for some j, $j\in\mathbb{J}$.

Let now the point x_0 be a boundary one: $x_0\in(Q/S_\varepsilon)\bigcap\Gamma$.

Consider the domain $G_{j_0}=\{x\in Q:\psi_i(x)<0$, if $\nu_i(x_0)=-\frac{\nabla\psi_i(x_0)}{\|\nabla\psi_i(x_0)\|}$, and $-\psi_i(x)<0$, if $\nu_i(x_0)=\frac{\nabla\psi_i(x_0)}{\|\nabla\psi_i(x_0)\|}, i\in I_{x_0}\}$, $j_0\in J_{x_0}$. Examine a solution of the Cauchy problem $\dot{x}=f_{j_0}(x,u_{j_0}(x))$, $x(0)=x_0$, $x\in V_{j_0}$. This solution can be given by $x(t)=x_0+f_{j_0}(x,u_{j_0}(x))t+\omega(x_0,t)$, where $\frac{\|\omega(x_0,t)\|}{t}\longrightarrow 0$ as $t\longrightarrow 0$.

By definition of the cone $\overset{\circ}{K}{}_x^*$ of possible directions at the point x_0 of the domain G_{j_0} [34, 35], there exists the moment of time $0<\tau$, such that $x_0+t(f_{j_0}(x,u_{j_0}(x))+\frac{\omega(x_0,t)}{t})\in G_{j_0}$ for each $0<t\le\tau$. Thus, each point of the boundary $x_0\in(Q/S_\varepsilon)\bigcap\Gamma$ corresponds uniquely to the domain G_{j_0} and the control u_{j_0}. The solution $x(t)$ of the Cauchy problem

$$\dot{x}=f_{j_0}(x,u_{j_0}(x)),\quad x(0)=x_0,\ x\in V_{j_0}$$

is the trajectory of system (5) with an arbitrary initial condition $x(0)=x_0$, $x_0\in\Gamma\bigcap Q/S_\varepsilon$. A unique solution exists on the time segment $[0,\tau]$, and $x(t)\in G_{j_0}$ for each $0<t\le\tau$. □

Definition 2.1 The trajectory $x(t)$ reaches the boundary Γ, iff there exists the sequence of moments of time $t_s\longrightarrow T$ as $s\longrightarrow\infty$, such that $x(t_s)\in G_j$ for some j, and $x(t_s)\longrightarrow x\in\Gamma$ as $s\longrightarrow\infty$.

Definition 2.2 Call a point $x\in\Gamma$ the pass point, iff the trajectory of system (5) with an arbitrary initial condition $x(0)=x_0$, $x_0\in Q/S_\varepsilon$, which crosses Γ at the point x at some moment of time $T>0$, transfers from one domain G_j to another.

Lemma 2.2 *Consider the control process given by* (1), *where*

$$f(x,u) = f_j(x,u), \quad x \in G_j,\ u \in \Omega_j \subset \mathbb{R}.$$

Let conditions A *and* B *be satisfied.*

If the trajectory of the system $\dot{x} = f(x, u(x))$, *starting at an arbitrary point* $x_0 \in Q/S_\varepsilon$, *reaches the boundary* Γ *at some moment of time* $T > 0$, *i.e.* $x = x(T) \in \Gamma$, *then* x *is the pass point.*

Proof Actually the trajectory of system (5) with an arbitrary initial condition $x(0) = x_0$, $x_0 \in Q/S_\varepsilon$, can not slide along surfaces Γ_i due to the previous remark. The solution of the Cauchy problem

$$\dot{x} = f(x, u(x)), \quad x(T) = x,\ x \in \Gamma,$$

is equal to the solution $x(t)$ of the Cauchy problem

$$\dot{x} = f_{j_0}(x, u_{j_0}(x)), \quad x(T) = x,\ x \in V_{j_0},\ j_0 \in J_x;$$

$x(t)$ belongs to the domain G_{j_0} for each $T < t \le T + \tau$ for some $\tau > 0$.

Show that there exists $0 < \tau' \le T$, such that the trajectory $x(t)$ does not belong to the domain G_{j_0} for $T - \tau' \le t < T$. Assume the converse. Then $x(t)$ is the solution of the Cauchy problem

$$\dot{x} = f_{j_0}(x, u_{j_0}(x)), \quad x(T) = x$$

for $T - \tau' \le t < T$. Therefore, we obtain that $x(T - t) = x(T) - t f_{j_0}(x(T), u_{j_0}(x(T))) + \omega(x(T), t) = x + t(-f_{j_0}(x, u_{j_0}(x)) + \frac{\omega(x,t)}{t})$, $\frac{\|\omega(x(T),t)\|}{t} \to 0$ as $t \to 0$. Under our assumptions $x(T - t) \in G_{j_0}$ for $0 < t \le \tau'$. This means that $(-f_{j_0}(x, u_{j_0}(x)), \nu_i(x)) \ge 0$ for each $i \in I_x$. Since $x \in \Gamma$, observe, that $f_{j_0}(x, u_{j_0}(x)) \in \overset{\circ}{K}{}^*_x$; i.e., $(f_{j_0}(x, u_{j_0}(x)), \nu_i(x)) > 0$, $\forall\, i \in I_x$, due to the condition *B2*. The contradiction proves that the point $x \in \Gamma$ is the pass point of the trajectory of system (5) with an arbitrary initial condition $x_0 \in Q/S_\varepsilon$. □

Assume that the following condition holds.

Condition C Suppose that, for each point $x_0 \in (Q/S_\varepsilon) \bigcap \Gamma$, the control $u_{j_0}(x)$, $x \in V_{j_0}$, be such that there exists a time segment $[0, \tau_\varepsilon]$, where the solution of the Cauchy problem $\dot{x} = f_{j_0}(x, u_{j_0}(x))$, $x(0) = x_0$, belongs to the domain G_{j_0}/S_ε. Specifically, either $x(t) \in G_{j_0}/S_\varepsilon$ for $0 < t \le \tau_\varepsilon$, or $x(t) \in S_\varepsilon$ for some $0 < t \le \tau_\varepsilon$.

3 Sufficient Condition for Solvability of Synthesis Problem

Consider the control process

$$\dot{x} = f(x,u), \quad f(x,u) = (f_1(x,u), \dots, f_i(x,u), \dots, f_n(x,u))^*,$$

with the constraints $u \in \Omega \subset \mathbb{R}$. Describe the main idea of the controllability function method from [8]. Assume that the ancillary function $\Theta = \Theta(x)$ (controllability function) exists. The function satisfies the following conditions:

(1) $\Theta(x) > 0$ for $x \neq 0$, $\Theta(0) = 0$;
(2) $\Theta(x)$ is continuous everywhere and continuously differentiable everywhere except for the origin;
(3) there exists a number $c > 0$, such that the set

$$Q = \{x \in \mathbb{R}^n : \Theta(x) \leq c\}$$

is bounded.

Suppose also that there exists the control $u = \widetilde{u}(x, \Theta(x))$, such that the differential equation

$$\sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x), \quad \alpha \geq 1, \ \beta > 0$$

holds. This means that a movement takes in the direction of decrease of the function $\Theta(x)$. Since the inequality holds we see that the trajectory gets to the origin due to the control $u(x)$ in a finite time.

If the equality $\sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) = -1$ holds, then $\Theta(x) = T(x)$, where $T(x)$ is the time to get to the origin from x. If

$$\min_{u \in \Omega} \sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u) = \sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) = -1,$$

then for $\omega(x) = -\Theta(x)$ one obtains the Bellman equation

$$\max_{u \in \Omega} \sum_{i=1}^{n} \frac{\partial \omega}{\partial x_i} f_i(x, u) = 1.$$

The choice of the control based on the Bellman equation can be interpreted as the minimization of the controllability function $\Theta(x)$. In the controllability function method this angle is not necessarily minimal. From the above equation, for $\alpha = \infty$, it follows that $\Theta(x) = V(x)$ ($V(x)$ being the Lyapunov function). The traditional Lyapunov function is defined explicitly, while $\Theta(x)$ is defined implicitly through the equation $\Phi(\Theta, x) = 0$. The optimal time of movement for the linear time optimal problem is also designed implicitly [3].

Definition 3.1 The trajectory $x(t)$ ends in the origin in a finite time, iff there exist two sequences of numbers $\varepsilon_s > 0$, $\varepsilon_s \longrightarrow 0$ and t_s, $t_s \longrightarrow T$, $T < \infty$ as $s \longrightarrow \infty$, such that $x(t_s) \in S_{\varepsilon_s}$.

Now formulate and prove a new sufficient condition for the solvability of the synthesis problem.

Theorem 3.1 *Consider the control process described by* (1), *where* $f(x,u) = f_j(x,u)$, $x \in G_j$, $f_j(x,u) = (f_{1j}(x,u), \dots, f_{nj}(x,u))^*$, $u \in \Omega_j \subset \mathbb{R}$. *Let conditions* A, B, C *hold. Define the function* $u(x) = u_j(x)$ *in each domain* G_j. *Let the controllability function* $\Theta(x)$ *exist. For* $x \in G_j$ *inequalities*

$$-\beta_1 \Theta^{1-\frac{1}{\alpha_1}}(x) \le \sum_{i=1}^{n} \frac{\partial \Theta}{\partial x_i} f_{ij}(x, u_j(x)) \le -\beta_2 \Theta^{1-\frac{1}{\alpha_2}}(x) \tag{7}$$

are true for some positive β_1, β_2, α_1, α_2. *Then, the trajectory of the system* $\dot{x} = f(x, u(x))$, *starting at an arbitrary point* $x_0 \in Q$, *ends in the origin at the time* $(\alpha_1/\beta_1)\Theta^{\frac{1}{\alpha_1}}(x_0) \le T(x_0) \le (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)$.

Proof Let us show that the trajectory $x(t)$ of system (5) with an arbitrary initial condition $x_0 \in Q$ gets to the boundary of the ball S_ε at the moment of time, satisfying estimation

$$(\alpha_1/\beta_1)\Theta^{\frac{1}{\alpha_1}}(x_0) \le T(x_0) \le (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0). \tag{8}$$

Let $x_0 \in \overline{G}_j$ for some j. If $x_0 \in G_j$, then the solution of system (5) with the initial condition x_0 exists on some time segment $[0, \tau]$, $\tau > 0$, $x(t) \in G_j/S_\varepsilon$ for $t \in [0, \tau]$. If $x_0 \in \partial G_j$, then $x(t) \in G_{j_0}$ for $0 < t \le \tau$, $\tau > 0$. Since inequality (7) holds, we obtain that the trajectory $x(t)$ of system (5) belongs to the domain Q. The solution continues on some segment $[0, \tau_1]$, $\tau_1 > \tau$.

Both cases are possible: either $x(\tau_1) \in S_\varepsilon$ due to (7) or $x(\tau_1) \in \Gamma$. If $x(\tau_1)$ belongs to the boundary Γ, then the trajectory $x(t)$ for $\tau_1 < t \le \tau_1 + \tau_\varepsilon$ belongs to the domain G_{j_1}/S_ε. Then the solution of system (5) is defined on some maximum segment $[\tau_1, \tau_2]$, $\tau_2 - \tau_1 \ge \tau_\varepsilon$, and $x(t) \in G_{j_1}/S_\varepsilon$ for $\tau_1 \le t < \tau_2$. Observe that $x(\tau_2) \in \Gamma$, $x(\tau_2)$ does not belong to S_ε. The trajectory $x(t)$ for $t > \tau_2$ belongs to the new domain G_{j_2} etc. Hence, the trajectory $x(t)$ intersects the boundary Γ in no more than a countable set of points $0 = \tau_0 \le \tau_1 < \tau_2 < \cdots$. The trajectory is defined for $t \le T$, and $x(t)$ does not belong to S_ε for $t \le T$. Let us show that the number of moments τ_i, for which $x(\tau_i)$ belongs to one of the surfaces of Γ, is finite. We also establish that the trajectory $x(t)$ gets to the boundary of S_ε at the moment of time $T(x_0)$, $T(x_0) \ge \tau_p$, satisfying estimation (8).

Let p be an arbitrary natural number. Consider one of time segments $[\tau_i, \tau_{i+1}]$, $i \in \{1, \dots, p-1\}$. The solution $x(t)$ belongs to the domain G_j for $t \in (\tau_i, \tau_{i+1})$. Since condition C holds, we see that the following inequality $\tau_{i+1} - \tau_i \ge \tau_\varepsilon$ is true. From inequality (7) it follows that

$$\Theta^{\frac{1}{\alpha_2}}(x(\tau_{i+1})) \le \Theta^{\frac{1}{\alpha_2}}(x(\tau_i)) - \frac{\beta_2(\tau_{i+1} - \tau_i)}{\alpha_2}, \quad i = 0, \dots, p-1, \tag{9}$$

$$\Theta^{\frac{1}{\alpha_2}}(x(t)) \le \Theta^{\frac{1}{\alpha_2}}(x(\tau_i)) - \frac{\beta_2(t - \tau_i)}{\alpha_2}, \quad i = 0, \dots, p-1,\ t > \tau_p. \tag{10}$$

Since $\Theta(x_0) > \Theta(x(\tau_1)) > \cdots > \Theta(x(\tau_p))$, we have

$$\Theta^{\frac{1}{\alpha_2}}(x(\tau_p)) \le \Theta^{\frac{1}{\alpha_2}}(x_0) - \frac{\beta_2\, \tau_p}{\alpha_2}. \tag{11}$$

If $\tau_p > (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)$, then the right-hand side of inequality (11) is negative, but it is impossible because $\Theta^{\frac{1}{\alpha_2}}(x(\tau_p)) \geq 0$.

Observe, that $p \leq \frac{(\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)}{\tau_\varepsilon}$, because $\tau_{i+1} - \tau_i \geq \tau_\varepsilon$, $i = 0, \ldots, p-1$.

If $x(\tau_p) \in S_\varepsilon$, then the necessary result is established. Let $x(\tau_p)$ not belong to S_ε, then from inequality (10) it follows that

$$\Theta^{\frac{1}{\alpha_2}}(x(t)) \leq \Theta^{\frac{1}{\alpha_2}}(x_0) - \frac{\beta_2 t}{\alpha_2}. \tag{12}$$

At time $t \leq (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)$ the trajectory $x(t)$ gets to the boundary of the ball S_ε. Otherwise, if $t > (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)$, then the right-hand side of inequality (12) is negative and the left-hand side is positive. There exists a segment $[0, T]$ such that $x(T) \in S_\varepsilon$, and $T \leq (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)$. It follows that $x(T) \in S_\varepsilon$ as in [16]: $T \leq (\alpha_2/\beta_2)\Theta^{\frac{1}{\alpha_2}}(x_0)$ and $\lim_{t\to T(x_0)} x(t) = 0$. To get the lower estimate for the moment of time $T(x_0)$, by analogy, we can take the inequality

$$\Theta^{\frac{1}{\alpha_1}}(x_0) - \frac{\beta_1 t}{\alpha_1} \leq \Theta^{\frac{1}{\alpha_1}}(x(t)).$$

Hence, $T(x_0) \geq (\alpha_1/\beta_1)\Theta^{\frac{1}{\alpha_1}}(x_0)$ because $\lim_{t\to T(x_0)} \Theta(x(t)) = 0$. □

Remark 3.1 The requirement of continuous differentiability and even continuity of the controllability function $\Theta(x)$ at the boundary is restrictive enough. However, this requirement could be avoided.

4 Solving of Synthesis Problem for Control System with Discontinuous Right-Hand Side

Let some closed neighborhood of the origin $\widehat{Q}$ be broken on a finite number of open sets G_j, $j = 1, \ldots, k$. Consider the method to solve the synthesis problem, when the control system of differential equations is given by

$$\dot{x} = a(x) + \gamma(x, u)b(x), \tag{13}$$

$a(x) = a_j(x) = (a_{1j}(x), \ldots, a_{nj}(x))^*$, $b(x) = b_j(x) = (b_{1j}(x), \ldots, b_{nj}(x))^*$, $\gamma(x, u) = \gamma_j(x, u)$ for $x \in G_j$. Here the symbol * means transposition. Assume that functions $a_{ij}(x)$, $b_{ij}(x)$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ be continuously differentiable n times for $x \in V_j$. $\gamma_j(x, u)$ is continuously differentiable with respect to the pair of variables, ${\gamma_j}'_u(x, u) \geq \gamma_0$, $\gamma_0 > 0$, $x \in V_j$, $u \in \Omega_j$, $j = 1, \ldots, k$. If $0 \in \overline{G}_j$ then $a_j(0) = 0$, $\gamma_j(0, 0) = 0$.

The nonlinear system on each domain G_j is given by

$$\dot{x} = a_j(x) + \gamma_j(x, u)b_j(x), \tag{14}$$

with constraints on the control $|u| \le d_j$, $d_j > 0$.

Let system (13) be such that there exists a smooth transformation of variables $z = \Phi(x)$, $\Phi(0) = 0$, $x \in \bigcup_{j=1}^{k} V_j$. In each domain G_j (14) is given by

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \dots, \dot{z}_{n-1} = z_n, \quad \dot{z}_n = \phi_j(z_1, z_2, \dots, z_n, u). \tag{15}$$

Denote by H_j and σ_i, respectively, the images of sets G_j and Γ_i under the mapping Φ, $H_j = \Phi(G_j), \sigma_i = \Phi(\Gamma_i)$.

As in [20], define the transformation of variables $\Phi(x)$ in the following way. Let the continuously differentiable $n+1$ times scalar function $\varphi(x) = \varphi(x_1, x_2, \dots, x_n)$ exist for $x \in \bigcup_{j=1}^{k} V_j$. Denote by $L_a\varphi = \varphi_x a$, where $\varphi_x = (\varphi_{x_1}, \dots, \varphi_{x_n})$. Consider Lie brackets

$$ad_a^0 b = b, \quad ad_a^1 b = [a, b] = b_x a - a_x b, \quad ad_a^p b = [a, ad_a^{p-1} b],$$
$$p = 1, \dots, n-1.$$

Let vectors $b(x)$, $ad_a^1 b(x)$, $\dots$, $ad_a^{n-1} b(x)$ form the basis in the space $\mathbb{R}^n$ for each $x \in \bigcup_{j=1}^{k} V_j$, and

$$\varphi_x b = 0, \quad \varphi_x ad_a^1 b = 0, \quad \dots, \quad \varphi_x ad_a^{n-2} b = 0, \quad \varphi_x ad_a^{n-1} b \neq 0. \tag{16}$$

Vectors $a_j(x)$ and $b_j(x)$, $j = 1, 2, \dots, k$, for $x \in V_j$ satisfy

$$\varphi_x b_j = 0, \quad \varphi_x ad_{a_j}^1 b_j = 0, \quad \dots, \quad \varphi_x ad_{a_j}^{n-2} b_j = 0, \quad \varphi_x ad_{a_j}^{n-1} b_j \neq 0,$$

and $L_{a_j}^i \varphi = L_{a_m}^i \varphi$, $i = 1, \dots, n-1$, $j, m = 1, 2, \dots, k$.

In each domain G_j, we take the transformation of variables $z = \Phi(x)$

$$z_1 = \varphi(x), \quad z_2 = L_{a_j}\varphi(x), \quad \dots, \quad z_n = L_{a_j}^{n-1}\varphi(x). \tag{17}$$

For every sequence $z_1, z_2, \dots, z_n$, equalities (17) are assumed to be uniquely solvable by x_1, $x_2, \dots, x_n$. Denote by

$$\phi_j(z_1, z_2, \dots, z_n, u) = L_{a_j}^n \varphi(x) + \gamma_j(x, u) L_{b_j} L_{a_j}^{n-1} \varphi(x).$$

Then in each domain H_j system (14) is given by (15).

Suppose there exists a constant $c > 0$, such that, for $L_b L_a^{n-1}\varphi(x)$ and for $x \in V_j$, the inequality

$$|L_b L_a^{n-1} \varphi(x)| \ge c \tag{18}$$

be true.

Remark 4.1 If some additional conditions be true, as in [36], then it is sufficient to require belonging to the class C^1 from functions $a_j(x)$, $b_j(x)$ to transform system (14) to system (15), using non-degenerate transformation of variables $z = \Phi(x)$.

Theorem 4.1 *Let the control system of differential equations with discontinuous right-hand side be given by* (13). *Conditions* (16) *and* (18) *are true. Then, the problem of the admissible feedback synthesis can be solved in some closed and bounded neighborhood of the origin.*

Proof In each domain G_j, $j=1,2,\dots,k$, make transformation of variables (17). System (14) is given by (15).

Introduce the new control $v=\phi_j(z_1,z_2,\dots,z_n,u)$, $z\in H_j$, $j=1,\dots,k$. System (15) takes form

$$\dot{y}=A_0y+b_0v,$$
$$A_0=\begin{pmatrix}0&1&\dots&0\\ \dots&\dots&\dots&\dots\\ 0&0&\dots&1\\ 0&0&\dots&0\end{pmatrix},\qquad b_0=\begin{pmatrix}0\\ \dots\\ 0\\ 1\end{pmatrix}. \tag{19}$$

The control u satisfies the given constraints: $|u|\le d_j$, $d_j>0$, in the domain G_j. If $x\in G_j$, then $z=\Phi(x)\in H_j$, $H_j\subset Q_0$, where $Q_0=\Phi(\widehat{Q})$ is the closed neighborhood of the origin in the space of variables z. Let the constraints on the control v be given in the form $|v|\le d_0$, $d_0>0$. The constant d_0 is defined below.

One of the methods of constructing the controllability function, presented in [16], consists in the following. The function $\Theta(z)$ is defined as a unique positive solution of equation

$$2a_0\Theta=(F(\Theta)z,z), \tag{20}$$

for each z, where $F^{-1}(\Theta)=\left(\frac{(-1)^{2n-i-j}\Theta^{2n-i-j+1}}{(n-i)!(n-j)!(2n-i-j+1)(2n-i-j+2)}\right)_{i,j=1}^{n}$. a_0 is a positive number which satisfies conditions $a_0\le\frac{2d_0^2}{f_{nn}}$, f_{nn} is the lower right angular element of the matrix $F(1)$.

The control v is chosen as

$$v(y)=-\frac{1}{2}b_0^*F(\Theta(z))z. \tag{21}$$

Control (21) solves the problem of the admissible feedback synthesis in the neighborhood Q_0 [16], $|v|\le d_0$.

Find the control $u_j(z)$, $j=1,2,\dots,k$, $z\in H_j$, from the equation $v(z)=\phi_j(z_1, z_2,\dots,z_n,u)=L_{a_j}^n\ \varphi(\Phi^{-1}(z))+\gamma_j(\Phi^{-1}(z),u_j)\ L_{b_j}L_{a_j}^{n-1}\ \varphi(\Phi^{-1}(z))$. Since the function $L_{a_j}^n\varphi(\Phi^{-1}(z))$ is continuous and $L_{a_j}^n\varphi(0)=0$, if $0\in\overline{H}_j$, there exists some constant c_1, such that the inequality $|L_{a_j}^n\varphi(\Phi^{-1}(z))|\le c_1$ holds in some bounded neighborhood $Q_1\subset Q_0$.

Hence, $|\gamma_j(\Phi^{-1}(z),u)|\le\frac{d_0+c_1}{c}$. Moreover,

$$2\frac{d_0+c_1}{c}\ge|\gamma_j(\Phi^{-1}(z),u)-\gamma_j(\Phi^{-1}(z),0)|=|\gamma_{j}{}'_u(\Phi^{-1}(z),u')||u|\ge\gamma_0|u|.$$

If $\gamma_j(\Phi^{-1}(z),0)=0$ for each $z\in Q_1$, then

$$\frac{d_0+c_1}{c}\geq|\gamma_j(\Phi^{-1}(z),u)|\geq\gamma_0|u|,\quad |u|\leq\frac{d_0+c_1}{c\gamma_0}.$$

Let now c_1 and d_0 be such, that inequalities $2\frac{d_0}{\gamma_0\,c}\leq\frac{d}{2}$, $2\frac{c_1}{\gamma_0\,c}\leq\frac{d}{2}$, $d=\min d_j$, $j=1,\dots,k$, be true, then the control $u_j(z)$ for $z\in H_j\bigcap Q_1$ satisfies the given constraints: $|u_j|\leq d_j$.

Choose the domain $Q_1\subset Q_0$ in the form $\{z\in\mathbb{R}^n:\Theta(z)\leq c_2(c_1)\}$.

If the point $z\in\bigcup_{i=1}^m\sigma_i$ (the common boundary point of domains H_j, $j\in J_z$), then the value of control $u(z)$ can be chosen from one of equalities

$$v(z)=\phi_j(z_1,z_2,\dots,z_n,u_j),\quad j\in J_z.$$

The trajectory of system (15) is smooth for all z. The trajectory $x(t)$, found from equalities (17) with right-hand sides $z_i(t)$, $i=1,\dots,n$, is also smooth for all $0\leq t\leq T(z)$, where $T(z)$ is the moment of time to get to the origin from an arbitrary point z of Q_1.

The problem of the admissible feedback synthesis is solved in the domain $Q=\{x\in\mathbb{R}^n:\Theta(\Phi(x))\leq c_2(c_1)\}\subset\widehat{Q}$ in a finite time $T(\Phi(x))$. The control satisfies the constraints $|u|\leq d_j$ for $x\in G_j$. □

Remark 4.2 The trajectory $x(t)$ of the system

$$\dot{x}=a(x)+\gamma(x,u(x))b(x),$$

where $a(x)=a_j(x)$, $b(x)=b_j(x)$, $\gamma(x,u(x))=\gamma_j(x,u_j(x))$ for $x\in G_j$, can slide along surfaces Γ_i, $i=1,\dots,m$.

5 Examples

Example 5.1 In the domain $G_1=\{(x_1,x_2):x_1>0\}$, let the system

$$\dot{x}_1=x_1^3+x_1+\frac{u}{3x_1^2+1},\dot{x}_2=\frac{u}{3x_1^2+1},\tag{22}$$

be given; i.e., $a_1(x)=\binom{x_1^3+x_1}{0}$, $b_1(x)=\binom{1}{1}$,
$\gamma_1(x,u)=\frac{u}{3x_1^2+1}$, $|u|\leq1$. In the domain $G_2=\{(x_1,x_2):x_1<0\}$, let the system

$$\dot{x}_1=x_1^3+u,\ \dot{x}_2=-x_1+u,\tag{23}$$

$a_2(x)=\binom{x_1^3}{-x_1}$, $b_2(x)=\binom{1}{1}$, $\gamma_2(x,u)=u$, $|u|\leq1$, be given. Here $\psi_1(x)=x_1$.

If the function $\varphi(x)$ is given by $\varphi(x_1,x_2)=x_1-x_2$, then conditions (16) hold, i.e. $\varphi_xb_1=0$, $\varphi_xb_2=0$, $L_{a_1}\varphi=L_{a_2}\varphi=x_1^3+x_1$. The transformation of variables $y=\Phi(x)$ is given by

$$z_1=\varphi(x_1,x_2)=x_1-x_2,z_2=L_a\varphi(x_1,x_2)=x_1^3+x_1.\tag{24}$$

The second equation of (24) defines $x_1 = h(z_2)$, $h(0) = 0$, then

$$x_2 = h(z_2) - z_1.$$

Here $h(z_2) = \frac{-2\cdot 3^{1/3} + 2^{1/3}(9z_2 + \sqrt{12 + 81z_2^2})^{2/3}}{6^{2/3}(9z_2 + \sqrt{12 + 81z_2^2})^{1/3}}$.

The domain G_1 transforms to the domain $H_1 = \{(z_1, z_2) : z_2 > 0\}$, the domain G_2 transforms to $H_2 = \{(z_1, z_2) : z_2 < 0\}$. In the domain H_1 system (22) is given by

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = (3h^2(z_2) + 1)z_2 + u. \tag{25}$$

In the domain H_2 system (23) takes form

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = (3h^2(z_2) + 1)(h^3(z_2) + u). \tag{26}$$

Let us introduce the new control

$$v = \begin{cases} (3h^2(z_2) + 1)z_2 + u, & z \in H_1, \\ (3h^2(z_2) + 1)(h^3(z_2) + u), & z \in H_2. \end{cases}$$

Then systems (25) and (26) take form $\dot{z}_1 = z_2$, $\dot{z}_2 = v$, $|v| \le d_0$, where d_0 is sufficiently small, the constraints on d_0 are described below.

Solve the synthesis problem using the controllability function $\Theta = \Theta(z)$. Define the function $\Theta(z)$ for $z \neq 0$ as a unique positive solution of (20). The matrix $F(\Theta)$ is given by $\begin{pmatrix} \frac{36}{\Theta^3} & \frac{12}{\Theta^2} \\ \frac{12}{\Theta^2} & \frac{6}{\Theta} \end{pmatrix}$. Equation (20) is given by $2a_0\Theta^4 = 36z_1^2 + 24\Theta z_1 z_2 + 6\Theta^2 z_2^2$, $a_0 \le \frac{2d_0^2}{6}$.

Control

$$v(z) = -\frac{6}{\Theta^2(z_1, z_2)} z_1 - \frac{3}{\Theta(z_1, z_2)} z_2 \tag{27}$$

solves the synthesis problem; i.e., it moves any point $(z_1^0, z_2^0) \in \mathbb{R}^n$ to the origin along the trajectory of the system at the finite time $\Theta(z_1^0, z_2^0)$ and satisfies the constraints $|v| \le d_0$.

The trajectory can be found as the solution of the Cauchy problem

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = -\frac{6z_1}{\Theta^2(t)} - \frac{3z_2}{\Theta(t)}, \quad \dot{\Theta} = -1,$$
$$z_1(0) = z_1^0, \quad z_2(0) = z_2^0, \quad \Theta(0) = \Theta(z_1^0, z_2^0).$$

Observe that

$$z_1(t) = (\Theta_0 - t)^2(c_1 \cos\omega(t) + c_2 \sin\omega(t)),$$
$$z_2(t) = (\Theta_0 - t)((2c_1 + \sqrt{2}c_2)\cos\omega(t) + (-\sqrt{2}c_1 + 2c_2)\sin\omega(t)).$$

Here $\omega(t)=\sqrt{2}\ln(\Theta_0-t)$,

$$c_1=\frac{1}{2\Theta_0^2}(2z_1^0\cos(\sqrt{2}\ln\Theta_0)+\sqrt{2}(2z_1^0+z_2^0\Theta_0)\sin(\sqrt{2}\ln\Theta_0)),$$
$$c_2=\frac{1}{\sqrt{2}\Theta_0^2}(-(2z_1^0+z_2^0\Theta_0)\cos(\sqrt{2}\ln\Theta_0)+\sqrt{2}z_1^0\sin(\sqrt{2}\ln\Theta_0)).$$

The control $v(z)$ is given by

$$v(z_1(t),z_2(t))=3\sqrt{2}(c_2\cos\omega(t)-c_1\sin\omega(t)).$$

The control solving the local feedback synthesis problem is given by

$$u(x_1,x_2)=-\frac{6(x_1-x_2)}{\Theta^2(x_1-x_2,x_1+x_1^3)}-\frac{3(x_1^3+x_1)}{\Theta(x_1-x_2,x_1+x_1^3)}-(3x_1^2+1)(x_1^3+x_1),$$

for $(x_1,x_2)\in G_1$, and for $(x_1,x_2)\in G_2$:

$$u(x_1,x_2)=\frac{1}{3x_1^2+1}\left(-\frac{6(x_1-x_2)}{\Theta^2(x_1-x_2,x_1+x_1^3)}-\frac{3(x_1^3+x_1)}{\Theta(x_1-x_2,x_1+x_1^3)}\right)-x_1^3.$$

On the boundary of domains; i.e., for $x_1=0$, the controls are equal.

The trajectory of the given system satisfies the equations

$$\begin{aligned}
&x_1(t)-x_2(t)=(\Theta_0-t)^2(c_1\cos\omega(t)+c_2\sin\omega(t)),\\
&x_1^3(t)+x_1(t)=(\Theta_0-t)((2c_1+\sqrt{2}c_2)\cos\omega(t)+(-\sqrt{2}c_1+2c_2)\sin\omega(t)),\\
&z_1(0)=z_1^0=x_1(0)-x_2(0)=x_1^0-x_2^0,\\
&z_2(0)=z_2^0=x_1^3(0)+x_1(0)=x_1^{0^3}+x_1^0.
\end{aligned}$$

Control (27) solves the synthesis problem in some neighborhood of the origin Q_0. Between x_1 and x_2 and z_1, z_2 dependence (24) exists. The closed neighborhood Q_0 in the space of variables z_1, z_2 corresponds to the closed neighborhood $\widehat{Q}=\Phi^{-1}(Q_0)$ in the space of variables x_1, x_2. Let Q_0 contained in a ball of radius R_0 and the neighborhood $\widehat{Q}$ contained in a ball of radius $\widehat{R}$. As $d_0\to 0$, $R_0\to 0$ and $\widehat{R}\to 0$.

In the domain $H_1\bigcap Q_0$ the control

$$|u|\le|v-(3h^2(z_2)+1)z_2|\le d_0+|(3h^2(z_2)+1)z_2|.$$

Let $d_0\le 1/2$. If $|z_2|\le 1/3$, then the function $(3h^2(z_2)+1)z_2$ satisfies the constraints $|(3h^2(z_2)+1)z_2|\le 1/2$. In the domain

$$H_1\bigcap\{(z_1,z_2):\ |z_2|\ \le\ 1/3\}$$

the control u satisfies the given constraints $|u|\le 1$.

In $H_2 \bigcap Q_0$ the control u satisfies $|u| = |\frac{v-(3h^2(z_2)+1)h^3(z_2)}{3h^2(z_2)+1}|$:

$$u \le d_0 + |(3h^2(z_2) + 1)h^3(z_2)|.$$

If $|z_2| \le 1/3$ then $|u| \le 1$.

Let us consider the ellipsoid $Q_1 = \{(z_1, z_2) : \ \Theta(z_1, z_2) \le c_2\}$. Find a positive number c_2, such that the inclusion $Q_1 \subset \{(z_1, z_2) : |z_2| \le 1/3\}$ be true. Suppose that $d_0 = 1/2$, $a_0 = \frac{2d_0^2}{6} = 1/12$, then $c_2 = \sqrt{12}/3 = 2\sqrt{3}/3$.

The control $u(x_1, x_2)$ satisfies the given constraints $|u| \le 1$ and moves an arbitrary point (x_1^0, x_2^0) of the neighborhood of the origin

$$Q = \{(x_1, x_2) : \Theta(x_1 - x_2, x_1 + x_1^3) \le c_2\}$$

to zero in the finite time $\Theta(x_1^0 - x_2^0, x_1^0 + (x_1^0)^3)$.

Example 5.2 Consider the solution of the local synthesis problem for the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -\sin x_1 + u$; $|u| \le 2$, $x \in G_1 = \{x \in \mathbb{R}^2 : x_1 > 0\}$; and $\dot{x}_1 = x_2$, $\dot{x}_2 = a \sin x_1 + u$; $|u| \le 2$, $x \in G_2 = \{x \in \mathbb{R}^2 : x_1 < 0\}$; $a \in \mathbb{R}$, $|a| \le 1$. For construction of the controllability function use one of the methods from [16]. Find the function $\Theta(x)$ from the equation (see [16]) $\frac{2}{9}\Theta^4 - \Theta^2 x_2^2 - 2\Theta x_1 x_2 - 3x_1^2 = 0$. Introduce the new control

$$v(x) = \begin{cases} -\sin x_1 + u, & x_1 > 0, \\ a \sin x_1 + u, & x_1 \le 0. \end{cases}$$

Suppose that $-1 \le v \le 1$. Thus, $|u| = |v + \sin x_1| \le 2$ for $x_1 > 0$, and $|u| \le 2$, if $x_1 < 0$. The control $v(x) = -\frac{x_1}{\Theta^2(x_1,x_2)} - \frac{2x_2}{\Theta(x_1,x_2)}$ solves the problem of the admissible

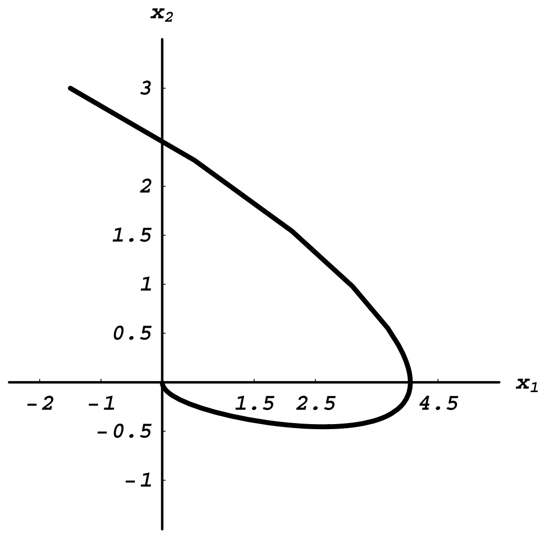

Fig. 1 Trajectory, starting at the point $(-1.5, 3)$, $a = 0.5$ (Example 5.2)

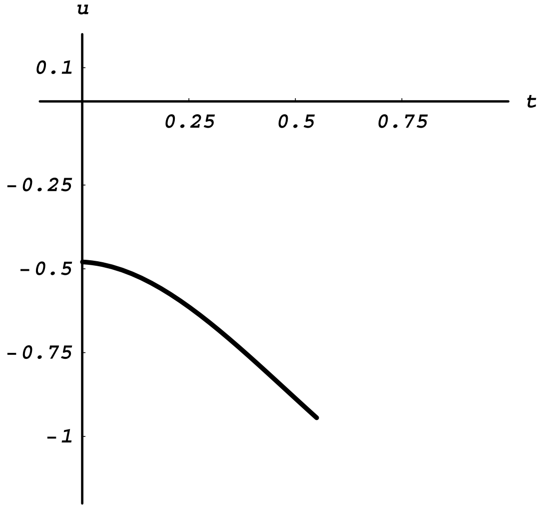

Fig. 2 Control $u(t)$ in domain G_2 (Example 5.2)

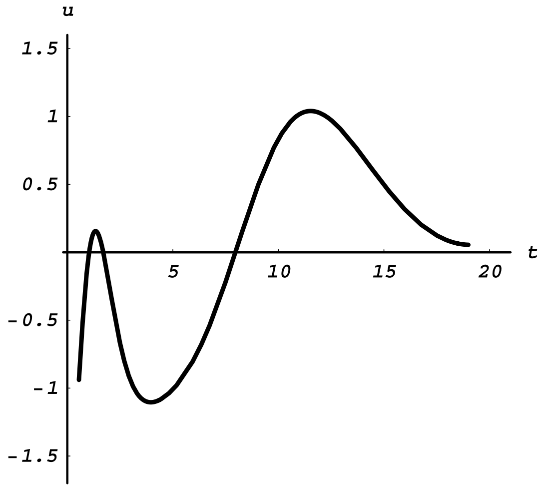

Fig. 3 Control $u(t)$ in domain G_1 (Example 5.2)

feedback synthesis for the system $\dot{x}_1 = x_2$, $\dot{x}_2 = v$ in any neighborhood of the origin Q_0 and satisfies the constraints $|v| \leq 1$.

On account of the system, the derivative of the function Θ (see [16]) is given by $\dot{\Theta} = -\frac{2x_1^2+2x_2^2\Theta^2}{12x_1^2+6x_1x_2\Theta+2x_2^2\Theta^2}$, $-\frac{2}{7-\sqrt{34}} \leq \dot{\Theta} \leq -\frac{2}{7+\sqrt{34}}$. The time of movement $T(x_1^0, x_2^0)$ from an arbitrary point $x_0 = (x_1^0, x_2^0)$ to the origin satisfies inequalities

$$\frac{7-\sqrt{34}}{2}\,\Theta(x_1^0, x_2^0) \leq T(x_1^0, x_2^0) \leq \frac{7+\sqrt{34}}{2}\,\Theta(x_1^0, x_2^0).$$

Let us denote $x_3(t) = \Theta(x(t))$, $x_3(t) > 0$ for $t \neq 0$. Then $x(t) = (x_1(t), x_2(t), x_3(t))$ is the solution of the system of differential equations $\dot{x}_1 = x_2$, $\dot{x}_2 = -\frac{x_1}{x_3^2} - \frac{2x_2}{x_3}$,

$\dot{x}_3 = -\frac{x_1^2+x_2^2x_3^2}{6x_1^2+3x_1x_2x_3+x_2^2x_3^2}$, with initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, $x_3(0) = \Theta(x_1^0, x_2^0)$. In the domain G_1 the programming control solving the synthesis problem is given by $u_1(t) = -\frac{x_1(t)}{x_3^2(t)} - \frac{2x_2(t)}{x_3(t)} + \sin x_1(t)$. In the domain G_2 the control is given by $u_2(t) = -\frac{x_1(t)}{x_3^2(t)} - \frac{2x_2(t)}{x_(t)} - a \sin x_1(t)$.

6 Concluding Remarks

The problem of the admissible feedback synthesis for nonlinear systems with x-discontinuous right-hand side is investigated. This problem has not been considered before. Sufficient conditions for solvability of this problem are proved in detail. The proof is based on the controllability function method. The feedback control is chosen, such that the corresponding trajectory cannot slide along surfaces Γ_i. We do not use properties of differential inclusions, since we assume that the trajectory of the closed system be unique. These conditions together with the condition of the existence of controllability function lead to the trajectory of the closed system to the origin in a finite time.

The problem of the admissible feedback synthesis is completely studied for control systems of the form $\dot{x} = a_j(x) + \gamma_j(x, u)b_j(x)$, where $u \in \Omega_j \subset \mathbb{R}$ for $x \in G_j$.

The obtained results are illustrated by examples.

References

1. Boltjanskii, V.G., Gamkrelidze, R.V., Mischenko, E.F., Pontryagin, L.S.: Mathematical Theory of Optimal Processes. Nauka, Moscow (1961)
2. Bellman, R.: Dynamic Programming. Princeton University Press, Princeton (1957)
3. Korobov, V.I., Sklyar, G.M.: Optimal speed and power moment problem. Mat. Sb. **134(176)**2(10), 186–206 (1987)
4. Korobov, V.I., Sklyar, G.M.: Time-optimal control and the trigonometric moment problem. Math. USSR, Izv., **53**(4), 868–885 (1989)
5. Boltjanskii, V.: Mathematical Methods of Optimal Control, 2nd edn., rev. and suppl. (1969)
6. Kurzhanski, A.B.: On synthesis of systems with impulse controls. Mechatronics Autom. Control **4**, 2–12 (2006)
7. Daryin, A.N., Kurzhanski, A.B.: Generalized functions of high order as feedback controls. Differ. Uravn. **43**(11), 1443–1453 (2007)
8. Korobov, V.I.: A general approach to the solution of a problem of restricted controls synthesis in a controllability problem. Mat. Sb. **109**(4), 582–606 (1979) (Russian, English translation)
9. Korobov, V.I.: Solution of a synthesis problem using a controllability function. Dokl. Akad. Nauk SSSR **248**(5), 1051–1055 (1979) (Russian, English translation)
10. Korobov, V.I.: Solution of a synthesis problem in differential games using controllability function. Dokl. Akad. Nauk SSSR **266**(2), 269–273 (1982) (Russian, English translation)
11. Korobov, V.I., Sklyar, G.M.: The solution of synthesis problem using a controllability functional in infinite-dimensional spaces. Dokl. Akad. Nauk Ukr. SSR, Ser. A **5**, 11–14 (1983) (Russian)
12. Korobov, V.I., Gavrilyako, V.M., Sklyar, G.M.: Synthesis of restricted control of dynamic systems in the entire space with the help of a controllability function. Avtom. Telemeh. **11**, 30–36 (1986) (Russian, English translation)

13. Korobov, V.I.: Solution of a synthesis problem for controllable processes with perturbations with the help of a controllability function. Differ. Uravn. **23**(2), 236–243 (1987) (Russian, English translation)
14. Korobov, V.I., Sklyar, G.M.: Methods for constructing of positional controls and an admissible maximum principle. Differ. Uravn. **26**(11), 1914–1924 (1990) (Russian, English translation)
15. Korobov, V.I., Sklyar, G.M., Skoryk, V.A.: Solution of the synthesis problem in Hilbert spaces. In: Pros. CD of the 14th Intern. Symposium of Mathematical Theory of Networks and Systems, Perpignan (2000)
16. Korobov, V.I.: Controllability Function Method. Monograph., M. (2007) (Russian)
17. Brockett, R.W.: Lie algebras and Lie groups in control theory. In: Geom. Meth. Syst. Theory. Dordrecht, Boston, pp. 43–82. (1973)
18. Respondek, W.: Linearization, feedback and Lie brackets. Sci. Pap. Inst. Tech. Cybern. Techn. University Wroclaw, Conf. **29**(70), 131–166 (1985)
19. Nijmeijer, H., Van der Schaft: Nonlinear Dynamical Control Systems. Springer, New York (1990)
20. Korobova, E.V., Sklyar, G.M.: Constructive method for mapping nonlinear systems onto linear systems. Teor. Funkc. Funkc. Anal. Prilozh. **55**(1), 68–74 (1991) Engl. Transl. J. Soviet Math. **59**(1) 631–635 (1992)
21. Korobov, V.I., Skoryk, V.A.: Positional synthesis of the bounded inertial systems with one-dimensional control. Differ. Uravn. **38**(3), 319–331 (2002)
22. Roxin, E.: On finite stability in control systems. Pendiconti del Circolo Matematico di Palermo **3**(3), 273–282 (1966)
23. Filippov, A.F.: Differential equations with discontinuous right-hand side. Mat. Sb. **51**(93), 99–128 (1960)
24. Levant, A.: Homogeneous quasi-continuous sliding mode control. In: Edwards, C., Colet, E.F., Fridman, L. (eds.) Advances of Variable Structure and Sliding Mode Control, pp. 143–168. Springer, Berlin (2006)
25. Hezinger, T.A.: The theory of hybrid systems. In: Proceedings of 11 th Annual Symposium on Logics in Computer Science, pp. 278–292. IEEE Computer Society, Los Alamitos (1996)
26. Goebel, R., Sanfelice, R.-G., Teel, A.-R.: Hybrid dynamical systems. IEEE Control Syst. Mag. **29**(2), 28–93 (2009)
27. Liberzon, D.: Swithing in Systems and Control. Birkhäuser, Boston (2003)
28. Moulay, E., Perruquetti, W.: Finite time stability of differential inclusions. IMA J. Math. Control Inf. **22**, 465–475 (2005)
29. Levant, A.: Homogeneity approach to high-order sliding mode design. Automatica **41**, 823–830 (2005)
30. Korobov, V.I., Gavrilyako, V.M.: The use of differential inequalities in the solution of a synthesis problem by the controllability function method. Vestnik Khar‘kov Univ. **277**, 7–13 (1985) (Russian)
31. Polyakov, A., Poznyak, A.: Lyapunov function design for finite-time convergence analysis of twisting and super-twisting second order sliding mode controllers. Automatica **45**, 444–418 (2009)
32. Boltjanskii, V.G.: The maximum principle for variable structure systems. Int. J. Control **77**, 1445–1451 (2004)
33. Boltjanskii, V.G.: The maximum principle for systems with discontinuous right hand. In: Proceeding Intelligent Systems and Control, vol. 446, pp. 384–387. Springer, Berlin (2004)
34. Dubovitski, A.Ya., Milutin, A.A.: Necessary conditions for a weak extremum in problems of optimal control with mixed constraints. Zh. Vychisl. Mat. Mat. Fiz. **8**(4), 725–779 (1968)
35. Girsanov, I.V.: Lectures on Mathematical Theory of Extremum Problems. Moscow State University, Moscow (1970)
36. Sklyar, G.M., Sklyar, K.V., Ignatovich, S.Y.: On the extension of the Korobov's class of linearizable triangular systems by nonlinear control systems of the class C^1. Syst. Control Lett. **54**, 1097–1108 (2005)