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Generalisations of the Harer–Zagier recursion for 1-point functions

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Abstract
Harer and Zagier proved a recursion to enumerate gluings of a $2d$-gon that result in an orientable genus g surface, in their work on Euler characteristics of moduli spaces of curves. Analogous results have been discovered for other enumerative problems, so it is natural to pose the following question: how large is the family of problems for which these so-called 1-point recursions exist? In this paper, we prove the existence of 1-point recursions for a class of enumerative problems that have Schur function expansions. In particular, we recover the Harer–Zagier recursion, but our methodology also applies to the enumeration of dessins d'enfant, to Bousquet-Mélou–Schaeffer numbers, to monotone Hurwitz numbers, and more. On the other hand, we prove that there is no 1-point recursion that governs single Hurwitz numbers. Our results are effective in the sense that one can explicitly compute particular instances of 1-point recursions, and we provide several examples. We conclude the paper with a brief discussion and a conjecture relating 1-point recursions to the theory of topological recursion.

Keywords Harer–Zagier formula · 1-point functions · Holonomic functions · Schur functions · Hurwitz numbers · Ribbon graphs · Dessins d'enfant

Mathematics Subject Classification 05A15 · 05E10 · 14N10

1 Introduction

For integers $g \geq 0$ and $d \geq 1$, let $a_g(d)$ denote the number of ways to glue the edges of a $2d$-gon in pairs to obtain an orientable genus g surface. The data of a surface constructed by gluing edges of polygons in pairs are often referred to in the literature

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as a *ribbon graph*. In their pioneering work, Harer and Zagier apply matrix model techniques to this enumeration of ribbon graphs with one face to deduce a formula for the virtual Euler characteristics of moduli spaces of curves. One consequence of their calculation is the fact that the numbers $a_g(d)$ satisfy the following recursion [38].

$$(d+1)\, a_g(d) = 2(2d-1)\, a_g(d-1) + (2d-1)(d-1)(2d-3)\, a_{g-1}(d-2) \quad (1)$$

Despite the simple appearance of this formula, Zagier later stated [44]: "No combinatorial interpretation of the recursion… is known". The Harer–Zagier recursion has since attracted a great deal of interest, and there now exist several proofs, some of which are combinatorial in nature [1,11,34,45,49,57].

In more recent work of the second author and Norbury [20], as well as the subsequent work of Chekhov [14], an analogue of the Harer–Zagier recursion was deduced for the number of *dessins d'enfant* with one face. More precisely, let $b_g(d)$ denote the number of ways to glue the edges of a $2d$-gon, whose vertices are alternately coloured black and white, in pairs to obtain an orientable genus g surface. Of course, we impose the caveat that vertices may only be glued together if they share the same colour. The numbers $b_g(d)$ satisfy the following recursion.

$$(d+1)\, b_g(d) = 2(2d-1)\, b_g(d-1) + (d-1)^2(d-2)\, b_{g-1}(d-2) \qquad (2)$$

It is natural to embed the problem of calculating $a_g(d)$ into the more general enumeration of ways to glue the edges of n labelled polygons with $d_1, d_2, \ldots, d_n$ sides to obtain an orientable genus g surface. This problem then lends itself naturally to a simple combinatorial recursion, whose roots lie in the work of Tutte [61], but was first expressed by Walsh and Lehman [62]. The mechanism for such a recursion comes from removing an edge from the ribbon graph formed by the edges of the polygons, and observing that one is left with either a simpler ribbon graph or the disjoint union of two simpler ribbon graphs. The cost of combinatorial simplicity is the necessity to consider gluings of an arbitrary number of polygons, rather than gluings of just one polygon.

Recursions similar to those expressed in (1) and (2) have appeared in other contexts, such as random matrix theory [46]. However, it is not true in general that these recursions involve three terms, as in the examples above. In the context of enumerative geometry and mathematical physics, the analogues of $a_g(d)$ and $b_g(d)$ are known as *1-point invariants*, since they often arise as expansion coefficients of 1-point correlation functions. And more generally, the enumeration of ways to glue n polygons to obtain surfaces produces numbers known as *n-point invariants*. The preceding discussion motivates us to make the following definition.

Definition 1 We say that the collection of numbers $n_g(d) \in \mathbb{C}$ for integers $g \geq 0$ and $d \geq 1$ satisfies a *1-point recursion* if there exist integers $i_{\max}$, $j_{\max}$ and complex polynomials p_{ij}, not all equal to zero, such that

$$\sum_{i=0}^{i_{\max}} \sum_{j=0}^{j_{\max}} p_{ij}(d)\, n_{g-i}(d-j) = 0, \qquad (3)$$

whenever all terms in the equation are defined.

The current work is motivated by the following interrelated questions.

– What unified proofs of 1-point recursions exist, which encompass (1) and (2)?
– How universal is the notion of a 1-point recursion?

We partially answer these questions by first observing that the enumeration of both ribbon graphs and dessins d'enfant can be expressed in terms of Schur functions. This suggests that 1-point recursions may exist more generally for problems that may be defined in an analogous way. Thus, we consider *double Schur function expansions* of the following form.

$$Z(\mathbf{p};\mathbf{q};\hbar) = \sum_{\lambda\in\mathscr{P}} s_\lambda(p_1,p_2,\ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots\right) \prod_{\Box\in\lambda} G(c(\Box)\hbar)$$
$$= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} N_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\,p_{d_1}p_{d_2}\cdots p_{d_n}\right] \tag{4}$$

The precise meaning of all terms appearing in this equation will be discussed in Sect. 2, particularly in Definition 2. It currently suffices to observe that the "enumerative problem" is stored in the numbers $N_{g,n}(d_1, d_2, \ldots, d_n)$ appearing in the second line. Note that the quantity $N_{g,n}(d_1, d_2, \ldots, d_n)$ depends on the parameters $q_1, q_2, \ldots$, although we leave this dependence implicit to avoid overloading the notation. These Schur function expansions were originally introduced by Harnad and Orlov [39], and the numbers $N_{g,n}(d_1, d_2, \ldots, d_n)$ were subsequently given a combinatorial interpretation and referred to as *weighted Hurwitz numbers* by Guay-Paquet and Harnad [36,37]. More recently, Alexandov, Chapuy, Eynard and Harnad have studied these objects in the context of topological recursion [2],

The primary contribution of this paper is an approach to proving 1-point recursions for such "enumerative problems". In particular, our main result is the following.

Theorem 1 *Let $G(z) \in \mathbb{C}(z)$ be a rational function with $G(0) = 1$ and suppose that finitely many terms of the sequence $q_1, q_2, q_3, \ldots$ of complex numbers are nonzero. Then, the numbers $n_g(d) = d\, N_{g,1}(d)$ defined by (4) satisfy a 1-point recursion.*

The proof of this theorem will be taken up in Sect. 4, where we use the theory and language of holonomic sequences and functions. The basic observation is Lemma 11, which states that a 1-point recursion exists for $n_g(d)$ if and only if the sequence $n_d = \sum_g n_g(d)\, \hbar^{2g-1}$ is holonomic over $\mathbb{C}(\hbar)$.

Example 1 If we take $G(z) = 1 + z$ and $\mathbf{q} = (0, 1, 0, 0, \ldots)$ in (4), then we recover the enumeration of ribbon graphs introduced earlier. In other words, we have $n_g(d) = a_g(d)$, so Theorem 1 asserts the existence of a 1-point recursion for the numbers $a_g(d)$.

Analogously, if we take $G(z) = (1 + z)^2$ and $\mathbf{q} = (1, 0, 0, \ldots)$ in (4), then we recover the enumeration of dessins d'enfant introduced earlier. In other words, we have $n_g(d) = b_g(d)$, so Theorem 1 asserts the existence of a 1-point recursion for the numbers $b_g(d)$.

One of the features of the theory of holonomic sequences and functions is that there are readily available algorithms to carry out computations, such as those found in the `gfun` package for MAPLE [58]. Our proof of Theorem 1 not only asserts the existence of 1-point recursions, but also yields an algorithm to produce them. We use this to determine explicit 1-point recursions for:

- the enumeration of 3-hypermaps and 3-BMS numbers (see Proposition 5); and
- the enumeration of monotone Hurwitz numbers (see Proposition 8).

Example 2 The monotone Hurwitz numbers satisfy the following 1-point recursion.

$$d\, m_g(d) = 2(2d-3)\, m_g(d-1) + d(d-1)^2\, m_{g-1}(d)$$

As a partial converse to Theorem 1, we prove that there are enumerative problems governed by double Schur function expansions that do not satisfy a 1-point recursion. Of particular note is the case of single Hurwitz numbers, which arise from (4) by taking $G(z) = \exp(z)$ and $\mathbf{q} = (1, 0, 0, \ldots)$.

Proposition 1 *The single Hurwitz numbers do not satisfy a 1-point recursion.*

Underlying our work are the related notions of integrability and topological recursion. Regarding the former, we only remark that the double Schur function expansions of (4) are examples of hypergeometric tau-functions for the Toda integrable hierarchy [56]. The topological recursion can be used to produce enumerative invariants from a spectral curve, which is essentially a plane algebraic curve satisfying some mild conditions and equipped with certain extra data. From the work of Alexandrov, Chapuy, Eynard and Harnad [2], we know that the assumptions of Theorem 1 lead to numbers $N_{g,n}(d_1, d_2, \ldots, d_n)$ in (4) that can be calculated via the topological recursion. Furthermore, the associated spectral curve is an explicit rational curve, which depends on the particular choice of $G(z)$ and $\mathbf{q}$. Combining Theorem 1 with the aforementioned work of Alexandrov et al. suggests the following conjecture, whose precise statement will later appear as 2.

Conjecture 1 Topological recursion on a rational spectral curve produces invariants that satisfy a 1-point recursion.

In practice, one may only be interested in 1-point functions, as is the case for the problem originally studied by Harer and Zagier [38]. Calculating these via the topological recursion requires the knowledge of n-point functions for all positive integers n. Thus, a 1-point recursion can provide an effective tool for calculation, from both the practical and theoretical perspectives. For instance, a 1-point recursion can lead to direct information regarding the structure of 1-point invariants—see 1 for an example of this phenomenon.

We conclude the paper with some evidence towards the conjecture above as well as a brief discussion on the related notion of quantum curves. In the context of the double Schur function expansions studied in this paper, quantum curves arise from a specialisation of (4) that reduces the summation over all partitions to a summation over 1-part partitions. On the other hand, we will observe that 1-point recursions arise from a different specialisation that reduces it to a summation over hook partitions.

The structure of the paper is as follows.

- In Sect. 2, we introduce four classes of enumerative problems that will provide motivation for and examples of our main results. These are: the enumeration of ribbon graphs and dessins d'enfant; Bousquet-Mélou–Schaeffer numbers; Hurwitz numbers; and monotone Hurwitz numbers. A common thread between these problems is that their so-called partition functions have double Schur function expansions.
- In Sect. 3, we precisely define double Schur function expansions and deduce an expression for their 1-point invariants. We also present certain evaluations of Schur functions that will subsequently prove useful.
- In Sect. 4, we recall the notion of holonomicity and relate it to the existence of 1-point recursions. This is used to prove Theorem 1 on the existence of 1-point recursions, which then leads to an algorithm for 1-point recursions.
- In Sect. 5, we return to the four classes of enumerative problems introduced in Sect. 2. For three of these, we present examples of 1-point recursions, but for the case of single Hurwitz numbers, we prove that no such recursion exists. We also demonstrate how 1-point recursions can be used to prove structural results, and sometimes explicit formulas, for 1-point invariants.
- In Sect. 6, we discuss relations between our work and the theory of topological recursion. In particular, we formulate a precise statement of 1, which loosely states that is a 1-point recursion for the invariants arising from topological recursion applied to a rational spectral curve. Some evidence towards this conjecture is presented, along with some remarks on the similarity between our calculation of 1-point recursions and the calculation of quantum curves.

2 Enumerative problems

Our work is primarily motivated by the Harer–Zagier formula for the enumeration of ribbon graphs with one face [38], as well as the analogue for the enumeration of dessins d'enfant with one face [14,20]. Apart from the obvious similarities between these two problems, they also both arise from double Schur function expansions. So we propose to study the broad class of "enumerative problems" stored in double Schur function expansions of the general form

$$Z(\mathbf{p};\mathbf{q};\hbar) = \sum_{\lambda\in\mathscr{P}} s_\lambda(p_1,p_2,\ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots\right)\, F_\lambda(\hbar).$$

Here, $\mathscr{P}$ is the set of all partitions (including the empty partition), $s_\lambda(p_1,p_2,\ldots)$ denotes the Schur function expressed in terms of power sum symmetric functions, and $F_\lambda(\hbar)$ is a formal power series in $\hbar$ for each partition λ. We use the shorthand $\mathbf{p} = (p_1,p_2,p_3,\ldots)$ and $\mathbf{q} = (q_1,q_2,q_3,\ldots)$ throughout the paper. Following the mathematical physics literature, we will refer to such power series as *partition functions* (although we note that this name does not refer to the integer partitions that appear in the equation above).

Remark 1 We emphasise that the notation for Schur functions used here differs from that appearing, for example, in the book of Macdonald [48]. In that reference, an infinite set of commuting variables $x_1, x_2, x_3, \ldots$ is introduced and each Schur function is a symmetric function in these variables. On the other hand, we use $s_\lambda(p_1, p_2, p_3, \ldots)$ to denote the Schur function corresponding to the partition λ, expressed as a polynomial in the power sum symmetric functions $p_1, p_2, p_3, \ldots$, where $p_k = x_1^k + x_2^k + x_3^k + \cdots$. This notation is commonly used in the literature on integrable hierarchies, in which case $p_1, p_2, p_3, \ldots$ correspond to flow parameters. As a concrete example, the Schur function corresponding to the partition (2) in the variables $x_1, x_2, x_3, \ldots$ is given by

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + \cdots) &+ (x_1x_2 + x_1x_3 + x_2x_3 + \cdots) \\ &= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + \cdots) + \frac{1}{2}(x_1 + x_2 + x_3 + \cdots)^2.\end{aligned}$$

So in the notation of the present paper, we have $s_2(p_1, p_2, p_3, \ldots) = \frac{1}{2}p_2 + \frac{1}{2}p_1^2$.

For our applications, we will take $F_\lambda(\hbar)$ to have the so-called *content product* form

$$F_\lambda(\hbar) = \prod_{\Box\in\lambda} G(c(\Box)\hbar).$$

Here, the product is over the boxes in the Young diagram for λ, $G(z) \in \mathbb{C}[[z]]$ is a formal power series normalised to have constant term 1, and $c(\Box)$ denotes the content of the box. Recall that the *content* of a box in row i and column j of a Young diagram is the integer $j - i$.

The partition function can be expressed as

$$\begin{aligned}Z(\mathbf{p}; \mathbf{q}; \hbar) &= \sum_{\lambda\in\mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots\right) \prod_{\Box\in\lambda} G(c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} N_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1}p_{d_2}\cdots p_{d_n}\right],\end{aligned} \tag{5}$$

where $N_{g,n}(d_1, d_2, \ldots, d_n) \in \mathbb{C}[q_1, q_2, \ldots]$. For various natural choices of the formal power series $G(z)$ and the weights $q_1, q_2, q_3, \ldots$, the quantity $N_{g,n}(d_1, d_2, \ldots, d_n)$ enumerates objects of combinatorial interest.

Definition 2 Let $G(z) \in \mathbb{C}[[z]]$ be a formal power series with constant term 1 and let $q_1, q_2, q_3, \ldots$ be a set of commuting variables. For positive integers $d_1, d_2, \ldots, d_n$, define $N_{g,n}(d_1, d_2, \ldots, d_n) \in \mathbb{C}[q_1, q_2, q_3, \ldots]$ via the series expansions of the two expressions in (5) in the variables $p_1, p_2, p_3, \ldots$.

Remark 2 We suppress the explicit dependence on $G(z)$ and $q_1, q_2, q_3, \ldots$ to avoid overloading the notation in Definition 2. Any particular choices of or assumptions on $G(z)$ and $q_1, q_2, q_3, \ldots$ will usually be clear from the context.

Due to the homogeneity properties of Schur functions, we know that the quantity $N_{g,n}(d_1, d_2, \ldots, d_n) \in \mathbb{C}[q_1, q_2, q_3, \ldots]$ is a weighted homogeneous polynomial of degree $d_1 + d_2 + \cdots + d_n$ in $q_1, q_2, q_3, \ldots$, where the weight of q_i is i. In fact, these have been referred to in the literature as *weighted Hurwitz numbers* and are known to enumerate certain paths in the Cayley graph of $S_{|\mathbf{d}|}$ generated by transpositions [2,36,37,39]. Furthermore, it is known that the partition function $Z(\mathbf{p}; \mathbf{q}; \hbar)$ is a hypergeometric tau-function for the Toda integrable hierarchy [56].

We now proceed to examine four classes of combinatorial problems that arise from double Schur function expansions. Readers looking for the general description of double Schur function expansions and their 1-point recursions may wish to skip directly to Sect. 3.

2.1 Ribbon graphs and dessins d'enfant

A ribbon graph—also known as a map, embedded graph, fat graph or rotation system—can be thought of as the 1-skeleton of a cell decomposition of an oriented compact surface. Ribbon graphs arise naturally in various areas of mathematics, including topological graph theory, moduli spaces of Riemann surfaces, and matrix models [44]. A more formal definition is the following.

Definition 3 A *ribbon graph* is a finite connected graph equipped with a cyclic ordering of the half-edges meeting at each vertex. An *isomorphism* between two ribbon graphs is a bijection between their sets of half-edges that preserves all adjacencies, as well as the cyclic ordering of the half-edges meeting at each vertex.

The underlying graph of a ribbon graph is precisely the 1-skeleton of a cell decomposition of a compact connected orientable surface. The cyclic ordering of the half-edges meeting at every vertex allows one to reconstruct the 2-cells and hence, the underlying oriented compact surface. Thus, one can assign a genus to a ribbon graph.

Alternatively, one can encode a ribbon graph as a pair of permutations (τ_0, τ_1) such that τ_1 has cycle type $(2, 2, \ldots, 2)$, and τ_0 and τ_1 generate a transitive subgroup of the symmetric group. We think of these permutations as acting on the half-edges of the ribbon graph, where τ_0 rotates half-edges anticlockwise around their adjacent vertex and τ_1 swaps half-edges belonging to the same underlying edge. More generally, one can consider an *m-hypermap* as a pair of permutations (τ_0, τ_1) such that τ_1 has cycle type $(m, m, \ldots, m)$, and τ_0 and τ_1 generate a transitive subgroup of the symmetric group. For further information on these topics, one may consult the book of Lando and Zvonkin [44].

Definition 4 Define the *ribbon graph number* $A_{g,n}(d_1, d_2, \ldots, d_n)$ to be the weighted count of ribbon graphs of genus g with n labelled faces of degrees $d_1, d_2, \ldots, d_n$. The weight of a ribbon graph Γ is $\frac{1}{|\mathrm{Aut}\ \Gamma|}$, where Aut Γ denotes the group of face-preserving automorphisms. The corresponding 1-point invariant is denoted $a_g(d) = 2d\, A_{g,1}(2d)$.

(The factor of $2d$ in this definition provides agreement with the work of Harer and Zagier [38] and produces a simpler 1-point recursion.)

We similarly define $A^m_{g,n}(d_1, d_2, \ldots, d_n)$ to be the weighted count of m-hypermaps of genus g with n labelled faces of degrees $d_1, d_2, \ldots, d_n$. The corresponding 1-point invariant is denoted $a^m_g(d) = md\, A^m_{g,1}(md)$.

The following result is a consequence of the work of Alexandrov, Lewanski and Shadrin, in which they show an equivalence between counting hypermaps and the notion of strictly monotone orbifold Hurwitz numbers [3].

Lemma 1 *The ribbon graph numbers arise from taking* $\mathbf{q} = (0, 1, 0, \ldots)$ *and* $G(z) = 1 + z$ *in* (4). *In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(0, \tfrac{1}{\hbar}, 0, \ldots) \prod_{\Box \in \lambda} (1 + c(\Box)\hbar)$$

$$= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} A_{g,n}(d_1, d_2, \ldots, d_n) \frac{\hbar^{2g-2+n}}{n!} p_{d_1} p_{d_2} \cdots p_{d_n}\right].$$

More generally, the enumeration of m-hypermaps arises from keeping $G(z) = 1 + z$, *but taking* $q_m = 1$ *and* $q_i = 0$ *for* $i \neq m$.

Ribbon graphs can be considered as special cases of the more general notion of dessins d'enfant.

Definition 5 A *dessin d'enfant* is a ribbon graph whose vertices are coloured black and white such that every edge is adjacent to one vertex of each colour. An *isomorphism* between two dessins d'enfant is an isomorphism between their underlying ribbon graphs that preserves the vertex colouring.

One obtains the notion of a ribbon graph by considering dessins d'enfant in which every black vertex has degree two. In that case, one can simply remove the degree two vertex and amalgamate the adjacent two edges into a single edge, to obtain a ribbon graph. Similarly, dessins d'enfant in which every black vertex has degree m give rise to m-hypermaps.

Definition 6 The *dessin d'enfant number* $B_{g,n}(d_1, d_2, \ldots, d_n)$ is the weighted count of dessins d'enfant of genus g with n labelled faces of degrees $2d_1, 2d_2, \ldots, 2d_n$. The weight of a dessin d'enfant Γ is $\frac{1}{|\mathrm{Aut}\,\Gamma|}$, where Aut Γ denotes the group of face-preserving automorphisms. The corresponding 1-point invariant is denoted $b_g(d) = d\, B_{g,1}(d)$.

More generally, we can refine the enumeration by weighting with parameters that record the degrees of the black vertices.

Definition 7 Define the *double dessin d'enfant number* $\overline{B}_{g,n}(d_1, d_2, \ldots, d_n)$ to be the analogous weighted count of dessins d'enfant, where the weight of a dessin d'enfant Γ with black vertices of degrees $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ is $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}\,\Gamma|}$. The corresponding 1-point invariant is denoted $\overline{b}_g(d) = d\, \overline{B}_{g,1}(d)$.

d	g	$a_g(d)$	$\overline{b}_g(d)$
1	0	1	q_1
2	0	2	$q_2+q_1^2$
2	1	1	0
3	0	5	$q_3+3q_2q_1+q_1^3$
3	1	10	q_3
4	0	14	$q_4+4q_3q_1+2q_2^2+6q_2q_1^2+q_1^4$
4	1	70	$5q_4+4q_3q_1+q_2^2$
4	2	21	0
5	0	42	$q_5+5q_4q_1+5q_3q_2+10q_3q_1^2+10q_2^2q_1+10q_2q_1^3+q_1^5$
5	1	420	$15q_5+25q_4q_1+15q_3q_2+10q_3q_1^2+5q_2^2q_1$
5	2	483	$8q_5$
6	0	132	$q_6+6q_5q_1+6q_4q_2+15q_4q_1^2+3q_3^2+30q_3q_2q_1+20q_3q_1^3+5q_2^3+30q_2^2q_1^2+15q_2q_1^4+q_1^6$
6	1	2310	$35q_6+90q_5q_1+60q_4q_2+75q_4q_1^2+25q_3^2+90q_3q_2q_1+20q_3q_1^3+10q_2^3+15q_2^2q_1^2$
6	2	6468	$84q_6+48q_5q_1+24q_4q_2+12q_3^2$
6	3	1485	0

Fig. 1 Table of ribbon graph numbers and double dessin d'enfant numbers

We have $\overline{B}_{g,n}(d_1, d_2, \ldots, d_n) \in \mathbb{Q}[q_1, q_2, q_3, \ldots]$, since $q_1, q_2, q_3, \ldots$ are indeterminates in the above definition. The following result generalises Lemma 1.

Lemma 2 *The double dessin d'enfant numbers arise from taking* $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ *and* $G(z) = 1 + z$ *in* (4). *In other words, we have*

$$\begin{aligned} Z(\mathbf{p}; \mathbf{q}; \hbar) &= \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots\right) \prod_{\Box \in \lambda} (1 + c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} \overline{B}_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n}\right]. \end{aligned}$$

One obtains the usual dessin d'enfant enumeration by setting $\mathbf{q} = (1, 1, 1, \ldots)$ in the double dessin d'enfant enumeration.

$$\begin{aligned} Z(\mathbf{p}; \mathbf{q}; \hbar) &= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} B_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n}\right] \\ &= \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{1}{\hbar}, \tfrac{1}{\hbar}, \tfrac{1}{\hbar}, \ldots\right) \prod_{\Box \in \lambda} (1 + c(\Box)\hbar) \\ &= \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{1}{\hbar}, 0, 0, \ldots\right) \prod_{\Box \in \lambda} (1 + c(\Box)\hbar)^2 \end{aligned}$$

The second equality here relies on the fact that $s_\lambda(\frac{1}{\hbar}, \frac{1}{\hbar}, \frac{1}{\hbar}, \ldots) = s_\lambda(\frac{1}{\hbar}, 0, 0, \ldots) \prod(1+c(\Box)\hbar)$, which is a direct corollary of the hook-length and the hook-content formulas—see (9).

2.2 Bousquet-Mélou–Schaeffer numbers

One can encode a dessin d'enfant via a pair (σ_1, σ_2) of permutations acting on the edges. Here, σ_1 acts by rotating each edge anticlockwise around its adjacent black vertex and σ_2 acts by rotating each edge anticlockwise around its adjacent white vertex. The connectedness of the dessin d'enfant is encoded in the fact that the two permutations generate a transitive subgroup of the symmetric group. For more details, one can consult the extensive literature on dessins d'enfant [44]. More generally, one has the notion of Bousquet-Mélou–Schaeffer numbers [10].

Definition 8 For m a positive integer, the *Bousquet-Mélou–Schaeffer (BMS) number* $B^m_{g,n}(d_1, d_2, \ldots, d_n)$ is equal to $\frac{1}{|\mathbf{d}|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of permutations in $S_{|\mathbf{d}|}$ such that

- $\sum_{i=1}^{m}(|\mathbf{d}| - k(\sigma_i)) = 2g - 2 + n + |\mathbf{d}|$, where $k(\sigma)$ denotes the number of cycles in σ;
- $\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_m$ has n labelled cycles with lengths $d_1, d_2, \ldots, d_n$; and
- $\sigma_1, \sigma_2, \ldots, \sigma_m$ generate a transitive subgroup of the symmetric group.

The corresponding 1-point invariant is denoted $b^m_g(d) = d\, B^m_{g,1}(d)$.

Lemma 3 *The m-BMS numbers arise from taking* $\mathbf{q} = (1, 0, 0, \ldots)$ *and* $G(z) = (1 + z)^m$ *in* (4). *In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{1}{\hbar}, 0, 0, \ldots\right) \prod_{\Box \in \lambda} (1 + c(\Box)\hbar)^m$$
$$= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} B^m_{g,n}(d_1, d_2, \ldots, d_n) \frac{\hbar^{2g-2+n}}{n!} p_{d_1} p_{d_2} \cdots p_{d_n}\right].$$

By the Riemann existence theorem, one can consider $B^m_{g,n}(d_1, d_2, \ldots, d_n)$ to be the weighted count of connected genus g branched covers $f : (C; p_1, p_2, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- $f^{-1}(\infty) = d_1 p_1 + d_2 p_2 + \cdots + d_n p_n$;
- all other ramification occurs at the mth roots of unity.

The weight of a branched cover $f : C \to \mathbb{CP}^1$ is $\frac{1}{|\mathrm{Aut}\, f|}$, where an automorphism of f is a Riemann surface automorphism $\phi : C \to C$ such that $f \circ \phi = f$.

More generally, we can refine the enumeration by weighting by parameters that record the ramification profile at one of the roots of unity.

Definition 9 The *double Bousquet-Mélou–Schaeffer number* $\overline{B}^m_{g,n}(d_1, d_2, \ldots, d_n)$ is the weighted count of genus g connected branched covers $f : (C; p_1, p_2, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- $f^{-1}(\infty) = d_1 p_1 + d_2 p_2 + \cdots + d_n p_n$;
- all other ramification occurs at the mth roots of unity.

d	g	$\overline{b}_g^3(d)$
1	0	q_1
2	0	$q_2+2q_1^2$
2	1	q_2
3	0	$q_3+6q_2q_1+5q_1^3$
3	1	$8q_3+12q_2q_1+q_1^3$
3	2	$3q_3$
4	0	$q_4+8q_3q_1+4q_2^2+28q_2q_1^2+14q_1^4$
4	1	$30q_4+96q_3q_1+34q_2^2+100q_2q_1^2+10q_1^4$
4	2	$93q_4+88q_3q_1+34q_2^2+16q_2q_1^2$
4	3	$20q_4$
5	0	$q_5+10q_4q_1+10q_3q_2+45q_3q_1^2+45q_2^2q_1+120q_2q_1^3+42q_1^5$
5	1	$80q_5+400q_4q_1+280q_3q_2+770q_3q_1^2+560q_2^2q_1+700q_2q_1^3+70q_1^5$
5	2	$901q_5+1990q_4q_1+1290q_3q_2+1405q_3q_1^2+1055q_2^2q_1+380q_2q_1^3+8q_1^5$
5	3	$1650q_5+1200q_4q_1+820q_3q_2+180q_3q_1^2+140q_2^2q_1$
5	4	$248q_5$

Fig. 2 Table of double BMS-3 numbers

To a branched cover with ramification profile $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ over $\exp(\frac{2\pi i}{m})$, we assign the weight $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}\ f|}$. The corresponding 1-point invariant is denoted $\overline{b}_g^m(d) = d\,\overline{B}_{g,1}^m(d)$.

These numbers arise from taking $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ and $G(z) = (1+z)^{m-1}$ in (4).

2.3 Hurwitz numbers

Hurwitz numbers enumerate branched covers of the Riemann sphere. They were first studied by Hurwitz [40] in the late nineteenth century, although interest in Hurwitz numbers has been revived in recent decades due to connections to enumerative geometry [24,54], integrability [55], and topological recursion [9,27].

Definition 10 The *single Hurwitz number* $H_{g,n}(d_1, d_2, \ldots, d_n)$ is the weighted count of genus g connected branched covers $f : (C; p_1, p_2, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- $f^{-1}(\infty) = d_1 p_1 + d_2 p_2 + \cdots + d_n p_n$; and
- the only other ramification is simple and occurs at the mth roots of unity.

The weight of a branched cover f is $\frac{1}{m!\,|\mathrm{Aut}\ f|}$, where we have $m = 2g - 2 + n + |\mathbf{d}|$ from the Riemann–Hurwitz formula. The corresponding 1-point invariant is denoted $h_g(d) = d\,H_{g,1}(d)$.

Again, the Riemann existence theorem allows one to encode a branched cover via its monodromy representation, which makes connection with permutation factorisations. The result is the following algebraic description of single Hurwitz numbers.

Proposition 2 *The single Hurwitz number* $H_{g,n}(d_1, d_2, \ldots, d_n)$ *is* $\frac{1}{m!\,|\mathbf{d}|!}$ *multiplied by the number of tuples* $(\tau_1, \tau_2, \ldots, \tau_m)$ *of transpositions in* $S_{|\mathbf{d}|}$ *such that*

– $m = 2g - 2 + n + |\mathbf{d}|$;
– $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ *has labelled cycles of lengths* $d_1, d_2, \ldots, d_n$*; and*
– $\tau_1, \tau_2, \ldots, \tau_m$ *generate a transitive subgroup of the symmetric group.*

This algebraic description of single Hurwitz numbers then leads naturally to the following result [55].

Lemma 4 *The single Hurwitz numbers arise from taking* $\mathbf{q} = (1, 0, \ldots)$ *and* $G(z) = \exp(z)$ *in* (4)*. In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{1}{\hbar}, 0, 0, \ldots\right) \prod_{\square \in \lambda} \exp(c(\square)\hbar)$$
$$= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} H_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n}\right].$$

As with the enumerations considered previously in this section, one can consider a generalisation of the Hurwitz enumeration to its "double" counterpart [17].

Definition 11 The *double Hurwitz number* $\overline{H}_{g,n}(d_1, d_2, \ldots, d_n)$ is the weighted count of genus g connected branched covers $f : (C; p_1, p_2, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

– $f^{-1}(\infty) = d_1 p_1 + d_2 p_2 + \cdots + d_n p_n$;
– the ramification profile over 0 is arbitrary; and
– the only other ramification is simple and occurs at the mth roots of unity.

The weight of a branched cover f with ramification profile $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ over 0 is $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{m!\,|\mathrm{Aut}\ f|}$.

Remark 3 The notion of double Hurwitz number from Definition 11 differs from, but is closely related to, the notion of double Hurwitz number that appears in various places elsewhere in the literature [33,55]. In these references, double Hurwitz numbers count branched covers of $\mathbb{CP}^1$ in which both the ramification profiles over 0 and ∞ are specified. Up to a simple renormalisation, such numbers arise as coefficients of the polynomial $\overline{H}_{g,n}(d_1, d_2, \ldots, d_n) \in \mathbb{C}[q_1, q_2, q_3, \ldots]$ defined above. The idea of packaging these numbers together as in Definition 11 has proven to be useful in the context of showing that double Hurwitz numbers are governed by topological recursion [5,17]. Furthermore, the notion of double Hurwitz number that we adopt is consistent with the use of the word "double" for other contexts in this paper, such as in Definitions 7 and 9 .

Again, we have a natural double Schur function expansion for double Hurwitz number partition function [55].

d	g	$h_g(d)$	$\overline{h}_g(d)$
1	0	1	q_1
1	1	0	0
1	2	0	0
2	0	1	$q_2 + q_1^2$
2	1	$\frac{1}{6}$	$\frac{1}{2} q_2 + \frac{1}{6} q_1^2$
2	2	$\frac{1}{120}$	$\frac{1}{24} q_2 + \frac{1}{120} q_1^2$
3	0	$\frac{3}{2}$	$q_3 + 3 q_2 q_1 + \frac{3}{2} q_1^3$
3	1	$\frac{9}{8}$	$3 q_3 + \frac{9}{2} q_2 q_1 + \frac{9}{8} q_1^3$
3	2	$\frac{27}{80}$	$\frac{9}{4} q_3 + \frac{81}{40} q_2 q_1 + \frac{27}{80} q_1^3$
4	0	$\frac{8}{3}$	$q_4 + 4 q_3 q_1 + 2 q_2^2 + 8 q_2 q_1^2 + \frac{8}{3} q_1^4$
4	1	$\frac{16}{3}$	$10 q_4 + 24 q_3 q_1 + \frac{28}{3} q_2^2 + \frac{80}{3} q_2 q_1^2 + \frac{16}{3} q_1^4$
4	2	$\frac{208}{45}$	$\frac{82}{3} q_4 + \frac{216}{5} q_3 q_1 + \frac{244}{15} q_2^2 + \frac{1456}{45} q_2 q_1^2 + \frac{208}{45} q_1^4$
5	0	$\frac{125}{24}$	$q_5 + 5 q_4 q_1 + 5 q_3 q_2 + \frac{25}{2} q_3 q_1^2 + \frac{25}{2} q_2^2 q_1 + \frac{125}{6} q_2 q_1^3 + \frac{125}{24} q_1^5$
5	1	$\frac{3125}{144}$	$25 q_5 + \frac{250}{3} q_4 q_1 + \frac{125}{2} q_3 q_2 + \frac{3125}{24} q_3 q_1^2 + \frac{625}{6} q_2^2 q_1 + \frac{3125}{24} q_2 q_1^3 + \frac{3125}{144} q_1^5$
5	2	$\frac{15625}{384}$	$\frac{2125}{12} q_5 + \frac{1250}{3} q_4 q_1 + \frac{6875}{24} q_3 q_2 + \frac{21875}{48} q_3 q_1^2 + \frac{3125}{9} q_2^2 q_1 + \frac{15625}{48} q_2 q_1^3 + \frac{15625}{384} q_1^5$

Fig. 3 Table of single Hurwitz numbers and double Hurwitz numbers

Lemma 5 *The double Hurwitz numbers arise from taking* $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ *and* $G(z) = \exp(z)$ *in* (4). *In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots\right) \prod_{\Box \in \lambda} \exp(c(\Box)\hbar)$$
$$= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} \overline{H}_{g,n}(d_1, d_2, \ldots, d_n) \frac{\hbar^{2g-2+n}}{n!} p_{d_1} p_{d_2} \cdots p_{d_n} \right].$$

2.4 Monotone Hurwitz numbers

Monotone Hurwitz numbers first appeared in a series of papers by Goulden, Guay-Paquet and Novak, in which they arose as coefficients in the large N asymptotic expansion of the Harish-Chandra–Itzykson–Zuber matrix integral over the unitary group $U(N)$ [30–32]. Their definition resembles that of Hurwitz numbers, but with a monotonicity constraint imposed on the transpositions. This monotonicity condition is rather natural from the standpoint of the Jucys–Murphy elements in the symmetric group algebra $\mathbb{C}[S_{|\mathbf{d}|}]$. Monotone Hurwitz numbers are known to obey several analogous properties to Hurwitz numbers. For instance, there is a polynomial structure theorem [31], they are governed by topological recursion [16], there is a quantum curve [16], and there is an ELSV-type formula [3,15].

Definition 12 The *monotone Hurwitz number* $M_{g,n}(d_1, d_2, \ldots, d_n)$ is $\frac{1}{|\mathbf{d}|!}$ multiplied by the number of tuples $(\tau_1, \tau_2, \ldots, \tau_m)$ of transpositions in $S_{|\mathbf{d}|}$ such that

- $m = 2g - 2 + n + |\mathbf{d}|$;
- $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has labelled cycles of lengths $d_1, d_2, \ldots, d_n$;

- $\tau_1, \tau_2, \ldots, \tau_m$ generate a transitive subgroup of the symmetric group; and
- if $\tau_i = (a_i\ b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

The corresponding 1-point invariant is denoted $m_g(d) = d\, M_{g,1}(d)$.

Lemma 6 *The monotone Hurwitz numbers arise from taking* $\mathbf{q} = (1, 0, 0, \ldots)$ *and* $G(z) = \frac{1}{1-z}$ *in* (4). *In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{1}{\hbar}, 0, 0, \ldots\right) \prod_{\Box \in \lambda} \frac{1}{1 - c(\Box)\hbar}$$
$$= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} M_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n}\right].$$

Again, one can consider a generalisation of the monotone Hurwitz enumeration to its "double" counterpart.

Definition 13 Let the *double monotone Hurwitz number* $\overline{M}_{g,n}(d_1, d_2, \ldots, d_n)$ be the weighted count of tuples $(\sigma, \tau_1, \tau_2, \ldots, \tau_m)$ of transpositions in $S_{|\mathbf{d}|}$ such that

- $m = 2g - 2 + n + k(\sigma)$, where $k(\sigma)$ denotes the number of cycles in σ;
- $\sigma \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has labelled cycles of lengths $d_1, d_2, \ldots, d_n$;
- $\sigma, \tau_1, \tau_2, \ldots, \tau_m$ generate a transitive subgroup of the symmetric group; and
- if $\tau_i = (a_i\ b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

The weight of such a tuple with σ of cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is $\frac{1}{|\mathbf{d}|!}\, q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}$. The corresponding 1-point invariant is denoted $\overline{m}_g(d) = d\, \overline{M}_{g,1}(d)$.

Lemma 7 *The double monotone Hurwitz numbers arise from taking* $\mathbf{q} = (q_1, q_2, \ldots)$ *and* $G(z) = \frac{1}{1-z}$ *in* (4). *In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots\right) \prod_{\Box \in \lambda} \frac{1}{1 - c(\Box)\hbar}$$
$$= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} \overline{M}_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n}\right].$$

3 Double Schur function expansions

3.1 Partition functions and 1-point invariants

In the previous section, we established that for various choices of the formal power series $G(z)$ and the parameters $q_1, q_2, q_3, \ldots$, certain enumerative problems of geo-

d	g	$m_g(d)$	$\overline{m}_g(d)$
1	0	1	q_1
1	1	1	0
1	2	1	0
2	0	1	$q_2+q_1^2$
2	1	1	$q_2+q_1^2$
2	2	1	$q_2+q_1^2$
3	0	2	$q_3+3q_2q_1+2q_1^3$
3	1	10	$5q_3+15q_2q_1+10q_1^3$
3	2	42	$21q_3+63q_2q_1+42q_1^3$
4	0	5	$q_4+4q_3q_1+2q_2^2+10q_2q_1^2+5q_1^4$
4	1	70	$15q_4+60q_3q_1+25q_2^2+140q_2q_1^2+70q_1^4$
4	2	735	$161q_4+644q_3q_1+252q_2^2+1470q_2q_1^2+735q_1^4$
5	0	14	$q_5+5q_4q_1+5q_3q_2+15q_3q_1^2+15q_2^2q_1+35q_2q_1^3+14q_1^5$
5	1	420	$35q_5+175q_4q_1+140q_3q_2+490q_3q_1^2+420q_2^2q_1+1050q_2q_1^3+420q_1^5$
5	2	8778	$777q_5+3885q_4q_1+2835q_3q_2+10605q_3q_1^2+8505q_2^2q_1+21945q_2q_1^3+8778q_1^5$

Fig. 4 Table of monotone Hurwitz numbers and double monotone Hurwitz numbers

metric interest are stored in the partition function via the following equation.

$$\begin{aligned} Z(\mathbf{p};\mathbf{q};\hbar) &= \sum_{\lambda\in\mathscr{P}} s_\lambda(p_1,p_2,\ldots)\, s_\lambda\left(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots\right) \prod_{\Box\in\lambda} G(c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} N_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\,p_{d_1}p_{d_2}\cdots p_{d_n}\right] \end{aligned} \tag{6}$$

We will be primarily concerned with the 1-point invariants that arise when $n=1$. In particular, we consider the numbers $n_g(d) = d\,N_{g,1}(d)$ stored in the partition function, with the goal of determining whether or not there exists a 1-point recursion governing these numbers.[1] In order to obtain information about these numbers, we deform the partition function via a parameter s that keeps track of the unweighted degree in $p_1, p_2, p_3, \ldots$ and then extract the 1-point invariants by differentiation.

$$\begin{aligned} \left[\frac{\partial}{\partial s} Z(s\mathbf{p};\mathbf{q};\hbar)\right]_{s=0} &= \sum_{\lambda\in\mathscr{P}} \left[\frac{\partial}{\partial s} s_\lambda(sp_1,sp_2,\ldots)\right]_{s=0} s_\lambda\left(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots\right)\prod_{\Box\in\lambda} G(c(\Box)\hbar) \\ &= \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,1}(d)\,\hbar^{2g-1}\,p_d \end{aligned}$$

[1] The extra factor of d in the definition of $n_g(d)$ will have little bearing on our results, but is introduced here for consistency with the original Harer–Zagier recursion and other results in the literature. We remark that the 1-point recursions are generally simpler with this normalisation, as can be witnessed from (1) and (2).

At this stage, it is natural to introduce the so-called *principal specialisation* $p_d = x^d$ to record the degree via the single variable x.

$$\begin{aligned}
&\left[\frac{\partial}{\partial s} Z(sx, sx^2, sx^3, \dots; \mathbf{q}; \hbar)\right]_{s=0} \\
&= \sum_{\lambda \in \mathscr{P}} \left[\frac{\partial}{\partial s} s_\lambda(sx, sx^2, sx^3, \dots)\right]_{s=0} s_\lambda\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \dots\right) \prod_{\Box \in \lambda} G(c(\Box)\hbar) \qquad (7) \\
&= \sum_{g=0}^{\infty} \sum_{d=1}^{\infty} N_{g,1}(d)\, \hbar^{2g-1}\, x^d \qquad (8)
\end{aligned}$$

3.2 Schur function evaluations

In this section, we deduce some facts about Schur functions that will be required at a later stage. We begin with the crucial observation that the evaluation of the Schur function appearing in (7) is zero unless λ is a hook partition. Here, and throughout the paper, a *hook partition* refers to a partition of the form $(k, 1^{d-k})$, where $1 \le k \le d$.

Lemma 8

$$\left[\frac{\partial}{\partial s} s_\lambda(sx, sx^2, sx^3, \dots)\right]_{s=0} = \begin{cases} (-1)^{d-k} \frac{x^d}{d}, & \text{if } \lambda = (k, 1^{d-k}) \text{ is a hook partition,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof The lemma follows from the hook-content formula [48], which states that

$$s_\lambda(s, s, s, \dots) = \prod_{\Box \in \Lambda} \frac{s + c(\Box)}{h(\Box)}, \qquad (9)$$

where $c(\Box)$ and $h(\Box)$ denote the content and hook-length of a box in the Young diagram for λ, respectively.

If λ is a non-empty partition that is not a hook, then its Young diagram contains at least two boxes with content 0. So the hook-content formula implies that $s_\lambda(s, s, s, \dots)$ is a polynomial divisible by s^2 and it follows that

$$\left[\frac{\partial}{\partial s} s_\lambda(sx, sx^2, sx^3, \dots)\right]_{s=0} = 0.$$

If $\lambda = (k, 1^{d-k})$ is a hook partition, then its hook-lengths are $\{1, 2, \dots, k-1\} \cup \{1, 2, \dots, d-k\} \cup \{d\}$, while its contents are $\{1, 2, \dots, k-1\} \cup \{-1, -2, \dots, -(d-k)\} \cup \{0\}$. Thus, we obtain

$$s_\lambda(s, s, s, \dots) = (-1)^{d-k} \frac{(s+k-1)(s+k-2)\cdots(s+k-d)}{d(k-1)!(d-k)!}.$$

By directly differentiating with respect to s and evaluating at $s = 0$, we obtain

$$\left[\frac{\partial}{\partial s} s_\lambda(s, s, \ldots)\right]_{s=0} = \frac{(-1)^{d-k}}{d}.$$

The powers of x appearing in the statement of the lemma can be reinstated, using the fact that Schur functions are weighted homogeneous. □

Now use Lemma 8 in (7) to obtain the following.

$$\begin{aligned}&\left[\frac{\partial}{\partial s} Z(sx, sx^2, sx^3, \ldots; \mathbf{q}; \hbar)\right]_{s=0}\\ &= \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,1}(d)\hbar^{2g-1}x^d\\ &= \sum_{d=1}^{\infty}\sum_{k=1}^{d}(-1)^{d-k}\frac{x^d}{d}\, s_{(k,1^{d-k})}\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots\right) \prod_{\Box\in\lambda} G(c(\Box)\hbar)\end{aligned}$$

Extracting the x^d coefficient yields the following result.

Lemma 9 *The 1-point invariants $n_g(d) = d\, N_{g,1}(d)$ defined by (4) satisfy*

$$\sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1} = \sum_{k=1}^{d}(-1)^{d-k}\, s_{(k,1^{d-k})}\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots\right) \prod_{i=1}^{d} G((k-i)\hbar),$$

for every positive integer d.

We will later be interested in setting the parameter $q_i = 0$ for i sufficiently large. In this case, we write $s_\lambda(\frac{q_1}{\hbar}, \frac{q_2}{\hbar}, \ldots, \frac{q_r}{\hbar})$ to mean the Schur function $s_\lambda(\frac{q_1}{\hbar}, \frac{q_2}{\hbar}, \ldots)$ evaluated at $q_{r+1} = q_{r+2} = \cdots = 0$.

We complete the section by presenting the following relations concerning Schur functions, which will be useful for the next section [48].

Lemma 10 *The Schur function indexed by the hook $(k, 1^{d-k})$ can be expressed as*

$$s_{(k,1^{d-k})}(\mathbf{p}) = \sum_{j=1}^{k}(-1)^{j+1}\, h_{k-j}(\mathbf{p})\, e_{d-k+j}(\mathbf{p}).$$

Here, h_n and e_n, respectively, denote the homogeneous and elementary symmetric functions, which can in turn be expressed in terms of power sum symmetric functions via

$$\sum_{n=0}^{\infty} h_n(\mathbf{p})\, x^n = \exp\left[\sum_{k=1}^{\infty}\frac{p_k}{k}x^k\right] \quad \textit{and} \quad \sum_{n=0}^{\infty} e_n(\mathbf{p})\, x^n = \exp\left[\sum_{k=1}^{r}(-1)^{k-1}\frac{p_k}{k}x^k\right].$$

In the case $p_1 = s$ and $p_k = 0$ for $k \geq 2$, the above expression evaluates to

$$s_{(k,1^{d-k})}(s, 0, 0, \dots) = \binom{d-1}{k-1} \frac{s^d}{d!}.$$

4 Recursions for 1-point functions

4.1 Holonomic sequences and functions

A sequence $a_0, a_1, a_2, \dots$ is said to be *holonomic over* $\mathbb{K}$ if the terms satisfy a nonzero linear difference equation of the form

$$p_r(d)\, a_{d+r} + p_{r-1}(d)\, a_{d+r-1} + \cdots + p_1(d)\, a_{d+1} + p_0(d)\, a_d = 0, \qquad (10)$$

where $p_0, p_1, \dots, p_r$ are polynomials over the field $\mathbb{K}$ of characteristic 0. Moreover, a formal power series $A(x) = \sum_{d=0}^{\infty} a_d\, x^d$ is said to be *holonomic over* $\mathbb{K}$ if it satisfies a nonzero linear differential equation of the form

$$\left[P_r(x) \frac{\partial^r}{\partial x^r} + P_{r-1}(x) \frac{\partial^{r-1}}{\partial x^{r-1}} + \cdots + P_1(x) \frac{\partial}{\partial x} + P_0(x) \right] A(x) = 0, \qquad (11)$$

where $P_0, P_1, \dots, P_r$ are polynomials over $\mathbb{K}$. The dual use of the term "holonomic" is due to the elementary fact that the sequence $a_0, a_1, a_2, \dots$ is holonomic over $\mathbb{K}$ if and only if the formal power series $a_0 + a_1 x + a_2 x^2 + \cdots$ is holonomic over $\mathbb{K}$. For our applications, we will use the ground field $\mathbb{K} = \mathbb{C}(\hbar)$.

Lemma 11 *A 1-point recursion exists for the numbers $n_g(d)$ in the sense of Definition 1 if and only if the formal power series*

$$F(x, \hbar) = \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^d$$

is holonomic over $\mathbb{C}(\hbar)$.

Proof If $F(x, \hbar)$ is holonomic, then there exist polynomials $P_0, P_1, \dots, P_r$ with coefficients in $\mathbb{C}(\hbar)$ such that

$$\left[P_r(x) \frac{\partial^r}{\partial x^r} + P_{r-1}(x) \frac{\partial^{r-1}}{\partial x^{r-1}} + \cdots + P_1(x) \frac{\partial}{\partial x} + P_0(x) \right] F(x, \hbar) = 0.$$

One can assume that the coefficients of $P_0, P_1, \ldots, P_r$ actually lie in $\mathbb{C}[\hbar]$, by clearing denominators in the equation above. Thus, the equation takes the form

$$\left[\sum_{i,j,k=0}^{\text{finite}} C_{ijk}\,\hbar^i x^j \frac{\partial^k}{\partial x^k}\right] F(x,\hbar) = 0, \tag{12}$$

for some complex constants C_{ijk}. Applying $C_{ijk}\,\hbar^i x^j \frac{\partial^k}{\partial x^k}$ to a term $n_g(d)\,\hbar^{2g-1}\,x^d$ in the expansion for $F(x,\hbar)$ has the effect of shifting the powers of $\hbar$ and x, and introducing a factor that is polynomial in d. So after collecting terms in the resulting equation, one obtains a relation of the form of (3). Therefore, there exists a 1-point recursion for the numbers $n_g(d)$.

Conversely, suppose that there exists a 1-point recursion for the numbers $n_g(d)$, so there exists a relation of the form of (3). Multiplying both sides by $\hbar^{2g-1}\,x^d$ and summing over g and d yields

$$\sum_{d=1}^{\infty}\sum_{g=0}^{\infty}\sum_{i=0}^{i_{\max}}\sum_{j=0}^{j_{\max}} p_{ij}(d)\,n_{g-i}(d-j)\,\hbar^{2g-1}\,x^d = 0.$$

Now replace $p_{ij}(d)\,x^d$ with $p_{ij}\big(x\frac{\partial}{\partial x}\big)\,x^d$ and reindex the summations over d and g to obtain

$$\sum_{d=1}^{\infty}\sum_{g=0}^{\infty}\sum_{i=0}^{i_{\max}}\sum_{j=0}^{j_{\max}} p_{ij}\big(x\tfrac{\partial}{\partial x}\big)\,\hbar^{2i}\,x^j\,n_g(d)\hbar^{2g-1}\,x^d = 0$$

$$\Rightarrow \quad \left[\sum_{i=0}^{i_{\max}}\sum_{j=0}^{j_{\max}} p_{ij}\big(x\tfrac{\partial}{\partial x}\big)\,\hbar^{2i}\,x^j\right] F(x,\hbar) = 0.$$

This final equation can be expressed in the form of (12) by applying the commutation relation $[\frac{\partial}{\partial x}, x] = 1$. It then follows that $F(x,\hbar)$ is holonomic over $\mathbb{C}(\hbar)$. □

The following result lists some closure properties, which provide standard tools to prove holonomicity [41].

Proposition 3 *Let* $A(x) = \sum_{d=0}^{\infty} a_d\,x^d$ *and* $B(x) = \sum_{d=0}^{\infty} b_d\,x^d$ *be holonomic over a field* $\mathbb{K}$ *of characteristic zero. Then,*

(a) $\alpha A(x) + \beta B(x)$ *is holonomic for all* $\alpha, \beta \in \mathbb{K}$*;*
(b) the Cauchy product $A(x)\,B(x)$ *and the Hadamard product* $\big(a_n b_n\big)_{n=0,1,2,\ldots}$ *are holonomic;*
(c) the derivative $a'(x)$ *and the forward shift* $\big(a_{n+1}\big)_{n=0,1,2,\ldots}$ *are holonomic; and*
(d) the integral $\int^x A(x)\,\mathrm{d}x$ *and the indefinite sum* $\big(\sum_{k=0}^{n} a_k\big)_{n=0,1,2,\ldots}$ *are holonomic.*

Definition 14 We define the *order* and *degree* of the difference equation

$$p_r(d)\,a_{d+r} + p_{r-1}(d)\,a_{d+r-1} + \cdots + p_1(d)\,a_{d+1} + p_0(d)\,a_d = 0$$

to be r (assuming $p_r(d) \neq 0$) and $\max\{\deg p_0, \deg p_1, \ldots, \deg p_r\}$, respectively. Similarly, we define the *order* and *degree* of the differential equation

$$\left[P_r(x)\,\frac{\partial^r}{\partial x^r} + P_{r-1}(x)\,\frac{\partial^{r-1}}{\partial x^{r-1}} + \cdots + P_1(x)\,\frac{\partial}{\partial x} + P_0(x)\right] A(x) = 0$$

to be r (assuming $P_r(x) \neq 0$) and $\max\{\deg P_0, \deg P_1, \ldots, \deg P_r\}$, respectively.

Note that for a fixed holonomic sequence or function, there are difference or differential operators of many possible orders and degrees that annihilate it. Furthermore, it is not generally true that there exists such an operator that simultaneously minimises both the order and the degree. Thus, one does not usually refer to the order and degree of a holonomic sequence or function itself, but to the order and degree of a particular operator.

4.2 Multivariate holonomic functions

There are competing ways in which the notion of holonomicity may be generalised to the case of many variables, but the following is well-suited to our purposes. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and let $\mathbb{K}[[\mathbf{x}]] = \mathbb{K}[[x_1, x_2, \ldots, x_n]]$. A multivariate formal power series $A(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]$ is said to be *holonomic over* $\mathbb{K}$—also commonly known as *D-finite*—if the set of derivatives

$$\left\{ \frac{\partial^{i_1+i_2+\cdots+i_n}}{\partial x_1^{i_1}\,\partial x_2^{i_2}\,\cdots\,\partial x_n^{i_n}} A(\mathbf{x}) \;\middle|\; i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0} \right\}$$

lie in a finite-dimensional vector space over $\mathbb{K}(\mathbf{x})$. This is equivalent to the fact that $A(\mathbf{x})$ satisfies a system of linear partial differential equations of the form

$$\left[P_{i,r}(\mathbf{x})\,\frac{\partial^r}{\partial x_i^r} + P_{i,r-1}(\mathbf{x})\,\frac{\partial^{r-1}}{x_i^{r-1}} + \cdots + P_{i,0}(\mathbf{x})\right] A(\mathbf{x}) = 0, \qquad \text{for } i = 1, 2, \ldots, n, \tag{13}$$

where $P_{ij}(x) \in \mathbb{K}[x]$. Clearly, the case $n = 1$ recovers the definition of a holonomic function described earlier.

Definition 15 For $A(\mathbf{x}) = \sum_{i_1, i_2, \ldots, i_n} a(i_1, i_2, \ldots, i_n)\, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \mathbb{K}[[\mathbf{x}]]$ and integers $1 \leq k < \ell \leq n$, define the *primitive diagonal*

$$I_{k\ell}(A(\mathbf{x})) = \sum_{i_1, \ldots, \widehat{i_\ell}, \ldots, i_n} a(i_1, \ldots, i_k, \ldots, i_k, \ldots, i_n)\, x_1^{i_1} \cdots x_k^{i_k} \cdots \widehat{x_\ell^{i_\ell}} \cdots x_n^{i_n},$$

where the hats denote omission of the index i_ℓ and the term $x_\ell^{i_\ell}$.

For example, taking $k = 1$, $\ell = 2$ and $n = 4$ leads to

$$I_{12}(A(x_1, x_2, x_3, x_4)) = \sum_{i_1, i_3, i_4} a(i_1, i_1, i_3, i_4)\, x_1^{i_1} x_3^{i_3} x_4^{i_4}.$$

The following result lists some closure properties for multivariate holonomic functions [47,60].

Proposition 4 *Let*

$$A(\mathbf{x}) = \sum a(i_1, i_2, \ldots, i_n)\, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \textit{and} \quad B(\mathbf{x}) = \sum b(i_1, i_2, \ldots, i_n)\, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

be holonomic functions over a field $\mathbb{K}$ of characteristic zero. Then,

(a) the primitive diagonal $I_{k\ell}(A(\mathbf{x}))$ is holonomic for all $1 \le k < \ell \le n$;
(b) the Cauchy product $A(\mathbf{x})\, B(\mathbf{x})$ is holonomic;
(c) the Hadamard product $A(\mathbf{x}) * B(\mathbf{x}) = \sum a(i_1, i_2, \ldots, i_n)\, b(i_1, i_2, \ldots, i_n)\, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ is holonomic; and
(d) the formal power series

$$\sum_{(i_1, i_2, \ldots, i_n) \in C} a(i_1, i_2, \ldots, i_n)\, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

is holonomic if $C \subseteq \mathbb{Z}^n_{\ge 0}$ is defined by a finite set of inequalities of the form $\sum a_k i_k + b \ge 0$, where $a_1, a_2, \ldots, a_n, b \in \mathbb{Z}$.

4.3 Existence of 1-point recursions

We begin by proving the existence of 1-point recursions in the *single* case when $\mathbf{q} = (1, 0, 0, \ldots)$. (The word "single" has been ported from the context of Hurwitz numbers to this more general setting.)

Theorem 2 *Let $G(z) \in \mathbb{C}(z)$ be a rational function and let $\mathbf{q} = (1, 0, 0, \ldots)$. Define the numbers $n_g(d) = d\, N_{g,1}(d)$ via (4). Then, the numbers $n_g(d)$ satisfy a 1-point recursion in the sense of Definition 1.*

Proof We define $n(d)$ and calculate it as follows.

$$\begin{aligned} n(d) &= \sum_{g=0}^{\infty} n_g(d)\,\hbar^{2g-1} \\ &= \sum_{k=1}^{d} (-1)^{d-k}\, s_{(k,1^{d-k})}\left(\tfrac{1}{\hbar}, 0, 0, \ldots\right) \prod_{i=1}^{d} G((k-i)\hbar) && \text{(Lemma 9)} \\ &= \frac{1}{d!\,\hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \prod_{i=1}^{d} G((k-i)\hbar) && \text{(Lemma 10)} \\ &= \frac{1}{d\,\hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \frac{1}{(k-1)!\,(d-k)!} \prod_{i=1}^{d} G((k-i)\hbar) && (14) \end{aligned}$$

Define the sequences

$$u_k = \frac{1}{(k-1)!\,\hbar^k} \prod_{i=1}^{k} G((i-1)\hbar) \qquad \text{and} \qquad v_k = \frac{(-1)^k}{k!\,\hbar^k} \prod_{i=1}^{k} G(-i\hbar).$$

These are holonomic over $\mathbb{C}(\hbar)$ since the ratios $\frac{u_{k+1}}{u_k} = \frac{G(k\hbar)}{k\hbar}$ and $\frac{v_{k+1}}{v_k} = -\frac{G(-(k+1)\hbar)}{(k+1)\hbar}$ are rational functions of k with coefficients from $\mathbb{C}(\hbar)$. Hence, parts (b) and (c) of Proposition 3 imply that the sequence

$$n(d) = \frac{1}{d} \sum_{k=1}^{d} u_k\, v_{d-k}$$

is holonomic over $\mathbb{C}(\hbar)$. So Lemma 11 guarantees the existence of a 1-point recursion for $n_g(d)$. □

To tackle the case of general weights $\mathbf{q} = (q_1, q_2, \ldots, q_r, 0, 0, \ldots)$, we use the following lemma.

Lemma 12 *If a_d, b_d, u_d, v_d are holonomic sequences, then so is*

$$s_d = \sum_{k=1}^{d} a_k b_{d-k} \sum_{\ell=0}^{k-1} u_\ell v_{d-\ell}.$$

Proof Define the generating functions

$$A(x_1) = \sum_{n=1}^{\infty} a_n x_1^n, \quad B(x_2) = \sum_{n=0}^{\infty} b_n x_2^n, \quad U(x_3, x_4) = \sum_{n=0}^{\infty} u_n (x_3 x_4)^n,$$
$$V(x_4) = \sum_{n=1}^{\infty} v_n x_4^n.$$

Observe that each of these is a holonomic function in the appropriate variables. Since Cauchy products preserve holonomicity—see part (b) of 4—we know that

$$H(x_3, x_4) = \frac{x_3}{1 - x_3}\, U(x_3, x_4)\, V(x_4) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\sum_{\ell=0}^{k-1} u_\ell v_{n-\ell} \right) x_3^k x_4^n$$

is holonomic. (We interpret the inner summation by discarding any terms that involve $v_{n-\ell}$ with $n - \ell \leq 0$.) By part (d) of 4, restricting to the terms satisfying $n - k \geq 0$, we obtain the holonomic function

$$\widehat{H}(x_3, x_4) = \sum_{n \geq k \geq 1} \left(\sum_{\ell=0}^{k-1} u_\ell v_{n-\ell} \right) x_3^k x_4^n.$$

Then,

$$\begin{aligned} L(x_1, x_2, x_3, x_4) &= A(x_1)\, B(x_2)\, \widehat{H}(x_3, x_4) \\ &= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{n \geq k \geq 1} a_i b_j \left(\sum_{\ell=0}^{k-1} u_\ell v_{n-\ell} \right) x_1^i x_2^j x_3^k x_4^n \end{aligned}$$

is holonomic by closure under Cauchy products. Invoking part (a) of 4, we know that

$$I_{13}(L(x_1, x_2, x_3, x_4)) = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \sum_{n=k}^{\infty} a_k b_j \left(\sum_{\ell=0}^{k-1} u_\ell v_{n-\ell} \right) x_1^k x_2^j x_4^n$$

is holonomic. Now use part (d) of 4 with the inequalities $j + k - n \geq 0$ and $-j - k + n \geq 0$—in other words, restricting to $j = n - k$—to deduce holonomicity of

$$\widehat{L}(x_1, x_2, x_4) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} a_k b_{n-k} \left(\sum_{\ell=0}^{k-1} u_\ell v_{n-\ell} \right) x_1^k x_2^{n-k} x_4^n.$$

By evaluating this formal power series at $x_1 = 1$, $x_2 = 1$ and $x_4 = x$—which clearly preserves holonomicity—we obtain the desired result. □

We are now in a position to prove Theorem 1, which we restate in the following way.

Theorem 3 *Let $G(z) \in \mathbb{C}(z)$ be a rational function satisfying $G(0) = 1$ and let $\mathbf{q} = (q_1, q_2, \ldots, q_r, 0, 0, \ldots)$. Define the numbers $n_g(d) = d\, N_{g,1}(d)$ via (4). Then, the generating function*

$$\sum_{d=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^d \tag{15}$$

is holonomic over $\mathbb{C}(\hbar)$, so the numbers $n_g(d)$ satisfy a 1-point recursion in the sense of Definition 1.

Proof We calculate the coefficient $n(d)$ of x^d in (15) as follows.

$$\begin{aligned}
n(d) &= \sum_{g=0}^{\infty} n_g(d)\,\hbar^{2g-1} \\
&= \sum_{k=1}^{d} (-1)^{d-k}\, s_{(k,1^{d-k})}\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right) \prod_{i=1}^{d} G((k-i)\hbar) \\
&= \sum_{k=1}^{d} (-1)^{d-k} \prod_{i=1}^{d} G((k-i)\hbar) \sum_{j=1}^{k} (-1)^{j+1}\, h_{k-j}\left(\tfrac{q_1}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right) e_{d-k+j}\left(\tfrac{q_1}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right) \\
&= \sum_{k=1}^{d} \prod_{i=1}^{k} G((i-1)\hbar) \prod_{i=1}^{d-k} G(-i\hbar) \sum_{\ell=0}^{k-1} h_\ell\left(\tfrac{q_1}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right) (-1)^{d-\ell+1}\, e_{d-\ell}\left(\tfrac{q_1}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right)
\end{aligned}$$

Now define the sequences

$$\begin{aligned}
a_n &= \prod_{i=1}^{n} G((i-1)\hbar), & u_n &= h_n\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right), \\
b_n &= \prod_{i=1}^{n} G(-i\hbar), & v_n &= (-1)^{n+1}\, e_n\left(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_r}{\hbar}\right).
\end{aligned}$$

The first two are holonomic over $\mathbb{C}(\hbar)$ since the ratios $\frac{a_{n+1}}{a_n} = G(n\hbar)$ and $\frac{b_{n+1}}{b_n} = G(-(n+1)\hbar)$ are rational functions of n with coefficients from $\mathbb{C}(\hbar)$. The last two are holonomic over $\mathbb{C}(\hbar)$ due to Lemma 10, from which we deduce that

$$\left[\hbar \frac{\partial}{\partial x} - \sum_{k=1}^{r} q_k x^{k-1}\right] \left(\sum_{n=0}^{\infty} u_n x^n\right) = 0$$

and

$$\left[\hbar \frac{\partial}{\partial x} + \sum_{k=1}^{r} (-1)^k q_k x^{k-1}\right] \left(\sum_{n=0}^{\infty} v_n x^n\right) = 0.$$

Hence, Lemma 12 implies that the sequence

$$n(d) = \sum_{k=1}^{d} a_k b_{d-k} \sum_{\ell=0}^{k-1} u_\ell v_{d-\ell}$$

is holonomic over $\mathbb{C}(\hbar)$. It then follows from Lemma 11 that there exists a 1-point recursion for the numbers $n_g(d)$. □

4.4 Algorithms for 1-point recursions

One of the features of the theory of holonomic sequences and functions is the fact that theoretical results can often be turned into effective algorithms. Although Theorem 3 only asserts the existence of 1-point recursions, its proof can be converted into an algorithm to calculate them from the initial data of the rational function $G(z)$ and the positive integer r that records the number of nonzero weights $\mathbf{q} = (q_1, q_2, \ldots, q_r)$. For example, a naive though feasible approach would be to express the putative 1-point recursion as

$$\sum_{i=0}^{D} \sum_{j=0}^{R} a_{ij}\, d^i\, n(d-j) = 0,$$

and treat this as a linear system in the $(D+1)(R+1)$ variables $a_{ij} \in \mathbb{C}(\hbar)$. One obtains a linear constraint for each positive integer d, so a finite number of these allows for the computation of the 1-point recursion.

To implement this approach, one requires explicit and simultaneous bounds on the degree D and the order R of such a recursion. We remark that it is possible to obtain such bounds in terms of the degree of $G(z)$ and the positive integer r. Begin with the operators that annihilate the generating functions for the sequences a_n, b_n, u_n, v_n that appear in the proof of Theorem 3. Then use known bounds for the degree and order of operators that annihilate functions obtained by the holonomicity closure properties used in the proof—namely Cauchy product, taking diagonals, restricting summations, and evaluation. We do not pursue these calculations in the current work.

There are more efficient algorithms for computing with holonomic functions that are implemented in the `gfun` package for Maple [58]. The following example concerning monotone Hurwitz numbers demonstrates how our results lead to effective algorithms to produce previously unknown 1-point recursions.

Example 3 The proof of Theorem 2 implies that monotone Hurwitz numbers satisfy the relation

$$m(d) = \sum_{g=0}^{\infty} m_g(d)\, \hbar^{2g-1} = \frac{1}{d} \sum_{k=1}^{d} u_k\, v_{d-k},$$

where $\frac{u_{k+1}}{u_k} = \frac{G(k\hbar)}{k\hbar}$ and $\frac{v_{k+1}}{v_k} = -\frac{G(-(k+1)\hbar)}{(k+1)\hbar}$. So the sequence $m(d)$ can be obtained by taking the Cauchy product of u_k and v_k, and then taking the Hadamard product of the result and the sequence $\frac{1}{k}$. In Appendix 1, we demonstrate several lines of Maple code that explicitly implements the above procedure to produce the following previously unknown 1-point recursion for monotone Hurwitz numbers.

$$d\, m_g(d) = 2(2d-3)\, m_g(d-1) + d(d-1)^2\, m_{g-1}(d)$$

5 Examples and applications

In this section, we return our attention to the enumerative problems introduced in Sect. 2. In particular, we apply the methodology developed in Sect. 4 to deduce 1-point recursions for the enumeration of hypermaps, Bousquet-Mélou–Schaeffer numbers and monotone Hurwitz numbers. For the case of single Hurwitz numbers, the weight generating function $G(z)$ is not a rational function, so Theorem 1 ceases to apply. As a partial converse to this theorem, we show that single Hurwitz numbers do not satisfy a 1-point recursion. We furthermore demonstrate how our calculations may yield explicit formulas and polynomial structure results for 1-point invariants.

5.1 Hypermaps and Bousquet-Mélou–Schaeffer numbers

The methodology of Sect. 4 allows one to recover the 1-point recursions for the enumeration of ribbon graphs and dessins d'enfant, stated as (1) and (2), respectively. Recall that these two examples inspired the current work. It is possible to use the methodology developed in Sect. 4 to deduce other 1-point recursions, although the results are often rather lengthy to state. The following result provides two examples.

Proposition 5 *The 3-hypermap enumeration satisfies the following 1-point recursion.*

$$\begin{aligned}
2d(2d+1)\,a_g^3(d) &= 3(3d-1)(3d-2)\,a_g^3(d-1)\\
&\quad + (3d-1)(3d-2)(9d^2-8d+2)\,a_{g-1}^3(d-1)\\
&\quad - (d-1)(3d-1)(3d-2)(3d-4)(3d-5)(6d-7)\,a_{g-2}^3(d-2)\\
&\quad + (d-1)(d-2)(3d-1)(3d-2)(3d-4)(3d-5)\\
&\quad (3d-7)(3d-8)\,a_{g-3}^3(d-3)
\end{aligned}$$

The 3-BMS numbers satisfy the following 1-point recursion.

$$\begin{aligned}
2d(2d+1)(3d-4)\,b_g^3(d) &= 3(3d-1)(3d-2)(3d-4)\,b_g^3(d-1)\\
&\quad + (d-1)(3d+1)(9d^3-22d^2+14d-2)\,b_{g-1}^3(d-1)\\
&\quad - (d-1)^2(d-2)(18d^4-93d^3+172^2-127d+26)\\
&\quad b_{g-2}^3(d-2)\\
&\quad + (d-1)^2(d-2)^5(d-3)(3d-1)\,b_{g-3}^3(d-3)
\end{aligned}$$

5.2 Hurwitz numbers

Observe that Theorem 1 does not apply in the case of Hurwitz numbers, since the weight generating function $G(z)=\exp(z)$ is not rational. Thus, the following result provides a partial converse to our main theorem.

Proposition 6 *The single Hurwitz numbers do not satisfy a 1-point recursion.*

Proof By Lemma 11, we know that the single Hurwitz numbers satisfy a 1-point recursion if and only if the sequence

$$h(d) = \frac{1}{d!\,\hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \exp(d(2k-d-1)\hbar/2)$$
$$= \frac{1}{d!\,\hbar^d} \exp(-d(d+1)\hbar/2)\,(\exp(d\hbar)-1)^{d-1}$$

is holonomic over $\mathbb{C}(\hbar)$. However, if this were the case, then we could evaluate at $\hbar = 1$ to deduce that the sequence

$$\frac{1}{d!} \exp(-d(d+1)/2)\,(\exp(d)-1)^{d-1}$$

is holonomic over $\mathbb{C}$. It is known that holonomic sequences $a_1, a_2, a_3, \ldots$ over $\mathbb{C}$ must satisfy the asymptotic growth condition $a_d = O(d!^{\alpha})$ for some constant α. On the other hand, we have

$$\frac{1}{d!} \exp(-d(d+1)/2)\,(\exp(d)-1)^{d-1} \sim \frac{1}{d!} \exp(d(d-3)/2).$$

Applying Stirling's formula, we see that this grows too fast to be holonomic. So it follows that the single Hurwitz numbers do not satisfy a 1-point recursion. □

Equation (14) still applies to this case though, so the 1-part Hurwitz numbers satisfy

$$\sum_{g=0}^{\infty} h_g(d)\,\hbar^{2g-1} = \frac{1}{d!\,\hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \exp\left(d(2k-d-1)\hbar/2\right).$$

By extracting coefficients of $\hbar$ on both sides, we recover the following formula.

Proposition 7 *The 1-part single Hurwitz numbers are given by*

$$h_g(d) = \frac{(d/2)^{d+2g-1}}{d!\,(d+2g-1)!} \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} (d-1-2k)^{d+2g-1}.$$

In particular, it follows that $h_g(d) = \frac{d^d}{d!} p_g(d)$*, where* p_g *is a polynomial of degree* $3g-1$*. One can make sense of this statement in the case* $g = 0$ *by taking* $p_0(d) = \frac{1}{d}$.

We remark that the polynomial structure derived here is a direct corollary of the more general polynomial structure for single Hurwitz numbers with any number of parts. This in turn follows from the ELSV formula, which relates single Hurwitz numbers to intersection theory on moduli spaces of curves [24]. The formula of Proposition 7 is not new either, but first appeared in the work of Shapiro, Shapiro and Vainshtein [59]. The result and proof here may generalise to other settings, as we will observe in the context of monotone Hurwitz numbers.

5.3 Monotone Hurwitz numbers

In 4.4, we observed that the following 1-point recursion for monotone Hurwitz numbers could be deduced from several lines of MAPLE code. As with the Harer–Zagier recursion, it would be of interest to have an independent and purely combinatorial proof of this statement.

Proposition 8 *The 1-part monotone Hurwitz numbers satisfy the 1-point recursion*

$$d\, m_g(d) = 2(2d-3)\, m_g(d-1) + d(d-1)^2\, m_{g-1}(d).$$

In the context of monotone Hurwitz numbers, (14) implies that

$$\begin{aligned}\sum_{g=0}^{\infty} m_g(d)\, \hbar^{2g-1} &= \frac{1}{d!\, h^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \prod_{j=1}^{d} \frac{1}{1-(k-j)\hbar} \\ &= \frac{(2d-2)!}{d!\,(d-1)!} \prod_{k=-d+1}^{d-1} \frac{1}{1-k\hbar}.\end{aligned}$$

The identity that leads to the second equality can be established by considering the residue at $\hbar = \frac{1}{k}$ for $-d+1 \leq k \leq d-1$. By extracting coefficients of $\hbar$ on both sides, we recover the following formula.

Corollary 1 *The 1-part monotone Hurwitz numbers satisfy the equation*

$$\begin{aligned}m_g(d) &= \frac{(2d-2)!}{d!(d-1)!} \sum_{k_1+\cdots+k_{d-1}=g} \prod_{i=1}^{d-1} i^{2k_i} \\ &= \frac{(2d-2)!}{d!(d-1)!} \sum_{1 \leq m_1 \leq m_2 \leq \cdots \leq m_g \leq d-1} (m_1 m_2 \cdots m_g)^2.\end{aligned}$$

From the latter summation, it follows that $m_g(d) = \binom{2d}{d} p_g(d)$*, where* p_g *is a polynomial of degree* $3g-1$*. One can make sense of this statement in the case* $g=0$ *by taking* $p_0(d) = \frac{1}{d}$*.*

This polynomial structure is a particular case of the more general result derived by Goulden, Guay-Paquet and Novak [31], who prove that monotone Hurwitz numbers satisfy

$$M_{g,n}(d_1, d_2, \ldots, d_n) = \prod_{i=1}^{n} \binom{2d_i}{d_i} \times P_{g,n}(d_1, d_2, \ldots, d_n),$$

where $P_{g,n}$ is a polynomial of degree $3g-3+n$. One wonders whether the techniques of this paper can be used to prove this more general structure theorem.

6 Relations to topological recursion and quantum curves

6.1 Topological recursion

In this section, we aim to address the question: how universal is the notion of a 1-point recursion? Thus, one seeks a natural class of "enumerative" problems for which 1-point recursions exist. Such a class should include not only the ribbon graph and dessin d'enfant enumerations, but also those families of problems encompassed by Theorem 3—namely, those arising from the double Schur expansion of (4) with $\mathbf{q} = (q_1, q_2, \ldots, q_r, 0, 0, \ldots)$ and a rational weight generating function $G(z)$. We claim that a natural candidate is the class of problems governed by the topological recursion that we subsequently discuss.

The topological recursion of Chekhov, Eynard and Orantin was originally inspired by the loop equations in the theory of matrix models [12,25]. It has since found widespread applications to various problems across mathematics and physics. For example, it is known to govern the enumeration of maps on surfaces [4,19,21,23,42,51], various flavours of Hurwitz problems [7,9,16,18,27], the Gromov–Witten theory of $\mathbb{P}^1$ [22,53] and toric Calabi–Yau threefolds [8,26,28]. There are also conjectural relations to knots invariants [6,35]. Much of the power of the topological recursion lies in its universality—in other words, its wide applicability across broad classes of problems—and its ability to reveal commonality among such problems.

The topological recursion can naively be thought of as a vast generalisation of Tutte's recursion for the enumeration of ribbon graphs. It calculates n-point functions in a recursive manner, starting from the input data of a spectral curve. For our purposes, we restrict to the class of *rational spectral curves*, that are given by a pair $(x(z), y(z))$ of rational functions satisfying some mild assumptions. For more information on the theory of the topological recursion, one should consult the relevant literature [25].

The following result asserts that the weighted Hurwitz numbers—essentially, the $N_{g,n}(d_1, d_2, \ldots, d_n)$ of (4)—are governed by the topological recursion.

Theorem 4 *(Alexandrov, Chapuy, Eynard and Harnad [2]) The rational spectral curve given by*

$$x(z) = \frac{z}{G(Q(z))} \quad \text{and} \quad y(z) = \frac{Q(z)}{z} G(Q(z)), \quad \text{where } Q(z) = q_1 z + q_2 z^2 + \cdots + q_r z^r,$$

produces correlation differentials that satisfy

$$\omega_{g,n} = \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} N_{g,n}(d_1, d_2, \ldots, d_n) \prod_{i=1}^{n} d_i x_i^{d_i - 1} \, \mathrm{d}x_i.$$

This lends credence to the following conjecture, which states that 1-point recursions exist for rational spectral curves in general.

Conjecture 2 Consider a rational spectral curve given by the pair of rational functions $(x(z), y(z))$. Suppose that the correlation differentials produced by the topological recursion applied to this spectral curve have an expansion of the form

$$\omega_{g,n} = \sum_{d_1,d_2,\ldots,d_n=1}^{\infty} N_{g,n}(d_1, d_2, \ldots, d_n) \prod_{i=1}^{n} d_i x_i^{d_i-1} \, \mathrm{d}x_i.$$

Then the numbers $n_g(d) = d\, N_{g,1}(d)$ satisfy a 1-point recursion.

We conclude this section with an example of a problem that is governed by topological recursion and satisfies a 1-point recursion, but does not satisfy the conditions of Theorem 3. Thus, one can consider this as further evidence towards the conjecture above.

Example 4 Chekhov and Norbury [13] consider topological recursion applied to the spectral curve $x^2y^2 - 4y^2 - 1 = 0$ given by the rational parametrisation

$$x(z) = z + \frac{1}{z} \qquad \text{and} \qquad y(z) = \frac{z}{z^2 - 1}.$$

The resulting correlation differentials can be expressed as

$$\omega_{g,n} = \sum_{d_1,d_2,\ldots,d_n=1}^{\infty} J_{g,n}(d_1, d_2, \ldots, d_n) \prod_{i=1}^{n} d_i z_i^{d_i-1} \, \mathrm{d}z_i.$$

These are derivatives of the correlation functions for the Legendre ensemble, which arise from a particular Hermitian matrix model, as well as related models from conformal field theory. In the latter context, Gaberdiel, Klemm and Runkel use null vectors for Virasoro highest weight representations to deduce an equation [29, equation (4.18)] that is equivalent to a 1-point recursion for the numbers $j_g(d) = d\, J_{g,1}(d)$. In summary, the 1-point invariants produced by the topological recursion on the rational spectral curve above satisfy a 1-point recursion.[2]

Kontsevich and Soibelman have recently provided an alternative and more general formulation of the topological recursion [43]. It allows one to calculate n-point functions using a technique that is ostensibly more algebraic and less analytic. So it may provide a promising approach to 2.

6.2 Quantum curves

The notion of quantum curves is closely related to that of topological recursion [52]. In short, they are non-commutative deformations of spectral curves that are used as the input to the topological recursion. Although it is not currently clear when they exist, the quantum curve phenomenon has been proven or observed in many instances of the topological recursion.

[2] Observe that we are here expanding in z, while 2 has been expressed in terms of x. However, since they are related by a rational change of coordinates, this does not affect the existence of a 1-point recursion.

A quantum curve can be viewed as a differential operator $\widehat{P}(\widehat{x}, \widehat{y})$ that annihilates the so-called principal specialisation of the partition function.

$$\widehat{P}(\widehat{x}, \widehat{y})\ Z(\mathbf{p}; \hbar)|_{p_i = x^i} = 0$$

We use here the operators $\widehat{x} = x$ and $\widehat{y} = -\hbar \frac{\partial}{\partial x}$. The quantum curve phenomenon is the fact that there is a natural choice of the operator $\widehat{P}(\widehat{x}, \widehat{y})$ whose semi-classical limit—obtained by setting $\hbar = 0$ and allowing x and y to commute—recovers the spectral curve $P(x, y) = 0$.

In the context of the double Schur expansions considered in this paper, the principle specialisation of the wave function is given by

$$\Psi(x; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathscr{P}} s_\lambda(x, x^2, x^3, \ldots)\, s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) \prod_{\Box \in \lambda} G(c(\Box)\hbar).$$

As in Sect. 3, the hook-content formula stated in (9) may be invoked to simplify the expression to obtain

$$\begin{aligned}\Psi(x; \mathbf{q}; \hbar) &= \sum_{d=0}^{\infty} x^d\, s_{(d)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) \prod_{k=1}^{d-1} G(k\hbar) \\ &= \sum_{d=0}^{\infty} x^d \prod_{k=1}^{d-1} G(k\hbar) [y^d] \exp\left(\sum_{k=1}^{r} \frac{q_k}{k\hbar} y^k \right).\end{aligned}$$

Here, $[y^d]$ denotes extraction of the coefficient of y^d.

We simply remark here that our calculation of the 1-point invariants from the partition function in Sect. 3 bears a strong resemblance to the calculation of the quantum curve from the partition function [2,3,50]. In the former case, the partition function reduces to a sum over hook partitions, while in the latter case, it reduces to a sum over 1-part partitions. One may wonder whether there may be a deeper connection here.

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Example MAPLE code for 1-point recursions

In Example 3, we asserted that a 1-point recursion for monotone Hurwitz numbers could be derived from several lines of code, using the `gfun` package for MAPLE [58]. We reproduce such code below, which may be adapted for other enumerative problems.

```
> with(gfun):
> G(z) := 1/(1-z):
> rec1:={d*hbar*m(d+1)-G(d*hbar)*m(d)=0, m(0)=0, m(1)=1}:
> rec2:={(d+1)*hbar*m(d+1)+G(-(d+1)*hbar)*m(d)=0, m(1)=-G(-hbar)}:
> rec3:={(d+1)*m(d+1)-d*m(d)=0, m(1)=1}:
```

```
> recprod:={cauchyproduct(rec1, rec2, m(d))=0}:
> finalrec:=`rec*rec`(recprod, rec3, m(d));
```

$$\{(-2+4*d)*m(d)+(-d-1+hbar^2*d^3+hbar^2*d^2)*m(d+1), m(0)=0, m(1)=\_C[0]\}$$

We next provide some explanatory notes to indicate how the code above produces the desired 1-point recursion. Recall that monotone Hurwitz numbers satisfy the relation

$$m(d)=\sum_{g=0}^{\infty} m_g(d)\,\hbar^{2g-1}=\frac{1}{d}\sum_{k=1}^{d} u_k\, v_{d-k},$$

where $\frac{u_{k+1}}{u_k}=\frac{G(k\hbar)}{k\hbar}$ and $\frac{v_{k+1}}{v_k}=-\frac{G(-(k+1)\hbar)}{(k+1)\hbar}$.

- Line 1 loads the `gfun` package into MAPLE.
- Line 2 defines the weight generating function $G(z)$ that produces monotone Hurwitz numbers.
- Line 3 expresses the recursion satisfied by the sequence $u_0, u_1, u_2, \ldots$ above.
- Line 4 expresses the recursion satisfied by the sequence $v_0, v_1, v_2, \ldots$ above.
- Line 5 expresses the recursion satisfied by the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$.
- Line 6 determines a recursion for the Cauchy product of the sequences $u_0, u_1, u_2, \ldots$ and $v_0, v_1, v_2, \ldots$.
- Line 7 determines a recursion for the Hadamard product of the Cauchy product from the previous line and the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$.
- The output asserts that

  $$(-2\hbar+4d\hbar)\,m(d)+(-d-1+\hbar^2d^3+\hbar^2d^2)\,m(d+1)=0.$$

  By collecting the coefficient of h^{2g-1} and shifting the index, we obtain the 1-point recursion

  $$d\,m_g(d)=2(2d-3)\,m_g(d-1)+d(d-1)^2\,m_{g-1}(d).$$

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