J Algebr Comb (2019) 49:179–191
https://doi.org/10.1007/s10801-018-0825-3

On 2-distance-transitive circulants

Jiyong Chen[1] · **Wei Jin**[2,3] · **Cai Heng Li**[4]

Received: 17 March 2017 / Accepted: 9 April 2018 / Published online: 22 May 2018

Abstract A complete classification is given of 2-distance-transitive circulants, which shows that a 2-distance-transitive circulant is a cycle, a Paley graph of prime order, a regular complete multipartite graph, or a regular complete bipartite graph of order twice an odd integer minus a 1-factor.

Keywords 2-Distance-transitive · Circulant · Cayley graph

Supported by NSF of China (11661039,11231008,11771200,11561027) and NSF of Jiangxi (20171BAB201010, 20171BCB23046, GJJ170321).

✉ Wei Jin
jinwei@jxufe.edu.cn

Jiyong Chen
cjy1988@pku.edu.cn

Cai Heng Li
lich@sustc.edu.cn

1 School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

2 School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi 330013, People's Republic of China

3 Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi 330013, People's Republic of China

4 Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, People's Republic of China

1 Introduction

In this paper, all graphs are finite, simple, and undirected. An ordered pair of adjacent vertices is called an *arc*. A graph Γ is called *arc-transitive* if all arcs are equivalent under automorphisms of the graph. For a graph Γ and two vertices u and v, the *distance* between u and v in Γ is denoted by $d(u, v)$, which is the smallest length of paths between u and v. The *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximum distance occurring over all pairs of vertices. An arc-transitive graph Γ is said to be 2*-distance-transitive* if Γ is not complete, and any two vertex pairs of vertices (u_1, v_1) and (u_2, v_2) with $d(u_1, v_1) = d(u_2, v_2) = 2$ are equivalent under automorphisms.

A 2*-arc* is a triple of distinct vertices (u, v, w) such that v is adjacent to both u and w. A regular graph is called 2*-arc-transitive* if all 2-arcs are equivalent under automorphisms. A 2-arc-transitive graph is obviously 2-distance-transitive.

The concept of 2-distance-transitive graph generalizes the concepts of distance-transitive graph and 2-arc-transitive graph. Both distance-transitive graphs and 2-arc-transitive graphs have been extensively studied, see [3,15]. The investigation of 2-distance-transitive graphs was initiated recently, see [4–6].

A vertex-transitive graph with n vertices is called a *circulant* if it has an automorphism of order n which acts freely on the set of vertices. Alspach et al. [1] classified 2-arc-transitive circulants; Miklavič and Potočnik [13] classified distance-regular circulants; Kovács [10] and Li [11] gave a characterization of arc-transitive circulants, see Theorem 2.1. The purpose of this paper is to give a complete classification of 2-distance-transitive circulants, stated in the following main theorem.

Theorem 1.1 *The class of* 2*-distance-transitive graphs consists of cycles, Paley graphs of prime order, regular complete multipartite graphs, and regular complete bipartite graphs of order twice an odd integer minus a* 1*-factor.*

By definition, a 2-distance-transitive circulant is an arc-transitive circulant. Thus, to prove Theorem 1.1, we only need to determine which of the arc-transitive circulants described in [10,11] are 2-distance-transitive. However, this is unexpectedly nontrivial (see Lemmas 2.3–2.10), which motivates some interesting problems that we explain below.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ of G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. It is known that a graph Γ is a Cayley graph of G if and only if Γ has an automorphism group which is isomorphic to G and regular on the vertex set, see [2, Lemma 16.3] and [17]. For a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, if G is a normal subgroup of $\mathrm{Aut}\Gamma$, then Γ is called a *normal Cayley graph*. The study of normal Cayley graphs was initiated by Xu [18] and has been done under various additional conditions, see [7,16]. A circulant is thus a Cayley graph of a cyclic group, and if further it is a normal Cayley graph of a cyclic group, then it is called a *normal circulant*. Many interesting examples of arc-transitive graphs and 2-arc-transitive graphs are constructed as normal Cayley graphs, due to the fact that the arc-transitivity and 2-arc-transitivity of a graph are equivalent to the local-transitivity and local-2-transitivity, respectively. The status for 2-distance-transitive graphs is, however, different. To our best knowledge, the known examples of 2-distance-transitive graphs are either distance-transitive or

2-arc-transitive. For instance, 2-distance-transitive normal circulants are cycles and Paley graphs by Theorem 1.1, which are distance-transitive. The following is a curious question.

Question 1.2 *Is there a normal Cayley graph which is* 2*-distance-transitive, but neither distance-transitive nor* 2*-arc-transitive?*

Another interesting problem is to characterize 2-distance-transitive Cayley graphs of certain special classes of groups, such as abelian groups and metacyclic groups.

2 Proof

We first introduce the classification of arc-transitive circulants given in [10,11], which we need to prove Theorem 1.1.

Let $\Gamma = (V, E)$ be a connected graph with vertex set V and edge set E. Its *complement graph* $\overline{\Gamma}$ is the graph with vertex V such that two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. Then $\Gamma = \Gamma_1[\Gamma_2]$ denotes the *lexicographic product* of Γ_1 and Γ_2, where the vertex set of Γ is $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in Γ if either u_1 and v_1 are adjacent in Γ_1, or $u_1 = v_1$ and u_2, v_2 are adjacent in Γ_2.

For a positive integer b, let K_b be the complete graph with b vertices. For a graph Γ, the graph consisting of b vertex disjoint copies of Γ is denoted by $b\Gamma$, and $\Gamma[\overline{\mathrm{K}_b}]-b\Gamma$ is the graph whose vertex set is the same as $\Gamma[\overline{\mathrm{K}_b}]$ and edge set equals the edge set of $\Gamma[\overline{\mathrm{K}_b}]$ minus the edge set of $b\Gamma$.

Theorem 2.1 [11, Theorem 1.3] *Let Γ be a connected arc-transitive circulant of order n which is not a complete graph. Then either*

(1) *Γ is a normal circulant, or*
(2) *there exists an arc-transitive circulant Σ of order m such that $mb = n$ with $b, m > 1$ and*

$$\Gamma = \begin{cases} \Sigma\left[\overline{\mathrm{K}_b}\right], & \textit{or} \\ \Sigma\left[\overline{\mathrm{K}_b}\right] - b\Sigma, & \textit{where } (m, b) = 1. \end{cases}$$

Our proof of Theorem 1.1 is to analyze which graphs satisfying Theorem 2.1 are 2-distance-transitive. We now describe all examples of 2-distance-transitive circulants, which consist of cycles and the following three families:

Example 2.2 (1) Let $\Gamma = \mathrm{K}_{m[b]}$ be a complete multipartite graph which has m parts of size b. Clearly, $\mathrm{K}_{m[b]} = \mathrm{K}_m[\overline{\mathrm{K}_b}]$. Then $\mathrm{Aut}\Gamma = \mathrm{S}_b \wr \mathrm{S}_m$ is 2-distance-transitive on Γ and has a cyclic subgroup which is regular on the vertex set. Thus Γ is a 2-distance-transitive circulant.

(2) Let $\Gamma = \mathrm{K}_{b,b} - b\mathrm{K}_2$ where b is an odd integer, namely, a complete bipartite graph minus a 1-factor. Then Γ is of valency $b - 1$ and of diameter 3, and $\mathrm{Aut}\Gamma = \mathrm{S}_b \times \mathrm{S}_2$ is distance-transitive and 2-arc-transitive. It follows that Γ is a 2-distance-transitive circulant.

(3) Let $q = p^e$ be a prime power such that $q \equiv 1 \pmod 4$. Let $\mathbb{F}_q$ be the finite field of order q. Then the *Paley graph* $\mathsf{P}(q)$ is the graph with vertex set $\mathbb{F}_q$, and two distinct vertices u, v are adjacent if and only if $u - v$ is a nonzero square in $\mathbb{F}_q$. The congruence condition on q implies that -1 is a square in $\mathbb{F}_q$, and hence $\mathsf{P}(q)$ is an undirected graph. This family of graphs was first defined by Paley in 1933, see [14]. Note that the field $\mathbb{F}_q$ has $(q-1)/2$ elements which are nonzero squares, so $\mathsf{P}(q)$ has valency $(q-1)/2$. Moreover, $\mathsf{P}(q)$ is a Cayley graph for the additive group $G = \mathbb{F}_q^+ \cong \mathbb{Z}_p^e$. Let w be a primitive element of $\mathbb{F}_q$. Then $S = \{w^2, w^4, \ldots, w^{q-1} = 1\}$ is the set of nonzero squares of $\mathbb{F}_q$, and $\mathsf{P}(q) = \mathrm{Cay}(G, S)$. If $q = p$ is a prime, then $\mathsf{P}(p)$ is a circulant of prime order. By [12], $\mathrm{Aut}\mathsf{P}(p) = \mathbb{F}_p^+ : \langle w^2 \rangle$. This implies that $\mathsf{P}(p)$ is a 2-distance-transitive normal circulant.

Proof of Theorem 1.1 By Example 2.2, cycles, regular complete multipartite graphs, regular complete bipartite graphs of order twice an odd integer minus a 1-factor, and Paley graphs of prime order are all distance-transitive and so 2-distance-transitive.

Conversely, let $\Gamma = \mathrm{Cay}(G, S)$ be a connected 2-distance-transitive circulant, and $G \cong \mathbb{Z}_n$. Then Γ is arc-transitive, and so Γ satisfies Theorem 2.1. If Γ is of valency 2, then $\Gamma \cong \mathrm{C}_n$. Assume that Γ has valency at least 3. Since Γ is arc-transitive, either Γ is a normal circulant, or Γ satisfies part (2) of Theorem 2.1. We shall treat these two cases in two subsections, respectively. In Sect. 2.1, we prove that if Γ is a normal circulant, then Γ is a Paley graph of prime order, stated in Lemma 2.8. In Sect. 2.2, we deal with nonnormal circulants and show that, if Γ is not a normal circulant, then Γ is a regular complete multipartite graph, or a regular complete bipartite graph of order twice an odd integer minus a 1-factor. □

We now introduce a few notations which we will use later. For a graph Γ and a vertex v, we denote by $\Gamma_i(v)$ the i-th neighborhood of v in Γ, that is, the set of vertices which are at distance i from v. A sequence of vertices $v_0, v_1, \ldots, v_s$ is called an *s-geodesic* if $\{v_i, v_{i+1}\}$ is an edge for all $0 \leqslant i \leqslant s-1$ and $d(v_0, v_s) = s$. We sometimes need to consider distances of the same pair of vertices in different graphs, so let $d_\Gamma(u, v)$ denote the distance of u and v in the graph Γ.

2.1 2-Distance-transitive normal circulants

Consider the cyclic group

$$G = \langle g \rangle \cong \mathbb{Z}_n.$$

Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected 2-distance-transitive circulant of valency $k \geqslant 3$. Then $G = \langle S \rangle$. Assume further that Γ is a normal circulant. Let $A = \mathrm{Aut}\Gamma$, and let u be the vertex of Γ corresponding to the identity of the group G. Then $\Gamma(u) = S$ and $A = G{:}A_u$. By [9, Lemma 2.1], we have

$$A_u = \mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}.$$

Moreover, as $\mathrm{Aut}(G)$ is abelian and $A_u = \mathrm{Aut}(G, S)$ is transitive and faithful on S, it implies that A_u is regular on S, and

$$|A_u| = |S| = k.$$

We establish a series of lemmas to prove that Γ is a Paley graph.

Lemma 2.3 *Γ has girth* 3.

Proof Suppose that the girth of Γ is greater than 4. Then, for each pair of vertices θ, θ' with distance $d(\theta, \theta') = 2$, there is a unique 2-arc between θ and θ'. Hence Γ being 2-distance-transitive implies that it is 2-arc-transitive. By the classification given in [1, Theorem 1.1], the graph Γ has girth 4, which is a contradiction.

Assume that Γ has girth 4. Then, for any vertex $v \in \Gamma(u) = S$, we have $|\Gamma_2(u) \cap \Gamma(v)| = k - 1$. Thus there are $k(k-1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|A_u| = k$ and A is 2-distance-transitive on Γ, we conclude that A_u acts transitively on $\Gamma_2(u)$, and the size $|\Gamma_2(u)|$ is a divisor of $|A_u| = k$. Let $w \in \Gamma_2(u) \cap \Gamma(v)$. Then $|S \cap \Gamma(w)| \leqslant |\Gamma(w)| = k$. Note that $|S \cap \Gamma(w)| \cdot |\Gamma_2(u)| = k(k-1)$. Hence $|S \cap \Gamma(w)| = k-1$ and $k = |\Gamma_2(u)| = |S|$. It follows that $|\Gamma_3(u) \cap \Gamma(w)| + |\Gamma_2(u) \cap \Gamma(w)| = 1$. Set $\Gamma(u) = \{v = v_1, \ldots, v_k\}$ and $\Gamma_2(u) = \{w = w_1, \ldots, w_k\}$. Let $\Gamma_2(u) \cap \Gamma(v) = \{w_1, \ldots, w_{k-1}\}$ and $\Gamma(u) \cap \Gamma(w) = \{v_1, \ldots, v_{k-1}\}$.

Assume that $|\Gamma_3(u) \cap \Gamma(w)| = 0$. Then $|\Gamma_2(u) \cap \Gamma(w)| = 1$. Since Γ is 2-distance-transitive, the stabilizer A_u is transitive on $\Gamma_2(u)$, so the induced subgraph $[\Gamma_2(u)] \cong (k/2)\mathrm{K}_2$. As Γ has girth 4, it follows that $[\Gamma_2(u) \cap \Gamma(v)]$ is an empty graph, and so w_1 is adjacent to w_k. Thus $[\{w_2, \ldots, w_{k-1}\}] \cong (k/2 - 1)\mathrm{K}_2$. However, $\{w_2, \ldots, w_{k-1}\} \subset \Gamma_2(u) \cap \Gamma(v)$, contradicting the fact that $[\Gamma_2(u) \cap \Gamma(v)]$ is an empty graph. Therefore, $|\Gamma_3(u) \cap \Gamma(w)| = 1$, say $\Gamma_3(u) \cap \Gamma(w) = \{z\}$.

By the transitivity of A_u on the set $\Gamma(u)$, we have $|\Gamma_2(u) \cap \Gamma(v_k)| = |\Gamma_2(u) \cap \Gamma(v_1)| = k-1$. Then $\Gamma_2(u) \cap \Gamma(v_k) = \{w_2, \ldots, w_k\}$ since v_k and w_1 are not adjacent in Γ. Note that $(u, w), (v, z), (v_k, z)$ are all vertex pairs of distance 2 and Γ is 2-distance-transitive. We have $|\Gamma(u) \cap \Gamma(w)|$, $|\Gamma(v) \cap \Gamma(z)|$ and $|\Gamma(v_k) \cap \Gamma(z)|$ are all equal to $k-1$. Hence $\Gamma(v) \cap \Gamma(z) = \Gamma(v) \cap \Gamma_2(u)$ and $\Gamma(v_k) \cap \Gamma(z) = \Gamma(v_k) \cap \Gamma_2(u)$. Thus $\Gamma(z) = \Gamma_2(u)$ and $\Gamma_3(u) = \{z\}$, so Γ has diameter 3 and is distance-transitive. Therefore, $\{u\} \cup \Gamma_3(u)$ is a block of imprimitivity of $\mathrm{Aut}\Gamma$ on the vertex set V, and $\Gamma \cong \mathrm{K}_{k+1,k+1} - (k+1)\mathrm{K}_2$. It is clear that $\mathrm{K}_{k+1,k+1} - (k+1)\mathrm{K}_2$ is not a normal circulant, which is a contradiction. Hence the girth of Γ is 3. □

Lemma 2.4 *The order $|G| = n$ is odd.*

Proof Suppose that $|G| = n$ is an even integer. Since $G = \langle S \rangle$ and all elements of S have the same order, it follows that S consists of generators of G. Without loss of generality, we assume that $g \in S$. By Lemma 2.3, Γ has girth 3. Then there is $g^i \in S$ such that (g, g^i) is an arc of Γ. This implies that $g^{i-1} = g^i g^{-1}$ belongs to set S. Since Γ is a normal circulant, each element in S is a generator of G. This means that $1, i-1, i$ are all relatively prime to n. This contradicts the fact that n is an even number. □

Lemma 2.5 *The second neighborhood $\Gamma_2(u)$ consists of generators of G.*

Proof Suppose that $\Gamma_2(u)$ contains an element which is not a generator of G. Then $|G|$ is not a prime. Since $A_u = \mathrm{Aut}(G, S)$ is transitive on $\Gamma_2(u)$, none of the elements in $\Gamma_2(u)$ is a generator.

As $\langle S\rangle = G$ and all elements of S are conjugate in $\mathrm{Aut}(G)$, we may assume that $g \in S$. By Lemma 2.4, the order $|G|$ is an odd integer. Thus g^2 is a generator, and $g^2 \notin \Gamma_2(u)$. Since $g^2 \in \Gamma(u) \cup \Gamma_2(u)$, $g^2 \in S = \Gamma(u)$. Assume that $g^r \in S$ where $r \leqslant n-3$. Then $g^{r+1} = g^r g \in \Gamma(u) \cup \Gamma_2(u)$. Thus either g^{r+1} is a generator of G and $g^{r+1} \in S$, or g^{r+1} is not a generator and $g^{r+1} \in \Gamma_2(u)$. Similarly, we have $g^{r+2} = g^r g^2 \in \Gamma(u)\cup\Gamma_2(u)$. Therefore, either g^{r+2} is a generator of G and $g^{r+2} \in S$, or g^{r+2} is not a generator and $g^{r+2} \in \Gamma_2(u)$.

Let p be the smallest prime divisor of $n = |G|$. Then $g, g^2, \ldots, g^{p-1} \in S$, and $g^p \in \Gamma_2(u)$. Suppose that G is not a p-group. Let q be the second smallest prime divisor of n. By the deduction above, we have $g^\lambda \in S$ for any $1 \leqslant \lambda \leqslant q-1$ with $(\lambda, p) = 1$. Noting that p is coprime to at least one of $q-2$ and $q-1$. If p and $q-2$ are coprime, then $g^{q-2} \in S$, so $g^q = g^{q-2}g^2 \in \Gamma_2(u)$, as $g^2 \in S$; if p and $q-1$ are coprime, then $g^{q-1} \in S$, so $g^q = g^{q-1}g \in \Gamma_2(u)$, as $g \in S$. Thus, g^q is in $\Gamma_2(u)$. However Γ is 2-distance-transitive and normal, which means that all elements of $\Gamma_2(u)$ have the same order. This contradicts the fact that $o(g^p) \neq o(g^q)$ and $g^p, g^q \in \Gamma_2(u)$. Thus G is a p-group.

Suppose that $|G| = p^r$. If $r \geqslant 3$, then by a similar argument as the previous paragraph, we have $g^\lambda \in S$ for any $1 \leqslant \lambda \leqslant p^r - 1$ with $(\lambda, p) = 1$. Hence $g^p, g^{p^2} \in \Gamma_2(u)$. This is impossible since $o(g^p) \neq o(g^{p^2})$ and all elements of $\Gamma_2(u)$ have the same order.

Therefore, we get $n = p^2$. Furthermore, $S = \{g^\lambda | 1 \leqslant \lambda \leqslant p^2 - 1, (p, \lambda) = 1\}$ and $\Gamma_2(u) = \{g^{\mu p} | 1 \leqslant \mu \leqslant p-1\}$. Thus $\Gamma \cong \mathrm{K}_{p[p]}$.

Note that $\mathrm{K}_{p[p]}$ is not normal. We have that $\Gamma_2(u)$ has no nongenerators of G. This means $\Gamma_2(u)$ consists of generators of G. □

Let

$$R = \Gamma_2(u),$$

the second neighborhood of the vertex u (corresponding to the identity of G).

Lemma 2.6 *The stabilizer A_u is regular on R, and $|R| = |S|$ divides $p-1$ for each prime divisor p of $|G|$.*

Proof Since A is 2-distance-transitive on Γ, A_u is transitive on $R = \Gamma_2(u)$. As $A_u = \mathrm{Aut}(G, S)$ is abelian and R consists of generators of G, A_u is faithful on R. Thus A_u is regular on $R = \Gamma_2(u)$, and so $|R| = |A_u| = |S|$.

Let $n = p_1^{t_1} p_2^{t_2} \ldots p_\ell^{t_\ell}$, where $p_1 < p_2 < \cdots < p_\ell$ are distinct primes. Let $G = \langle x_1\rangle \times \cdots \times \langle x_\ell\rangle$, where $o(x_i) = p_i^{t_i}$ for $1 \leqslant i \leqslant \ell$ and $g = x_1 \ldots x_\ell$. Then

$$A_u \leqslant \mathrm{Aut}(G) = \mathrm{Aut}(\langle x_1\rangle) \times \cdots \times \mathrm{Aut}(\langle x_\ell\rangle).$$

Set

$$B_j = \mathrm{Aut}(\langle x_1\rangle) \times \cdots \times \mathrm{Aut}(\langle x_{j-1}\rangle) \times \mathrm{Aut}(\langle x_{j+1}\rangle) \times \cdots \times \mathrm{Aut}(\langle x_\ell\rangle),$$

where $1 \leqslant j \leqslant \ell$. We claim that $A_u \cap B_j = \{1\}$ and $A_u \cong A_u B_j / B_j \lesssim \mathbb{Z}_{p_j - 1}$.

Assume that $A_u \cap B_j \neq \{1\}$. Then there exists $\sigma \in A_u \cap B_j$ such that $\sigma \neq 1$. Hence $g^\sigma = (x_1 \ldots x_\ell)^\sigma = (x_1 \ldots x_{j-1})^\sigma x_j (x_{j+1} \ldots x_\ell)^\sigma$, and $g^\sigma g^{-1} \neq 1$ is not a generator of G. Observing that g and g^σ are in S, we have $g^\sigma g^{-1} \in S \cup R$, which contradicts the fact that all elements in S and R are generators of G. Hence $A_u \cap B_j = \{1\}$ and

$$A_u \cong A_u B_j / B_j \lesssim \mathrm{Aut}(\langle x_j \rangle) \cong \mathrm{Aut}(\mathbb{Z}_{p_j^{t_j}}) \cong \mathbb{Z}_{(p_j - 1) p_j^{t_j - 1}}.$$

If p_j divides $|A_u|$, then there exists $\sigma \in A_u$ such that $o(\sigma) = p_j$. Furthermore, $x_j^\sigma = x_j^{\lambda p_j + 1} \neq x_j$ for some integer λ. Thus $g^\sigma g^{-1} \in \langle x_1 \rangle \times \cdots \times \langle x_{j-1} \rangle \times \langle x_j^{p_j} \rangle \times \langle x_{j+1} \rangle \times \ldots \langle x_\ell \rangle$ is not a generator of G. This contradicts the fact that $g^\sigma g^{-1} \in S \cup R$. Therefore, $A_u \lesssim \mathbb{Z}_{p_j - 1}$ for $1 \leqslant j \leqslant \ell$. Noting that A_u acts regularly on S, $|S| = |A_u|$. Hence $|S|$ divides $p_i - 1$ for $1 \leqslant i \leqslant \ell$. □

By virtue of Lemma 2.6, we can assume that

$$A_u = \langle \sigma \rangle \text{ and } g^\sigma = g^\lambda,$$

where λ is coprime to n. Let g^μ be an element of R and τ be an automorphism of G such that $g^\tau = g^\mu$. Then $(g^\mu)^\tau = (g^\tau)^\mu = g^{\mu^2}$. Let $k = |S| = |A_u|$. We have

$$\begin{aligned} S &= g^{\langle \sigma \rangle} = \{g, g^\lambda, g^{\lambda^2}, \ldots, g^{\lambda^{k-1}}\}, \\ R &= (g^\mu)^{\langle \sigma \rangle} = \{g^\mu, g^{\mu\lambda}, g^{\mu\lambda^2}, \ldots, g^{\mu\lambda^{k-1}}\} = S^\tau, \\ R^\tau &= (g^\mu)^{\langle \sigma \rangle \tau} = (g^\mu)^{\tau \langle \sigma \rangle} = (g^{\mu^2})^{\langle \sigma \rangle} = \{g^{\mu^2}, g^{\mu^2 \lambda}, g^{\mu^2 \lambda^2}, \ldots, g^{\mu^2 \lambda^{k-1}}\}. \end{aligned}$$

Let

$$\Sigma = \Gamma^\tau = \mathrm{Cay}(G, S)^\tau = \mathrm{Cay}(G, S^\tau) = \mathrm{Cay}(G, R).$$

Then Σ and Γ are isomorphic. (Two graphs are *isomorphic* if there exists a bijection between their vertex sets which preserves the adjacency and the nonadjacency.)

Lemma 2.7 *Let $x, y \in G$. Then $\mathrm{d}_\Gamma(x, y) = 2$ if and only if $\mathrm{d}_\Gamma(xy^{-1}, u) = 2$, and the following conditions are equivalent:*

(i) $xy^{-1} \in R$;
(ii) $\mathrm{d}_\Sigma(x, y) = 1$;
(iii) $\mathrm{d}_\Gamma(x, y) = \mathrm{d}_\Gamma(xy^{-1}, u) = 2$.

Proof For any $y \in G$, let $\sigma_{y^{-1}}$ be the right translation by y^{-1}. Then $\sigma_{y^{-1}}$ is an automorphism of Γ since Γ is a Cayley graph of G. Thus for any $x \in G$, we have

$$d_\Gamma(x, y) = d_\Gamma(x^{\sigma_{y^{-1}}}, y^{\sigma_{y^{-1}}}) = d_\Gamma(xy^{-1}, u).$$

Noting that $R = \Gamma_2(u) = \Sigma(u)$, we have $xy^{-1} \in R$ if and only if $d_\Gamma(xy^{-1}, u) = d_\Gamma(x, y) = 2$. By the same argument, we also have $xy^{-1} \in R$ if and only if $d_\Sigma(xy^{-1}, u) = d_\Sigma(x, y) = 1$. □

For two sets B_1, B_2, we use $B_1 \sqcup B_2$ to denote $B_1 \cup B_2$ when $B_1 \cap B_2 = \emptyset$. We denote $\Gamma_{\geqslant i}(u) = \Gamma_i(u) \cup \Gamma_{i+1}(u) \cup \cdots \cup \Gamma_{\mathrm{diam}(\Gamma)}(u)$.

Lemma 2.8 *The graph Γ is a Paley graph $\mathsf{P}(p)$, where $p \equiv 1 \pmod 4$ is prime.*

Proof By Lemma 2.5, $\Gamma_2(u) = R$ contains generators of G. If the diameter of Γ is 2, then all the elements in $G \setminus \{u\}$ are generators of G, and so n is an odd prime p. By Lemma 2.6, $|S| = |R|$, so $|S| = |R| = (p-1)/2$. Thus S is either the set of square elements or the set of nonsquare elements of $G \setminus \{u\}$, and Γ is the Paley graph $\mathsf{P}(p)$, see also [8, Lemma 2.2].

In the remainder, we suppose that Γ has diameter at least 3. Let (u, z, v, w) be a 3-geodesic of Γ. We set $k = |S| = |R|$, $a_1 = |\Gamma(z) \cap S|$, $b_1 = |\Gamma(z) \cap R|$ and $c_2 = |\Gamma(v) \cap S|$. Let N be the number of edges in Γ with one end in S and the other end in R. Then

$$N = b_1|z^{A_u}| = b_1 k = c_2|v^{A_u}| = c_2 k.$$

Hence $b_1 = c_2$.

Note that all S, R, and R^τ are orbits of A_u. We will argue in two cases.

Case 1 $R^\tau \neq S$.

Since $\Sigma_2(u) = R^\tau$ and $R^\tau \neq S$, it follows that $S \subseteq \Sigma_{\geqslant 3}(u)$. Thus, for each $y \in R$ and $x \in S$, $d_\Sigma(x, y) \neq 1$, and it follows from Lemma 2.7 that $d_\Gamma(x, y) \neq 2$.

Let $w \in R^\tau$. Then there exist vertices $z \in S$ and $v \in R$, such that (u, v, w) and (u, z, v) are 2-geodesics in Σ and Γ, respectively. Hence, by Lemma 2.7, $d_\Gamma(w, v) = 2$ (Fig. 1).

Since Γ is 2-distance-transitive, there exists $\eta \in A_v$ such that $u^\eta = w$. Thus

$$b_1 = c_2 = |\Gamma(v) \cap \Gamma(u)| = |\Gamma(v^\eta) \cap \Gamma(u^\eta)| = |\Gamma(v) \cap \Gamma(w)|.$$

Since $\Gamma(v) \cap \Gamma(w) \subseteq R \cup \Gamma_3(u)$, we have

$$k = |\Gamma(v)| = |\Gamma(v) \cap S| + |\Gamma(v) \cap (R \cup \Gamma_3(u))| \geqslant 2b_1. \quad (1)$$

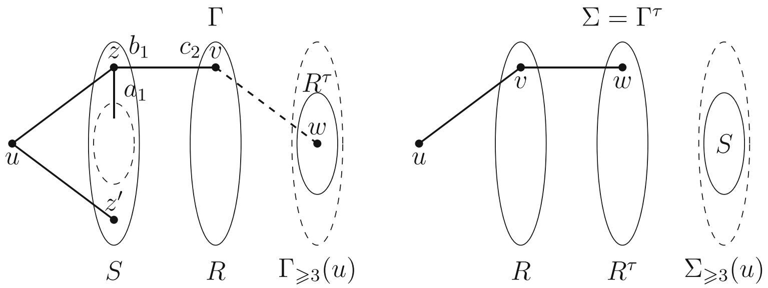

Fig. 1 Case 1

For any $x \in S \cap \Gamma(z)$, (v, z, x) is a 2-arc. Since $d_\Gamma(v, x) \neq 2$, $d_\Gamma(v, x) = 1$ and $x \in \Gamma(v)$. Thus

$$\{z\} \sqcup (S \cap \Gamma(z)) \subseteq \Gamma(v) \cap S \tag{2}$$

and $a_1 + 1 \leqslant b_1$. Consider the valency of the vertex z, we have

$$k = |\Gamma(z)| = 1 + a_1 + b_1 \leqslant 2b_1. \tag{3}$$

By inequalities (1) and (3), $k = 2b_1$, so (2) can be modified into

$$\{z\} \sqcup (\Gamma(z) \cap S) = \Gamma(v) \cap S. \tag{4}$$

By the same deduction, for any $y \in \Gamma(z) \cap R$, we have

$$\{z\} \sqcup (\Gamma(z) \cap S) = \Gamma(y) \cap S. \tag{5}$$

Similarly, for each $x \in \Gamma(z) \cap S \subset \Gamma(v) \cap S$,

$$\{x\} \sqcup (\Gamma(x) \cap S) = \Gamma(v) \cap S = \{z\} \sqcup (S \cap \Gamma(z)). \tag{6}$$

Equalities (5) and (6) indicate that for each $x \in \Gamma(z) \backslash \{u\}$, $\Gamma(x) \cap S \subset \{z\} \sqcup (\Gamma(z) \cap S)$.

Let $z' \in S \setminus (\{z\} \cup \Gamma(z))$. Then $d_\Gamma(z, z') = 2$. There is an automorphism $\eta \in A$ such that $z^\eta = u$ and $(z')^\eta = v$. Let (z, x, z') be a 2-arc in Γ. Then $z' \in \Gamma(x) \cap S$ and $x \in \Gamma(z)$. Thus $x = u$ and

$$1 = |\Gamma(z) \cap \Gamma(z')| = |\Gamma(u) \cap \Gamma(v)| = b_1.$$

Hence $k = 2$. This contradicts the assumption that Γ has valency at least 3.

Case 2 $R^\tau = S$.

Since $R^\tau = S$, it follows that $\Gamma_{\geqslant 3}(u) = \Sigma_{\geqslant 3}(u)$. For each $x \in R$ and $y \in \Sigma_{\geqslant 3}(u)$, we have $d_\Sigma(x, y) \neq 1$. This implies $d_\Gamma(x, y) \neq 2$. Thus $\Gamma_{\geqslant 3}(u) = \Gamma_3(u)$ and $\text{diam}(\Gamma) = 3$ (Fig. 2).

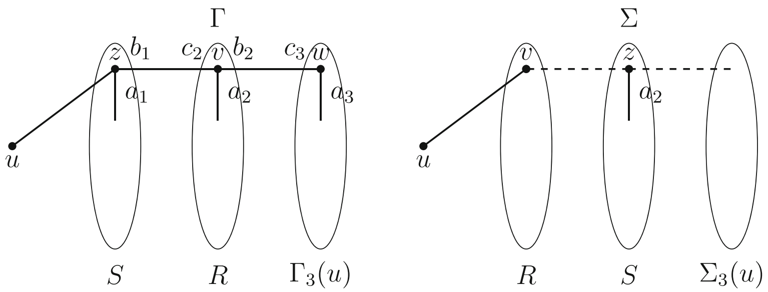

Fig. 2 Case 2

Let $w \in \Gamma_3(u)$. Then there exist $z \in S$, $v \in R$ such that (u, z, v, w) is a 3-geodesic. Let

$$b_2 = |\Gamma(v) \cap \Gamma_3(u)|, \quad c_3 = |\Gamma(w) \cap R|,$$
$$a_2 = |\Gamma(v) \cap R|, \quad a_3 = |\Gamma(w) \cap \Gamma_3(u)|.$$

Then $k = 1 + a_1 + b_1 = a_2 + b_2 + c_2 = a_3 + c_3$.

Let p be the smallest prime factor of n, and let N' be the number of edges in Γ with one end in R and the other end in $\Gamma_3(u)$. Then

$$k(k-1) \geqslant N' = kb_2 \geqslant |\Gamma_3(u)| = n - 2k - 1.$$

Hence $n \leqslant k^2 + k + 1$. By Lemma 2.6, k is a divisor of $p - 1$, and thus $k + 1 < p + 1$. Then $n \leqslant k^2 + k + 1 = k(k+1) + 1 < (p-1)(p+1) + 1 = p^2$, and so n is a prime. This implies that w is also a generator of G and $|w^{A_u}| = |A_u| = k$. Furthermore,

$$N' = |v^{A_u}|b_2 = kb_2 \geqslant |w^{A_u}|c_3 = kc_3.$$

This means

$$c_3 \leqslant b_2. \tag{7}$$

For any $x \in \Gamma(w) \cap \Gamma_3(u)$, (v, w, x) is a 2-arc. Since $d_\Gamma(v, x) \neq 2$, we have $d_\Gamma(v, x) = 1$ and $x \in \Gamma(v)$. Thus

$$\{w\} \sqcup (\Gamma(w) \cap \Gamma_3(u)) \subseteq \Gamma(v) \cap \Gamma_3(u),$$

and

$$a_3 + 1 \leqslant b_2. \tag{8}$$

By inequalities (7) and (8), we have

$$k = c_3 + a_3 \leqslant b_2 + b_2 - 1. \tag{9}$$

For any $x \in S \setminus \{z\}$, (z, u, x) is a 2-arc in Γ, and $d_\Gamma(z, x) \leqslant 2$. Thus

$$S = \{z\} \sqcup (\Gamma(z) \cap S) \sqcup (\Gamma_2(z) \cap S).$$

Note that $\Gamma_2(z) \cap S = \Sigma(z) \cap S$ and

$$|\Sigma(z) \cap S| = |\Sigma^{\tau^{-1}}(z^{\tau^{-1}}) \cap S^{\tau^{-1}}| = |\Gamma(v') \cap R| = a_2$$

for some $v' \in R$ where the graph isomorphism τ is defined in the paragraph before Lemma 2.7. We have

$$k = 1 + |\Gamma(z) \cap S| + |\Gamma_2(z) \cap S| = 1 + |\Gamma(z) \cap S| + |\Sigma(z) \cap S| = 1 + a_1 + a_2.$$

Hence $k = 1 + a_1 + b_1 = 1 + a_1 + a_2$, and

$$b_1 = a_2. \tag{10}$$

For any $x \in \Gamma(z) \cap S$, (v, z, x) is a 2-arc in Γ. Thus we have $d_\Gamma(x, v) \leqslant 2$. Then

$$\begin{aligned}\{z\} \sqcup (\Gamma(z) \cap S) &\subseteq (\Gamma(v) \cap S) \sqcup (\Gamma_2(v) \cap S)\\ &= (\Gamma(v) \cap S) \sqcup (\Sigma(v) \cap S)\end{aligned}$$

and

$$1 + a_1 \leqslant b_1 + b_1.$$

Thus $b_1 = k - (1 + a_1) \geqslant k - 2b_1 = b_2$. By inequality (9),

$$k = 2b_1 + b_2 > 2b_2 - 1 \geqslant k.$$

This is a contradiction.

Therefore, Γ is of diameter 2, and is a Paley graph as observed above. □

2.2 2-Distance-transitive nonnormal circulants

Let $\Gamma = (V, E)$ be an arc-transitive circulant which is not a normal circulant. By Theorem 2.1, there exists an arc-transitive circulant Σ of order m such that $mb = n$, and

$$\Gamma = \begin{cases} \Sigma\left[\overline{\mathrm{K}_b}\right], & \textit{or} \\ \Sigma\left[\overline{\mathrm{K}_b}\right] - b\Sigma, & \text{where } (m, b) = 1. \end{cases}$$

We next determine which of these graphs are 2-distance-transitive.

The vertex set V of Γ is partitioned into m parts of size b, and thus we may label the vertices as

$$V = \{v_{i,j} \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant b\}$$

such that

$$B_i := \{v_{i,1}, v_{i,2}, \ldots, v_{i,b}\}, \text{ where } 1 \leqslant i \leqslant m$$

are blocks for $\mathrm{Aut}\Gamma$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$, the corresponding block system for $\mathrm{Aut}\Gamma$ acting on V. The *quotient graph* $\Gamma_\mathcal{B}$ is the graph with vertex set $V_\mathcal{B} = \mathcal{B}$ such that two vertices $B_1, B_2 \in \mathcal{B}$ are adjacent if and only if there exist $u_1 \in B_1$ and $u_2 \in B_2$ which are adjacent in Γ. Then $\Gamma_\mathcal{B} \cong \Sigma$, and each element $g \in \mathrm{Aut}\Gamma$ naturally induces a permutation $\overline{g}$ on set $\mathcal{B}$ which is an automorphism of the graph $\Gamma_\mathcal{B}$.

Lemma 2.9 *Let u be an arbitrary vertex in Γ. Then, except for the case $\Gamma = \Sigma[\overline{\mathrm{K}_2}]-2\Sigma$, the subset $\{u\} \cup \Gamma_2(u) \subset V$ is a block of size b.*

Proof It is clear that B_1 is a block of size b for $\mathrm{Aut}\Gamma$. Without loss of generality, set $u = v_{1,1}$, since Γ is vertex-transitive. Thus, we only need to show that $B_1 = \{u\} \cup \Gamma_2(u)$.

Let $w = v_{1,2} \in B_1$. We have $d_\Gamma(u, w) \geqslant 2$ since the induced subgraph $[B_1] \cong \overline{\mathrm{K}_b}$. Suppose that B_1, B_2 are two vertices of $\Gamma_{\mathcal{B}}$ which are adjacent. If $\Gamma = \Sigma[\overline{\mathrm{K}_b}]$, then $(u, v_{2,1}, w)$ is a 2-geodesic of Γ. If $\Gamma = \Sigma[\overline{\mathrm{K}_b}]-b\Sigma$, where $b \geqslant 3$, then $(u, v_{2,3}, w)$ is a 2-geodesic of Γ. Thus in either case, $w \in \Gamma_2(u)$. By the same deduction, for any $w' \in B_1 \setminus \{u\}$, we have $w' \in \Gamma_2(u)$. Hence $B_1 \subseteq \{u\} \cup \Gamma_2(u)$.

Let $A = \mathrm{Aut}\Gamma$ and A_u be the stabilizer of vertex u. Since Γ is 2-distance-transitive and B_1 is a block of V for A, we have $w^{A_u} = \Gamma_2(u) \subseteq B_1$. Thus $\{u\} \cup \Gamma_2(u) = B_1$, and it is a block of size b on V for A. □

Lemma 2.10 *Let Γ be a 2-distance-transitive circulant which is not a normal circulant. Then $\Gamma = \mathrm{K}_{m[b]}$ or $\mathrm{K}_{b,b} - b\mathrm{K}_2$.*

Proof Assume first that $m = 2$. Since Γ is of valency at least 3 by our assumption, either $b \geqslant 3$ and $\Gamma = \mathrm{K}_{b,b}$, or $b \geqslant 5$ and $\Gamma = \mathrm{K}_2[\overline{\mathrm{K}_b}] - b\mathrm{K}_2 = \mathrm{K}_{b,b} - b\mathrm{K}_2$. We next consider the case where $m \geqslant 3$.

Assume that $\Gamma = \Sigma[\overline{\mathrm{K}_b}]$ with $m \geqslant 3$. Let $u = v_{1,1} \in B_1$. By Lemma 2.9, $\{u\} \cup \Gamma_2(u) = B_1$ is a block for $A = \mathrm{Aut}\Gamma$. Thus there is no vertex $w \in V \setminus B_1$ at distance 2 with u in Γ. It follows that Σ is a complete graph, and so $\Gamma \cong \Sigma[\overline{\mathrm{K}_b}] \cong \mathrm{K}_{m[b]}$.

Now, let $\Gamma = \Sigma[\overline{\mathrm{K}_b}] - b\Sigma$ with $m \geqslant 3$ and $b \geqslant 3$. Again, by Lemma 2.9, $\{u\} \cup \Gamma_2(u) = B_1$ is a block for $A = \mathrm{Aut}\Gamma$. Similarly, there is no vertex $w \in V \setminus B_1$ at distance 2 with u in Γ, and Σ is a complete graph. Therefore, $(u, v_{2,2}, v_{3,1})$ is a 2-geodesic in Γ. This contradicts the fact that $v_{3,1} \notin B_1 = \{u\} \cup \Gamma_2(u)$.

Finally, assume that $\Gamma = \Sigma[\overline{\mathrm{K}_2}] - 2\Sigma$ with $m \geqslant 3$. According to Lemma 2.1, m is relatively prime to b. Hence m is an odd integer. If Σ is a complete graph, then $\Gamma = \mathrm{K}_{m[2]} - 2\mathrm{K}_m$. Note that $\mathrm{K}_{m[2]} - 2\mathrm{K}_m$ is isomorphic to $\mathrm{K}_{2[m]} - m\mathrm{K}_2$. We have $\Gamma \cong \mathrm{K}_{2[m]} - m\mathrm{K}_2$. If Σ is not a complete graph, then it is clear that the quotient graph $\Gamma_{\mathcal{B}}$ is also 2-distance-transitive. By the argument above we have $\Gamma_{\mathcal{B}}$ is isomorphic to C_m, $\mathsf{P}(p)$, or $\mathrm{K}_{m'[b']}$. When $p = 5$, the Paley graph $\mathsf{P}(5)$ is isomorphic to C_5. If $\Gamma_{\mathcal{B}} \cong \mathrm{C}_m$ then $\Gamma \cong \mathrm{C}_{2m}$ is normal, a contradiction. If $\Gamma_{\mathcal{B}} \cong \mathsf{P}(p)$ for $p > 5$, there is a triangle in $\Gamma_{\mathcal{B}}$. If $\Gamma_{\mathcal{B}} \cong \mathrm{K}_{m'[b']}$, then there exists a triangle in $\Gamma_{\mathcal{B}}$ too. In either case, let (B_1, B_2, B_3) be a triangle in $\Gamma_{\mathcal{B}}$ and (B_1, B_2, B_4) be a 2-geodesic in $\Gamma_{\mathcal{B}}$. Since $\Gamma = \Sigma[\overline{\mathrm{K}_2}] - 2\Sigma$, it follows that the vertex $v_{2,2}$ is adjacent to $v_{1,1}$, $v_{3,1}$, and $v_{4,1}$, and the vertex $v_{1,1}$ is not adjacent to $v_{3,1}$ and $v_{4,1}$. Hence $(v_{1,1}, v_{2,2}, v_{3,1})$ and $(v_{1,1}, v_{2,2}, v_{4,1})$ are two 2-geodesics in Γ. Thus there exists an automorphism $\sigma \in \mathrm{Aut}\Gamma$ such that $v_{1,1}^\sigma = v_{1,1}$ and $v_{3,1}^\sigma = v_{4,1}$. This is impossible since σ induces an automorphism of $\Gamma_{\mathcal{B}}$ and $1 = d_{\Gamma_{\mathcal{B}}}(B_1, B_3) \neq d_{\Gamma_{\mathcal{B}}}(B_1, B_4) = 2$. □

References

1. Alspach, B., Conder, M., Marušič, D., Xu, M.Y.: A classification of 2-arc transitive circulants. J. Algebraic Combin. **5**, 83–86 (1996)

2. Biggs, N.: Algebraic Graph Theory. Cambridge Mathematical Library, 2nd edn. Cambridge University Press, Cambridge (1993)
3. Brouwer, A.E., Cohen, A.M., Neumaier, A.: Distance-Regular Graphs. Springer, Berlin, Heidelberg, New York (1989)
4. Corr, B., Jin, W., Schneider, C.: Finite two-distance-transitive graphs. J. Graph Theory (2017). https://doi.org/10.1002/jgt.22112
5. Devillers, A., Giudici, M., Li, C.H., Praeger, C.E.: Locally s-distance transitive graphs. J. Graph Theory **69**(2), 176–197 (2012)
6. Devillers, A., Giudici, M., Li, C.H., Praeger, C.E.: Locally s-distance transitive graphs and pairwise transitive designs. J. Combin. Theory Ser. A **120**, 1855–1870 (2013)
7. Devillers, A., Jin, W., Li, C.H., Praeger, C.E.: On normal 2-geodesic transitive Cayley graphs. J. Algebraic Combin. **39**, 903–918 (2014)
8. Devillers, A., Jin, W., Li, C.H., Praeger, C.E.: Finite 2-geodesic transitive graphs of prime valency. J. Graph Theory **80**, 18–27 (2015)
9. Godsil, C.D.: On the full automorphism group of a graph. Combinatorica **1**, 243–256 (1981)
10. Kovács, I.: Classifying arc-transitive circulants. J. Algebraic Combin. **20**, 353–358 (2004)
11. Li, C.H.: Permutation groups with a cyclic regular subgroup and arc transitive circulants. J. Algebraic Combin. **21**, 131–136 (2005)
12. Lim, T.K., Praeger, C.E.: On generalized Paley graphs and their automorphism groups. Michigan Math. J. **58**, 293–308 (2009)
13. Miklavič, Š., Potočnik, P.: Distance-regular circulants. European J. Combin. **24**(7), 777–784 (2003)
14. Paley, R.E.A.C.: On orthogonal matrices. J. Math. Phys. **12**, 311–320 (1933)
15. Praeger, C.E.: An O'Nan Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs. J. Lond. Math. Soc. **47**(2), 227–239 (1993)
16. Praeger, C.E.: Finite normal edge-transitive Cayley graphs. Bull. Aust. Math. Soc. **60**, 207–220 (1999)
17. Sabidussi, G.: Vertex-transitive graphs. Monatsh. Math. **68**, 426–438 (1964)
18. Xu, M.Y.: Automorphism groups and isomorphisms of Cayley digraphs. Discrete Math. **182**, 309–319 (1998)