J Dyn Diff Equat (2019) 31:81–91
https://doi.org/10.1007/s10884-018-9653-3

A Lipschitz Refinement of the Bebutov–Kakutani Dynamical Embedding Theorem

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Received: 25 April 2017 / Revised: 26 September 2017 / Published online: 1 March 2018
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Abstract We prove that an $\mathbb{R}$-action on a compact metric space embeds equivariantly in the space of one-Lipschitz functions $\mathbb{R} \to [0, 1]$ if its fixed point set can be topologically embedded in the unit interval. This is a refinement of the classical Bebutov–Kakutani theorem (1968).

Keywords Compact universal flow · Dynamical embedding · Lipschitz functions · Local section · Bebutov–Kakutani theorem

Mathematics Subject Classification 37B05 · 54H20

1 Introduction

The purpose of this short paper is to refine a classical theorem of Bebutov [2] and Kakutani [4] on dynamical systems. We call (X, T) a **flow** if X is a compact metric space and

$$T : \mathbb{R} \times X \to X, \quad (t, x) \mapsto T_t x$$

Y.G. was partially supported by the NCN (National Science Center, Poland) grant 2016/22/E/ST1/00448 and the Marie Curie grant PCIG12-GA-2012-334564. Y.G. and L.J. were partially supported by the National Science Center (Poland) Grant 2013/08/A/ST1/00275. M.T. was supported by John Mung Program of Kyoto University.

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is a continuous action of $\mathbb{R}$. We define $\mathrm{Fix}(X, T)$ (sometimes abbreviated to $\mathrm{Fix}(X)$) as the set of $x \in X$ satisfying $T_t x = x$ for all $t \in \mathbb{R}$. We define $C(\mathbb{R})$ as the space of continuous maps $\varphi : \mathbb{R} \to [0, 1]$. It is endowed with the topology of uniform convergence over compact subsets of $\mathbb{R}$, namely the topology given by the distance

$$\sum_{n=1}^{\infty} 2^{-n} \max_{|t|\le n} |\varphi(t) - \psi(t)|, \quad (\varphi, \psi \in C(\mathbb{R})). \tag{1.1}$$

The group $\mathbb{R}$ continuously acts on it by the translation:

$$\mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R}), \quad (s, \varphi(t)) \mapsto \varphi(t + s). \tag{1.2}$$

A continuous map $f : X \to C(\mathbb{R})$ is called an **embedding of a flow** (X, T) if f is an $\mathbb{R}$-equivariant topological embedding. Bebutov [2] and Kakutani [4] found that the $\mathbb{R}$-action on $C(\mathbb{R})$ has the following remarkable "universality":

Theorem 1.1 *(*Bebutov–Kakutani*) A flow* (X, T) *can be equivariantly embedded in* $C(\mathbb{R})$ *if and only if* $\mathrm{Fix}(X, T)$ *can be topologically embedded in the unit interval* $[0, 1]$.

The "only if" part is trivial because the set of fixed points of $C(\mathbb{R})$ is homeomorphic to $[0, 1]$. So the main statement is the "if" part.

Although the Bebutov–Kakutani theorem is clearly a nice theorem, it has one drawback: The space $C(\mathbb{R})$ is not compact (nor locally compact). So it is not a "flow" in the above definition. This poses the following problem:

Problem 1.2 Is there a *compact* invariant subset of $C(\mathbb{R})$ satisfying the same universality?

The purpose of this paper is to solve this problem affirmatively. Let $L(\mathbb{R})$ be the set of maps $\varphi : \mathbb{R} \to [0, 1]$ satisfying the one-Lipschitz condition:

$$\forall s, t \in \mathbb{R} : \quad |\varphi(s) - \varphi(t)| \le |s - t|. \tag{1.3}$$

$L(\mathbb{R})$ is a subset of $C(\mathbb{R})$. It is compact with respect to the distance (1.1) by Ascoli–Arzela's theorem. The $\mathbb{R}$-action (1.2) preserves $L(\mathbb{R})$. So it becomes a flow. Our main result is the following. This solves [3, Question 4.1].

Theorem 1.3 *A flow* (X, T) *can be equivariantly embedded in* $L(\mathbb{R})$ *if and only if* $\mathrm{Fix}(X, T)$ *can be topologically embedded in the unit interval* $[0, 1]$.

As in the case of the Bebutov–Kakutani theorem, the "only if" part is trivial because the fixed point set $\mathrm{Fix}(L(\mathbb{R}))$ is homeomorphic to $[0, 1]$. Since $L(\mathbb{R})$ is compact, it is a more reasonable choice of such a "universal flow".

The proof of Theorem 1.3 is based on the techniques originally used in the proof of the Bebutov–Kakutani theorem (in particular, the idea of *local section*). A main new ingredient is the topological argument given in Sect. 2, which has some combinatorial flavor.

Remark 1.4 Problem 1.2 asks us to find a universal flow *smaller than* $C(\mathbb{R})$. If we look for a universal flow *larger than* $C(\mathbb{R})$, then it is much easier to find an example. Let $L^\infty(\mathbb{R})$ be the set of L^∞-functions $\varphi : \mathbb{R} \to [0, 1]$. (We identify two functions which are equal to each other almost everywhere.) We consider the weak* topology on it. Namely a sequence $\{\varphi_n\}$ in $L^\infty(\mathbb{R})$ converges to $\varphi \in L^\infty(\mathbb{R})$ if for every L^1-function $\psi : \mathbb{R} \to \mathbb{R}$

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n(t)\psi(t)\, dt = \int_{\mathbb{R}} \varphi(t)\psi(t)\, dt.$$

Then $L^\infty(\mathbb{R})$ is compact and metrizable by Banach–Alaoglu's theorem and the separability of the space of L^1-functions, respectively. The group $\mathbb{R}$ acts continuously on it by translation. So it becomes a flow. Note that Fix $(L^\infty(\mathbb{R}))$ is homeomorphic to [0, 1] and that the natural inclusion map $C(\mathbb{R}) \subset L^\infty(\mathbb{R})$ is an equivariant continuous injection. Then the Bebutov–Kakutani theorem implies the universality of $L^\infty(\mathbb{R})$: A flow (X, T) can be equivariantly embedded in $L^\infty(\mathbb{R})$ if and only if $\mathrm{Fix}(X, T)$ can be topologically embedded in [0, 1].

2 Topological Preparations

Let a be a positive number. We define $L[0, a]$ as the space of maps $\varphi : [0, a] \to [0, 1]$ satisfying

$$\forall s, t \in [0, a]: \quad |\varphi(s) - \varphi(t)| \le |s - t|.$$

$L[0, a]$ is endowed with the distance $\|\varphi - \psi\|_\infty = \max_{0\le t\le a} |\varphi(t) - \psi(t)|$. We define $F_L[0, a] \subset L[0, a]$ as the space of constant functions $\varphi : [0, a] \to [0, 1]$, which is homeomorphic to [0, 1].

Let (X, d) be a compact metric space. We define $C(X, L[0, a])$ as the space of continuous maps $f : X \to L[0, a]$, which is endowed with the distance

$$\max_{x\in X} \|f(x) - g(x)\|_\infty .$$

Lemma 2.1 *Let $f \in C(X, L[0, a])$ and suppose there exists $0 < \tau < 1$ satisfying*

$$\forall x \in X, \forall s, t \in [0, a]: \quad |f(x)(s) - f(x)(t)| \le \tau|s - t|. \tag{2.1}$$

Then for any $\delta > 0$ there exists $g \in C(X, L[0, a])$ satisfying

(1) $\max_{x\in X} \|f(x) - g(x)\|_\infty < \delta$.
(2) $g(x)(0) = f(x)(0)$ *and* $g(x)(a) = f(x)(a)$ *for all* $x \in X$.
(3) $g(X) \cap F_L[0, a] = \emptyset$.

Proof We take $0 < b < c < a$ satisfying $b = a - c < \delta/4$. We take an open covering $\{U_1, \ldots, U_M\}$ of X satisfying

$$\forall 1 \le m \le M: \quad \mathrm{diam} f(U_m) < \min\left(\frac{\delta}{4}, \frac{(1-\tau)b}{2}\right). \tag{2.2}$$

We take a point $p_m \in U_m$ for each m. We choose a natural number N satisfying

$$N > M, \quad \Delta \overset{\mathrm{def}}{=} \frac{c-b}{N-1} < \frac{\delta}{4}.$$

We divide the interval $[b, c]$ into $(N - 1)$ intervals of length Δ:

$$b = a_1 < a_2 < \cdots < a_N = c, \quad a_{n+1} - a_n = \Delta \quad (\forall 1 \le n \le N - 1).$$

Set $A = \{a_1, \ldots, a_N\}$ and define a vector $e \in \mathbb{R}^A$ by $e = (1, 1, \ldots, 1)$. Notice that $f(p_m)|_A$ is an element of $[0, 1]^A$. Since $N > M$ we can choose $u_1, \ldots, u_M \in [0, 1]^A$ satisfying

(1) $|f(p_m)(a_n) - u_m(a_n)| < \min(\delta/4, (1-\tau)b/2)$ for all $1 \le m \le M$ and $1 \le n \le N$.
(2) $|u_m(a_{n+1}) - u_m(a_n)| < \Delta$ for all $1 \le m \le M$ and $1 \le n \le N - 1$.
(3) The $(M + 1)$ vectors $e, u_1, \ldots, u_M$ are linearly independent.

Let $\{h_m\}_{m=1}^M$ be a partition of unity on X satisfying $\operatorname{supp} h_m \subset U_m$ for all m. For $x \in X$ we define a piecewise linear function $g(x) : [0, a] \to [0, 1]$ as follows. (We set $a_0 = 0$ and $a_{N+1} = a$.)

- $g(x)(0) = f(x)(0)$ and $g(x)(a) = f(x)(a)$.
- $g(x)(a_n) = \sum_{m=1}^M h_m(x) u_m(a_n)$ for $1 \le n \le N$.
- We extend $g(x)$ linearly. Namely, for $t = (1-\lambda)a_n + \lambda a_{n+1}$ with $0 \le \lambda \le 1$ and $0 \le n \le N$ we set $g(x)(t) = (1-\lambda)g(a_n) + \lambda g(a_{n+1})$.

□

Claim 2.2 $g(x) \in L[0, a]$ *and* $\|g(x) - f(x)\|_\infty < \delta$.

Proof For proving $g(x) \in L[0, a]$ it is enough to show $|g(x)(a_{n+1}) - g(x)(a_n)| \le |a_{n+1} - a_n|$ for all $0 \le n \le N$. For $1 \le n \le N-1$, this is a direct consequence of the property (2) of u_m. So we consider the case of $n = 0$. (The case of $n = N$ is the same).

$$
\begin{aligned}
|g(x)(b) - f(x)(0)| \le & \sum_{m=1}^M h_m(x)|u_m(b) - f(p_m)(b)| + \sum_{m=1}^M h_m(x)|f(p_m)(b) - f(x)(b)| \\
& + |f(x)(b) - f(x)(0)|.
\end{aligned}
$$

We apply to each term of the right-hand side the property (1) of u_m, $\operatorname{diam} f(U_m) < (1-\tau)b/2$ in (2.2) and $|f(x)(b) - f(x)(0)| \le \tau b$ in (2.1) respectively. Then this is bounded by

$$
\frac{(1-\tau)b}{2} + \frac{(1-\tau)b}{2} + \tau b = b.
$$

This proves $g(x) \in L[0, a]$.

Next we show $|g(x)(a_n) - f(x)(a_n)| < \delta/2$ for all $0 \le n \le N+1$. For $n = 0, N+1$, this is trivial. For $1 \le n \le N$, we can bound $|g(x)(a_n) - f(x)(a_n)|$ from above by

$$
\begin{aligned}
& \sum_{m=1}^M h_m(x)|u_m(a_n) - f(p_m)(a_n)| + \sum_{m=1}^M h_m(x)|f(p_m)(a_n) - f(x)(a_n)| \\
& < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} \quad \left(\text{by the property (1) of } u_m \text{ and } \operatorname{diam} f(U_m) < \frac{\delta}{4} \text{ in (2.2)}\right).
\end{aligned}
$$

Finally, let $a_n < t < a_{n+1}$. We can bound $|g(x)(t) - f(x)(t)|$ by

$$
\begin{aligned}
& |g(x)(t) - g(x)(a_n)| + |g(x)(a_n) - f(x)(a_n)| + |f(x)(a_n) - f(x)(t)| \\
& < 2(a_{n+1} - a_n) + \frac{\delta}{2} \quad (\text{by } f(x), g(x) \in L[0, a]) \\
& < \delta \quad \left(\text{by } a_{n+1} - a_n \le \max(b, \Delta) < \frac{\delta}{4}\right).
\end{aligned}
$$

□

For every $x \in X$, the function $g(x) : [0, a] \to [0, 1]$ is a non-constant function because

$$
g(x)|_\Lambda = \sum_{m=1}^M h_m(x) u_m \notin \mathbb{R}e \quad (\text{by the property (3) of } u_m).
$$

This proves the statement. □

We need two lemmas on linear algebra. For $u = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ we set

$$Du = (x_2 - x_1, x_3 - x_2, \dots, x_{n+1} - x_n) \in \mathbb{R}^n.$$

Lemma 2.3 *Let $l \geq m+1$ and set $e = (\underbrace{1, 1, \dots, 1}_{l}) \in \mathbb{R}^l$. The set of $(u_1, \dots, u_m) \in \mathbb{R}^{l+1} \times \cdots \times \mathbb{R}^{l+1} = \left(\mathbb{R}^{l+1}\right)^m$ such that*

$$\text{the vectors } e, Du_1, Du_2, \dots, Du_m \text{ are linearly independent} \tag{2.3}$$

is open and dense in $\left(\mathbb{R}^{l+1}\right)^m$.

Proof The condition (2.3) defines a Zariski open set in $\left(\mathbb{R}^{l+1}\right)^m$. So it is enough to show that the set is non-empty because a non-empty Zariski open set is always dense in the Euclidean topology. We set

$$u_i = (\underbrace{-1, \dots, -1}_{i}, \underbrace{0, \dots, 0}_{l+1-i}), \quad (1 \leq i \leq m).$$

Then

$$Du_i = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{l-i}).$$

The vectors $e, Du_1, \dots Du_m$ are linearly independent. □

Lemma 2.4 *Let $n > l \geq 2m$. The set of $(u_1, \dots, u_m) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n = (\mathbb{R}^n)^m$ such that, for any integer α with $2 \leq \alpha \leq n - l + 1$,*

$$u_1|_1^l,\ u_1|_\alpha^{\alpha+l-1},\ u_2|_1^l,\ u_2|_\alpha^{\alpha+l-1}, \dots, u_m|_1^l,\ u_m|_\alpha^{\alpha+l-1} \text{ are linearly independent in } \mathbb{R}^l \tag{2.4}$$

is open and dense in $(\mathbb{R}^n)^m$. Here for $u_i = (x_{i1}, \dots, x_{in})$

$$u_i|_1^l = (x_{i1}, \dots, x_{il}), \quad u_i|_\alpha^{\alpha+l-1} = (x_{i,\alpha}, \dots, x_{i,\alpha+l-1}).$$

Proof The defined set $A = \cap_{\alpha=2}^{n-l+1} A_\alpha$, where

$$A_\alpha = \{(u_1, \dots, u_m) \in \left(\mathbb{R}^n\right)^m \mid (2.4) \text{ is satisfied}\},$$

is a Zariski open set in $(\mathbb{R}^n)^m$. Hence it is enough to show that every A_α is nonempty. For each fixed $2 \leq \alpha \leq n - l + 1$, we define $u_i = (x_{i1}, \dots, x_{in})$ $(1 \leq i \leq m)$ by

$$x_{ij} = 1 \quad (j = i, \alpha + l - i), \quad x_{ij} = 0 \quad (\text{otherwise}).$$

Then it is direct to check that $(u_1, \dots, u_m) \in A_\alpha$. One can also use a proof from [5, Lemma 5.5]. □

Lemma 2.5 *Let $f \in C(X, L[0, a])$ and suppose there exists $0 < \tau < 1$ satisfying (2.1). Then for any $\delta > 0$ there exists $g \in C(X, L[0, a])$ satisfying*

(1) $\max_{x \in X} \|f(x) - g(x)\|_\infty < \delta$.
(2) *$g(x)(0) = f(x)(0)$ and $g(x)(a) = f(x)(a)$ for all $x \in X$.*
(3) *If $x, y \in X$ and $0 \leq \varepsilon \leq a/2$ satisfy*

$$\forall t \in [0, a - \varepsilon] : g(x)(t + \varepsilon) = g(y)(t)$$

then $\varepsilon = 0$ and $d(x, y) < \delta$.

Proof Except for the use of the above two lemmas on linear algebra, the proof is close to Lemma 2.1. We take $0 < b < c < a$ with $b = a - c < \min(\delta/4, a/4)$. We take an open covering $\{U_1, \dots, U_M\}$ satisfying $\operatorname{diam} U_m < \delta$ and $\operatorname{diam} f(U_m) < \min(\delta/4, (1-\tau)b/2)$ for all $1 \le m \le M$. Take $p_m \in U_m$ for each m. Let $N \ge 2$ be a natural number and set $\Delta = (c-b)/(N-1)$. We introduce a partition $b = a_1 < a_2 < \cdots < a_N = c$ by $a_n = b + (n-1)\Delta$. We set $A = \{a_1, \dots, a_N\}$ and $\Lambda = A \cap [b, a/4] = \{a_1, \dots, a_L\}$. We also set $e = (\underbrace{1, 1, \dots, 1}_{L}) \in \mathbb{R}^L$. We choose N sufficiently large so that

$$\Delta < \frac{\delta}{4}, \quad N > L \ge 2M.$$

Since $L \ge 2M \ge M + 1$, by using Lemmas 2.3 and 2.4, we can choose $u_1, \dots, u_M \in [0,1]^A$ satisfying

(1) $|f(p_m)(a_n) - u_m(a_n)| < \min(\delta/4, (1-\tau)b/2)$ for all $1 \le m \le M$ and $1 \le n \le N$.
(2) $|u_m(a_{n+1}) - u_m(a_n)| < \Delta$ for all $1 \le m \le M$ and $1 \le n \le N-1$.
(3) Define $D_L u_m = (u_m(a_2) - u_m(a_1), \dots, u_m(a_{L+1}) - u_m(a_L)) \in \mathbb{R}^L$. Then the $(M+1)$ vectors $e, D_L u_1, \dots, D_L u_M$ in $\mathbb{R}^L$ are linearly independent.
(4) For any $\varepsilon > 0$ with $\varepsilon + \Lambda \subset A$,

$$u_1|_\Lambda,\ u_1|_{\varepsilon+\Lambda},\ u_2|_\Lambda,\ u_2|_{\varepsilon+\Lambda}, \dots, u_m|_\Lambda,\ u_m|_{\varepsilon+\Lambda} \text{ are linearly independent in } \mathbb{R}^\Lambda$$

For $x \in X$ we define $g(x) : [0, a] \to [0, 1]$ in the same way as in the proof of Lemma 2.1. Namely, we set $g(x)(0) = f(x)(0)$, $g(x)(a) = f(x)(a)$ and $g(x)(a_n) = \sum_{m=1}^M h_m(x) u_m(a_n)$ for $1 \le n \le N$, where $\{h_m\}$ is a partition of unity satisfying $\operatorname{supp} h_m \subset U_m$. We extend $g(x)$ to $[0, a]$ by linearity. It follows that $g(x) \in L[0, a]$ and $\|g(x) - f(x)\|_\infty < \delta$ as before. We need to check the property (3) of the statement. Suppose there exist $x, y \in X$ and $0 \le \varepsilon \le a/2$ satisfying $g(x)(t + \varepsilon) = g(y)(t)$ for all $0 \le t \le a - \varepsilon$.

First we show $\varepsilon + \Lambda \subset A$. Otherwise, $(\varepsilon + \Lambda) \cap A = \emptyset$. Then it follows from the piecewise linearity that the function $g(y)(t)$ becomes differentiable at every $t \in \Lambda$, which implies

$$g(y)(a_{n+1}) - g(y)(a_n) = g(y)(a_{n+2}) - g(y)(a_{n+1}) \quad (1 \le n \le L-1),$$

and hence

$$\sum_{m=1}^M h_m(y)\,(u_m(a_{n+1}) - u_m(a_n)) = \sum_{m=1}^M h_m(y)\,(u_m(a_{n+2}) - u_m(a_{n+1})) \quad (1 \le n \le L-1).$$

This means that $\sum_{m=1}^M h_m(y) D_L u_m \in \mathbb{R}e$, which contradicts the property (3) of u_m. So we must have $\varepsilon + \Lambda \subset A$.

The equation $g(x)(t + \varepsilon) = g(y)(t)$ $(0 \le t \le a - \varepsilon)$ implies

$$\sum_{m=1}^M h_m(x) u_m|_{\varepsilon+\Lambda} = \sum_{m=1}^M h_m(y) u_m|_\Lambda.$$

It follows from the property (4) of u_m that $\varepsilon = 0$ and $h_m(x) = h_m(y)$ for all $1 \le m \le M$. Then $x, y \in U_m$ for some m and hence $d(x, y) \le \operatorname{diam} U_m < \delta$. □

3 Proof of Theorem 1.3

Let (X, T) be a flow. Set $F = \mathrm{Fix}(X, T)$. We define $F_L = \mathrm{Fix}\,(L(\mathbb{R}))$. Namely F_L is the space of constant maps $\varphi : \mathbb{R} \to [0, 1]$, which is homeomorphic to $[0, 1]$. Suppose there exists a topological embedding $h : F \to F_L$. We would like to show that there exists an equivariant embedding $f : X \to L(\mathbb{R})$ with $f|_F = h$. We define $C_{T,h}\,(X, L(\mathbb{R}))$ as the space of equivariant continuous maps $f : X \to L(\mathbb{R})$ satisfying $f|_F = h$, which is endowed with the compact-open topology. For $f \in C_{T,h}\,(X, L(\mathbb{R}))$ we define $\mathrm{Lip}(f)$ as the supremum of

$$\frac{|f(x)(t) - f(x)(s)|}{|s - t|}$$

over all $x \in X$ and $s, t \in \mathbb{R}$ with $s \neq t$.

Lemma 3.1 *The space $C_{T,h}\,(X, L(\mathbb{R}))$ is not empty. Moreover for any $\delta > 0$ there exists $f \in C_{T,h}\,(X, L(\mathbb{R}))$ satisfying $\mathrm{Lip}(f) \leq \delta$.*

Proof Consider the map

$$F \ni x \to h(x)(0) \in [0, 1].$$

By the Tietze extension theorem, we can extend this function to a continuous map $h_0 : X \to [0, 1]$. Let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth function satisfying

$$\int_{-\infty}^{\infty} \varphi(t)\, dt = 1, \quad \int_{-\infty}^{\infty} \varphi'(t)\, dt \leq \min\,(1, \delta)\,.$$

For $x \in X$ we define $f(x) : \mathbb{R} \to [0, 1]$ by

$$f(x)(t) = \int_{-\infty}^{\infty} \varphi(t - s) h_0(T_s x)\, ds.$$

Then $|f(x)'(t)| \leq \min(1, \delta)$ and $f = h$ on F. Hence $f \in C_{T,h}\,(X, L(\mathbb{R}))$ and $\mathrm{Lip}(f) \leq \delta$. □

We borrow the next lemma from Auslander [1, p. 186, Corollary 6].

Lemma 3.2 *Let $p \in X \backslash F$. There exist $a > 0$ and a closed set $S \subset X$ containing p such that the map*

$$[\,-a, a] \times S \to X, \quad (t, x) \mapsto T_t x \tag{3.1}$$

*is a continuous injection whose image contains an open neighborhood of p in X. We call (a, S) a **local section** around p and denote the image of (3.1) by $[-a, a] \cdot S$.*

Proof We explain the proof for the convenience of readers. We can find $c < 0$ and a continuous function $h : X \to [0, 1]$ satisfying $T_c p \notin \mathrm{supp}\, h$ and $h = 1$ on a neighborhood of p. We define $f : X \to \mathbb{R}$ by

$$f(x) = \int_c^0 h(T_t x) dt.$$

We choose $0 < a < |c|$ and a closed neighborhood A of p satisfying

$$\bigcup_{|t| \leq a} T_t(A) \subset \{h = 1\}, \quad \bigcup_{|t| \leq a} T_{t+c}(A) \cap \mathrm{supp}\, h = \emptyset.$$

It follows that $f(T_t x) = f(x) + t$ for any $x \in A$ and $|t| \le a$. Set $S = \{x \in A |\, f(x) = f(p)\}$. Then (a, S) becomes a local section. Indeed if $x, y \in S$ and $s, t \in [-a, a]$ satisfy $T_s x = T_t y$, then $s + f(p) = f(T_s x) = f(T_t y) = t + f(p)$ and hence $s = t$ and $x = y$. Thus the map (3.1) is injective. We take $0 < b < a$ and an open neighborhood U of p satisfying $\bigcup_{|t|<b} T_t(U) \subset A$. Then the set $[-a, a] \cdot S$ contains

$$\{x \in U | -b < f(x) - f(p) < b\} \tag{3.2}$$

because if $x \in U$ satisfies $t \overset{\text{def}}{=} f(x) - f(p) \in (-b, b)$ then $f(T_{-t}x) = f(x) - t = f(p)$ (i.e. $T_{-t}x \in S$) and $x = T_t(T_{-t}x) \in [-a, a] \cdot S$. The set (3.2) is an open neighborhood of p. □

Lemma 3.3 *For any point $p \in X \backslash F$ there exists a closed neighborhood A of p in X such that the set*

$$G(A) = \left\{ f \in C_{T,h}(X, L(\mathbb{R})) \mid f(A) \cap F_L = \emptyset \right\} \tag{3.3}$$

is open and dense in the space $C_{T,h}(X, L(\mathbb{R}))$.

Proof Take a local section (a, S) around p. For $x \in X$ we define $H(x) \subset \mathbb{R}$ (the set of **hitting times**) as the set of $t \in \mathbb{R}$ satisfying $T_t x \in S$. Any two distinct $s, t \in H(x)$ satisfy $|s - t| > a$. Notice that if $x \in F$ then $H(x) = \emptyset$. We denote by Int $([-a, a] \cdot S)$ the interior of $[-a, a] \cdot S$. We choose a closed neighborhood A_0 of p in S satisfying $A_0 \subset \text{Int}\,([-a, a] \cdot S)$. We define a closed neighborhood A of p in X by

$$A = \bigcup_{|t| \le a} T_t(A_0).$$

We choose a continuous function $q : S \to [0, 1]$ satisfying $q = 1$ on A_0 and $\text{supp}\, q \subset \text{Int}\,([-a, a] \cdot S)$.

The set $G(A)$ defined in (3.3) is obviously open. So it is enough to prove that it is dense. Take $f \in C_{T,h}(X, L(\mathbb{R}))$ and $0 < \delta < 1$. By Lemma 3.1 we can find $f_0 \in C_{T,h}(X, L(\mathbb{R}))$ satisfying $\text{Lip}(f_0) \le 1/2$. We define $f_1 \in C_{T,h}(X, L(\mathbb{R}))$ by

$$f_1(x)(t) = (1 - \delta) f(x)(t) + \delta f_0(x)(t).$$

It follows $\text{Lip}(f_1) \le 1 - \delta/2 < 1$. We apply Lemma 2.1 to the map

$$X \ni x \mapsto f_1(x)|_{[0,a]} \in L[0, a].$$

Then we find $g \in C(X, L[0, a])$ satisfying

(1) $|g(x)(t) - f_1(x)(t)| < \delta$ for all $x \in X$ and $0 \le t \le a$.
(2) $g(x)(0) = f_1(x)(0)$ and $g(x)(a) = f_1(x)(a)$ for all $x \in X$.
(3) $g(X) \cap F_L[0, a] = \emptyset$.

We set $u(x)(t) = g(x)(t) - f_1(x)(t)$ for $x \in X$ and $0 \le t \le a$. We define $g_1 \in C_{T,h}(X, L(\mathbb{R}))$ as follows: Let $x \in X$.

- For each $s \in H(x)$, we set

$$g_1(x)(t) = f_1(x)(t) + q(T_s x) \cdot u(T_s x)(t - s) \quad \text{for } t \in [s, s + a].$$

- For $t \in \mathbb{R} \backslash \bigcup_{s \in H(x)} [s, s + a]$, we set $g_1(x)(t) = f_1(x)(t)$.

This satisfies

$$|g_1(x)(t) - f(x)(t)| \leq |g_1(x)(t) - f_1(x)(t)| + |f_1(x)(t) - f(x)(t)| \leq 3\delta$$

for all $x \in X$ and $t \in \mathbb{R}$. If $x \in A$ then there exists $s \in [-a, a]$ with $T_s x \in A_0$ and hence

$$g_1(x)(s+t) = g(T_s x)(t) \quad \text{for } t \in [0, a].$$

It follows from the property (3) of g that the function $g_1(x)$ is not constant. Thus $g_1 \in G(A)$. Since f and δ are arbitrary, this proves that $G(A)$ is dense in $C_{T,h}(X, L(\mathbb{R}))$. □

Lemma 3.4 *For any two distinct points p and q in $X \backslash F$ there exist closed neighborhoods B and C of p and q in X respectively such that the set*

$$G(B, C) = \left\{ f \in C_{T,h}(X, L(\mathbb{R})) \mid f(B) \cap f(C) = \emptyset \right\} \tag{3.4}$$

is open and dense in $C_{T,h}(X, L(\mathbb{R}))$.

Proof Take local sections (a, S_1) and (a, S_2) around p and q respectively. We can assume that $[-a, a] \cdot S_1$ and $[-a, a] \cdot S_2$ are disjoint with each other. For $x \in X$ we define $H(x)$ as the set of $t \in \mathbb{R}$ satisfying $T_t x \in S_1 \cup S_2$. We choose closed neighborhoods B_0 of p in S_1 and C_0 of q in S_2 respectively satisfying $B_0 \subset \operatorname{Int}([-a, a] \cdot S_1)$ and $C_0 \subset \operatorname{Int}([-a, a] \cdot S_2)$. We take a continuous function $\tilde{q} : X \to [0, 1]$ satisfying $\tilde{q} = 1$ on $B_0 \cup C_0$ and $\operatorname{supp} \tilde{q} \subset \operatorname{Int}([-a, a] \cdot S_1) \cup \operatorname{Int}([-a, a] \cdot S_2)$. We define closed neighborhoods B and C of p and q respectively by

$$B = \bigcup_{|t| \leq a/4} T_t(B_0), \quad C = \bigcup_{|t| \leq a/4} T_t(C_0).$$

The set $G(B, C)$ defined in (3.4) is open. We show that it is dense. Take $f \in C_{T,h}(X, L(\mathbb{R}))$ and $0 < \delta < 1$. We can assume that

$$\delta < d(B_0, C_0) \stackrel{\text{def}}{=} \min_{x \in B_0, y \in C_0} d(x, y). \tag{3.5}$$

We define $f_1 \in C_{T,h}(X, L(\mathbb{R}))$ exactly in the same way as in the proof of Lemma 3.3. It satisfies $\operatorname{Lip}(f_1) \leq 1 - \delta/2$ and $|f(x)(t) - f_1(x)(t)| \leq 2\delta$ for all $x \in X$ and $t \in \mathbb{R}$.

We apply Lemma 2.5 to the map

$$X \ni x \mapsto f_1(x)|_{[0,a]} \in L[0, a].$$

Then we find $g \in C(X, L[0, a])$ satisfying

(1) $|g(x)(t) - f_1(x)(t)| < \delta$ for all $x \in X$ and $0 \leq t \leq a$.
(2) $g(x)(0) = f_1(x)(0)$ and $g(x)(a) = f_1(x)(a)$ for all $x \in X$.
(3) If $x, y \in X$ and $0 \leq \varepsilon \leq a/2$ satisfy

$$\forall t \in [0, a - \varepsilon] : g(x)(t + \varepsilon) = g(y)(t)$$

then $d(x, y) < \delta$.

We set $u(x)(t) = g(x)(t) - f_1(x)(t)$ for $x \in X$ and $0 \leq t \leq a$. We define $g_1 \in C_{T,h}(X, L(\mathbb{R}))$ as before. Namely, for $x \in X$,

- For each $s \in H(x)$, we set

$$g_1(x)(t) = f_1(x)(t) + \tilde{q}(T_s x) \cdot u(T_s x)(t - s) \quad \text{for } t \in [s, s + a].$$

- For $t \in \mathbb{R}\setminus\bigcup_{s\in H(x)}[s, s+a]$, we set $g_1(x)(t) = f_1(x)(t)$.

This satisfies $|g_1(x)(t) - f(x)(t)| \le |g_1(x)(t) - f_1(x)(t)| + |f_1(x)(t) - f(x)(t)| \le 3\delta$.

We would like to show $g_1(B) \cap g_1(C) = \emptyset$. Suppose $x \in B$ and $y \in C$ satisfy $g_1(x) = g_1(y)$. There exist $|s_1| \le a/4$ and $|s_2| \le a/4$ satisfying $T_{s_1}x \in B_0$ and $T_{s_2}y \in C_0$. We can assume $s_1 \le s_2$ without loss of generality. Set $\varepsilon = s_2 - s_1 \in [0, a/2]$. We have

$$g_1(x)(s_1 + t) = g(T_{s_1}x)(t) \text{ and } g_1(y)(s_2 + t) = g(T_{s_2}y)(t) \quad \text{for } t \in [0, a].$$

$g_1(x) = g_1(y)$ implies that

$$g(T_{s_1}x)(t + \varepsilon) = g(T_{s_2}y)(t) \quad \text{for } t \in [0, a - \varepsilon].$$

It follows from the property (3) of g that $d(T_{s_1}x, T_{s_2}y) < \delta$. Since $\delta < d(B_0, C_0) \le d(T_{s_1}x, T_{s_2}y)$, this is a contradiction. Therefore $g_1(B) \cap g_1(C) = \emptyset$. This proves the lemma. □

Now we can prove Theorem 1.3. Note that X and $X \times X$ are hereditarily Lindelöf (that means that every open cover of a subspace has a countable subcover). Using these facts and applying Lemma 3.3 to each point in $X\setminus F$ and Lemma 3.4 to every pair of distinct points in $X\setminus F$, there exist countable families of closed sets $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ of $X\setminus F$ such that

- $X\setminus F = \bigcup_{n=1}^{\infty} A_n$ and $(X\setminus F) \times (X\setminus F)\setminus\{(x, x) : x \in X\} = \bigcup_{n=1}^{\infty} B_n \times C_n$.
- $G(A_n)$ are open and dense in the space $C_{T,h}(X, L(\mathbb{R}))$ for all $n \ge 1$.
- $G(B_n, C_n)$ are open and dense in the space $C_{T,h}(X, L(\mathbb{R}))$ for all $n \ge 1$.

By the Baire category theorem, the set

$$\bigcap_{n=1}^{\infty} G(A_n) \cap \bigcap_{n=1}^{\infty} G(B_n, C_n)$$

is dense and G_δ in $C_{T,h}(X, L(\mathbb{R}))$. In particular it is not empty. Any element f in this set gives an embedding of the flow (X, T) in $L(\mathbb{R})$.

Remark 3.5 The proof of the Bebutov–Kakutani theorem in [1,4] used the idea of "constructing large derivative". It is possible to prove Theorem 1.3 by adapting this idea to the setting of one-Lipschitz functions. But this approach seems a bit tricky and less flexible than the proof given above. The above proof possibly has a wider applicability to different situations (e.g. other function spaces).

Acknowledgements This paper was written when the third named author stayed in the Einstein Institute of Mathematics in the Hebrew University of Jerusalem. He would like to thank the institute for its hospitality.

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