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Twisted Poincaré series and zeta functions on finite quotients of buildings

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Abstract
In the case where $G = \mathrm{SL}_2(F)$ for a non-archimedean local field F and Γ is a discrete torsion-free cocompact subgroup of G, there is a known relationship between the Ihara zeta function for the quotient of the Bruhat–Tits tree of G by the action of Γ, and an alternating product of determinants of twisted Poincaré series for parabolic subgroups of the affine Weyl group of G. We show how this can be generalized to other split simple algebraic groups of rank two over F and formulate a conjecture about how this might be generalized to groups of higher rank.

Keywords Building · Ihara zeta function · Coxeter group · Poincaré series

1 Introduction

The classical Ihara zeta function [9] is a counting function associated with a discrete torsion-free cocompact subgroup Γ of $G = \mathrm{PGL}_2(F)$, where F is a non-archimedean local field with the discrete valuation v and q elements in its residue field, defined as follows:

$$Z(\Gamma, u) = \prod_{[\gamma]} \left(1 - u^{\ell([\gamma])}\right)^{-1}.$$

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Here, $[\gamma]$ runs through all primitive conjugacy classes in Γ and ℓ is the length function defined as follows. For $\gamma \in G$ with eigenvalues λ_1 and λ_2, define

$$\ell(\gamma) = \max\left\{v\left(\frac{\lambda_1}{\lambda_2}\right), v\left(\frac{\lambda_2}{\lambda_1}\right)\right\} \quad \text{and} \quad \ell([\gamma]) = \min\left\{\ell\left(\gamma'\right) : \gamma' \in [\gamma]\right\}.$$

Here whenever $\lambda_1, \lambda_2 \notin F$, we set $\ell(\gamma) = 1$. On the other hand, one can also regard $Z(\Gamma, u)$ as a geometric counting function associated with the quotient of the Bruhat–Tits tree of G by the action of Γ, denoted by $\mathscr{B}_\Gamma$, such that

$$Z(\mathscr{B}_\Gamma, u) = \prod_c \left(1 - u^{\ell(c)}\right)^{-1}.$$

Here, c runs through all primitive closed geodesics (which are equivalence classes of primitive closed tailless and backtrack-less walks) in $\mathscr{B}_\Gamma$ and $\ell(c)$ is the path length of c in graph theory.

Ihara showed that the zeta function is indeed a rational function given by the following formula

$$Z\left(\mathscr{B}_\Gamma, u\right) = \frac{\left(1 - u^2\right)^{\chi(\mathscr{B}_\Gamma)}}{\det\left(1 - \mathrm{Au} + qu^2\right)}. \tag{1.1}$$

Here, $\chi\left(\mathscr{B}_\Gamma\right)$ is the Euler characteristic and A is the adjacency matrix of the finite graph $\mathscr{B}_\Gamma$, which is also a Hecke operator on G acting on spherical vectors of $L^2(\Gamma\backslash G)$.

Hashimoto [6] showed that there is an easy way to see the rationality of Ihara zeta function by considering the edge adjacency operator A_E, which is an Iwahori–Hecke operator of G acting on Iwahori spherical vectors of $L^2(\Gamma\backslash G)$. Hashimoto showed that

$$Z(\mathscr{B}_\Gamma, u) = \frac{1}{\det(1 - A_E u)}. \tag{1.2}$$

It is natural to ask if one can generalize Ihara and Hashimoto's result to other reductive groups. However, there is no canonical way to define Ihara zeta functions on finite quotients of higher-dimensional buildings. Thus, one must find a new interpretation of Eq. (1.1).

For example, the term $\det(1 - \mathrm{Au} + qu^2)^{-1}$ can be regarded as the Langlands L-function associated with the unramified subrepresentation $L^2(\Gamma\backslash G)$ of G (see [12] for details). In this case, the right-hand side of Eq. (1.1) is known in general and it remains to figure out the left-hand side in terms of geometric zeta functions. For this viewpoint, see [10,11] for the case of PGL_3, [4] for PGSP_4, [3] for the case of PGL_n over the 1-adic field, and [12] for the case of rank two algebraic groups over the 1-adic field. Roughly speaking, they obtain an identity of the following form.

Zeta identity for groups of adjoint type:

the unramified Langlands L-function
= the alternating product of geometric zeta functions of various dimensions.

Note that in Ihara's original result and the above generalizations, G is always of adjoint type. On the other hand, the building of G only depends on its root system so that there are other groups having the same building as G. For instance, $\mathrm{SL}_2(F)$ and $\mathrm{PGL}_2(F)$ have the same associated buildings, but $\mathrm{SL}_2(F)$ is simply connected instead of being of adjoint type. In this case, Hashimoto [7] gave a different way to express Ihara's identity.

Let W be the affine Weyl group of $G = \mathrm{SL}_2(F)$ with the standard generating set S consisting of elements of order two and K be the Iwahori subgroup of G. For $w \in W$, let e_w be the Iwahori Hecke operator associated with w. Then, for each parabolic subgroup W_I generating by $I \subset S$, one can consider its Poincaré series $W_I(\rho, u)$ attached to a representation ρ of the Iwahori Hecke algebra $H(G, K)$ as

$$W_I(\rho, u) = \sum_{w \in W_I} \rho(e_w) u^{\ell(w)}.$$

where $\ell(\cdot)$ is the word length with respect to S. Let π_Γ be the representation of $H(G, K)$ acting on $L^2(\Gamma \backslash G/K)$. Hashimoto's result implies that

$$Z(\mathscr{B}_\Gamma, u) = \prod_{I \subset S} \det W_I(\pi_\Gamma, u)^{(-1)^{|I|}}. \tag{1.3}$$

The main goal of this paper is to generalize Ihara's result along this direction to simple simply connected split algebraic groups of rank two. Note that the right-hand side of the above equation can be naturally defined for simple simply connected split algebraic groups of higher ranks. It remains to figure out the left-hand side. Comparing to the result of adjoint type, we expect to obtain an identity of the following form.

Zeta identity for simply connected groups:

the geometric zeta function of top dimension

= the alternating product of twisted Poincaré series of parabolic subgroups.

Now suppose G is a simple simply connected split algebraic group of rank two. In this case, the building $\mathscr{B}$ of G and its finite quotient $\mathscr{B}_\Gamma$ are two dimensional complexes. Instead of counting closed geodesics in a finite quotient $\mathscr{B}_\Gamma$ which lift to straight lines in $\mathscr{B}$, we count closed geodesic strips in $\mathscr{B}_\Gamma$ which lift to straight strips in $\mathscr{B}$. One can define the zeta function of closed geodesic strips $Z(\mathscr{B}_\Gamma, u)$ by the same manner as zeta function of closed geodesics (see Sect. 4.6 for details). The main theorem of the paper is the following.

Theorem 1.1 *Let π_Γ be the representation of the Iwahori Hecke algebra $H(G, K)$ acting on $L^2(\Gamma \backslash G/K)$. There are two Iwahori–Hecke operators e_{w_1} and e_{w_2}, such that*

$$Z(\mathscr{B}_\Gamma, u) = \frac{1}{\det\left(I - \pi_\Gamma(e_{w_1})u^{\ell(w_1)}\right) \det\left(I - \pi_\Gamma(e_{w_2})u^{\ell(w_2)}\right)}. \tag{1.4}$$

Moreover, we also have

$$Z(\mathscr{B}_\Gamma, u) = \prod_{I \subset S} \det W_I(\pi_\Gamma, u)^{(-1)^{|I|+|S|}}. \tag{1.5}$$

Here, $\pi_\Gamma(e_{w_1})$ and $\pi_\Gamma(e_{w_1})$ are indeed chamber adjacency operators which play the role of A_E in Hashimoto's formula (see Sect. 4.6 for details).

Remark The zeta function of closed geodesic strips in this paper is indeed the geometric zeta function of top dimension (which is called chamber zeta function or gallery zeta function) occurred in the works [4,10,11].

When G is of rank $n > 2$, one may consider the zeta function $Z(\mathscr{B}_\Gamma, u)$ counting closed geodesic tube in $\mathscr{B}_\Gamma$ which lift to straight tubes in $\mathscr{B}$ by the same manner (however we will not define the zeta function of closed geodesic tubes in this paper). If the above two theorems still hold, then there should exist some Iwahori Hecke operators $e_{w_1}, \ldots, e_{w_n}$ so that

$$\begin{aligned} Z(\mathscr{B}_\Gamma, u) &= \frac{1}{\det(I - \pi_\Gamma(e_{w_1})u^{\ell(w_1)}) \cdots \det(I - \pi_\Gamma(e_{w_n})u^{\ell(w_n)})} \\ &= \prod_{I \subset S} \det W_I(\pi_\Gamma, u)^{(-1)^{|I|+|S|}}. \end{aligned}$$

In the end of the paper, we verify the second part of the above identity with π_Γ replaced by the trivial representation ρ_0. In this case, $W_I(\rho_0, u) = W_I(u)$ is the usual (un-twisted) Poincaré series and the second part of the above identity becomes the following.

$$\prod_{I \subset S} W_I(u)^{(-1)^{|I|+|S|}} = \frac{1}{\left(1 - u^{\ell(w_1)}\right) \cdots \left(1 - u^{\ell(w_n)}\right)}.$$

We obtain the following result.

Theorem 1.2 *Let W be the affine Coxeter group of rank $n+1$. Then the alternating product $\prod_{I \subset S} W_I(u)^{(-1)^{|I|+|S|}}$ is the reciprocal of a polynomial of the form*

$$\left(1 - u^{d_1}\right) \cdots \left(1 - u^{d_n}\right)$$

where d_i are positive integers.

Note that for a general Coxeter group W, $\prod_{I \subset S} W_I(u)^{(-1)^{|I|+|S|}}$ is a rational function. In fact, by direct computation, one can show that for an irreducible Coxeter group with three generators, this alternating product is a reciprocal of a polynomial if and only if it is an affine Coxeter group.

The paper is organized as follows. In Sect. 2, we review the definition of Poincaré series and their twistings and state Hashimoto's interpretation of Ihara's formula.

Especially, the result is based on a length-preserving decomposition of the affine Weyl group of SL_2. In Sect. 3, we study such decomposition for affine Weyl groups of rank three and discuss their relation with geometric zeta functions in Sect. 4. The Proof of Theorem 1.1 is given in the end of Sect. 4. In Sect. 5, we state our conjecture about higher-dimensional cases and prove Theorem 1.2.

2 Twisted Poincaré series and Ihara zeta functions

2.1 Poincaré series

Let (W, S) be a Coxeter system with a generating set S consisting of elements of order two. For $w \in W$, let $\ell(w)$ be the shortest length of a word consisting of elements of S whose product is equal to w. The Poincaré series associated with (W, S) is a power series with integer coefficients defined as

$$W(u) = \sum_{w \in W} u^{\ell(w)}.$$

For a subset D of W, we also define $D(u) = \sum_{w \in D} u^{\ell(w)}$. Especially, we are interested in the case where $D = W_I$, the subgroup generated by some subset $I \subset S$. For instance, when W is finite,

$$\sum_{I \subset S} (-1)^{|I|} \frac{W(u)}{W_I(u)} = u^{\ell(w_0)}.$$

Here, w_0 is the unique element of maximal length. When W is infinite,

$$\sum_{I \subset S} (-1)^{|I|} \frac{W(u)}{W_I(u)} = 0,$$

which implies that

$$W(u) = \left(\sum_{I \subsetneq S} \frac{(-1)^{|I|+|S|+1}}{W_I(u)} \right)^{-1}. \tag{2.1}$$

See [8, Sections 1.11 and 5.12] for the proof of these statements.

2.2 Hecke algebras

For a Coxeter system (W, S) and a formal parameter q, there is an associative $\mathbb{C}$-algebra $H_q(W, S)$, called a Hecke algebra, with generators $\{e_w\}_{w \in W}$. The multiplication of $H_q(W, S)$ is characterized by the following the relations:

$$\begin{aligned} (e_s + 1)(e_s - q) &= 0, &&\text{if } s \in S; \\ e_w e_v &= e_{wv}, &&\text{if } \ell(wv) = \ell(w) + \ell(v). \end{aligned}$$

Especially, if we set $q = 1$, then $H_q(W, S)$ is isomorphic to the group algebra $\mathbb{C}[W]$. Recall that, one can identify the (affine) Hecke algebra $H_q(W, S)$ and the Iwahori–Hecke algebra $H(G, K)$ by mapping e_w to KwK.

2.3 Twisted Poincaré series

For a representation (ρ, V_ρ) of $H_q(W, S)$, consider the following power series

$$D(\rho, u) = \sum_{w \in D} \rho(e_w) u^{\ell(w)} \in \mathrm{End}(V_\rho)[[u]],$$

called the Poincaré series of D twisted by ρ. Note that when D contains the identity element (e.g. $D = W_I$), the constant term of $D(\rho, u)$ is the identity operator and $D(\rho, u)$ has an inverse in $\mathrm{End}(V_\rho)[[u]]$.

Note that $e_s \mapsto q$ for all $s \in S$ induces a one-dimensional representation ρ_1 of $H_q(W, S)$. In this case,

$$D(\rho_1, u) = \sum_{w \in D} \rho_1(e_w) u^{\ell(w)} = \sum_{w \in D} q^{\ell(w)} u^{\ell(w)} = D(qu).$$

Therefore, one can regard the usual Poincaré series as a special case of twisted Poincaré series.

2.4 Ihara zeta functions

Let W be the affine Weyl group of $G = \mathrm{SL}_2$ over a local field F with p^n elements in its residue field. In this case, $S = \{s_1, s_2\}$ and $s_1 s_2$ has order infinity. Let the formal parameter q be equal to p^n, then the Hecke algebra $H_q(W, S)$ is isomorphic to the Iwahori–Hecke algebra of G. Fix a discrete torsion-free cocompact subgroup Γ of G. Then, the quotient of the Bruhat–Tits tree $\mathscr{B}$ of G by Γ is a bipartite finite $(q + 1)$-regular graph $\mathscr{B}_\Gamma$. The Ihara zeta function of $\mathscr{B}_\Gamma$ is defined as

$$Z(\mathscr{B}_\Gamma, u) = \prod_c (1 - u^{\ell(c)})^{-1} \in \mathbb{Z}[[u]].$$

Here, c runs over all primitive closed geodesics in $\mathscr{B}_\Gamma$ and $\ell(c)$ is the usual length in graph theory.

Let π_Γ be the complex representation of $H_q(W, S)$ on the space of Iwahori-fixed vectors of $L^2(\Gamma \backslash G)$. In this case, elements in $H_q(W, S)$ can be regarded as operators on edges of $\mathscr{B}_\Gamma$ and Hashimoto [6] showed that

Theorem 2.1 $Z(\mathscr{B}_\Gamma, u) = \det(1 - \pi_\Gamma(e_{s_2 s_1}) u^2)^{-1}$.

On the other hand, one can factor the group W as a product of three subsets as follows.

$$W = \langle s_1 \rangle \times \{(s_2 s_1)^m\}_{m=0}^{\infty} \times \langle s_2 \rangle. \tag{2.2}$$

Here, for three subsets X, Y, Z of W, we say X factors into a product of Y and Z, denoted by $X = Y \times Z$, if every element of X can be written as yz for unique some $y \in Y$ and $z \in Z$. Similar notations apply to factorization involving more than two sets.

Note that all elements in the above product are reduced words. Therefore, for any representation ρ of $H_q(W, S)$, we have

$$W_{\{s_1,s_2\}}(\rho, u) = W_{\{s_1\}}(\rho, u)\left(\sum_{i=1}^{\infty} \rho(e_{s_2}e_{s_1})^m u^{2m}\right)W_{\{s_2\}}(\rho, u)$$
$$= W_{\{s_1\}}(\rho, u)\left(I - \rho(e_{s_2}e_{s_1})u^2\right)^{-1} W_{\{s_2\}}(\rho, u)$$

Moreover, when ρ is a finite dimensional representation over a field K, one can consider the determinant of $W_I(\rho, u)$, which is an invertible element in the power series $K[[u]]$ (since its constant term is equal to one). For convenience, define

$$\mathrm{Alt}(W)(u) = \prod_{I \subset S} W_I(u)^{(-1)^{|I|+|S|}}$$

and

$$\det \mathrm{Alt}(W)(\rho, u) = \prod_{I \subset S} \det W_I(\rho, u)^{(-1)^{|I|+|S|}}.$$

It was pointed out in [15] that one can rewrite Hashimoto's result as the following.

Theorem 2.2 *As a power series,*

$$Z(\mathscr{B}_\Gamma, u) = \det \mathrm{Alt}(W)(\pi_\Gamma, u). \tag{2.3}$$

Note that the above theorem is the Eq. (1.3) in the introduction. Besides, this interpretation suggests a possible way to generalize Ihara's theorem to higher rank cases.

3 Affine Coxeter groups of rank three

3.1 Alternating products of Poincaré series

Suppose (W, S) is an irreducible affine Coxeter system of rank three where $S = \{s_1, s_2, s_3\}$. Let m_{ij} be the order of $s_i s_j$. There are three types of such Coxeter systems up to isomorphism, characterized as the following:

- Type $\tilde{A}_2$: $(m_{12}, m_{23}, m_{13}) = (3, 3, 3)$.
- Type $\tilde{C}_2$: $(m_{12}, m_{23}, m_{13}) = (4, 2, 4)$.
- Type $\tilde{G}_2$: $(m_{12}, m_{23}, m_{13}) = (6, 2, 3)$.

Especially, every such Coxeter group is isomorphic to an affine Weyl group of some irreducible root system. To evaluate $\mathrm{Alt}(W)(u)$, note that W_I is a dihedral group or a cyclic group of order two when I is a non-empty proper subset of S. In this case, $W_I(u)$ can be written down directly. On the other hand, one can apply Eq. (2.1) to compute the Poincaré series of $W(u)$ and obtain the following results.

$$\mathrm{Alt}(W)(u)^{-1} = \begin{cases} (1-u^3)^2, & \text{if } (W,S) \text{ is of type } \tilde{A}_2; \\ (1-u^4)(1-u^3), & \text{if } (W,S) \text{ is of type } \tilde{C}_2; \\ (1-u^5)(1-u^3), & \text{if } (W,S) \text{ is of type } \tilde{G}_2. \end{cases}$$

We shall show that the above identities can be extended to the case of twisted Poincaré series of Hecke algebras.

3.2 Coxeter complexes

For the affine Coxeter system (W, S), fix a geometric realization of (W, S) on a real vector space V endowed with a Euclidean metric invariant under W.

The hyperplane H_s fixed by an affine reflection s in W is called a wall. The set of walls gives V a simplicial structure and the resulting simplicial complex is called the Coxeter complex $\mathscr{A}$ of W. Connected components of V with all walls removed are open 2-simplices, called alcoves or chambers.

The unique chamber whose boundary is contained in $H_{s_1} \cup H_{s_2} \cup H_{s_3}$ is the fundamental chamber $\mathscr{C}$. Label the edge (1-simplex) of $\mathscr{C}$ by i if it is contained in H_{s_i}. Then, one can extend this labeling to a W-invariant labeling on all edges in $\mathscr{A}$ uniquely.

Besides, there is a bijection between W and chambers in $\mathscr{A}$ given by $w \mapsto w\mathscr{C}$. Moreover, if $w = s_{i_1} \cdots s_{i_n}$, then the chambers $\mathscr{C}$ and $w\mathscr{C}$ can be linked by the gallery

$$C_0 = \mathscr{C} \to C_1 = s_{i_1}\mathscr{C} \to C_2 = s_{i_1}s_{i_2}\mathscr{C} \to \cdots \to C_n = s_{i_1}\cdots s_{i_n}\mathscr{C}$$

such that the edge $\bar{C}_k \cap \bar{C}_{k+1}$ is labeled by i_{k+1}. In other words, starting from the fundamental chamber $\mathscr{C}$, one can cross the edges labeled by $i_1, \ldots, i_n$ sequentially to arrive at the chamber $w\mathscr{C}$.

3.3 Straight strip

Let $v_i = H_{s_{i+1}} \cap H_{s_{i+2}}$ be the vertex of the fundamental chamber $\mathscr{C}$ where the subscripts are read modulo 3. For convenience, we may assume that v_3 is the origin of V. Then the stabilizer of v_3 in W is the linear subgroup W_0 generated by s_1 and s_2, which is the Weyl group (this follows from our special choice of the ordering of m_{ij} in the beginning of the section). For the affine transformation $w \in W$, we can uniquely write $w = w_0 w_t$ where $w_0 \in W_0$ is the linear part of w and w_t is a translation.

Now for $i = 1, 2$, removing all walls parallel to the vector v_i from V, the connected component containing $\mathscr{C}$ in the resulting set is called the fundamental straight strip $\mathscr{T}_i$ in the direction v_i, which is the gray region in Figs. 1, 2, 3 and 4. For $w = w_0 w_t \in W$, $w\mathscr{T}_i$ is a straight strip in the direction of $w_0 v_i$ and all such strips are called of type i.

Fig. 1 Type $\tilde{A}_2$

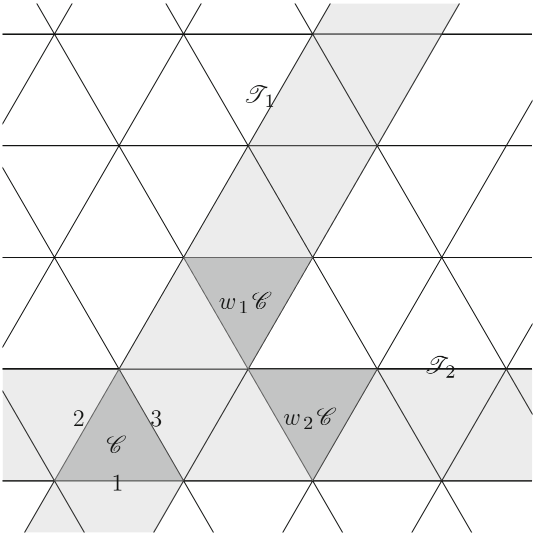
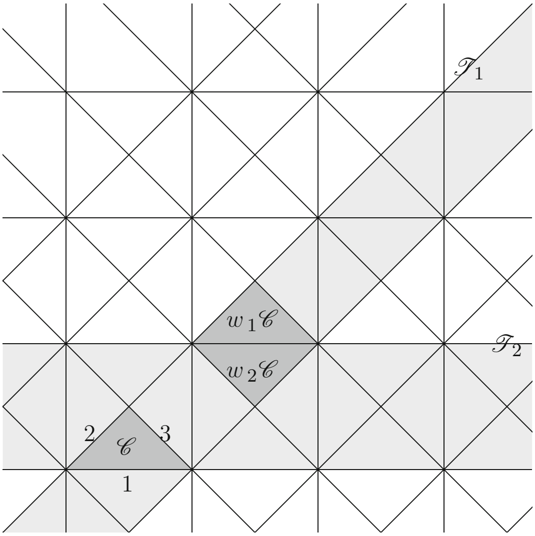


Fig. 2 Type $\tilde{C}_2$



3.4 Stabilizer of $\mathscr{T}_i$

Let $\mathrm{Stab}(\mathscr{T}_i)$ be the stabilizer of $\mathscr{T}_i$ consisting of elements mapping $\mathscr{T}_i$ to itself and preserving the direction v_i. Let $\mathscr{L}$ be the middle line of $\mathscr{T}_i$, then for $w \in \mathrm{Stab}(\mathscr{T}_i)$, w must preserve $\mathscr{L}$ and its action on $\mathscr{L}$ has to be a translation (in the direction of v_i or $-v_i$). If the action of w on $\mathscr{L}$ is trivial, w fixes $\mathscr{L}$ point-wisely and it is the identity element or an affine reflection whose reflection axis is $\mathscr{L}$. On the other hand, $\mathscr{L}$ is not a wall (by the construction of $\mathscr{T}_i$) and w must therefore be the identity element in this case. Therefore, elements in $\mathrm{Stab}(\mathscr{T}_i)$ are uniquely determined by their actions on $\mathscr{L}$.

Fig. 3 $\mathscr{T}_1$ in type $\tilde{G}_2$

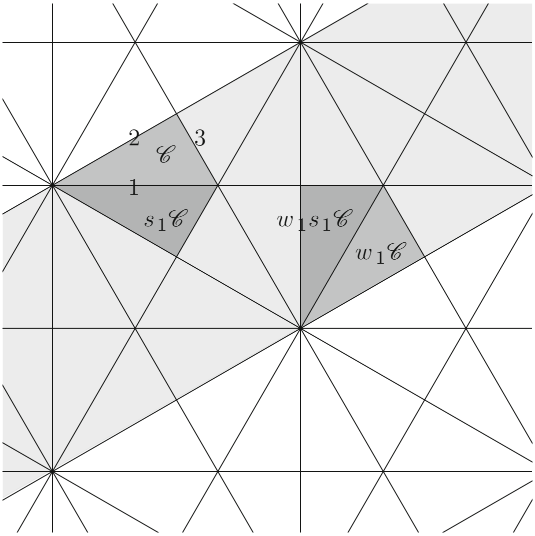
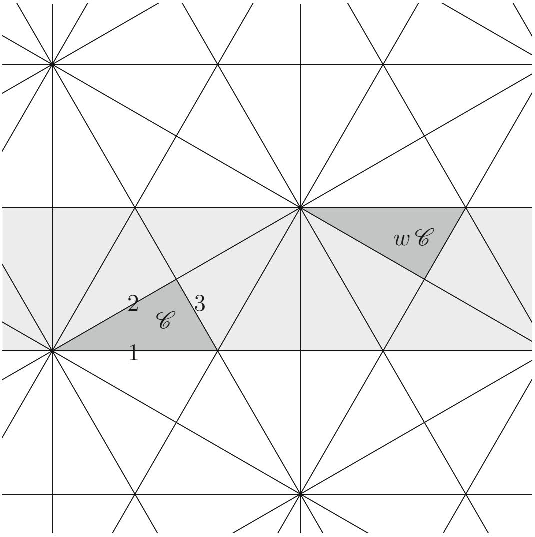


Fig. 4 $\mathscr{T}_2$ in type $\tilde{G}_2$



Let w_i be the element in $\mathrm{Stab}(\mathscr{T}_i)$ such that its action on $\mathscr{L}$ is the minimal translation in the direction of v_i. Then, we have the following theorem.

Theorem 3.1 *The stabilizer $Stab(\mathscr{T}_i)$ is a cyclic group generated by w_i.*

By direct computation, we obtain the following.

1. When (W, S) is of type $\tilde{A}_2$, we have $w_1 = s_3 s_2 s_1$ and $w_2 = s_3 s_1 s_2$.
2. When (W, S) is of type $\tilde{C}_2$, we have $w_1 = s_3 s_1 s_2 s_1$ and $w_2 = s_3 s_1 s_2$.
3. When (W, S) is of type $\tilde{G}_2$, we have $w_1 = s_3 s_1 s_2 s_3 s_1$ and $w_2 = s_3 s_1 s_2 s_1 s_2$.

One can check case by case to see that except for w_1 in the case of type $\tilde{G}_2$,

- the length of $(w_i)^k$ is equal to $|k|$ times of the length of w_i.

In general, an element in W satisfying the above property is called straight (the notion of straight element was first introduced by [13]).

Now, in the case of $\tilde{G_2}$, we shall replace w_1 by

$$w_1' = s_1 w_1 s_1 = (s_1 s_3 s_1) s_2 s_3 = (s_3 s_1 s_3) s_2 s_3 = s_3 s_1 (s_3 s_2 s_3) = s_3 s_1 s_2.$$

which is a generator of the stabilizer of $s_1 \mathscr{T}_1$. Then, w_1' will be straight. To abuse the notation, we will also denote w_1' by w_1 (in the case of the group $\tilde{G_2}$) in the rest of paper.

Let $H_i = \{(w_i)^k\}_{k \in \mathbb{Z}_{\geq 0}}$. Note that

$$H_i(u) = \sum_{i=0}^{\infty} u^{\ell(w_i^k)} = \sum_{i=0}^{\infty} u^{k\ell(w_i)} = \left(1 - u^{\ell(w_i)}\right)^{-1}.$$

Therefore, we can rewrite the result in Sect. 3.1 as

Proposition 3.2 *For any affine Coxeter group (W, S) of rank three, the following holds.*

$$H_1(u) H_2(u) = \mathrm{Alt}(W)(u).$$

3.5 Factorization of Coxeter groups

Next, we will factor W in terms of W_I and H_i. To do so, we need the following lemma.

Lemma 3.3 *Let $D_1, \ldots, D_m$ be non-empty subsets of a Coxeter group W. If $W = D_1 \times \cdots \times D_m$ and $D_1(u) \cdots D_m(u) = W(u)$, then this factorization is length-preserving, i.e. for $w_i \in D_i$, $\ell(w_1 \cdots w_m) = \ell(w_1) + \cdots + \ell(w_m)$.*

Proof Since $\ell(w_1 \cdots w_m) \leq \ell(w_1) + \cdots + \ell(w_m)$, for any integer k we have

$$\begin{aligned} \Omega_k &:= \{(w_1, \ldots, w_m) \in D_1 \times \cdots \times D_m : \ell(w_1) + \cdots + \ell(w_m) \leq k\} \\ &\subseteq \{(w_1, \ldots, w_m) \in D_1 \times \cdots \times D_m : \ell(w_1 \cdots w_m) \leq k\} \\ &= \{w \in W : \ell(w) \leq k\} =: \Omega_k'. \end{aligned}$$

On the other hand,

$$D_1(u) \cdots D_m(u) = \prod_{i=1}^{m} \left(\sum_{w_i \in D_i} u^{\ell(w_i)} \right) = |\Omega_0| + \sum_{i=1}^{\infty} (|\Omega_i| - |\Omega_{i-1}|)\, u^i$$

and

$$W(u) = |\Omega'_0| + \sum_{i=1}^{\infty} \left(|\Omega'_i| - |\Omega'_{i-1}|\right) u^i$$

From $D_1(u) \cdots D_m(u) = W(u)$, we conclude that $|\Omega_i| = |\Omega'_i|$ for all i. Therefore, $\Omega_k = \Omega'_k$ and $\ell(w_1 \cdots w_m) = \ell(w_1) + \cdots + \ell(w_m)$ for $w_i \in D_i$. □

Note that when $\ell(w_1 \cdots w_m) = \ell(w_1) + \cdots + \ell(w_m)$, for any representation ρ of $H_q(W, S)$, we have

$$\rho(e_{w_1 \cdots w_m}) = \rho(e_{w_1}) \cdots \rho(e_{w_m}).$$

Together with the above lemma, we have

Theorem 3.4 *If $W = D_1 \times \cdots \times D_m$ and $D_1(u) \cdots D_m(u) = W(u)$, then for any representation ρ of $H_q(W, S)$*

$$W(\rho, u) = D_1(\rho, u) \cdots D_m(\rho, u).$$

Let us give some examples of the above theorem. For subsets $I \subset J \subset S$, define

$$W_{J/I} = \{w \in W_J : \ell(ws) > \ell(w), \forall s \in I\} \quad \text{and}$$
$$W_{I\backslash J} = \{w \in W_J : \ell(sw) > \ell(w), \forall s \in I\}$$

which are the sets of left minimal length W_I-coset representatives and right minimal length W_I-coset representatives of W_J respectively (remark: more common notations for left and right minimal length coset representative are W_J^I and ${}^I W_J$. However, such notations may cause confusion when we consider the product of several such sets).

Theorem 3.5 [8, Sect. 1.11] *For subsets $I \subset J \subset S$, the following hold.*

1. $W_J = W_{J/I} \times W_I = W_I \times W_{I\backslash J}$.
2. $W_J(u) = W_{J/I}(u) W_I(u) = W_I(u) W_{I\backslash J}(u)$.

For convenience, we also denote the set of generators $I = \{s_{i_1}, \ldots, s_{i_k}\}$ by its set of indices $\{i_1, \ldots, i_k\}$. Next, let us prove the following theorem for affine Coxeter groups of rank three.

Theorem 3.6 *Let $D_1, \ldots, D_m$ be subsets of W given by the following.*

- $(D_1, \ldots, D_5) = (W_{\{1,2\}/\{2\}}, H_1, W_{\{2,3\}/\{3\}}, H_2, W_{\{1\}\backslash\{1,3\}})$ *if (W, S) is of type $\tilde{A}_2$ or $\tilde{C}_2$.*
- $(D_1, \ldots, D_4) = (W_{\{2,1\}/\{1\}}, H_2 H_1 W_{\{1,3\}})$ *if (W, S) is of type $\tilde{G}_2$.*

Then $D_1(u) \cdots D_m(u) = W(u)$ and $W = D_1 \times \cdots \times D_m$.

Proof Note that for type $\tilde{A}_2$ or $\tilde{C}_2$, by Proposition 4.5,

$$\begin{aligned} D_1(u)\cdots D_5(u) &= W_{\{1,2\}/\{2\}}(u)H_1(u)W_{\{2,3\}/\{3\}}(u)H_2(u)W_{\{1\}\backslash\{1,3\}}(u) \\ &= \frac{W_{\{1,2\}}(u)W_{\{2,3\}}(u)W_{\{1,3\}}(u)}{W_{\{1\}}(u)W_{\{2\}}(u)W_{\{3\}}(u)}\mathrm{Alt}(W)(u) = W(u). \end{aligned}$$

For type $\tilde{G}_2$, note that $W_{\{2,3\}} = \{e, s_2, s_3, s_2s_3\} = W_{\{2\}} \times W_{\{3\}}$. Therefore,

$$W_{\{2,3\}}(u) = W_{\{2\}}(u)W_{\{3\}}(u)$$

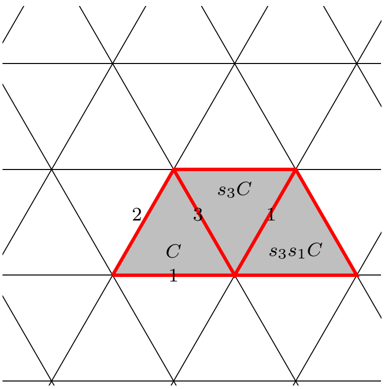
and

$$\begin{aligned} D_1(u)\cdots D_4(u) &= W_{\{2,1\}/\{1\}}(u)H_2(u)H_1(u)W_{\{1,3\}}(u) \\ &= \frac{W_{\{2,1\}}(u)W_{\{1,3\}}(u)}{W_{\{1\}}(u)}\mathrm{Alt}(W)(u)\cdot\frac{W_{\{2,3\}}(u)}{W_{\{2\}}(u)W_{\{3\}}(u)} = W(u). \end{aligned}$$

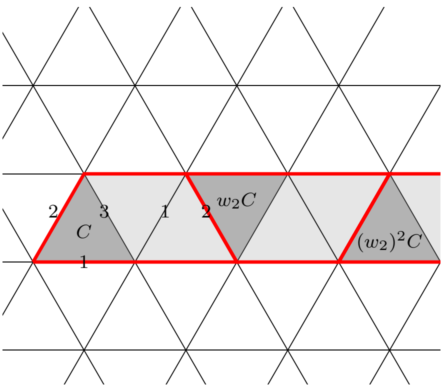
It remains to show that $W = D_1 \times \cdots \times D_m$. Our strategy is to study the geometric interpretation of this factorization. As we mentioned before, for each $w = s_{i_1}\cdots s_{i_k} \in W$, one can start from the fundamental chamber C and cross the edges labeled by $i_1, \ldots, i_n$ sequentially to arrive the chamber wC. Therefore, one can find the collections of chambers $\mathfrak{C}_{m+1} = \{C\}$, $\mathfrak{C}_m = D_m(\mathfrak{C}_{m+1}), \cdots, \mathfrak{C}_1 = D_1(\mathfrak{C}_2)$ step by step.

More precisely, for the case of type $\tilde{A}_2$, we have

1. $\mathfrak{C}_5 = D_5(C) = \{C, s_3C, s_3s_1C\}$.
2. $\mathfrak{C}_4 = D_4(\mathfrak{C}_5) = \{(w_2)^i\mathfrak{C}_5, i = 0 \sim \infty\}$, where $w_2 = s_3s_1s_2$ is a glide reflection.

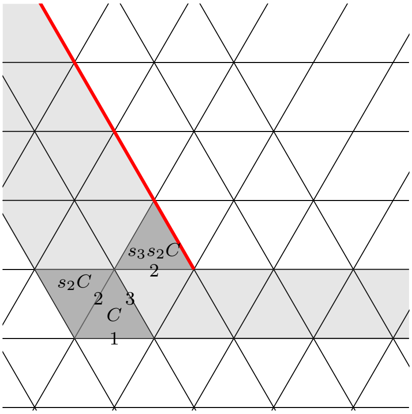


$\mathfrak{C}_5$

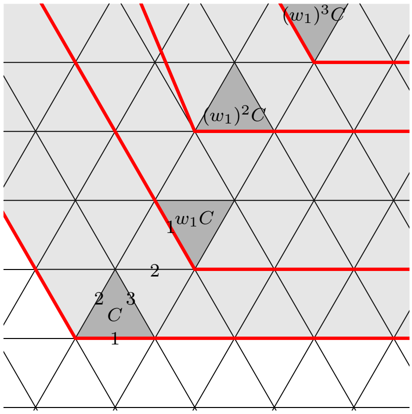


Form $\mathfrak{C}_5$ to $\mathfrak{C}_4$

3. $\mathfrak{C}_3 = D_3(\mathfrak{C}_4) = \{\mathfrak{C}_4, s_2\mathfrak{C}_4, s_3s_2\mathfrak{C}_4\}$. Here, s_3s_2 is a rotation.
4. $\mathfrak{C}_2 = D_2(\mathfrak{C}_3) = \{(w_1)^i\mathfrak{C}_3 : i = 0 \sim \infty\}$, where $w_1 = s_3s_2s_1$ is a glide reflection.

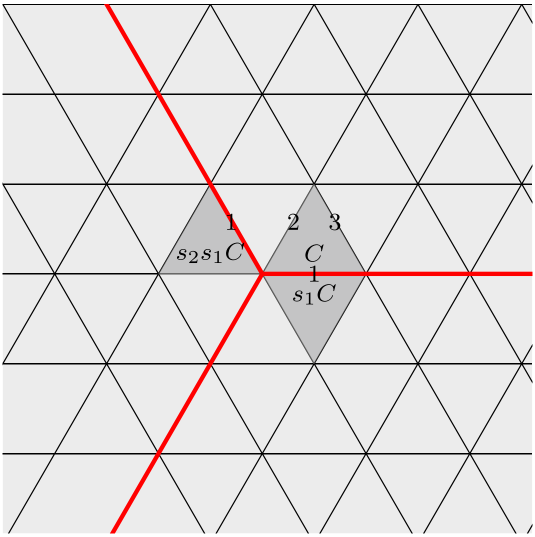


Form $\mathfrak{C}_4$ to $\mathfrak{C}_3$



Form $\mathfrak{C}_3$ to $\mathfrak{C}_2$

5. $\mathfrak{C}_1 = D_1(\mathfrak{C}_2) = \{\mathfrak{C}_1, s_1\mathfrak{C}_1, s_2s_1\mathfrak{C}_1\}$. □

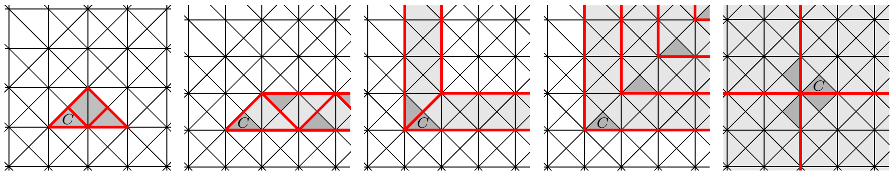


Form $\mathfrak{C}_2$ to $\mathfrak{C}_1$

Here, at each stage, the whole gray area is the region $\mathfrak{C}_i$; connected components with red boundaries are copies of $\mathfrak{C}_{i+1}$. Moreover, the set of dark gray triangles is the image of $D_i(C)$. From these figures, we see that $\mathfrak{C}_1 = D_1 \cdots D_m(C)$ and $W(C)$ are bijective. We conclude that $W = D_1 \times \cdots \times D_m$.

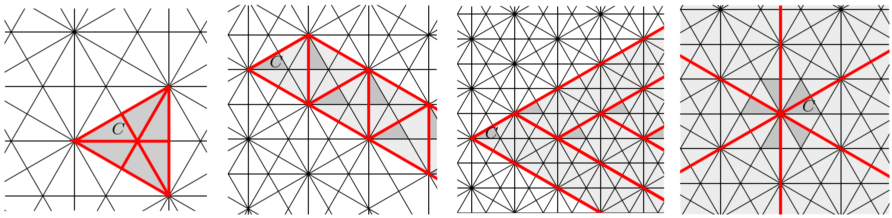
The arguments are similar for type $\tilde{C}_2$ and $\tilde{G}_2$, so we just draw the following figures.

3.6 Type $\tilde{C}_2$



$\mathfrak{C}_5$ $\mathfrak{C}_4$ $\mathfrak{C}_3$ $\mathfrak{C}_2$ $\mathfrak{C}_1$

3.7 Type $\tilde{G}_2$



$\mathfrak{C}_4$ $\mathfrak{C}_3$ $\mathfrak{C}_2$ $\mathfrak{C}_1$

□

Combining Theorems 3.4 and 3.6, we obtain the main result of this section.

Theorem 3.7 *Let (W, S) be an affine Coxeter system of rank three and ρ be a finite dimensional representation of $H_q(W, S)$. Then*

$$\det H_1(\rho, u) \det H_2(\rho, u) = \det \mathrm{Alt}(W)(\rho, u).$$

4 Zeta functions of closed geodesic strips

In this section, we will pass the result from affine Coxeter systems in the previous section to algebraic groups.

4.1 Straight strips in the building

Let G be a simply connected connected split simple algebraic group over a local field F with q elements in its residue field, whose affine Weyl group W is a Coxeter group of rank three with the generating set S. Let $\mathscr{B}$ be the Bruhat–Tits building of G whose chambers are parametrized by the cosets G/K, where K is the Iwahori subgroup, which is the stabilizer of a fixed choice of the fundamental chamber. We shall identify the Coxeter complex $\mathscr{A}$ of the Coxeter system (W, S) as the fundamental apartment so that the fundamental chamber $\mathscr{C}$ in $\mathscr{A}$ is represented by the Iwahori subgroup K.

Now for $g \in G$ and the strip $\mathscr{T}_i$ defined Sect. 3.4 (which is a subset of $\mathscr{A}$) whose stabilizer is generated by w_i, we call $g\mathscr{T}_i$ a straight strip of type w_i.

Fix a discrete torsion-free cocompact subgroup Γ of G, the quotient $\mathscr{B}_\Gamma$ is a finite complex with the fundamental group Γ. We will give a geometric interpretation of Theorem 3.7 as a zeta function of closed geodesic strips on $\mathscr{B}_\Gamma$.

4.2 Chambers in an apartment

In order to study straight strips in the building, we need a criterion to determine when a given set of chambers is contained in an apartment. Let $C_1 = g_1\mathscr{C}$ and $C_2 = g_2\mathscr{C}$ be two chambers in the building $\mathscr{B}$. Recall the Weyl distance of C_1 and C_2 is defined to be

$$\delta(C_1, C_2) = w \in W$$

where w satisfies $g_1^{-1}g_2 \in KwK$. Then, we have the following lemma.

Lemma 4.1 [1, Exercise 5.77] *Let $\mathscr{S}$ be a collection of chambers of $\mathscr{B}$. If for any three chambers C_1, C_2, and C_3 in $\mathscr{S}$, we have*

$$\delta(C_1, C_3) = \delta(C_1, C_2)\delta(C_2, C_3).$$

Then $\mathscr{S}$ is contained in an apartment in the maximal apartment system.

On the other hand, when the Iwahori subgroup is open and compact (which is always the case when F is a non-archimedean local field), the group G acts transitively on the maximal apartment systems (see [5, Sect. 17.7] for details). Together with the above lemma, we have the following useful theorem.

Theorem 4.2 *If $\gamma \in KwK$ and w is straight (i.e. $\ell(w)^k = k\ell(w)$ for all k), then $\{\gamma^k\mathscr{C} : k \in \mathbb{Z}\}$ is contained in $g\mathscr{A}$ for some $g \in G$.*

4.3 Stabilizer of straight strips

Let $\mathscr{T} := \mathscr{T}_i$ be the fundamental straight strip in the direction of $v = v_i$ defined in Sect. 3.4. Note that for a simplicial automorphism σ on $\mathscr{T}$, its action on the middle line $\mathscr{L}$ is either an affine reflection (which has a fixed point) or a translation by kv for some real number k. In the later case, when $k > 0$, $\sigma\big|_{\mathscr{L}}$ is called a positive translation (with respect to v). Note that $\mathrm{Aut}_1(\mathscr{T})$ is a cyclic generated by a minimal positive translation. Now let $\mathrm{Aut}(\mathscr{T})$ be the group of simplicial automorphisms of $\mathscr{T}$ and consider the following subgroup of $\mathrm{Aut}(\mathscr{T})$.

$$\mathrm{Aut}_1(\mathscr{T}) = \{\sigma \in \mathrm{Aut}(\mathscr{T}) : \sigma|_{\mathscr{L}} \text{ is a translation}\}$$

and

$$\mathrm{Aut}_2(\mathscr{T}) = \{\sigma \in \mathrm{Aut}(\mathscr{T}) : \sigma \text{ is type-preserving}\}.$$

Then, one can verify that $\mathrm{Aut}_1(\mathscr{T}) \cap \mathrm{Aut}_2(\mathscr{T})$ is a cyclic group generated by $w = w_i$, on a case-by-case basis from Figs. 1, 2, 3 and 4 in Sect. 2.

Since G is simply connected, the action of G on the building is type-preserving. Moreover, the assumption that Γ is discrete and torsion-free implies that no non-identity element can have a fixed point in the building. Thus, the setwise stabilizer of $\mathscr{T}$ in Γ acts faithfully on $\mathscr{T}$ and

$$\langle \gamma_0 \rangle = \mathrm{Stab}_\Gamma(\mathscr{T}) \cong \mathrm{Stab}_\Gamma(\mathscr{T})\big|_{\mathscr{T}} \subseteq \langle w \rangle$$

for some unique γ_0 with $\gamma_0\big|_{\mathscr{L}}$ being a positive translation. A similar result is true for any other straight strip in the building $\mathscr{B}$. Thus, immediately, we have the following lemma.

Proposition 4.3 *For $\gamma \in \Gamma$, if $\gamma(g\mathscr{T}) = g\mathscr{T}$, then $\gamma g\mathscr{C} = gw^k\mathscr{C}$ for some $k \in \mathbb{Z}$. Especially, it implies that $g^{-1}\gamma g \in Kw^kK$.*

Denote by $\mathrm{SConv}(\langle\gamma\rangle g\mathscr{C})$ the simplicial convex hull of $\{\gamma^n g\mathscr{C}\}_{n\in\mathbb{Z}}$, which is the smallest simplicial convex set containing this set.

Proposition 4.4 *For $\gamma \in \Gamma$ and $g \in G$, suppose that $g^{-1}\gamma g \in Kw^kK$ for some $k \in \mathbb{Z}$. Then $\mathrm{SConv}(\langle\gamma\rangle g\mathscr{C})$ is a straight strip.*

Proof Recall that $w = w_i$ is straight and so is w^k (see Sect. 3.4). By Theorem 4.2, the set $\{g^{-1}\gamma^k g\mathscr{C} : k \in \mathbb{Z}\}$ is contained in some apartment $g'\mathscr{A}$, and so are the set $\{\gamma^k g\mathscr{C} : k \in \mathbb{Z}\}$ and its simplicial convex hull $\mathrm{SConv}(\langle\gamma\rangle g\mathscr{C})$. Moreover, we may assume that $g'\mathscr{C} = g\mathscr{C}$ since the stabilizer of an apartment acts transitively on its chambers. Note that every chamber in the apartment $g'\mathscr{A}$ is uniquely determined by its Weyl distance to the chamber $g'\mathscr{C}$. On the other hand, we have

$$\delta\left(g'\mathscr{C}, g'w^{kn}\mathscr{C}\right) = w^{kn} = \delta\left(g\mathscr{C}, \gamma^n g\mathscr{C}\right)$$

for all $n \in \mathbb{Z}$. We conclude that $g\gamma^n\mathscr{C} = g'w^{kn}\mathscr{C}$ for all n and hence

$$\mathrm{SConv}\left(\langle\gamma\rangle g\mathscr{C}\right) = g'\mathrm{SConv}\left(\langle w^k\rangle\mathscr{C}\right) = g'\mathscr{T}.$$

Here, we use the fact that $\mathrm{SConv}\left(\langle w^k\rangle\mathscr{C}\right) = \mathscr{T}$ which can be checked case by case via Figs. 1, 2, 3 and 4. □

4.4 Pointed closed geodesic strips

Consider the following set

$$\begin{aligned}\mathscr{P} &= \left\{(\gamma, g\mathscr{C}, g\mathscr{T}) : g \in G, \gamma \in \Gamma, \gamma\left(g\mathscr{T}\right) = g\mathscr{T}, \gamma\ |_{g\mathscr{L}} \text{ is a positive translation}\right\}\\ &= \left\{(\gamma, g\mathscr{C}, g\mathscr{T}) : g \in G, \gamma \in \Gamma, \gamma\left(g\mathscr{T}\right) = g\mathscr{T}, g^{-1}\gamma g\ |_{\mathscr{L}} \text{ is a positive translation}\right\}\end{aligned}$$

endowed with an equivalence relation defined as follows: $(\gamma_1, g_1\mathscr{C}, g_1\mathscr{T}) \sim (\gamma_2, g_2\mathscr{C}, g_2\mathscr{T})$ if there exists some $\gamma \in \Gamma$, such that

$$\gamma g_1\mathscr{T} = g_2\mathscr{T}, \gamma g_1\mathscr{C} = g_2\mathscr{C}, \quad \gamma\gamma_1 g_1\mathscr{C} = \gamma_2 g_2\mathscr{C}.$$

The equivalence class of $(\gamma, g\mathscr{C}, g\mathscr{T})$ is given by

$$\left[(\gamma, g\mathscr{C}, g\mathscr{T})\right] = \left\{\left(\tilde{\gamma}\gamma\tilde{\gamma}^{-1}, \tilde{\gamma}g\mathscr{C}, \tilde{\gamma}g\mathscr{T}\right) : \quad \forall\tilde{\gamma} \in \Gamma\right\}.$$

We define the notion of a "pointed closed geodesic strip of type w" in $\mathscr{B}_\Gamma$ to be an equivalence class of $\mathscr{P}$. Note that when $\gamma(g\mathscr{T}) = g\mathscr{T}$, the projection of $g\mathscr{T}$ is a closed strip in $\mathscr{B}_\Gamma$, called a closed geodesic strip of $\mathscr{B}_\Gamma$. Therefore, each pointed closed geodesic strip can be regarded as a closed geodesic strip with a fixed choice chamber in $\mathscr{B}_\Gamma$. In other words, the pointed closed strips are analogue of the closed geodesics with fixed starting vertex/directed edge in graphs.

Combing Propositions 4.3 and 4.4, when $\gamma(g\mathscr{T}) = g\mathscr{T}$, $\mathrm{SConv}(\langle\gamma\rangle g\mathscr{C})$ is a also straight strip and it must be equal to $g\mathscr{T}$. Thus we have the following proposition.

Proposition 4.5 *For* $(\gamma, g\mathscr{C}, g\mathscr{T}) \in \mathscr{P}$*,* $(\gamma, g\mathscr{C}, g\mathscr{T}) = (\gamma, g\mathscr{C}, \mathrm{SConv}(\langle\gamma\rangle g\mathscr{C}))$.

Now let $g_1\mathscr{C}, \dots g_n\mathscr{C}$ be a complete list of liftings of chambers in $\mathscr{B}_\Gamma$.

Theorem 4.6 *The set* $\cup_{k=1}^{\infty}\mathscr{P}_k$ *forms a set of representatives of the equivalence classes in* $\mathscr{P}$*, where*

$$\mathscr{P}_k = \{(\gamma, g_i\mathscr{C}, \mathrm{SConv}(\langle\gamma\rangle g_i\mathscr{C}) : \gamma \in \Gamma, i \in \{1, 2, \dots n\}, g_i^{-1}\gamma g_i \in Kw^kK\}.$$

Proof For an element $p = (\gamma, g\mathscr{C}, g\mathscr{T})$ of $\mathscr{P}$, first we show that p is equivalent to some element in $\mathscr{P}_k$. Write $g\mathscr{C} = \delta g_i\mathscr{C}$ for some representative $g_i\mathscr{C}$ and $\delta \in \Gamma$. Then by Proposition 4.5,

$$p = (\gamma, g\mathscr{C}, \mathrm{SConv}(\langle\gamma\rangle g\mathscr{C})) = (\gamma, \delta g_i\mathscr{C}, \mathrm{SConv}(\langle\gamma\rangle\delta g_i\mathscr{C}))$$

which is equivalent to

$$p' = (\delta^{-1}\gamma\delta, g_i\mathscr{C}, \delta^{-1}\mathrm{SConv}(\langle\gamma\rangle\delta g_i\mathscr{C})) = (\gamma', g_i\mathscr{C}, \mathrm{SConv}(\langle\gamma'\rangle g_i\mathscr{C}))$$

where $\gamma' = \delta^{-1}\gamma\delta \in \Gamma$. Next, we show that p' is indeed an element of some $\mathscr{P}_k$. Since $g\mathscr{C} = \delta g_i\mathscr{C}$ and $\gamma(g\mathscr{T}) = g\mathscr{T}$, we have $ga = \delta g_i$ for some $a \in K$ and $g^{-1}\gamma g \in Kw^kK$ for some positive integer k. Therefore,

$$g_i^{-1}\gamma' g_i = g_i^{-1}\delta^{-1}\gamma\delta g_i = a^{-1}g^{-1}\gamma ga \in Kw^kI.$$

Thus, $p' \in \mathscr{P}_k$.

To complete the proof, it remains to show that all elements in $\cup_{k=1}^{\infty}\mathscr{P}_k$ are not equivalent. Suppose that $(\gamma_1, g_{i_1}\mathscr{C}, \mathrm{SConv}(\langle\gamma_1\rangle g_{i_1}\mathscr{C})$ are equivalent, $(\gamma_2, g_{i_2}\mathscr{C}, \mathrm{SConv}(\langle\gamma_2\rangle g_{i_2}\mathscr{C}))$. Then

$$\gamma g_{i_1}\mathscr{T} = g_{i_2}\mathscr{T}, \gamma g_{i_1}\mathscr{C} = g_{i_2}\mathscr{C}, \quad \gamma\gamma_1 g_{i_1}\mathscr{C} = \gamma_2 g_{i_2}\mathscr{C}$$

for some $\gamma \in \Gamma$. Since $\{g_i\mathscr{C}\}$ are representatives of Γ-orbits, we have $g = g_{i_1} = g_{i_2}$. Therefore $\gamma g\mathscr{C} = g\mathscr{C}$ which implies $\gamma \in \Gamma \cap gKg^{-1} = \{e\}$ (since the intersection is a discrete compact torsion-free subgroup). Thus, γ must be the identity element. Applying the third part of the above condition,

$$\gamma\gamma_1 g_{i_1}\mathscr{C} = \gamma_2 g_{i_2}\mathscr{C} \Rightarrow \gamma_1 g\mathscr{C} = \gamma_2 g\mathscr{C} \Rightarrow \gamma_1^{-1}\gamma_2 \in \Gamma \cap gKg^{-1} \Rightarrow \gamma_1 = \gamma_2.$$

We conclude that any two equivalent elements are always the same element. □

Now for a pointed closed geodesic strip p represented by some element in $\mathscr{P}_k$, we define its length $\ell(p)$ to be $k\ell(w)$ and its normalized length $\ell_0(p)$ to be k.

4.5 Counting pointed closed geodesic strips

Recall that one can identify the (affine) Hecke algebra $H_q(W, S)$ and the Iwahori–Hecke algebra $H(G, K)$ by mapping e_w to KwK. Let π_Γ be the natural representation of $H_q(W, S)$ acting on $L^2(\Gamma\backslash G/K)$ and let A_w be the matrix of $\pi_\Gamma(e_w)$ with respect to the basis of characteristic functions on $\Gamma g_1 K, \ldots, \Gamma g_n K$, denoted by $\{f_1, \ldots, f_n\}$. Write

$$Kw^kK = \bigsqcup_{\alpha\in\Omega_k} \alpha K,$$

Then

$$(A_w^k)_{i,j} = \#\{\alpha \in \Omega_k : \Gamma g_i K = \Gamma g_j \alpha K\}.$$

Theorem 4.7 *The cardinality of $\mathscr{P}_k$ is equal to the trace of $(A_w)^k$.*

Proof Note that when $\Gamma g_i K = \Gamma g_j \alpha K$, there exists some $\gamma \in \Gamma$, such that $\gamma g_i K = g_j \alpha K$ and such γ is unique since if there exists γ' satisfying the same condition, then again we have $\gamma'\gamma^{-1} \in \Gamma \cap g_i K g_i^{-1} = \{e\}$. Therefore, we can rewrite the above as

$$(A_w^k)_{i,j} = \#\{(\alpha, \gamma) \in \Omega_k \times \Gamma : \gamma g_i K = g_j \alpha K\}.$$

On the other hand, when $\gamma g_i K = g_j \alpha K$, we have $\gamma g_i K \subset g_j K w^k K$ and conversely, when $\gamma g_i K \subset g_j K w^k K$ there exists a unique $\alpha \in \Omega_k$ such that $\gamma g_i K = g_j \alpha K$. Therefore, we obtain

$$(A_w^k)_{i,j} = \#\{\gamma \in \Gamma : \gamma g_i K \subseteq g_j K w^k K\} = \#\{\gamma \in \Gamma : g_j^{-1}\gamma g_i \in K w^k K\}.$$

Consequently, we have

$$\mathrm{tr}((A_w)^k) = \#\{(g_i K, \gamma) : i = 1 \sim n, \gamma \in \Gamma, g_i^{-1}\gamma g_i \in K w^k K\} = \#\mathscr{P}_k.$$

□

4.6 Closed geodesic strips

We shall regard pointed closed geodesic strips as analogues of closed geodesics with a fixed starting vertex in finite graphs. In this subsection, we define the concept of closed geodesic strips, which are an analogue of closed geodesics without a fixed starting vertex in finite graphs. Set

$$\tilde{\mathscr{P}} = \{(\gamma, g\mathscr{T}) : g \in G, \gamma \in \Gamma, \gamma(g\mathscr{T}) = g\mathscr{T}, \gamma\ |_{g\mathscr{L}} \text{ is a positive translation}\}$$

and define $(\gamma_1, g_1\mathscr{T}) \sim (\gamma_2, g_2\mathscr{T})$ if there exists some chambers C_1 and C_2 such that $(\gamma_1, C_1, g_1\mathscr{T}) \sim (\gamma_2, C_2, g_2\mathscr{T})$. Equivalence classes of $\tilde{\mathscr{P}}$ are called closed straight strips in $\mathscr{B}_\Gamma$ (of type w).

Furthermore, a closed geodesic strip is called primitive if it is not a repetition of a shorter closed geodesic strip. It is clear that every closed geodesic strip is a repetition of a unique primitive closed geodesic strip.

Theorem 4.8 *The number of pointed closed geodesic strips mapped to a given closed geodesic strip is equal to the normalized length of its primitive closed geodesic strip.*

Proof Consider the canonical map σ from $\mathscr{P}$ to $\tilde{\mathscr{P}}$ given by $\sigma((\gamma, g\mathscr{C}, g\mathscr{T})) = (\gamma, g\mathscr{T})$. By Proposition 4.3, we have

$$\sigma^{-1}\left((\gamma, g\mathscr{T})\right) = \left\{\left(\gamma, gw^k\mathscr{C}, g\mathscr{T}\right) : k \in \mathbb{Z}\right\}.$$

Observe that the equivalence relation on $\tilde{\mathscr{P}}$ is induced form the equivalence relation on $\mathscr{P}$ and σ induces a map $\tilde{\sigma}$ from $\mathscr{P}/\sim$ to $\tilde{\mathscr{P}}/\sim$, which is the map from the set of closed geodesic strips to the set of pointed closed geodesic strips by dropping pointed chambers. The size of the preimage of the equivalence class of $[(\gamma, g\mathscr{T})]$ is given by

$$\#\tilde{\sigma}^{-1}([(\gamma, g\mathscr{T})]) = \#\left(\{(\gamma, gw^k\mathscr{C}, g\mathscr{T}) : k \in \mathbb{Z}\}/\sim\right).$$

To evaluate the left-hand side of the above equation, recall that the setwise stabilizer $\mathrm{Stab}_\Gamma(g\mathscr{T})$ is a cyclic group containing γ. Let γ_0 be the unique generator of $\mathrm{Stab}_\Gamma(g\mathscr{T})$ satisfying that $\gamma|_{g\mathscr{L}}$ is a positive translation. Then, $\gamma = (\gamma_0)^m$ for some positive integer m and $\gamma_0 g\mathscr{C} = gw^{k_0}\mathscr{C}$ for some positive integer k_0. Moreover, the closed strip $[(\gamma, g\mathscr{T})]$ is primitive. Now we have

$$\left\{\left(\gamma, gw^k\mathscr{C}, g\mathscr{T}\right) : k \in \mathbb{Z}\right\}/\sim = \left\{\left(\gamma, gw^k\mathscr{C}, g\mathscr{T}\right) : k \in \mathbb{Z}\right\}/\langle\gamma_0\rangle$$

which contains exactly k_0 elements. □

4.7 Zeta functions of closed geodesic strips

Set

$$Z_{w_i}(\mathscr{B}_\Gamma, u) = \prod_c (1-u^{\ell_0(c)})^{-1} = \prod_{[c]} (1+u^{\ell_0(c)}+u^{2\ell_0(c)}+\cdots) \in \mathbb{Z}[[u]]$$

where c runs through primitive closed geodesic strips of type w_i and $\ell_0(c)$ is its normalized length defined in Sect. 4.4. Note that there are only finite many primitive closed strips with a given length. Therefore, the infinite product in the above definition is well-defined.

Recall that if a closed strip c of type w_i has normalized length n, then its length $\ell(c) = n\ell(w_i)$. Define the zeta function of closed geodesic strips of $\mathscr{B}_\Gamma$ to be

$$Z(\mathscr{B}_\Gamma, u) = Z_{w_1}\left(\mathscr{B}_\Gamma, u^{\ell(w_i)}\right) Z_{w_2}\left(\mathscr{B}_\Gamma, u^{\ell(w_2)}\right).$$

4.8 The Proof of Theorem 1.1

Like the case of graph zeta functions, zeta functions of straight strips also have infinite sum expression.

Theorem 4.9 *As a function in the complex variable u, the zeta function $Z_{w_i}(\mathscr{B}_\Gamma, u)$ can be expressed as*

$$Z_{w_i}(\mathscr{B}_\Gamma, u) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n u^n}{n}\right) = \det\left(1 - A_{w_i} u\right)^{-1}$$

for $|u| < \|A_{w_i}\|^{-1}$. Here, $\|A_{w_i}\|$ is the operator norm of A_{w_i} and N_n is the number of pointed closed geodesic strips of type w_i of normalized length n.

Proof Since $\operatorname{tr}((A_{w_i})^n) = N_n$ and the Taylor series of $\log(1-u)$ converges when $|u| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{N_n u^n}{n} = \sum_{n=1}^{\infty} \frac{\operatorname{tr}\left(\left(A_{w_i}\right)^n\right) u^n}{n} = \log\left(\det\left(1 - A_{w_i} u\right)^{-1}\right) \quad \text{when } |u| < \|A_{w_i}\|^{-1}.$$

Let $\mathscr{P}_0$ be the set of primitive pointed closed geodesic strips of type w_i and $\tilde{\mathscr{P}}_0$ be the set of primitive closed geodesic strips of type w_i. By Theorem 4.8, when $|u| < \|A_{w_i}\|^{-1}$ we have

$$\sum_{n=1}^{\infty} \frac{N_n u^n}{n} = \sum_{c \in \mathscr{P}} \frac{u^{\ell(c)}}{\ell(c)} = \sum_{c \in \mathscr{P}_0} \sum_{m=1}^{\infty} \frac{u^{m\ell(c)}}{m\ell(c)} = \sum_{c \in \tilde{\mathscr{P}}_0} \sum_{m=1}^{\infty} \frac{u^{m\ell(c)}}{m}$$
$$= \sum_{c \in \tilde{\mathscr{P}}_0} \log \left(1 - u^{\ell(c)}\right)^{-1}$$
$$= \log \prod_{c \in \tilde{\mathscr{P}}_0} \left(1 - u^{\ell(c)}\right)^{-1}.$$

Here, we use the fact that $\sum_{n=1}^{\infty} \frac{N_n u^n}{n}$ converges absolutely when $|u| < \|A_{w_i}\|^{-1}$ so that we can change the order of summation. □

Note that $\det(1 - A_{w_i} u)$ is a polynomial and A_{w_i} is the matrix of $\pi_\Gamma(e_{w_i})$. Therefore, as a power series in u, we also have

$$Z(\mathscr{B}_\Gamma, u) = \det \left(1 - \pi_\Gamma\left(e_{w_1}\right) u^{\ell(w_1)}\right)^{-1} \det \left(1 - \pi_\Gamma\left(e_{w_2}\right) u^{\ell(w_2)}\right)^{-1}$$

which proves the first part of Theorem 1.1. On the other hand, by the definition of $H_i(\rho, u)$ in Sect. 3, we have

$$H_i(\pi_\Gamma, u) = (I - \pi_\Gamma(e_{w_i}) u^{\ell(w_i)})^{-1}.$$

Together with the above equation and Theorem 3.7, we have

$$Z(\mathscr{B}_\Gamma, u) = \det H_1(\pi_\Gamma, u) \det H_2(\pi_\Gamma, u) = \det \mathrm{Alt}(W)(\pi_\Gamma, u)$$

which proves the second part Theorem 1.1.

5 Alternating products of Poincaré series

Note that one can generalize the definition of straight strips to "straight tubes" in the higher rank cases in the following manner. Once again one identifies the apartment with a real vector space and the origin with a special vertex v , and then for each vertex other than v with a corresponding vector v_i one may consider the connected components of the complement of the union of all the hyperplanes which are invariant under translation by the vector v_i. In this way, one obtains the straight tubes in the direction v_i for each i, and one may study their stabilizers. We expect that there exists an analogue of Theorems 1.1 and 3.7 for higher ranks cases. More precisely, let (W, S) be an irreducible affine Coxeter group of rank $|S| = n + 1$, and set

$$\mathrm{Alt}(W)(u) = \prod_{I \subset S} W_I(u)^{(-1)^{|I|+|S|}}.$$

Then, we have the following conjecture.

Conjecture 5.1 *There exist straight elements $w_1, \ldots, w_n \in W$ such that for $H_i = \{w_i^k\}_{k \in \mathbb{Z}_{\geq 0}}$, the following two identities holds.*

1. *w_i is a generator of the stabilizer of some straight tube.*
2. $\mathrm{Alt}(W)(u) = H_1(u) \cdots H_n(u)$.
3. *For any finite dimensional representation ρ of $H_q(W, S)$,*

$$\prod_{i=1}^{n} \det H_i(\rho, u) = \prod_{I \subset S} \det W_I(\rho, u)^{(-1)^{|I|+|S|}}.$$

Note that the above conjecture implies that

$$\mathrm{Alt}(W)(u)^{-1} = \left(1 - u^{d_1}\right) \cdots \left(1 - u^{d_n}\right) \tag{5.1}$$

where $d_i = \ell(w_i)$ are positive integers.

In the rest of the paper, we examine Eq. (5.1) for all affine Coxeter groups.

Let R be an irreducible reduced crystallographic root system of rank n, and let W_0 be the Weyl group with generating set S_0, and let W be the affine Weyl group with generating set S. Let h be the Coxeter number of W_0. Denote by $[a, b]^k$ the multiset consisting integers between a and b with multiplicity k and denote it by $[a]^k$ when $a = b$. For example,

$$[3]^2 = \{3, 3\} \quad \text{and} \quad [2, 4]^3 = \{2, 2, 2, 3, 3, 3, 4, 4, 4\}.$$

The main result of this section is the following theorem which is the same theorem as Theorem 1.2 with the extra table of d_i.

Theorem 5.2 *For the affine Weyl group W, its power series $\mathrm{Alt}(W)(u)^{-1}$ is indeed a polynomial of the form*

$$\left(1 - u^{d_1}\right) \cdots \left(1 - u^{d_n}\right)$$

where d_i are integers with $n + 1 = d_1 \leq d_2 \leq \cdots \leq d_n \leq h$ as shown as in the following table.

Type	Coxeter number h	$\{d_1, \ldots, d_n\}$
A_n	$n + 1$	$[h]^n$
B_n	$2n$	$[n + 1, h]$
C_n	$2n$	$[n + 1, h]$
D_n	$2n - 2$	$[n + 1, h] \sqcup [h]^2$
E_6	12	$\{7, 9, 9, 11, 12, 12\}$
E_7	18	$\{8, 10, 11, 13, 14, 17, 18\}$
E_8	30	$\{9, 11, 13, 14, 17, 19, 23, 29\}$
F_4	12	$\{5, 7, 8, 11\}$
G_2	6	$\{3, 5\}$

Remark Note that d_n is always equal to h or $h-1$, and further the number of times h occurs in the set of all d_n is equal to the connection index minus one. Moreover, $d_n = h-1$ only when W is of type E_7, F_4 or G_2. These three types of groups are the only simple algebraic groups whose simply connected and adjoint forms are the same. Besides, the multiplicity of h in the above table is equal to the number of conjugacy classes of Coxeter elements.

5.1 MacDonald's formula

We shall prove the above theorem using MacDonald's formula [14] for Poincaré series of affine Weyl groups in terms of positive roots. Let us recall his result. Let R be a reducible root system in a finite dimensional real vector space V. Fix a set of simple roots $B = \{\alpha_1, \ldots, \alpha_n\}$ and let R^+ be the set of positive roots with respect to B. Let S_0 be the set of reflections corresponding to B and W_0 be the group generated by S_0, which is the Weyl group of R. In this case, (W_0, S_0) forms a finite Coxeter system. For a root $\alpha = \sum c_i\alpha_i$, its height is defined as

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{n} c_i.$$

Theorem 5.3 (MacDonald) *The following identities hold.*

$$W_0(u) = \prod_{a\in R^+} \frac{1-u^{\mathrm{ht}(a)+1}}{1-u^{\mathrm{ht}(a)}}$$

Next, let us consider the affine root system $\tilde{R} = R \times \mathbb{Z}$, whose elements $\lambda = (\alpha, k)$ are regarded as affine functions on V

$$\lambda(x) = (\alpha, x) + k.$$

For convenience, we also denote λ by $k+\alpha$ and regard R as a subset of $\tilde{R}$. Let $\tilde{B} = B \cup \{\alpha_0\}$, where $\alpha_0 = 1-\rho$ and ρ is the highest root. Let $\tilde{R}^+$ be the set of positive affine roots, which elements are non-negative integer linear combinations of $\tilde{B}$. Let S be the set of affine reflections corresponding to $\tilde{B}$ and W be the group generated by S, which is the affine Weyl group of R. Then, (W, S) is a Coxeter system associated with $\tilde{R}$. For $k+\alpha = \sum_{i=0}^{n} c_i\alpha_i \in \tilde{R}$, define its weight to be

$$\mathrm{ht}(k+\alpha) = \sum_{i=0}^{n} c_i.$$

Then, we also have

$$\mathrm{ht}(k+\alpha) = k\mathrm{ht}(1) + \mathrm{ht}(\alpha) = kh + \mathrm{ht}(\alpha).$$

Here, h is the Coxeter number which is equal to $\mathrm{ht}(1) = 1 + \mathrm{ht}(\rho)$.

Consider the set

$$P = \{\lambda \in \tilde{R} : 0 < \mathrm{ht}(\lambda) < h\} = R^+ \cup \{1 - \alpha : \alpha \in R^+\}.$$

Theorem 5.4 (MacDonald) *The following identities hold.*

$$W(u) = \frac{1}{\left(1-u^h\right)^n} \prod_{a \in P} \frac{1-u^{\mathrm{ht}(a)+1}}{1-u^{\mathrm{ht}(a)}}.$$

5.2 A Proof of Theorem 5.2

For a subset I of S, consider the Möbius function $\mu(I) = (-1)^{|I|}$ which satisfies the property

$$\sum_{I \subset J \subset S} \mu(J) = \begin{cases} \mu(S), & \text{if } \quad I = S; \\ 0, & \text{otherwise.} \end{cases}$$

Then, we can write the alternating product as:

$$\mathrm{Alt}(W)(u) = \prod_{I \subset S} W_I(u)^{(-1)^{|I|+|S|}} = \prod_{I \subset S} W_I(u)^{\mu(I)\mu(S)}.$$

Now for a subset I of S, let $\bar{B}_I$ be the subset of $\tilde{B}$ consisting positive simple affine roots corresponding to affine reflections in I. Let $\bar{R}_I$ be the subroot system of $\tilde{R}$ with the set of simple roots $\bar{B}_I$. Then, we have $\bar{R}_I^+ \subset \bar{R}_J^+$ if $I \subset J$. Note that every proper subroot system $\bar{R}_I$ is a (finite) reducible root system so that $W_I(u)$ satisfies Theorem 5.3.

Applying Theorems 5.3 and 5.4 to the above equation, we have

$$\begin{aligned} \mathrm{Alt}(W)(u) &= W(u) \prod_{I \subsetneq S} W_I(u)^{\mu(I)\mu(S)} \\ &= \frac{1}{(1-u^h)^n} \prod_{\lambda \in P} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right) \cdot \prod_{I \subsetneq S} \prod_{\lambda \in \bar{R}_I^+} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right)^{\mu(I)\mu(S)}. \end{aligned}$$

To simplify the above equation, for a positive affine root $\lambda = \sum_{i=0}^n c_i \alpha_i$ with $c_i \geq 0$, define its support to be

$$\mathrm{Supp}(\lambda) = \left\{ \alpha_i \in \tilde{B} : c_i \neq 0 \right\}.$$

Then, λ is a positive root in $\bar{R}_J$ when $\mathrm{Supp}(\lambda) \subset \bar{B}_J$. Thus

$$\prod_{I \subsetneq S} \prod_{\lambda \in \bar{R}_I^+} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right)^{\mu(I)\mu(S)} = \prod_{\lambda \in \bar{R}^+} \prod_{I:\mathrm{Supp}(\lambda) \subset \bar{B}_I \subsetneq \tilde{B}} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right)^{\mu(I)\mu(S)}.$$

On the other hand, it is easy to see that every positive root in the proper subroot system with height bounded by h, which implies such root is contained in P. Therefore, we can write the alternating product as:

$$\begin{aligned} \mathrm{Alt}(\tilde{W})(u) &= \frac{1}{(1-u^h)^n} \prod_{\lambda \in P} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right)^{\mu(S)^2} \\ &\quad \cdot \prod_{\lambda \in P} \prod_{\mathrm{Supp}(\lambda) \subset \bar{B}_I \subsetneq \tilde{B}} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right)^{\mu(I)\mu(S)} \\ &= \frac{1}{(1-u^h)^n} \prod_{\lambda \in P} \prod_{\mathrm{Supp}(\lambda) \subset \bar{B}_I \subset \tilde{B}} \left(\frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}} \right)^{\mu(I)\mu(S)} \\ &= \frac{1}{(1-u^h)^n} \prod_{\lambda \in P, \mathrm{Supp}(\lambda) = \bar{B}} \frac{1-u^{\mathrm{ht}(\lambda)+1}}{1-u^{\mathrm{ht}(\lambda)}}. \end{aligned}$$

Since $P = \{\lambda \in \tilde{R} : 0 < \mathrm{ht}(\lambda) < h\} = R^+ \cup \{1-\alpha : \alpha \in R^+\}$ and none of element in R^+ satisfies $\mathrm{Supp}(\alpha) = \bar{B}$, we can write the above result as

$$\mathrm{Alt}(W)(u) = \frac{1}{(1-u^h)^n} \prod_{\alpha \in R^+, \mathrm{Supp}(1-\alpha) = \bar{B}} \frac{1-u^{\mathrm{ht}(1-a)+1}}{1-u^{\mathrm{ht}(1-\alpha)}}.$$

Observe that $1-\alpha = (1-\rho) + (\rho - \alpha)$. Therefore, $\mathrm{Supp}(1-\alpha) = \bar{B}$ if and only if $\mathrm{Supp}(\rho - \alpha) = B$. On the other hand, one can find the complete list positive roots R^+ and the highest root ρ in [2]. Therefore, we can compute those elements with $\mathrm{Supp}(\lambda - \alpha) = B$ case-by-case.

For example, suppose R is of type B_n. We have

$$R^+ = \left\{ \sum_{i \le k \le j} \alpha_k : 1 \le i \le j \le n \right\} \bigcup \left\{ \sum_{i \le k < j} \alpha_k + 2 \sum_{j \le k \le n} \alpha_k : 1 \le i < j \le n \right\}$$

and

$$\rho = a_1 + 2a_2 + 2a_3 + \cdots + 2a_n.$$

Therefore, $I(1-\alpha)=B$ if and only if α is of the form

$$\sum_{i\le k\le j}\alpha_k,\ 1<i\le j\le n.$$

The set of weight of those α is

$$[1]^{n-1}\sqcup[2]^{n-2}\sqcup\cdots\sqcup[n-2]^2\sqcup[n-1].$$

Note that the Coxeter number $h=1+\mathrm{ht}(\rho)=2n$ and the set of weight of $1-\alpha$ is

$$[h-1]^{n-1}\sqcup[h-2]^{n-2}\sqcup\cdots\sqcup[n+2]^2\sqcup[n+1].$$

We can also rewrite the above set of weight of as

$$[n+1,h-1]\sqcup[n+2,h-1]\sqcup\cdots\sqcup[h-2,h-1]\sqcup[h-1].$$

Now we have

$$\begin{aligned}\mathrm{Alt}(W)(u)&=\frac{1}{(1-u^h)^n}\prod_{\alpha\in R^+,\mathrm{Supp}(\rho-\alpha)=B}\frac{1-u^{\mathrm{ht}(1-a)+1}}{1-u^{\mathrm{ht}(1-a)}}\\&=\frac{1}{(1-u^{2n})^n}\prod_{i=n+1}^{2n-1}\prod_{j=i}^{2n-1}\frac{1-u^{j+1}}{1-u^j}\\&=\frac{1}{(1-u^{2n})^n}\prod_{i=n+1}^{2n-1}\frac{1-u^{2n}}{1-u^i}\\&=\prod_{i=n+1}^{2n}\frac{1}{1-u^i}.\end{aligned}$$

which proves Theorem 5.2 for type B_n. For the rest of cases, the computations are similar so we only record the heights of those positive root α with $\mathrm{Supp}(\rho-\alpha)=B$ in the following table.

Type	Coxeter no. h	Heights of $1-\alpha$ with $\mathrm{Supp}(\rho-\alpha)=B$
A_n	$n+1$	Empty set
B_n	$2n$	$\bigsqcup_i[i,h-1],\ i\in[n+1,h-1]$
C_n	$2n$	$\bigsqcup_i[i,h-1],\ i\in[n+1,h-1]$
D_n	$2n-2$	$\bigsqcup_i[i,h-1],\ i\in[n+1,h-1]$
E_6	12	$\bigsqcup_i[i,h-1],\ i=7,9,9,11$
E_7	18	$\bigsqcup_i[i,h-1],\ i=8,10,11,13,14,17$
E_8	30	$\bigsqcup_i[i,h-1],\ i=9,11,13,14,17,19,23,29$
F_4	12	$\bigsqcup_i[i,h-1],\ i=5,7,8,11$
G_2	6	$\bigsqcup_i[i,h-1],\ i=3,5$

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