

Ford fundamental domains in symmetric spaces of rank one

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Abstract We show the existence of isometric (or Ford) fundamental regions for a large class of subgroups of the isometry group of any rank one Riemannian symmetric space of noncompact type. The proof does not use the classification of symmetric spaces. All hitherto known existence results of isometric fundamental regions and domains are essentially subsumed by our work.

Keywords Ford fundamental domains · Isometric fundamental regions · Rank one symmetric spaces · Isometric spheres · Cygan metric

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1 Introduction

Let D be a rank one Riemannian symmetric space of noncompact type and denote its full group of Riemannian isometries by G . Suppose that Γ is a subgroup of G . A subset Y of D is called a *fundamental region for Γ in D* if Y is open, the translates of Y by each two elements of Γ are disjoint, and D is covered by the family of Γ -translates of the closure of Y . If, in addition, Y is connected, then it is called a *fundamental domain* for Γ in D .

In this article, we show the existence of so-called isometric (or Ford) fundamental regions for a large class of subgroups of G , see Theorem 3.18, Corollary 3.20 and Proposition 4.38. In many situations, the fundamental regions will turn out to actually be fundamental domains (see Corollary 3.23 and Proposition 4.38). Moreover, in Sect. 4, we show that our results subsume all previously known existence results of isometric fundamental regions and domains.

Let Γ be an admissible subgroup of G . The distinctive trait of an isometric fundamental region for Γ is that it consists of two building blocks. One of them is a fundamental region \mathcal{F}_∞ for the stabilizer group Γ_∞ of ∞ . The other one is the common part of the exteriors

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of all isometric spheres of Γ (see Sect. 3 for a definition). Then the isometric fundamental region is the set

$$\mathcal{F} := \mathcal{F}_\infty \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g),$$

where $\text{ext } I(g)$ denotes the exterior of the isometric sphere $I(g)$ of $g \in \Gamma \setminus \Gamma_\infty$. These fundamental regions are of interest for several applications. For example, they reflect the geometry of Γ in a way particularly adjusted to the needs of the construction of a symbolic dynamics for the geodesic flow on the orbifold $\Gamma \backslash D$ (see [19]).

Our proof of the existence of isometric fundamental regions does not use the classification of rank one Riemannian symmetric spaces of noncompact type. This was made possible by the classification-free constructions of all these spaces provided by Cowling et al. in [6] and [7], respectively, by Korányi and Ricci in [15] and [16]. These constructions appear to be the first ones which are uniform both on the level of spaces and on the level of isometry groups. The approach in [6] and [7] is intimately connected with the restricted root space decomposition of the Lie algebra of the isometry group of the symmetric space, and hence it is the method of choice for considerations of algebraic nature. In contrast, the construction in [16] and [15] reflects the geometric side of the spaces. Both constructions are amazingly easy to work with. Moreover, one can effortlessly switch from one construction to the other and translate insights and advantages from one model to the other. We recall both constructions in Sect. 2.

Using their work we provide a uniform definition of the notion of an isometric sphere and its exterior in Sect. 3. The uniformity on the level of isometry groups then allows to stick to a classification-free treatment of the isometric spheres, which finally results in a classification-free proof of the existence of isometric fundamental regions.

For real, complex and quaternionic hyperbolic spaces, there already exist several (different and also non-equivalent) definitions of isometric spheres in the literature, e.g., in [10] for the hyperbolic plane, in [13] for the upper half plane model and the disk model of two-dimensional real hyperbolic space, in [17] for three-dimensional real hyperbolic space, in [1] and [2] for real hyperbolic spaces of arbitrary dimension, in [11, 18] and [12] for complex hyperbolic spaces, and in [14] for quaternionic hyperbolic spaces. Moreover, for certain subgroups of the isometry group of real and complex hyperbolic spaces, the existence of isometric fundamental regions was already known. More precisely, Apanasov [1] and [2] provides the hitherto most general treatment of groups acting on real hyperbolic spaces. In [10], Ford investigates the case of the hyperbolic plane. However, his definition of fundamental region is not equivalent to our definition. Therefore, his result cannot be compared to our one. Groups acting on complex hyperbolic spaces are considered by Kamiya in [12]. In Sect. 4, we will investigate which definitions of isometric spheres are subsumed by our uniform one, and we will show that the known isometric fundamental regions are special cases of Theorem 3.18.

Throughout we will use the following notation. If T is a topological space and U a subset of T , then the closure of U is denoted by \bar{U} or $\text{cl}(U)$ and its boundary is denoted by ∂U . Moreover, we write U° for the interior of U . The complement of U in T is denoted by $\complement U$ or $T \setminus U$.

For two arbitrary sets A and B , the complement of B in A is written as $A \setminus B$. If \sim is an equivalence relation on A , then A/\sim denotes the set of equivalence classes. Likewise, if Γ a group acting on A , then we write A/Γ for the space of right cosets. Moreover, if p is a point of A , then Γ_p denotes the stabilizer group $\{g \in \Gamma \mid gp = p\}$ of p in Γ .

2 Classification-free constructions

The basic objects of the classification-free construction of rank one Riemannian symmetric spaces of noncompact type in [6] and [7] are so-called H -type (Heisenberg type) algebras. In contrast, Korányi and Ricci [15] and [16] construct these symmetric spaces from so-called J^2C -module structures.

In this section, we introduce these notions and we briefly recall the two constructions. Mostly we duplicate, for the convenience of the reader, material from their work. Nevertheless, in parts it can be considered as complementary, e. g., Lemma 2.2 on the (non-)uniqueness of decompositions of H -type algebras, the notion of ordered decompositions, and a refined definition of an isomorphism between H -type algebras, respectively, between C -module structures (which corrects a minor inaccuracy). Moreover, we prove in detail that the Cayley transform is an isometry, and that the group M given in [6] and [7] and that in [15] and [16] are indeed the same.

All omitted proofs can be found in [6] or [7] for statements in the language of H -type algebras, and in [15] or [16] for those in the language of J^2C -modules. As long as no confusion can arise, each inner product is denoted by $\langle \cdot, \cdot \rangle$ and its associated norm by $|\cdot|$.

2.1 H -type algebras and the J^2 -condition

A vector space is called *Euclidean* if it is a finite-dimensional real vector space endowed with an inner product. A Lie algebra is called *Euclidean* if, in addition to being a Lie algebra, it is a Euclidean vector space. For a vector space \mathfrak{v} let $\text{End}_{\text{vs}}(\mathfrak{v})$ denote the group and vector space of endomorphisms of \mathfrak{v} . If \mathfrak{v} carries additional structures, then the elements of $\text{End}_{\text{vs}}(\mathfrak{v})$ are not required to be compatible with these structures. In particular, if \mathfrak{v} is Euclidean, then $\varphi \in \text{End}_{\text{vs}}(\mathfrak{v})$ need not be orthogonal.

Definition 2.1 Let \mathfrak{n} be a Euclidean Lie algebra. Then \mathfrak{n} is said to be an *H -type algebra* if

(H1) there are two subvector spaces $\mathfrak{v}, \mathfrak{z}$ of \mathfrak{n} (each of which may be trivial) such that

$$[\mathfrak{n}, \mathfrak{z}] = \{0\}, \quad [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z},$$

and \mathfrak{n} is the orthogonal direct sum of \mathfrak{z} and \mathfrak{v} ,

(H2) for all $X \in \mathfrak{v}$, all $Z \in \mathfrak{z}$ we have

$$|J(Z)X| = |Z| \cdot |X|$$

where $J: \mathfrak{z} \rightarrow \text{End}_{\text{vs}}(\mathfrak{v})$ is the \mathbb{R} -linear map defined by

$$\langle J(Z)X, Y \rangle = \langle Z, [X, Y] \rangle$$

for all $X, Y \in \mathfrak{v}$, all $Z \in \mathfrak{z}$. The map J is well-defined and unique by Riesz' Representation Theorem (or its finite-dimensional counterpart).

If (H1) holds, then \mathfrak{n} is either abelian or two-step nilpotent. In the first case we call \mathfrak{n} *degenerate*, in the second *non-degenerate*.

The construction of a symmetric space from an H -type algebra \mathfrak{n} depends on the choice of \mathfrak{z} and \mathfrak{v} in the decomposition $\mathfrak{z} \oplus \mathfrak{v}$ of \mathfrak{n} . We call the pair $(\mathfrak{z}, \mathfrak{v})$ an *ordered decomposition* of \mathfrak{n} . The following lemma shows that the ordered decomposition is unique unless \mathfrak{n} is non-trivial and abelian (which precisely is the reason for calling abelian H -type algebras degenerate). It will turn out that both possible ordered decompositions of a non-trivial abelian H -type algebra give rise to the same symmetric space, but in different models. Nevertheless,

this non-uniqueness calls for a careful notion of isomorphisms between H -type algebras, which we will discuss after the lemma.

We denote the center of a Lie algebra \mathfrak{g} by $Z(\mathfrak{g})$.

Lemma 2.2 *Let \mathfrak{n} be an H -type algebra. If \mathfrak{n} is non-abelian or $\mathfrak{n} = \{0\}$, then the ordered decomposition $(\mathfrak{z}, \mathfrak{v})$ of \mathfrak{n} is unique. In this case, we have $\mathfrak{z} = Z(\mathfrak{n})$ and $\mathfrak{v} = Z(\mathfrak{n})^\perp$. If \mathfrak{n} is abelian and $\mathfrak{n} \neq \{0\}$, then there are two ordered decompositions of \mathfrak{n} , namely $(\mathfrak{z}, \mathfrak{v}) = (Z(\mathfrak{n}), \{0\}) = (\mathfrak{n}, \{0\})$ and $(\mathfrak{z}, \mathfrak{v}) = (\{0\}, \mathfrak{n})$.*

Proof Suppose that $(\mathfrak{z}, \mathfrak{v})$ is an ordered decomposition of \mathfrak{n} . Then \mathfrak{v} is uniquely determined by \mathfrak{z} , namely $\mathfrak{v} = \mathfrak{z}^\perp$. Since $[\mathfrak{z}, \mathfrak{z}] = \{0\}$, we know that \mathfrak{z} is a subvector space of $Z(\mathfrak{n})$ (even a subalgebra). If $\mathfrak{z} = \{0\}$, then $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z} = \{0\}$. In this case, \mathfrak{n} is abelian and $(\mathfrak{z}, \mathfrak{v}) = (\{0\}, \mathfrak{n})$.

Suppose now that $\mathfrak{z} \neq \{0\}$. We have to prove that $\mathfrak{z} = Z(\mathfrak{n})$. For contradiction assume that $\mathfrak{z} \neq Z(\mathfrak{n})$, hence $\dim \mathfrak{z} < \dim Z(\mathfrak{n})$. Then there is a non-trivial element $X \in \mathfrak{v} \cap Z(\mathfrak{n})$. Fix some $Z \in \mathfrak{z}, Z \neq 0$. For all $Y \in \mathfrak{v}$ it follows that

$$\langle J(Z)X, Y \rangle = \langle Z, [X, Y] \rangle = 0.$$

Thus $J(Z)X = 0$. But then

$$|J(Z)X| = 0 \neq |Z| \cdot |X|,$$

which is a contradiction to (H2). Therefore, $\mathfrak{z} = Z(\mathfrak{n})$.

This shows that for non-abelian \mathfrak{n} or for $\mathfrak{n} = \{0\}$, the pair $(\mathfrak{z}, \mathfrak{v}) = (Z(\mathfrak{n}), Z(\mathfrak{n})^\perp)$ is the only candidate for an ordered decomposition of \mathfrak{n} . Because there is at least one by hypothesis, $(Z(\mathfrak{n}), Z(\mathfrak{n})^\perp)$ is indeed an ordered decomposition of \mathfrak{n} . For non-trivial abelian \mathfrak{n} we have the two candidates $(\{0\}, \mathfrak{n})$ and $(\mathfrak{n}, \{0\})$, which both clearly satisfy (H1) and (H2). \square

Let \mathfrak{n} be a non-trivial abelian H -type algebra. Then \mathfrak{n} admits the two ordered decompositions $(\mathfrak{n}, \{0\})$ and $(\{0\}, \mathfrak{n})$. The isomorphism $\text{id}_{\mathfrak{n}}$ of \mathfrak{n} as a Euclidean Lie algebra does not respect these decompositions. In Sect. 2.3 we will see that preserving the decompositions is essential for the bijection between H -type algebras and C -module structures. Therefore, from now on, we will always consider an H -type algebra \mathfrak{n} as being equipped with a (fixed) ordered decomposition and denote it by $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ or, briefly, by $(\mathfrak{z}, \mathfrak{v}, J)$. Although the map J is determined by \mathfrak{z} and \mathfrak{v} , we keep it in the triple to fix a notation for it.

Definition 2.3 Let $\mathfrak{n}_j = (\mathfrak{z}_j, \mathfrak{v}_j, J_j), j = 1, 2$, be H -type algebras. An *isomorphism* from \mathfrak{n}_1 to \mathfrak{n}_2 is a pair (φ, ψ) of isomorphisms of Euclidean vector spaces $\varphi: \mathfrak{z}_1 \rightarrow \mathfrak{z}_2$ and $\psi: \mathfrak{v}_1 \rightarrow \mathfrak{v}_2$ such that the diagram

$$\begin{array}{ccc} \mathfrak{z}_1 \times \mathfrak{v}_1 & \xrightarrow{J_1} & \mathfrak{v}_1 \\ \varphi \times \psi \downarrow & & \downarrow \psi \\ \mathfrak{z}_2 \times \mathfrak{v}_2 & \xrightarrow{J_2} & \mathfrak{v}_2 \end{array}$$

commutes.

The following lemma shows that isomorphism of H -type algebras is a refined notion of isomorphism of Euclidean Lie algebras.

Lemma 2.4 *Let $\mathfrak{n}_j = (\mathfrak{z}_j, \mathfrak{v}_j, J_j), j = 1, 2$, be H -type algebras and suppose that the map $(\varphi, \psi): \mathfrak{n}_1 \rightarrow \mathfrak{n}_2$ is an isomorphism. Then $\varphi \times \psi: \mathfrak{z}_1 \oplus \mathfrak{v}_1 \rightarrow \mathfrak{z}_2 \oplus \mathfrak{v}_2$ is an isomorphism of Euclidean Lie algebras.*

Proof For all $Z \in \mathfrak{z}_2$ and all $X, Y \in \mathfrak{v}_1$ we have

$$\begin{aligned} \langle Z, [\psi(X), \psi(Y)] \rangle &= \langle J_2(Z)(\psi(X)), \psi(Y) \rangle = \langle \psi(J_1(\varphi^{-1}(Z))X), \psi(Y) \rangle \\ &= \langle J_1(\varphi^{-1}(Z))X, Y \rangle = \langle \varphi^{-1}(Z), [X, Y] \rangle \\ &= \langle Z, \varphi([X, Y]) \rangle. \end{aligned}$$

Since $\langle \cdot, \cdot \rangle|_{\mathfrak{z}_2 \times \mathfrak{z}_2}$ is non-degenerate, it follows that $\varphi([X, Y]) = [\psi(X), \psi(Y)]$. Define $\chi := \varphi \times \psi$ and let $Z_1, Z_2 \in \mathfrak{z}_1, X_1, X_2 \in \mathfrak{v}_1$. Then

$$[Z_1 + X_1, Z_2 + X_2] = [X_1, X_2]$$

and

$$\begin{aligned} [\chi(Z_1 + X_1), \chi(Z_2 + X_2)] &= [\varphi(Z_1) + \psi(X_1), \varphi(Z_2) + \psi(X_2)] \\ &= [\psi(X_1), \psi(X_2)] = \varphi([X_1, X_2]) \\ &= \chi([Z_1 + X_1, Z_2 + X_2]). \end{aligned}$$

Finally, χ is clearly an isomorphism of Euclidean vector spaces. This completes the proof. \square

Remark 2.5 Suppose that $\mathfrak{n}_1, \mathfrak{n}_2$ are non-degenerate H -type algebras and let the map $\chi : \mathfrak{n}_1 \rightarrow \mathfrak{n}_2$ be an isomorphism between \mathfrak{n}_1 and \mathfrak{n}_2 as Euclidean Lie algebras. Then $\chi(Z(\mathfrak{n}_1)) = Z(\mathfrak{n}_2)$. Hence Lemma 2.2 implies that χ is an isomorphism of H -type algebras. In turn, Lemma 2.4 shows that the isomorphisms between \mathfrak{n}_1 and \mathfrak{n}_2 as H -type algebras coincide with the isomorphisms between \mathfrak{n}_1 and \mathfrak{n}_2 as Euclidean Lie algebras.

For an H -type algebra $(\mathfrak{g}, \mathfrak{v}, J)$ we often write $J_Z X$ instead of $J(Z)X$, and we abbreviate the set $\{J_Z \mid Z \in \mathfrak{z}\}$ with $J_{\mathfrak{z}}$.

Definition 2.6 An H -type algebra $\mathfrak{n} = (\mathfrak{g}, \mathfrak{v}, J)$ is said to satisfy the J^2 -condition if

$$\forall X \in \mathfrak{v} \forall Z_1, Z_2 \in \mathfrak{z} : ((Z_1, Z_2) = 0 \Rightarrow \exists Z_3 \in \mathfrak{z} : J_{Z_1} J_{Z_2} X = J_{Z_3} X). \tag{H3}$$

If \mathfrak{n} is abelian, then (H3) is trivially satisfied.

2.2 C -module structures and the J^2 -condition

A C -module structure is a triple (C, V, J) consisting of two Euclidean vector spaces C and V and an \mathbb{R} -bilinear map $J : C \times V \rightarrow V$ satisfying the following properties:

- (M1) there exists $e \in C \setminus \{0\}$ such that $J(e, v) = v$ for all $v \in V$,
- (M2) for all $\zeta \in C$ and all $v \in V$ we have $|J(\zeta, v)| = |\zeta||v|$.

The cases where $V = \{0\}$ or $C = \mathbb{R}e$ are not excluded. We refer to these as *degenerate*. If $V \neq \{0\}$ and $C \neq \mathbb{R}e$, then the C -module structure (C, V, J) is called *non-degenerate*. For brevity, a C -module structure (C, V, J) is sometimes called a C -module structure¹ on V . One easily proves the following lemma.

Lemma 2.7 Let (C, V, J) be a C -module structure. If $V \neq \{0\}$, then the element e in (M1) is uniquely determined and of unit length.

¹ The “ C ” is “ C -module structure” or in “ C -module structure on V ” does not refer to the Euclidean space C in the triple (C, V, J) . Hence, if (C', V, J') satisfies (M1) and (M2), then it is still called a C -module structure on V .

The construction of a symmetric space from a C -module structure (satisfying the J^2 -condition defined below) depends on the choice of e in (M1) and uses $|e| = 1$. If (C, V, J) is a C -module structure with $V = \{0\}$, then J vanishes everywhere. Thus every element $a \in C \setminus \{0\}$ satisfies $J(a, \cdot) = \text{id}_V$. In this case, we endow C with a distinguished vector e of unit length and fix it (sometimes) in the notation as (C, e, V, J) . The influence of the particular choice of e on the constructed symmetric space is much weaker than that of the different ordered decompositions of degenerate H -type algebras. In fact, the choice of e determines the orthogonal decomposition $C = \mathbb{R}e \oplus C'$ (see below). If e_1, e_2 are two choices for e , then there is an isomorphism between $\mathbb{R}e_1 \oplus C'_1$ and $\mathbb{R}e_2 \oplus C'_2$ as Euclidean vector spaces which respects the decompositions. In turn, (C, e_1, V, J) and (C, e_2, V, J) are isomorphic as C -module structures (see below for the definition of isomorphism).

If (C, V, J) is a non-degenerate C -module structure, then the element e in (M1) is unique by Lemma 2.7. For reasons of uniformity, also in this case, we will often use the notation (C, e, V, J) for (C, V, J) .

If (C, V, J) is a C -module structure, then we will use $J_\zeta v$ or ζv to abbreviate $J(\zeta, v)$. Further, we set $Cv := \{\zeta v \mid \zeta \in C\}$ for $v \in V$.

A C -module structure (C, V, J) is said to satisfy the J^2 -condition if

$$C(Cv) = Cv \quad \text{for all } v \in V. \tag{M3}$$

In this case, V is called² a J^2C -module and (C, V, J) a J^2C -module structure.

Let (C_1, e_1, V_1, J_1) and (C_2, e_2, V_2, J_2) be C -module structures. An isomorphism³ from (C_1, e_1, V_1, J_1) to (C_2, e_2, V_2, J_2) is a pair (φ, ψ) of isomorphisms of Euclidean vector spaces $\psi: V_1 \rightarrow V_2$ and $\varphi: C_1 \rightarrow C_2$ with $\varphi(e_1) = e_2$ such that the diagram

$$\begin{array}{ccc} C_1 \times V_1 & \xrightarrow{J_1} & V_1 \\ \varphi \times \psi \downarrow & & \downarrow \psi \\ C_2 \times V_2 & \xrightarrow{J_2} & V_2 \end{array}$$

commutes.

The requirement that $\varphi(e_1) = e_2$ is relevant only if one of the C -module structures is degenerate. In fact, if (C_1, e_1, V_1, J_1) and (C_2, e_2, V_2, J_2) are non-degenerate C -module structures and (φ, ψ) is a pair of isomorphisms of Euclidean vector spaces $\psi: V_1 \rightarrow V_2$ and $\varphi: C_1 \rightarrow C_2$ such that $J_2 \circ (\varphi \times \psi) = \psi \circ J_1$, then

$$J_2(\varphi(e_1), v) = \psi(J_1(e_1, \psi^{-1}(v))) = v$$

for each $v \in V$. The uniqueness of e_2 shows that $\varphi(e_1) = e_2$.

For a C -module structure (C, e, V, J) let $C' := e^\perp$ denote the orthogonal complement of $\mathbb{R}e$ in C . For $\zeta = ae + z \in C$ with $a \in \mathbb{R}$ and $z \in C'$ we set $\text{Re } \zeta := a$, the real part of ζ , and $\text{Im } \zeta := z$, the imaginary part⁴ of ζ . Further we set $\bar{\zeta} := ae - z$, the conjugate of ζ . We will use the identification $\mathbb{R}e \rightarrow \mathbb{R}, ae \mapsto a$, of Euclidean vector spaces.

² As with the “ C ” in “ C -module structure”, the “ J^2 ” in “ J^2 -condition” and “ J^2C -module” does not refer to the map J .

³ In [15] and [16] the condition $\varphi(e_1) = e_2$ is omitted from the definition. However, as discussed, this condition is needed and indeed this stronger notion of isomorphism is used in their work.

⁴ Note that, in contrast to the usual definition in complex analysis, if $C = \mathbb{C}$ and $\zeta = a + ib \in \mathbb{C}$, then one has here $\text{Im } \zeta = ib$.

2.3 Bijection between H -type algebras and C -module structures

Let $\mathfrak{n} = (\mathfrak{g}, \mathfrak{v}, J)$ be an H -type algebra. Endow \mathbb{R} with the standard inner product and consider the Euclidean direct sum $\mathfrak{c} := \mathbb{R} \oplus \mathfrak{g}$. The map $\tilde{J}: \mathfrak{c} \times \mathfrak{v} \rightarrow \mathfrak{v}$, defined by

$$\tilde{J}(t + Z, X) := tX + J_Z X$$

for all $t + Z \in \mathbb{R} \oplus \mathfrak{g}$, $X \in \mathfrak{v}$, is \mathbb{R} -bilinear. Since $\langle J_Z X, X \rangle = \langle Z, [X, X] \rangle = 0$, we further find

$$\begin{aligned} |\tilde{J}(t + Z, X)|^2 &= |tX + J_Z X|^2 = t^2|X|^2 + |J_Z X|^2 \\ &= t^2|X|^2 + |Z|^2|X|^2 = (t^2 + |Z|^2)|X|^2 \\ &= |t + Z|^2|X|^2, \end{aligned}$$

hence $|\tilde{J}(t + Z, X)| = |t + Z||X|$. Moreover, for each $X \in \mathfrak{v}$ we have

$$\tilde{J}(1, X) = X.$$

Therefore, $(\mathfrak{c}, \mathfrak{v}, \tilde{J})$ is a C -module structure with $e = 1$. The condition (H3) is easily seen to be equivalent to

$$\tilde{J}(c)\tilde{J}(c)X = \tilde{J}(c)X \quad \text{for all } X \in \mathfrak{v}. \tag{H3'}$$

Thus, $(\mathfrak{c}, \mathfrak{v}, \tilde{J})$ satisfies the J^2 -condition if and only if \mathfrak{n} does. This construction provides an assignment of a C -module structure to each H -type algebra (with fixed ordered decomposition).

Vice versa, let (C, e, V, J) be a C -module structure and let $[\cdot, \cdot]: V \times V \rightarrow C'$ be the map defined by

$$\langle z, [x, y] \rangle = \langle J(z, x), y \rangle \tag{2.1}$$

for all $z \in C'$, all $x, y \in V$. Riesz' Representation Theorem (or its finite-dimensional counterpart) shows that $[\cdot, \cdot]$ is well-defined. We extend $[\cdot, \cdot]$ to the Euclidean direct sum $C' \oplus V$ by

$$[z_1 + v_1, z_2 + v_2] := [v_1, v_2]$$

for all $z_j + v_j \in C' \oplus V$. This map is \mathbb{R} -bilinear. Since J_z is skew-symmetric for each $z \in C'$ (cf. Sect. 2.5), the (extended) map $[\cdot, \cdot]$ is anti-symmetric. Moreover, $[V, V] \subseteq C'$ and $[V, C'] = [C', C'] = \{0\}$ imply the Jacobi identity for $[\cdot, \cdot]$. Thus, $C' \oplus V$ endowed with $[\cdot, \cdot]$ is a Euclidean Lie algebra. Let $J': C' \rightarrow \text{End}_{\text{vs}}(V)$ denote the map $J'(z)(v) = J(z, v)$. Then (C', V, J') is an H -type algebra. Using the equivalence of (H3) and (H3'), we see that this H -type algebra satisfies the J^2 -condition if and only if (C, V, J) does so.

Using the identification $e = 1$ from Sect. 2.2, the construction of a C -module structure from an H -type algebra

$$(\mathfrak{g}, \mathfrak{v}, J) \mapsto (\mathbb{R} \oplus \mathfrak{g}, 1, \mathfrak{v}, \tilde{J})$$

and that of an H -type algebra from a C -module structure

$$(C, e, V, J) \mapsto (C', V, J')$$

are inverse to each other. Moreover, one easily sees that these constructions are equivariant under isomorphisms of C -module structures, respectively of H -type algebras. More precisely, if $(\varphi, \psi): (\mathfrak{g}_1, \mathfrak{v}_1, J_1) \rightarrow (\mathfrak{g}_2, \mathfrak{v}_2, J_2)$ is an isomorphism of H -type algebras, then

$$(\text{id} \times \varphi, \psi) : (\mathbb{R} \oplus \mathfrak{z}_1, 1, \mathfrak{v}_1, \tilde{J}_1) \rightarrow (\mathbb{R} \oplus \mathfrak{z}_2, 1, \mathfrak{v}_2, \tilde{J}_2)$$

is an isomorphism of C -module structures.

Conversely, if $(\varphi, \psi) : (C_1, e_1, V_1, J_1) \rightarrow (C_2, e_2, V_2, J_2)$ is an isomorphism of C -module structures, then $\varphi(C'_1) = C'_2$ and the map

$$(\varphi|_{C'_1}, \psi) : (C'_1, V_1, J'_1) \rightarrow (C'_2, V_2, J'_2)$$

is an isomorphism of H -type algebras.

2.4 The model D

Let $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ be an H -type algebra. Further let \mathfrak{a} be a one-dimensional Euclidean Lie algebra and fix an element H of unit length in \mathfrak{a} . Then \mathfrak{a} is spanned by H . We denote by \mathfrak{s} the Euclidean direct sum Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{v}$ endowed with the Lie bracket that is determined by requiring that

$$\begin{aligned} [H, X] &= \frac{1}{2}X \quad \text{for all } X \in \mathfrak{v} \\ [H, Z] &= Z \quad \text{for all } Z \in \mathfrak{z} \end{aligned}$$

and that equals the original Lie bracket on \mathfrak{n} , when restricted to \mathfrak{n} .

Let $\exp(\mathfrak{s})$ be the connected, simply connected Lie group with Lie algebra \mathfrak{s} . We identify the tangent space to $\exp(\mathfrak{s})$ at the identity with \mathfrak{s} . Further we endow $\exp(\mathfrak{s})$ with the left- $\exp(\mathfrak{s})$ -invariant Riemannian metric that coincides with the inner product on \mathfrak{s} at the identity of $\exp(\mathfrak{s})$. We parametrize $\exp(\mathfrak{s})$ by

$$\begin{cases} \mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v} \rightarrow \exp(\mathfrak{s}) \\ (t, Z, X) \mapsto \exp(Z + X) \exp((\log t)H) \end{cases}$$

and set $S := \mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v}$. By requiring this parametrization to be a diffeomorphism and an isometry, S inherits the structure of a connected, simply connected Lie group with Riemannian metric. The Campbell–Baker–Hausdorff formula for $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ shows that the group operations on $\exp(\mathfrak{s})$ correspond on S to the group operations

$$(t_1, Z_1, X_1) (t_2, Z_2, X_2) = \left(t_1 t_2, Z_1 + t_1 Z_2 + \frac{1}{2} t_1^{1/2} [X_1, X_2], X_1 + t_1^{1/2} X_2 \right) \tag{2.2}$$

$$(t, Z, X)^{-1} = (t^{-1}, -t^{-1}Z, -t^{-1/2}X) \tag{2.3}$$

for all $(t_j, Z_j, X_j), (t, Z, X) \in S$. The differential structure on S coincides with the differential structure on the open subset $\mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v}$ of some \mathbb{R}^m . Now let

$$D := \left\{ (t, Z, X) \in \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \mid t > \frac{1}{4}|X|^2 \right\}$$

and consider the bijection

$$\Theta : \begin{cases} \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \rightarrow \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \\ (t, Z, X) \mapsto \left(t + \frac{1}{4}|X|^2, Z, X \right). \end{cases} \tag{2.4}$$

Then $\Theta(S) = D$, so that we define the structure of a Riemannian manifold on D by requiring Θ to be an isometry. The differential structure on D is identical to that of D being an open

subset of $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$. Moreover, Θ induces a simply transitive action of S on D by defining

$$s \cdot p := \Theta (s\Theta^{-1}(p))$$

for $s \in S$ and $p \in D$. In coordinates $s = (t_s, Z_s, X_s)$ and $p = (t_p, Z_p, X_p)$ this action reads

$$s \cdot p = \left(t_s t_p + \frac{1}{4}|X_s|^2 + \frac{1}{2}t_s^{1/2}\langle X_s, X_p \rangle, Z_s + t_s Z_p + \frac{1}{2}t_s^{1/2}[X_s, X_p], X_s + t_s^{1/2}X_p \right) \tag{2.5}$$

Due to the definition of the Riemannian metric, S obviously acts by isometries. We call $o_D := (1, 0, 0)$ the *base point* of D . The geodesic inversion σ of D at o_D is given by

$$\sigma(t, Z, X) = \frac{1}{t^2 + |Z|^2} (t, -Z, (-t + J_Z)X)$$

for all $(t, Z, X) \in D$. Then σ is an isometry, and hence D a symmetric space, if and only if \mathfrak{n} satisfies the J^2 -condition. In this case, D has rank one and, if in addition \mathfrak{n} is non-trivial, then D is of noncompact type. If $\mathfrak{n} = \{0\}$, then the constructed space D is the rank one Euclidean symmetric space \mathbb{R} .

Conversely, let D be a rank one symmetric space of noncompact type. Suppose that \mathfrak{g} is the simple Lie algebra of the Lie group of Riemannian isometries of D . Let ϑ be a Cartan involution of \mathfrak{g} , and let \mathfrak{k} and \mathfrak{p} be its $+1$ - respectively -1 -eigenspace. Fix a maximal abelian subalgebra \mathfrak{a} of \mathfrak{p} and choose a vector $H \in \mathfrak{a}$ which spans \mathfrak{a} . Then the decomposition of \mathfrak{g} into restricted root spaces is

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha},$$

where

$$\mathfrak{g}_{\beta} := \{X \in \mathfrak{g} \mid [H, X] = \beta(H)X\}$$

for the linear functional $\beta : \mathfrak{a} \rightarrow \mathbb{R}$ and

$$\mathfrak{m} := \{X \in \mathfrak{k} \mid [H, X] = 0\}.$$

We suppose that H is normalized such that $\alpha(H) = \frac{1}{2}$. If we set $p := \dim \mathfrak{g}_{\alpha}$ and $q := \dim \mathfrak{g}_{2\alpha}$ and if we endow $\mathfrak{n} := \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{\alpha}$ with the inner product

$$\langle X, Y \rangle := -\frac{1}{p + 4q} B(X, \vartheta Y)$$

where B is the Killing form of \mathfrak{g} , then $\mathfrak{n} = (\mathfrak{g}_{2\alpha}, \mathfrak{g}_{\alpha}, J)$ is an H -type algebra with J^2 -condition. The symmetric space constructed from $(\mathfrak{g}_{2\alpha}, \mathfrak{g}_{\alpha}, J)$ is exactly D . This means that each rank one Riemannian symmetric space of noncompact type arises from the construction above.

2.5 The ball model B

Let (C, e, V, J) be a J^2C -module structure and let $W := C \oplus V$ be the Euclidean direct sum of C and V . Consider the unit disc

$$B := \{w \in W \mid |w| < 1\}$$

in W and endow it with the differential structure induced from W . In the following we will define a Riemannian metric on B with respect to which B is a rank one Riemannian symmetric space of noncompact type if $(C, V) \neq (\mathbb{R}e, \{0\})$.

Polarization of the equation in (M2) shows that we have

$$\langle \zeta u, \eta v \rangle + \langle \eta u, \zeta v \rangle = 2\langle \zeta, \eta \rangle \langle u, v \rangle \tag{2.6}$$

for all $\zeta, \eta \in C$ and all $u, v \in V$. For $\zeta \in C$ let J_ζ^* denote the adjoint of J_ζ . Then (2.6) implies

$$J_\zeta^* = J_{\bar{\zeta}}. \tag{2.7}$$

Moreover we have

$$J_{\bar{\zeta}} J_\zeta = |\zeta|^2 \text{id}_V = J_\zeta J_{\bar{\zeta}}. \tag{2.8}$$

Thus, if we set $\zeta^{-1} := |\zeta|^{-2} \bar{\zeta}$ for $\zeta \in C \setminus \{0\}$, then we have

$$\zeta^{-1} (\zeta v) = v = \zeta (\zeta^{-1} v) \tag{2.9}$$

for all $v \in V$. In Sect. 4.1 we will see that, if $V \neq \{0\}$, there is a multiplication on C such that ζ^{-1} is the inverse of ζ .

Definition 2.8 Let $(\zeta, v), (\eta, u) \in W \setminus \{0\}$. Then (ζ, v) is called *equivalent* to (η, u) , if either $\zeta = 0 = \eta$ and $u \in Cv$, or $\zeta \neq 0 \neq \eta$ and $\zeta^{-1}v = \eta^{-1}u$. In this case, we write $(\zeta, v) \sim (\eta, u)$.

One easily proves that \sim is an equivalence relation on $W \setminus \{0\}$. For an element $w \in W \setminus \{0\}$ let

$$Cw := \{w' \in W \setminus \{0\} \mid w' \sim w\} \cup \{0\}$$

denote the equivalence class of w together with the element $0 \in W$. Since B is an open subset of the real vector space W , we shall identify the tangent space $T_w B$ to the point $w \in B$ with W . The Riemannian metric $w \mapsto \langle \cdot, \cdot \rangle_{w-}$ on B is defined by

$$\langle X, Y \rangle_{0-} := 4\langle X, Y \rangle \text{ on } T_0 B \tag{2.10}$$

and, for $w \in B \setminus \{0\}$, by

$$\langle X, Y \rangle_{w-} := \begin{cases} 4 \frac{\langle X, Y \rangle}{(1-|w|^2)^2} & \text{if } X, Y \in Cw \\ 4 \frac{\langle X, Y \rangle}{1-|w|^2} & \text{if } X, Y \in (Cw)^\perp \\ 0 & \text{if } X \in Cw, Y \in (Cw)^\perp \text{ (or vice versa).} \end{cases} \tag{2.11}$$

2.6 The Cayley transform

Let $\mathfrak{n} = (\mathfrak{g}, \mathfrak{v}, J)$ be a non-trivial H -type algebra which satisfies the J^2 -condition. Further let \mathfrak{a} be a one-dimensional Euclidean Lie algebra with fixed unit length vector H . Suppose that (C, e, V, J) is the J^2C -module structure associated to $(\mathfrak{g}, \mathfrak{v}, J)$ (see Sect. 2.3). We consider the Riemannian symmetric spaces D and B which are constructed in Sect. 2.4, respectively, 2.5.

The Cayley transform (in $\mathbb{R} \times \mathfrak{g} \times \mathfrak{v}$ -coordinates, see [7, (2.10a)])

$$C: \begin{cases} B \rightarrow D \\ (t, Z, X) \mapsto \frac{1}{(1-t)^2+|Z|^2} (1-t^2-|Z|^2, 2Z, 2(1-t+J_Z)X) \end{cases}$$

is clearly a diffeomorphism from B to D . Its inverse (see [7, (2.10b)]) is given by

$$C^{-1}: \begin{cases} D \rightarrow B \\ (t, Z, X) \mapsto \frac{1}{(1+t)^2+|Z|^2} (-1 + t^2 + |Z|^2, 2Z, (1 + t - J_Z)X) . \end{cases}$$

Proposition 2.9 *The Cayley transform is an isometry.*

Proof In [7], the pullback of the Riemannian metric of D to B via C is described as follows: Let $p \in B$. Denote the tangent space to B at p by $T_p B$, which we identify with $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$. If $\|\mathfrak{x}\|_p$ denotes the norm of $\mathfrak{x} \in T_p B$ induced by the pullback of the Riemannian metric on D , then

$$\|\mathfrak{x}\|_0 = 2|\mathfrak{x}|$$

for all $\mathfrak{x} \in T_0 B$. For $p \in B \setminus \{0\}$ we have

$$\|\mathfrak{x}\|_p^2 = \begin{cases} 4 \frac{|\mathfrak{x}|^2}{1-|p|^2} & \text{if } \mathfrak{x} \in T_p^{(1)}, \\ 4 \frac{|\mathfrak{x}|^2}{(1-|p|^2)^2} & \text{if } \mathfrak{x} \in \mathbb{R}p \oplus T_p^{(2)} \end{cases}$$

where

$$T_p B = \mathbb{R}p \oplus T_p^{(2)} \oplus T_p^{(1)}$$

is a direct sum which is orthogonal w.r.t. Euclidean metric and Riemannian inner product on $T_p B$. Theorem 6.8 in [7] provides explicit formulas for $T_p^{(1)}$ and $\mathbb{R}p \oplus T_p^{(2)}$, which we will state in the following. For $X \in \mathfrak{v}$ let $j(X) := J_3 X$ and define $\mathfrak{k}(X)$ to be the orthogonal complement of $\mathbb{R}X$ in

$$j(X)^\perp = \{Y \in \mathfrak{v} \mid \forall Z \in \mathfrak{z}: \langle J_Z X, Y \rangle = 0\}.$$

For $p = (t, Z, X) \in B \setminus \{0\}$ we have

- (i) $T_p^{(1)} = \mathfrak{v}$ and $\mathbb{R}p \oplus T_p^{(2)} = \mathbb{R} \oplus \mathfrak{z}$ if $X = 0$,
- (ii) $T_p^{(1)} = \mathbb{R} \oplus \mathfrak{z} \oplus \mathfrak{k}(X)$ and $\mathbb{R}p \oplus T_p^{(2)} = j(X) \oplus \mathbb{R}X$ if $(t, Z) = (0, 0)$,
- (iii) in the remaining cases,

$$T_p^{(1)} = \mathfrak{k}(X) \oplus \{(|X|^2 u, |X|^2 W, -(u + J_W)(t - J_Z)W) \mid W \in \mathfrak{z}, u \in \mathbb{R}\}$$

and

$$\mathbb{R}p \oplus T_p^{(2)} = \{(t^2 + |Z|^2)u, (t^2 + |Z|^2)W, (u + J_W)(t - J_Z)X \mid W \in \mathfrak{z}, u \in \mathbb{R}\}.$$

Note that this subsumes the degenerate cases. On $T_0 B$, the Riemannian inner product (2.10) obviously coincides with this one. For $p \in B \setminus \{0\}$, (2.11) implies that it suffices to show that $\mathbb{R}p \oplus T_p^{(2)} = Cp$. To that end let $p = (\zeta, v) = (t, Z, X) \in B \setminus \{0\}$ (hence $\zeta = (t, Z) \in C = \mathbb{R} \times \mathfrak{z}$ and $v = X \in V = \mathfrak{v}$). If $v = 0$, then

$$Cp = C \times \{0\} = C = \mathbb{R} \times \mathfrak{z} = \mathbb{R}p \oplus T_p^{(2)}.$$

If $\zeta = 0$, then

$$Cp = Cv = (\mathbb{R} + J_3)X = \mathbb{R}p \oplus T_p^{(2)}.$$

If $\zeta \neq 0$ and $v \neq 0$, then $(\eta, u) \in Cp$ if and only if $\eta \neq 0$ and $\zeta^{-1}v = \eta^{-1}u$, or $(\eta, u) = 0$. This means that in both cases

$$u = J_\eta J_{\zeta^{-1}}v = |\zeta|^{-2} J_\eta J_{\bar{\zeta}}v = J_{|\zeta|^{-2}\eta} J_{\bar{\zeta}}v.$$

Hence

$$Cp = \left\{ \left(\eta, J_{|\zeta|^{-2}\eta} J_{\bar{\zeta}} v \right) \mid \eta \in C \right\} = \left\{ \left(|\zeta|^2 \xi, J_{\xi} J_{\bar{\zeta}} v \right) \mid \xi \in C \right\} = \mathbb{R}p \oplus T_p^{(2)}.$$

This completes the proof. □

Note that for $n = \{0\}$, respectively, $(C, V) = (\mathbb{R}e, \{0\})$ Proposition 2.9 shows that the model B is isometric to the Euclidean symmetric space \mathbb{R} .

2.7 The map β_2

Let (C, e, V, J) be a J^2C -module structure. In this section, we introduce a map

$$\beta_2: V \times V \rightarrow C,$$

which will be shown to be C -hermitian, i.e., β_2 is \mathbb{R} -bilinear and for all $u, v \in V$ we have $\beta_2(u, v) = \overline{\beta_2(v, u)}$. The map β_2 encodes the inner product and the Lie bracket on V . We define $\beta_2: V \times V \rightarrow C$ by⁵

$$\langle \beta_2(v, u), \zeta \rangle := \langle J_{\zeta} u, v \rangle \quad \text{for all } \zeta \in C.$$

Lemma 2.10 For $\zeta, \eta \in C$ we have $\langle \eta, \bar{\zeta} \rangle = \langle \bar{\eta}, \zeta \rangle$.

Proof Let $\zeta = a + x$ and $\eta = b + y$ ($a, b \in \mathbb{R}, x, y \in C'$) be the decompositions of ζ and η w.r.t. $C = \mathbb{R} \oplus C'$. Since \mathbb{R} and C' are orthogonal, we find

$$\begin{aligned} \langle \bar{\zeta}, \eta \rangle &= \langle a, b \rangle - \langle x, b \rangle + \langle a, y \rangle - \langle x, y \rangle \\ &= \langle a, b \rangle - \langle x, y \rangle \\ &= \langle a, b \rangle + \langle x, b \rangle - \langle a, y \rangle - \langle x, y \rangle \\ &= \langle \zeta, \bar{\eta} \rangle. \end{aligned}$$

This proves the lemma. □

Proposition 2.11 The map $\beta_2: V \times V \rightarrow C$ is \mathbb{R} -bilinear. Further we have

- (i) $\beta_2(v, u) = \overline{\beta_2(u, v)}$ for all $u, v \in V$,
- (ii) $\beta_2(v, v) = \langle v, v \rangle$ for all $v \in V$.

Proof One easily sees that β_2 is \mathbb{R} -bilinear. Using Lemma 2.10 we find

$$\langle \beta_2(v, u), \zeta \rangle = \langle J_{\zeta} u, v \rangle = \langle u, J_{\bar{\zeta}} v \rangle = \langle \beta_2(u, v), \bar{\zeta} \rangle = \langle \overline{\beta_2(u, v)}, \zeta \rangle$$

for all $u, v \in V$ and all $\zeta \in C$. Hence $\beta_2(v, u) = \overline{\beta_2(u, v)}$, which proves (i). Finally let $v \in V$. From $\beta_2(v, v) = \overline{\beta_2(v, v)}$ it follows that $\beta_2(v, v) \in \mathbb{R}$. Then

$$\beta_2(v, v) = \langle \beta_2(v, v), e \rangle = \langle ev, v \rangle = \langle v, v \rangle,$$

which shows (ii). □

Lemma 2.12 For $v_1, v_2 \in V$ we have

$$\operatorname{Re} \beta_2(v_1, v_2) = \langle v_1, v_2 \rangle \quad \text{and} \quad \operatorname{Im} \beta_2(v_1, v_2) = [v_2, v_1].$$

⁵ This is J^* in [16].

Proof Let $v_1, v_2 \in V$. By Proposition 2.11 we have

$$\begin{aligned} |v_1|^2 + 2\langle v_1, v_2 \rangle + |v_2|^2 &= |v_1 + v_2|^2 = \beta_2(v_1 + v_2, v_1 + v_2) \\ &= |v_1|^2 + \beta_2(v_1, v_2) + \beta_2(v_2, v_1) + |v_2|^2 \\ &= |v_1|^2 + \beta_2(v_1, v_2) + \overline{\beta_2(v_1, v_2)} + |v_2|^2 \\ &= |v_1|^2 + 2 \operatorname{Re} \beta_2(v_1, v_2) + |v_2|^2. \end{aligned}$$

Hence $\operatorname{Re} \beta_2(v_1, v_2) = \langle v_1, v_2 \rangle$. To show the second claim, note that Proposition 2.11 implies that

$$\operatorname{Im} \beta_2(v_1, v_2) = \frac{1}{2}\beta_2(v_1, v_2) - \frac{1}{2}\overline{\beta_2(v_1, v_2)} = \frac{1}{2}\beta_2(v_1, v_2) - \frac{1}{2}\beta_2(v_2, v_1).$$

For each $\zeta \in C'$ it follows that

$$\begin{aligned} \langle \zeta, \operatorname{Im} \beta_2(v_1, v_2) \rangle &= \frac{1}{2}\langle \zeta, \beta_2(v_1, v_2) \rangle - \frac{1}{2}\langle \zeta, \beta_2(v_2, v_1) \rangle \\ &= \frac{1}{2}\langle J_\zeta v_2, v_1 \rangle - \frac{1}{2}\langle J_\zeta v_1, v_2 \rangle \\ &= \frac{1}{2}\langle J_\zeta v_2, v_1 \rangle + \frac{1}{2}\langle v_1, J_\zeta v_2 \rangle \\ &= \langle J_\zeta v_2, v_1 \rangle = \langle \zeta, [v_2, v_1] \rangle. \end{aligned}$$

Since $\langle \cdot, \cdot \rangle|_{C' \times C'}$ is non-degenerate and $\operatorname{Im} \beta_2(v_1, v_2) \in C'$ and $[v_2, v_1] \in C'$, it follows that $\operatorname{Im} \beta_2(v_1, v_2) = [v_2, v_1]$. □

2.8 The isometry group

Let $\mathfrak{n} = (\mathfrak{g}, \mathfrak{v}, J)$ be a non-trivial H -type algebra which satisfies the J^2 -condition, and let \mathfrak{a} be a one-dimensional Euclidean Lie algebra. Construct the Euclidean Lie algebra \mathfrak{s} and the spaces S and D as in Sect. 2.4. We denote by (C, e, V, J) the J^2C -module structure $(\mathbb{R} \oplus \mathfrak{g}, 1, \mathfrak{v}, \tilde{J})$ which is isomorphic to \mathfrak{n} . Let G denote the full isometry group of D . Suppose that N , respectively, A are the connected, simply connected Lie groups with Lie algebra \mathfrak{n} , respectively, \mathfrak{a} . Then N and A are subgroups of S , more precisely, S is the semidirect product AN . In the parametrization of S , the groups N and A are given by

$$N = \{n_{(Z,X)} := (1, Z, X) \mid (Z, X) \in \mathfrak{z} \times \mathfrak{v}\}$$

and

$$A = \{a_t := (t, 0, 0) \mid t \in \mathbb{R}^+\}.$$

Let K be the stabilizer of the base point $o_D = (1, 0, 0)$ in G and let $M := Z_K(A)$ be the centralizer of A in K . Recall the geodesic inversion

$$\sigma(t, Z, X) = \frac{1}{t^2 + |Z|^2} (t, -Z, (-t + J_Z)X)$$

at the origin o_D from Sect. 2.4.

Theorem 2.13 (Theorem 6.4 in [7]) *The Lie group G has the Bruhat decomposition $MAN \cup N\sigma MAN$.*

Let $\mathcal{X} := \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$, where ∞ denotes the point at infinity. A compactification of D is then given by the closure of D in \mathcal{X} , namely

$$\begin{aligned} \overline{D}^g &= \left\{ (t, Z, X) \in \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \mid t \geq \frac{1}{4}|X|^2 \right\} \cup \{\infty\} \\ &= \left\{ (\zeta, v) \in C \times V \mid \operatorname{Re} \zeta \geq \frac{1}{4}|v|^2 \right\} \cup \{\infty\}. \end{aligned}$$

The space \overline{D}^g is precisely the geodesic compactification of D , see, e. g., [3, Sect. I.2] or [8, Proposition 1.7.6]. Let \overline{B} be the closed unit ball in $W = \mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$. Then the Cayley transform $C: B \rightarrow D$ extends (uniquely) to a homeomorphism $\overline{B} \rightarrow \overline{D}^g$. Therefore, [7, Corollary 6.2] amounts to the following proposition.

Proposition 2.14 *The action of G extends continuously to \overline{D}^g .*

The Bruhat decomposition of G implies that the stabilizer G_∞ of ∞ in G equals MAN .

For future purposes we need explicit formulas for the action of the groups M, A and N on \overline{D}^g . The action of N and A in $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$ -coordinates is already given in (2.5). Suppose that $(\zeta, v) \in \overline{D}^g \setminus \{\infty\}$. For $a_t \in A$ we have

$$a_t \infty = \infty \quad \text{and} \quad a_t(\zeta, v) = (t\zeta, t^{1/2}v).$$

For $n_{(Z,X)} \in N$ we get $n_{(Z,X)}\infty = \infty$ and

$$n_{(Z,X)}(\zeta, v) = \left(\frac{1}{4}|X|^2 + Z + \zeta + \frac{1}{2}\beta_2(v, X), X + v \right).$$

In $C \times V$ -coordinates, the geodesic inversion σ reads as

$$\sigma \infty = 0, \quad \sigma 0 = \infty,$$

and for $(\zeta, v) \in \overline{D}^g \setminus \{0, \infty\}$,

$$\begin{aligned} \sigma(\zeta, v) &= \frac{1}{(\operatorname{Re} \zeta)^2 + |\operatorname{Im} \zeta|^2} (\operatorname{Re} \zeta, -\operatorname{Im} \zeta, J_{-\operatorname{Re} \zeta + \operatorname{Im} \zeta} v) \\ &= |\zeta|^{-2} (\bar{\zeta}, J_{-\bar{\zeta}} v) = (\zeta^{-1}, J_{-|\zeta|^{-2}\bar{\zeta}} v) \\ &= (\zeta^{-1}, -\zeta^{-1}v) = \zeta^{-1}(1, -v). \end{aligned}$$

The Cayley transform in $C \times V$ -coordinates is

$$C : \begin{cases} B \rightarrow D \\ (\zeta, v) \mapsto |1 - \zeta|^{-2} (1 + 2 \operatorname{Im} \zeta - |\zeta|^2, 2(1 - \bar{\zeta})v) \end{cases}$$

with

$$C^{-1} : \begin{cases} D \rightarrow B \\ (\eta, u) \mapsto |1 + \eta|^{-2} (-1 + 2 \operatorname{Im} \eta + |\eta|^2, 2(1 + \bar{\eta})u). \end{cases}$$

The following proposition provides the explicit form of the group M .

Proposition 2.15 *The group M is the group of automorphisms of the J^2C -module structure (C, e, V, J) , that is, the elements of M are the pairs (φ, ψ) of orthogonal endomorphisms*

$\varphi: C \rightarrow C$ with $\varphi(e) = e$ and $\psi: V \rightarrow V$ such that the diagram

$$\begin{array}{ccc} C \times V & \xrightarrow{J} & V \\ \varphi \times \psi \downarrow & & \downarrow \psi \\ C \times V & \xrightarrow{J} & V \end{array}$$

commutes. If $(\varphi, \psi) \in M$, then its action on $\overline{D^g}$ is given by $(\varphi, \psi)(\infty) = \infty$ and $(\varphi, \psi)(\zeta, v) = (\varphi(\zeta), \psi(v))$ for $(\zeta, v) \in \overline{D^g} \setminus \{\infty\}$.

Proof Let $\tilde{G} := C^{-1}GC$. For each $g \in G$ set $\tilde{g} := C^{-1}gC$, and for each subset T of G let $\tilde{T} := C^{-1}TC$ be the corresponding subset of \tilde{G} . Clearly, \tilde{G} is the full isometry group of B . We will first characterize the centralizer $Z_{\tilde{K}}(\tilde{A})$ of \tilde{A} in \tilde{K} as a subgroup of \tilde{G} . Let \tilde{M} denote the automorphism group of (C, e, V, J) and define the action of $(\varphi, \psi) \in \tilde{M}$ on B by $(\varphi, \psi)(\zeta, v) := (\varphi(\zeta), \psi(v))$. We will show that $\tilde{M} = Z_{\tilde{K}}(\tilde{A})$. By [15, Proposition 4.1] we have that $\tilde{M} \subseteq \tilde{K}$. Let $a_t \in A$ and $(\zeta, v) \in B$. Then one easily calculates that

$$\begin{aligned} \tilde{a}_t(\zeta, v) &= C^{-1} \circ a_t \circ C(\zeta, v) \\ &= \left| |1 - \zeta|^2 + t(1 + 2 \operatorname{Im} \zeta - |\zeta|^2) \right|^{-2} \\ &\quad \times \left(-|1 - \zeta|^4 + 4t \operatorname{Im} \zeta + t^2 |1 + 2 \operatorname{Im} \zeta - |\zeta|^2|^2 \right), \\ &\quad 4t^{1/2} |1 - \zeta|^2 (|1 - \zeta|^2 + t(1 - \operatorname{Im} \zeta - |\zeta|^2)) ((1 - \bar{\zeta})v). \end{aligned}$$

Suppose that $\tilde{m} = (\varphi, \psi) \in \tilde{M}$. Since $\varphi(e) = e$, we have that $\varphi(\bar{\zeta}) = \overline{\varphi(\zeta)}$ and $\varphi(\operatorname{Im} \zeta) = \operatorname{Im} \varphi(\zeta)$ for each $\zeta \in C$. Moreover, $|\zeta| = |\varphi(\zeta)|$ for each $\zeta \in C$. Then the first component of $\tilde{m} \circ \tilde{a}_t(\zeta, v)$ is given by

$$\begin{aligned} &\varphi \left(\frac{-|1 - \zeta|^4 + 4t \operatorname{Im} \zeta + t^2 |1 + 2 \operatorname{Im} \zeta - |\zeta|^2|^2}{\left| |1 - \zeta|^2 + t(1 + 2 \operatorname{Im} \zeta - |\zeta|^2) \right|^2} \right) \\ &= \frac{-|1 - \zeta|^4 + 4t \operatorname{Im} \varphi(\zeta) + t^2 |1 + 2 \operatorname{Im} \zeta - |\zeta|^2|^2}{\left| |1 - \zeta|^2 + t(1 + 2 \operatorname{Im} \zeta - |\zeta|^2) \right|^2} \\ &= \frac{-|1 - \varphi(\zeta)|^4 + 4t \operatorname{Im} \varphi(\zeta) + t^2 |1 + 2 \operatorname{Im} \varphi(\zeta) - |\varphi(\zeta)|^2|^2}{\left| |1 - \varphi(\zeta)|^2 + t(1 + 2 \operatorname{Im} \varphi(\zeta) - |\varphi(\zeta)|^2) \right|^2}. \end{aligned}$$

This is the first component of $\tilde{a}_t \circ \tilde{m}(\zeta, v)$. The second component of $\tilde{m} \circ \tilde{a}_t(\zeta, v)$ reads as

$$\begin{aligned} &\psi \left(\frac{(|1 - \zeta|^2 + t(1 - 2 \operatorname{Im} \zeta - |\zeta|^2)) ((1 - \bar{\zeta})v)}{\left| |1 - \zeta|^2 + t(1 + 2 \operatorname{Im} \zeta - |\zeta|^2) \right|^2} \right) \\ &= \frac{\varphi(|1 - \zeta|^2 + t(1 - 2 \operatorname{Im} \zeta - |\zeta|^2)) \psi((1 - \bar{\zeta})v)}{\left| |1 - \zeta|^2 + t(1 + 2 \operatorname{Im} \zeta - |\zeta|^2) \right|^2} \\ &= \frac{(|1 - \zeta|^2 + t(1 - 2 \operatorname{Im} \varphi(\zeta) - |\zeta|^2)) (\varphi(1 - \bar{\zeta})\psi(v))}{\left| |1 - \zeta|^2 + t(1 + 2 \operatorname{Im} \zeta - |\zeta|^2) \right|^2} \\ &= \frac{(|1 - \varphi(\zeta)|^2 + t(1 - 2 \operatorname{Im} \varphi(\zeta) - |\varphi(\zeta)|^2)) ((1 - \overline{\varphi(\zeta)})\psi(v))}{\left| |1 - \varphi(\zeta)|^2 + t(1 + 2 \operatorname{Im} \varphi(\zeta) - |\varphi(\zeta)|^2) \right|^2}. \end{aligned}$$

Clearly, this is the second component of $\tilde{a}_t \circ \tilde{m}(\zeta, v)$. Hence, \tilde{m} commutes with each \tilde{a}_t . This shows that $\tilde{M} \subseteq Z_{\tilde{K}}(\tilde{A})$.

Conversely, suppose that $\tilde{m} \in Z_{\tilde{K}}(\tilde{A})$. We have to show that $\tilde{m}(1, 0) = (1, 0)$. Then [15, Proposition 4.1] implies that $\tilde{m} \in \tilde{M}$. For each $t \in \mathbb{R}^+$, we have

$$\tilde{a}_t(1, 0) = \lim_{n \rightarrow \infty} \tilde{a}_t \left(1 - \frac{1}{n}, 0 \right) = (1, 0).$$

Thus, $\tilde{a}_t(\tilde{m}(1, 0)) = \tilde{m}(1, 0)$. The only points on ∂B that are invariant under each \tilde{a}_t are $(1, 0)$ and $(-1, 0)$. Seeking a contradiction assume that $\tilde{m}(1, 0) = (-1, 0)$. Since $\tilde{\sigma} = -\text{id}$, we get $\tilde{\sigma} \circ \tilde{m}(1, 0) = (1, 0)$. By [15, Proposition 4.1] we have $\tilde{\sigma} \circ \tilde{m} \in \tilde{M}$. Our previous argument then shows that $\tilde{\sigma} \circ \tilde{m}$ commutes with all \tilde{a}_t . Therefore,

$$\tilde{a}_t \circ \tilde{\sigma} \circ \tilde{m} = \tilde{\sigma} \circ \tilde{m} \circ \tilde{a}_t = \tilde{\sigma} \circ \tilde{a}_t \circ \tilde{m},$$

which means that $\tilde{\sigma}$ commutes with each \tilde{a}_t . This is a contradiction. Hence it follows that $\tilde{m}(1, 0) = (1, 0)$, which by [15, Proposition 4.1] shows that $\tilde{m} \in \tilde{M}$. Therefore, $\tilde{M} = Z_{\tilde{K}}(\tilde{A})$. Now let $\tilde{m} = (\varphi, \psi) \in \tilde{M}$ and set $m := \mathcal{C} \circ \tilde{m} \circ \mathcal{C}^{-1}$. For $(\eta, u) \in D$ we find

$$\begin{aligned} m(\eta, u) &= \mathcal{C} \circ \tilde{m} \circ \mathcal{C}^{-1}(\eta, u) \\ &= \mathcal{C} \circ \tilde{m} (|1 + \eta|^{-2} (1 + 2 \operatorname{Im} \eta + |\eta|^2, 2(1 + \bar{\eta})u)) \\ &= \mathcal{C} (|1 + \varphi(\eta)|^{-2} (1 + 2 \operatorname{Im} \varphi(\eta) + |\varphi(\eta)|^2, 2(1 + \overline{\varphi(\eta)})\psi(u))) \\ &= (\varphi(\eta), \psi(u)). \end{aligned}$$

The remaining claim follows directly from continuity or, alternatively, from the extension of the Cayley transform to \overline{B} . □

3 Isometric fundamental regions

Throughout this section let (C, e, V, J) be a J^2C -module structure such that $(C, V) \neq (\mathbb{R}e, \{0\})$ and suppose that $(\mathfrak{z}, \mathfrak{v}, J)$ is the corresponding H -type algebra with J^2 -condition. Recall the model D of the rank one Riemannian symmetric space of noncompact type which is constructed from (C, e, V, J) , respectively, $(\mathfrak{z}, \mathfrak{v}, J)$ in Sect. 2.4. Let G denote the full isometry group of D .

The purpose of this section is to prove the existence of isometric fundamental regions for certain subgroups Γ of G . For this we first have to define the notion of the isometric sphere of $g \in G \setminus G_\infty$, which is a sphere w. r. t. Cygan metric. The Cygan metric is a metric on $\overline{D}^g \setminus \{\infty\}$ which arises from a certain group norm on $\overline{D}^g \setminus \{\infty\}$. This group norm is an extension of the Heisenberg pseudonorm.

3.1 The Cauchy–Schwarz Theorem for β_2

In our setting, the proof of the Cauchy–Schwarz Theorem as it is usually taught in linear algebra cannot be adapted to the map β_2 . Therefore, we provide an alternative proof for which the following two lemmas are needed.

Lemma 3.1 *Let $u, v \in V$. Then $\beta_2(v, u) = 0$ if and only if $v \in (Cu)^\perp$.*

Proof We have $\beta_2(v, u) = 0$ if and only if

$$\langle \beta_2(v, u), \zeta \rangle = 0 \quad \text{for all } \zeta \in C.$$

By the definition of β_2 , this holds if and only if

$$\langle \zeta u, v \rangle = 0 \quad \text{for all } \zeta \in C,$$

hence if and only if $v \in (Cu)^\perp$. □

Lemma 3.2 *Let $u \in V$ and $\lambda \in C$. Then $\beta_2(\lambda u, u) = |u|^2\lambda$.*

Proof By the definition of β_2 and the polarization (2.6) we have

$$\langle \beta_2(\lambda u, u), \zeta \rangle = \langle \zeta u, \lambda u \rangle = \langle \lambda, \zeta \rangle |u|^2 = \langle |u|^2\lambda, \zeta \rangle$$

for all $\zeta \in C$. Hence $\beta_2(\lambda u, u) = |u|^2\lambda$. □

Proposition 3.3 *Let $u, v \in V$. Then*

$$|\beta_2(u, v)| \leq |u||v|.$$

Equality holds if and only if $v \in Cu$ or $u \in Cv$.

Proof Let $(v_1, v_2) \in Cu \times (Cu)^\perp$ be the unique pair such that $v = v_1 + v_2$. Using that β_2 is C -hermitian (see Proposition 2.11), Lemma 3.1 yields

$$\beta_2(u, v) = \beta_2(u, v_1) + \beta_2(u, v_2) = \beta_2(u, v_1).$$

Then Lemma 3.2 shows

$$|\beta_2(u, v_1)|^2 = |u|^2|v_1|^2.$$

Clearly

$$|v_1|^2 \leq |v_1|^2 + |v_2|^2 = |v|^2,$$

where equality holds if and only if $v_2 = 0$, hence if and only if $v = v_1 \in Cu$. Thus,

$$|\beta_2(u, v)|^2 = |\beta_2(u, v_1)|^2 = |u|^2|v_1|^2 \leq |u|^2|v|^2,$$

where the inequality is an equality if and only if $u = 0$ or $v \in Cu$. This proves the claim. □

3.2 H-coordinates, Cygan metric, and isometric spheres

Let $z = (t, Z, X) \in \overline{D}^g \setminus \{\infty\}$ and recall the map Θ from (2.4). The *horospherical coordinates* or *H-coordinates* of z relative to the origin o_D of D are defined as

$$\Theta^{-1}(z) = \left(t - \frac{1}{4}|X|^2, Z, X \right).$$

To avoid confusion with the ordinary coordinates of \overline{D}^g , H-coordinates will be subscripted by h . Hence, the H-coordinates of z are denoted by $(t - \frac{1}{4}|X|^2, Z, X)_h$.

Since Θ induces a bijection between $\overline{D}^g \setminus \{\infty\}$ and the topological closure

$$\overline{S} = \mathbb{R}_0^+ \times \mathfrak{z} \times \mathfrak{v}$$

of S in $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$, there is a geometric characterization of H-coordinates. We elaborate on this in the following remark.

Remark and Definition 3.4 Let $z = (t, Z, X) = (\zeta, v) \in \overline{D}^g \setminus \{\infty\}$. The horosphere through z with center ∞ is the N -orbit of z . We extend the group A to the set

$$A^+ := A \cup \{a_0\},$$

where $a_0: \overline{D}^g \rightarrow \overline{D}^g$ is defined by $a_0\infty := \infty$ and $a_0z := 0$ for all $z \in \overline{D}^g \setminus \{\infty\}$. Then there is a unique pair $(a_s, n) \in A^+ \times N$ such that

$$na_s(o_D) = z.$$

The height of z is defined as

$$\text{ht}(z) := t - \frac{1}{4}|X|^2 = \text{Re } \zeta - \frac{1}{4}|v|^2.$$

Formula (2.5) shows that $s = \text{ht}(z)$ and $n = (1, Z, X)$. Hence the H-coordinates of z are

$$(\text{ht}(z), Z, X)_h = (\text{ht}(z), \text{Im } \zeta, v)_h.$$

This means that the H-coordinates are given by the height of the ∞ -centered horosphere on which z lies and the coordinates of z in the canonical parametrization of this horosphere.

The Cygan metric on D is a metric on $\overline{D}^g \setminus \{\infty\}$ which arises from a certain group norm. If (\mathcal{G}, \cdot) is a group with neutral element $1_{\mathcal{G}}$, then a map $p: \mathcal{G} \rightarrow \mathbb{R}_0^+$ is called a group norm if

- (GN1) $p(g) = 0$ if and only if $g = 1_{\mathcal{G}}$,
- (GN2) $p(g^{-1}) = p(g)$ for all $g \in \mathcal{G}$,
- (GN3) $p(gh) \leq p(g) + p(h)$ for all $g, h \in \mathcal{G}$.

Suppose that p is a group norm on \mathcal{G} , then the map $d: \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}, d(g, h) := p(g^{-1}h)$ is a metric on \mathcal{G} . It is called the metric induced by p .

A well-known example of a group norm and a pre-form of the group norm for the Cygan metric is the Heisenberg pseudonorm on N . More precisely, N inherits an inner product from its canonical bijection to $\mathfrak{z} \times \mathfrak{v}$. Then the Heisenberg group norm q (which is also known as the Heisenberg pseudonorm) on N is defined by

$$q(Z, X) := \left| \frac{1}{4}|X|^2 + Z \right|^{1/2} = \left(\frac{1}{16}|X|^4 + |Z|^2 \right)^{1/4}.$$

The last equality holds because \mathfrak{z} and $\mathbb{R} \cong \mathfrak{a}$ are orthogonal. Since each height level set of $\overline{D}^g \setminus \{\infty\}$ is isomorphic to N , the metric induced from the Heisenberg group norm measures the distance between two elements in the same height level set.

To be able to also measure the distance between elements in different height level sets, we extend the Heisenberg pseudonorm to the direct product of the groups $(\mathbb{R}, +)$ (“differences between height level sets”) and N by

$$p: \begin{cases} \mathbb{R} \times N & \rightarrow \mathbb{R} \\ (k, Z, X) & \mapsto \left| \frac{1}{4}|X|^2 + |k| + Z \right|^{1/2}. \end{cases}$$

Obviously, $p|_{\{0\} \times N} = q$.

Proposition 3.5 The map p is a group norm on $\mathbb{R} \times N$.

Proof The neutral element of $\mathbb{R} \times N$ is $(0, 0, 0)$, and \mathfrak{z} is orthogonal to \mathbb{R} . This implies that p satisfies (GN1). To prove (GN2) let $g = (k, Z, X) \in \mathbb{R} \times N$. Then

$$\begin{aligned} p(g^{-1}) &= p(-k, -Z, -X) = \left| \frac{1}{4}|X|^2 + |k| - Z \right|^{1/2} = \left(\left(\frac{1}{4}|X|^2 + |k| \right)^2 + |Z|^2 \right)^{1/4} \\ &= \left| \frac{1}{4}|X|^2 + |k| + Z \right|^{1/2} = p(g). \end{aligned}$$

The triangle equality (GN3) is shown in several steps. For each $g = (k, Z, X) \in \mathbb{R} \times N$ we have

$$\begin{aligned} p(g) &= \left| \frac{1}{4}|X|^2 + |k| + Z \right|^{1/2} = \left(\left(\frac{1}{4}|X|^2 + |k| \right)^2 + |Z|^2 \right)^{1/4} \\ &\geq \left(\left(\frac{1}{4}|X|^2 \right)^2 \right)^{1/4} = \frac{1}{2}|X|. \end{aligned}$$

This and Proposition 3.3 yield that

$$\frac{1}{4}|\beta_2(X_2, X_1)| \leq \frac{|X_1|}{2} \frac{|X_2|}{2} \leq p(k_1, Z_1, X_1)p(k_2, Z_2, X_2)$$

for all $(k_j, Z_j, X_j) \in \mathbb{R} \times N, j = 1, 2$. Further, for all $(Z, X) \in N$ and $k_1, k_2 \in \mathbb{R}$, we find

$$\begin{aligned} \left| \frac{1}{4}|X|^2 + |k_1 + k_2| + Z \right|^{1/2} &= \left(\left(\frac{1}{4}|X|^2 + |k_1 + k_2| \right)^2 + |Z|^2 \right)^{1/4} \\ &\leq \left(\left(\frac{1}{4}|X|^2 + |k_1| + |k_2| \right)^2 + |Z|^2 \right)^{1/4} \\ &= \left| \frac{1}{4}|X|^2 + |k_1| + |k_2| + Z \right|^{1/2}. \end{aligned}$$

Now let $g_j = (k_j, Z_j, X_j) \in \mathbb{R} \times N, j = 1, 2$. Then

$$\begin{aligned} p(g_1 g_2) &= p\left(k_1 + k_2, Z_1 + Z_2 + \frac{1}{2}[X_1, X_2], X_1 + X_2\right) \\ &= \left| \frac{1}{4}|X_1 + X_2|^2 + |k_1 + k_2| + Z_1 + Z_2 + \frac{1}{2}[X_1, X_2] \right|^{1/2} \\ &\leq \left| \frac{1}{4}|X_1 + X_2|^2 + |k_1| + |k_2| + Z_1 + Z_2 + \frac{1}{2}[X_1, X_2] \right|^{1/2} \\ &= \left| \frac{1}{4}|X_1|^2 + |k_1| + Z_1 + \frac{1}{4}|X_2|^2 + |k_2| + Z_2 + \frac{1}{2}(\langle X_1, X_2 \rangle + [X_1, X_2]) \right|^{1/2}. \end{aligned}$$

Recall from Lemma 2.12 that

$$\langle X_1, X_2 \rangle + [X_1, X_2] = \beta_2(X_2, X_1).$$

Then

$$\begin{aligned}
 p(g_1g_2) &\leq \left| \frac{1}{4}|X_1|^2 + |k_1| + Z_1 + \frac{1}{4}|X_2|^2 + |k_2| + Z_2 + \frac{1}{2}\beta_2(X_2, X_1) \right|^{1/2} \\
 &\leq \left[\left| \frac{1}{4}|X_1|^2 + |k_1| + Z_1 \right| + \left| \frac{1}{4}|X_2|^2 + |k_2| + Z_2 \right| + \frac{1}{2} |\beta_2(X_2, X_1)| \right]^{1/2} \\
 &= \left[p(g_1)^2 + p(g_2)^2 + \frac{1}{2}|\beta_2(X_2, X_1)| \right]^{1/2} \\
 &\leq [p(g_1)^2 + p(g_2)^2 + 2p(g_1)p(g_2)]^{1/2} \\
 &= p(g_1) + p(g_2).
 \end{aligned}$$

This completes the proof. □

The definition of the Cygan metric uses the H-coordinates on $\overline{D}^g \setminus \{\infty\}$. We use the map

$$\kappa : \begin{cases} \overline{D}^g \setminus \{\infty\} &\rightarrow \mathbb{R} \times N \\ z &\mapsto \Theta^{-1}(z) \end{cases}$$

to assign to each element of $\overline{D}^g \setminus \{\infty\}$ its H-coordinates. The advantage of using H-coordinates is that the points of $\overline{D}^g \setminus \{\infty\}$ get embedded into the group $\mathbb{R} \times N$. This in turn allows to perform the group operations of $\mathbb{R} \times N$ on $\kappa(z_1), \kappa(z_2)$ for elements $z_1, z_2 \in \overline{D}^g \setminus \{\infty\}$.

Definition 3.6 The Cygan metric on D is given by

$$\rho : \begin{cases} \overline{D}^g \setminus \{\infty\} \times \overline{D}^g \setminus \{\infty\} &\rightarrow \mathbb{R} \\ (g_1, g_2) &\mapsto p(\kappa(g_1)^{-1}\kappa(g_2)). \end{cases}$$

Since κ is injective, the Cygan metric is in fact a metric on $\overline{D}^g \setminus \{\infty\}$.

The following lemma provides worked out formulas for the Cygan metric. It can be proved by a straightforward calculation.

Lemma 3.7 Let $z_j = (\zeta_j, v_j) = (k_j, Z_j, X_j)_h$ be elements of $\overline{D}^g \setminus \{\infty\}$. Then the Cygan metric is given by

$$\begin{aligned}
 \rho(z_1, z_2) &= \left| \frac{1}{4}|X_1 - X_2|^2 + |k_1 - k_2| + Z_1 - Z_2 + \frac{1}{2} \text{Im } \beta_2(X_2, X_1) \right|^{1/2} \\
 &= \left| \frac{1}{4}|v_1|^2 + \frac{1}{4}|v_2|^2 + |\text{ht}(z_1) - \text{ht}(z_2)| + \text{Im } \zeta_1 - \text{Im } \zeta_2 - \frac{1}{2}\beta_2(v_1, v_2) \right|^{1/2}.
 \end{aligned}$$

Convention 3.8 Let $g \in G \setminus G_\infty$. Whenever in the following we write $g = n_1\sigma ma_1n_2$, it should be understood that $n_1, n_2 \in N, a_t \in A$ and $m \in M$.

Definition 3.9 Let $g \in G \setminus G_\infty$. Suppose that $g = n_1\sigma ma_1n_2$, and let $R(g) := t^{-1/4}$. Then the set

$$I(g) := \{z \in D \mid \rho(z, g^{-1}\infty) = R(g)\}$$

is called the *isometric sphere* of g . Further,

$$\text{ext } I(g) := \{z \in D \mid \rho(z, g^{-1}\infty) > R(g)\}$$

is called the *exterior* of $I(g)$, and

$$\text{int } I(g) := \{z \in D \mid \rho(z, g^{-1}\infty) < R(g)\}$$

is called the *interior* of $I(g)$. The value $R(g)$ is the *radius* of $I(g)$.

Lemma 3.10 *Let $g = n_1\sigma ma_1n_2 \in G \setminus G_\infty$ and $n_2 = (1, Z_{s2}, X_{s2})$. Then*

$$\rho((\zeta, v), g^{-1}\infty) = \left| \frac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \frac{1}{2}\beta_2(v, X_{s2}) \right|^{1/2}.$$

Further $\rho(\cdot, g^{-1}\infty)$ is unbounded on D , and hence $\text{ext } I(g) \neq \emptyset$.

Proof We have

$$g^{-1}\infty = n_2^{-1}0 = \left(\frac{1}{4}|X_{s2}|^2 - Z_{s2}, -X_{s2} \right).$$

Thus, the expression for $\rho((\zeta, v), g^{-1}\infty)$ follows immediately from Lemma 3.7. We now prove the second part of the claim. Let $(\zeta, v) \in D$ and $t > 1$. Then

$$\text{ht}((t\zeta, v)) = t \text{Re } \zeta - \frac{1}{4}|v|^2 > \text{Re } \zeta - \frac{1}{4}|v|^2 > 0.$$

Hence $(t\zeta, v) \in D$. Further

$$\begin{aligned} \rho((t\zeta, v), g^{-1}\infty)^2 &= \left| t\zeta + \frac{1}{4}|X_{s2}|^2 + Z_{s2} + \frac{1}{2}\beta_2(v, X_{s2}) \right| \\ &\geq t|\zeta| - \left| \frac{1}{4}|X_{s2}|^2 + Z_{s2} + \frac{1}{2}\beta_2(v, X_{s2}) \right|, \end{aligned}$$

which converges to ∞ for $t \rightarrow \infty$. Hence $\rho(\cdot, g^{-1}\infty)$ is unbounded. This completes the proof. □

Lemma 3.11 *Let $g \in G \setminus G_\infty$. Then*

- (i) $\text{ext } I(g)$ and $\text{int } I(g)$ are open,
- (ii) $\overline{\text{ext } I(g)} = \{z \in D \mid \rho(z, g^{-1}\infty) \geq R(g)\} = \mathbb{C} \cap \text{int } I(g)$,
- (iii) $\overline{\text{int } I(g)} = \{z \in D \mid \rho(z, g^{-1}\infty) \leq R(g)\} = \mathbb{C} \cap \text{ext } I(g)$.
- (iv) If $(\zeta, v) \in \overline{\text{int } I(g)}$, then $(\zeta - s, v) \in \text{int } I(g)$ for each $s \in (0, \text{ht}(\zeta))$.
- (v) If $(\zeta, v) \in \overline{\text{ext } I(g)}$, then $(\zeta + s, v) \in \text{ext } I(g)$ for each $s > 0$.

Proof Suppose that $g = n_1\sigma ma_1n_2$ with $n_2 = (1, Z_2, X_2)$. Then $R(g) = t^{-1/4}$ and, by Lemma 3.10,

$$\rho((\zeta, v), g^{-1}\infty) = \left| \frac{1}{4}|X_2|^2 + Z_2 + \zeta + \frac{1}{2}\beta_2(v, X_2) \right|^{1/2}$$

for each $(\zeta, v) \in D$. Since $\beta_2(\cdot, X_2): V \rightarrow \mathbb{C}$ is \mathbb{R} -linear (see Proposition 2.11), the map

$$f: \begin{cases} D & \rightarrow \mathbb{R} \\ (\zeta, v) & \mapsto \left| \frac{1}{4}|X_2|^2 + Z_2 + \zeta + \frac{1}{2}\beta_2(v, X_2) \right| \end{cases}$$

is continuous. Then

$$\text{ext } I(g) = f^{-1}((t^{-1/2}, \infty)) \quad \text{and} \quad \text{int } I(g) = f^{-1}((-\infty, t^{-1/2})).$$

This shows that $\text{ext } I(g)$ and $\text{int } I(g)$ are open. Moreover, it follows that $\overline{\text{int } I(g)}$ is contained in $f^{-1}((-\infty, t^{-1/2}])$. For the converse inclusion relation, it suffices to show that $f^{-1}(t^{-1/2}) \subseteq \overline{\text{int } I(g)}$. Let $z_0 = (\zeta_0, v_0) \in f^{-1}(t^{-1/2})$. Then

$$\begin{aligned} t^{-1/2} &= \left| \frac{1}{4}|X_2|^2 + Z_2 + \zeta_0 + \frac{1}{2}\beta_2(v_0, X_2) \right| \\ &= \left[\left| \frac{1}{4}|X_2 + v_0|^2 + \text{ht}(\zeta_0) \right|^2 + \left| Z_2 + \text{Im } \zeta_0 + \frac{1}{2} \text{Im } \beta_2(v_0, X_2) \right|^2 \right]^{1/2}. \end{aligned}$$

For each $s \in (0, \text{ht}(\zeta_0))$ it follows that

$$\begin{aligned} t^{-1/2} &> \left[\left| \frac{1}{4}|X_2 + v_0|^2 + \text{ht}(\zeta_0) - s \right|^2 + \left| Z_2 + \text{Im } \zeta_0 + \frac{1}{2} \text{Im } \beta_2(v_0, X_2) \right|^2 \right]^{1/2} \\ &= \left| \frac{1}{4}|X_2|^2 + Z_2 + \zeta_0 - s + \frac{1}{2}\beta_2(v_0, X_2) \right|. \end{aligned}$$

Thus, $(\zeta_0 - s, v_0) \in \text{int } I(g)$ for each $s \in (0, \text{ht}(\zeta_0))$. Hence

$$\lim_{s \searrow 0} (\zeta_0 - s, v_0) = (\zeta_0, v_0) \in \overline{\text{int } I(g)}.$$

This proves (iii) and (iv). The proof of (ii) is analogous to that of (iii), and the proof of (v) is analogous to that of (iv). □

Proposition 3.12 *We have*

$$\overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)} = \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{ext } I(g)} = D \setminus \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g).$$

Proof Lemma 3.11(iii) states that $\mathbb{C} \text{ext } I(g) = \overline{\text{int } I(g)}$ for each $g \in \Gamma \setminus \Gamma_\infty$. Therefore,

$$\mathbb{C} \left(\overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)} \right) = \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \mathbb{C} \text{ext } I(g) \right)^\circ = \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)} \right)^\circ.$$

By Lemma 3.11(i), the interior of each isometric sphere is open. Thus

$$\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g) = \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g) \right)^\circ,$$

and hence

$$\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g) \subseteq \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)} \right)^\circ.$$

In the following we will prove the converse inclusion relation. Let $z = (\zeta, v)$ be an element of $\left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)} \right)^\circ$ and pick $\varepsilon > 0$ such that

$$z_\varepsilon := z + \varepsilon = (\zeta + \varepsilon, v) \in \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)}.$$

Fix some $k \in \Gamma \setminus \Gamma_\infty$ such that $z_\varepsilon \in \overline{\text{int } I(k)}$. Lemma 3.11(iv) yields that $z = z_\varepsilon - \varepsilon$ is in $\text{int } I(k)$. Thus $z \in \bigcup \{\text{int } I(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$. This shows that

$$\left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)} \right)^\circ \subseteq \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g).$$

In turn,

$$\mathbb{C} \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g) = \overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)}.$$

Finally we have $\mathbb{C} \text{int } I(g) = \overline{\text{ext } I(g)}$ for each $g \in \Gamma \setminus \Gamma_\infty$ by Lemma 3.11(iii). Therefore,

$$\overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)} = \mathbb{C} \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g) = \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \mathbb{C} \text{int } I(g) = \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{ext } I(g)}.$$

□

Let $g \in G \setminus G_\infty$ and $(\zeta, v) \in \overline{D^g} \setminus \{\infty\}$. Suppose that $g = n_1 \sigma m a_t n_2$ with $n_j = (1, Z_{sj}, X_{sj})$ and $m = (\varphi, \psi)$. A lengthy but easy calculation shows that

$$g(\zeta, v) = \left(\frac{1}{4} |X_{s1}|^2 + Z_{s1} + xt^{-1} - \frac{1}{2} t^{-1/2} \beta_2(x\psi(X_{s2} + v), X_{s1}), \right. \\ \left. X_{s1} - xt^{-1/2} \psi(X_{s2} + v) \right) \tag{3.1}$$

where

$$x := \left[\varphi \left(\frac{1}{4} |X_{s2}|^2 + Z_{s2} + \zeta + \frac{1}{2} \beta_2(v, X_{s2}) \right) \right]^{-1}.$$

This explicit expression for the action of g on D is used in the following two lemmas. We denote the set of orthogonal endomorphisms of C , respectively of V , by $O(C)$, respectively $O(V)$.

Lemma 3.13 *If $g \in G \setminus G_\infty$, then g maps $\text{ext } I(g)$ onto $\text{int } I(g^{-1})$, and $I(g)$ onto $I(g^{-1})$.*

Proof Suppose that $g = n_1 \sigma m a_t n_2$ with $n_j = (1, Z_{sj}, X_{sj})$ and $m = (\varphi, \psi)$. Then we have $g^{-1} = n_2^{-1} \sigma m a_t n_1^{-1}$. Hence it follows that

$$R(g) = R(g^{-1}) = t^{-1/4}$$

and

$$g^{-1} I(g^{-1}) = \{z \in D \mid \rho(gz, g\infty) = R(g)\}.$$

In the following we will compare $\rho(gz, g\infty)$ to $\rho(z, g^{-1}\infty)$. Let (ζ, v) be in D . Set

$$x := \left[\varphi \left(\frac{1}{4} |X_{s2}|^2 + Z_{s2} + \zeta + \frac{1}{2} \beta_2(v, X_{s2}) \right) \right]^{-1}.$$

Then (3.1) and Lemma 3.10 yield that

$$\rho(g(\zeta, v), g\infty) = \left| \frac{1}{2} |X_{s1}|^2 + xt^{-1} - \frac{1}{2} \beta_2(X_{s1}, X_{s1}) \right|^{1/2}.$$

By Proposition 2.11, $\beta_2(X_{s1}, X_{s1}) = |X_{s1}|^2$. Thus

$$\rho(g(\zeta, v), g\infty) = |xt^{-1}|^{1/2} = |x|^{1/2}t^{-1/2}.$$

Since $\varphi \in O(C)$, it follows that

$$\begin{aligned} |x|^{1/2} &= \left| \varphi \left(\frac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \frac{1}{2}\beta_2(v, X_{s2}) \right) \right|^{-1/2} \\ &= \left| \frac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \frac{1}{2}\beta_2(v, X_{s2}) \right|^{-1/2} \\ &= \rho((\zeta, v), g^{-1}\infty)^{-1}. \end{aligned}$$

Therefore,

$$\rho((\zeta, v), g^{-1}\infty) \rho(g(\zeta, v), g\infty) = t^{-1/2}.$$

Hence, $(\zeta, v) \in I(g)$ if and only if

$$\rho(g(\zeta, v), g\infty) = t^{-1/4}.$$

This is equivalent to $g(\zeta, v) \in I(g^{-1})$. In turn, $gI(g) = I(g^{-1})$. Further, we have $(\zeta, v) \in \text{ext } I(g)$ if and only if

$$t^{-1/4} > \rho(g(\zeta, v), g\infty).$$

Therefore, $g \text{ ext } I(g) = \text{int } I(g^{-1})$. □

Lemma 3.14 *Let $n = (1, Z, X) \in N$, $a_s \in A$, and $m = (\varphi, \psi) \in M$.*

- (i) *If $n' := (1, sZ, s^{1/2}X)$, then $a_s n = n' a_s$.*
- (ii) *For all $u, v \in V$ we have $\varphi(\beta_2(v, u)) = \beta_2(\psi(v), \psi(u))$.*
- (iii) *If $n' := (1, \varphi(Z), \psi(X))$, then $mn = n'm$.*

Proof Claim (i) is shown by direct calculation. To prove (ii) recall that $\varphi \in O(C)$, $\psi \in O(V)$, and that $\psi(J(\eta, w)) = J(\varphi(\eta), \psi(w))$ for each $(\eta, w) \in C \oplus V$. For each $\zeta \in C$ we have

$$\begin{aligned} \langle \varphi(\beta_2(v, u)), \zeta \rangle &= \langle \beta_2(v, u), \varphi^{-1}(\zeta) \rangle = \langle J(\varphi^{-1}(\zeta), u), v \rangle \\ &= \langle \psi(J(\varphi^{-1}(\zeta), u)), \psi(v) \rangle = \langle J(\zeta, \psi(u)), \psi(v) \rangle \\ &= \langle \beta_2(\psi(v), \psi(u)), \zeta \rangle. \end{aligned}$$

Thus $\varphi(\beta_2(v, u)) = \beta_2(\psi(v), \psi(u))$. For the proof of (iii) let $(\zeta, v) \in D$. Then

$$\begin{aligned} m(n(\zeta, v)) &= \left(\varphi \left(\frac{1}{4}|X|^2 + Z + \zeta + \frac{1}{2}\beta_2(v, X) \right), \psi(X + v) \right) \\ &= \left(\frac{1}{4}|\psi(X)|^2 + \varphi(Z) + \varphi(\zeta) + \frac{1}{2}\varphi(\beta_2(v, X)), \psi(X) + \psi(v) \right). \end{aligned}$$

Now (ii) yields

$$\begin{aligned} m(n(\zeta, v)) &= \left(\frac{1}{4}|\psi(X)|^2 + \varphi(Z) + \varphi(\zeta) + \frac{1}{2}\beta_2(\psi(v), \psi(X)), \psi(X) + \psi(v) \right) \\ &= n'(\varphi(\zeta), \psi(v)) = n'(m(\zeta, v)). \end{aligned}$$

Hence $mn = n'm$. □

Proposition 3.15 *Let $g \in G_\infty$ and $h \in G \setminus G_\infty$. Then*

$$\begin{aligned} gI(h) &= I(ghg^{-1}), \\ g \operatorname{int} I(h) &= \operatorname{int} I(ghg^{-1}), \\ g \operatorname{ext} I(h) &= \operatorname{ext}(ghg^{-1}). \end{aligned}$$

Proof It suffices to prove the claim for the three cases $g \in A$, $g \in N$, and $g \in M$. This we will do separately. Let $h = n_1 \sigma m a_t n_2$ with $n_j = (1, Z_j, X_j)$.

Suppose first that $g = a_s$. Let $(\zeta, v) \in D$. For $j \in \{1, 2\}$ set $n'_j := (1, sZ_j, s^{1/2}X_j)$. Lemma 3.14 implies that

$$ghg^{-1} = a_s n_1 \sigma m a_t n_2 a_s^{-1} = n'_1 \sigma m a_s^{-1} a_t a_s^{-1} n'_2 = n'_1 \sigma m a_{s^{-2}t} n'_2.$$

Therefore, $R(ghg^{-1}) = s^{1/2}t^{-1/4} = s^{1/2}R(h)$. By Lemma 3.10 we have

$$\rho((\zeta, v), (ghg^{-1})^{-1}\infty) = \left| \frac{1}{4}s|X_2|^2 + sZ_2 + \zeta + \frac{1}{2}s^{1/2}\beta_2(v, X_2) \right|^{1/2}$$

and

$$\begin{aligned} \rho(a_s^{-1}(\zeta, v), h^{-1}\infty) &= \left| \frac{1}{4}|X_2|^2 + Z_2 + s^{-1}\zeta + \frac{1}{2}s^{-1/2}\beta_2(v, X_2) \right|^{1/2} \\ &= s^{-1/2}\rho((\zeta, v), (ghg^{-1})^{-1}\infty). \end{aligned}$$

Then

$$\begin{aligned} gI(h) &= \{z \in D \mid \rho(g^{-1}z, h^{-1}\infty) = R(h)\} \\ &= \{z \in D \mid \rho(z, (ghg^{-1})^{-1}\infty) = s^{1/2}R(h)\} \\ &= \{z \in D \mid \rho(z, (ghg^{-1})^{-1}\infty) = R(ghg^{-1})\} \\ &= I(ghg^{-1}). \end{aligned}$$

Analogously, we see that $g \operatorname{ext} I(h) = \operatorname{ext} I(ghg^{-1})$ and $g \operatorname{int} I(h) = \operatorname{int} I(ghg^{-1})$.

Now suppose that $g = n_3 = (1, Z_3, X_3)$. Then $ghg^{-1} = (n_3 n_1) \sigma m a_t (n_2 n_3^{-1})$ and (see Lemma 2.12)

$$n_2 n_3^{-1} = \left(1, Z_2 - Z_3 - \frac{1}{2} \operatorname{Im} \beta_2(X_3, X_2), X_2 - X_3 \right).$$

Therefore, by Lemmas 3.10 and 2.12,

$$\begin{aligned} &\rho((\zeta, v), (n_3 h n_3^{-1})^{-1}\infty) \\ &= \left| \frac{1}{4}|X_2|^2 + \frac{1}{4}|X_3|^2 - \frac{1}{2}\beta_2(X_3, X_2) + Z_2 - Z_3 + \zeta + \frac{1}{2}\beta_2(v, X_2 - X_3) \right|^{1/2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\rho \left(n_3^{-1}(\zeta, v), h^{-1}\infty \right) \\ &= \left| \frac{1}{4}|X_2|^2 + Z_2 + \frac{1}{4}|X_3|^2 - Z_3 + \zeta - \frac{1}{2}\beta_2(v, X_3) + \frac{1}{2}\beta_2(-X_3 + v, X_2) \right|^{1/2} \\ &= \left| \frac{1}{4}|X_2|^2 + \frac{1}{4}|X_3|^2 + Z_2 - Z_3 + \zeta - \frac{1}{2}\beta_2(X_3, X_2) + \frac{1}{2}\beta_2(v, X_2 - X_3) \right|^{1/2} \\ &= \rho \left((\zeta, v), n_3 h^{-1} n_3^{-1} \infty \right). \end{aligned}$$

Since $R(h) = R(ghg^{-1})$, the claim follows for this case.

Finally suppose that $g = m_2 = (\varphi, \psi)$ and set $n'_j := (1, \varphi(Z_j), \psi(X_j))$ for $j = 1, 2$. Lemma 3.14 shows that

$$ghg^{-1} = m_2 n_1 \sigma m a_1 n_2 m_2^{-1} = n'_1 \sigma m_2^{-1} m m_2^{-1} a_1 n'_2.$$

From this it follows that $R(h) = R(m_2 h m_2^{-1})$. Moreover, we have

$$\rho \left((\zeta, v), (ghg^{-1})^{-1}\infty \right) = \left| \frac{1}{4}|\psi(X_2)|^2 + \varphi(Z_2) + \zeta + \frac{1}{2}\beta_2(v, \psi(X_2)) \right|^{1/2}.$$

From Lemma 3.14 it follows that

$$\rho \left((\zeta, v), (ghg^{-1})^{-1}\infty \right) = \left| \frac{1}{4}|X_2|^2 + \varphi(Z_2) + \zeta + \frac{1}{2}\varphi(\beta_2(\psi^{-1}(v), X_2)) \right|^{1/2}.$$

On the other side we find

$$\begin{aligned} \rho \left(g^{-1}(\zeta, v), h^{-1}\infty \right) &= \rho \left((\varphi^{-1}(\zeta), \psi^{-1}(v)), h^{-1}\infty \right) \\ &= \left| \frac{1}{4}|X_2|^2 + Z_2 + \varphi^{-1}(\zeta) + \frac{1}{2}\beta_2(\psi^{-1}(v), X_2) \right|^{1/2} \\ &= \left| \frac{1}{4}|X_2|^2 + \varphi(Z_2) + \zeta + \frac{1}{2}\varphi(\beta_2(\psi^{-1}(v), X_2)) \right|^{1/2} \\ &= \rho \left((\zeta, v), (ghg^{-1})^{-1}\infty \right). \end{aligned}$$

From this it follows that $gI(h) = I(ghg^{-1})$, $g \text{ ext } I(h) = \text{ ext } I(ghg^{-1})$, and $g \text{ int } I(h) = \text{ int } I(ghg^{-1})$. □

An immediate consequence of Proposition 3.15 is the following corollary.

Corollary 3.16 *Let Γ be a subgroup of G and $g \in \Gamma_\infty$. Then*

$$g \bigcap_{h \in \Gamma \setminus \Gamma_\infty} \text{ ext } I(h) = \bigcap_{h \in \Gamma \setminus \Gamma_\infty} \text{ ext } I(h)$$

and

$$g \bigcup_{h \in \Gamma \setminus \Gamma_\infty} \text{ int } I(h) = \bigcup_{h \in \Gamma \setminus \Gamma_\infty} \text{ int } I(h).$$

Lemma 3.17 *Let $z \in D$ and $g \in G \setminus G_\infty$. Then*

$$\text{ht}(z) = \left(\frac{\rho(z, g^{-1}\infty)}{R(g)} \right)^4 \text{ht}(gz).$$

Proof Let $z = (\zeta, v)$ and $g = n_1 \sigma m a_t n_2$ with $n_j = (1, Z_j, X_j)$ and $m = (\varphi, \psi)$. We first evaluate $\text{ht}(gz)$. Set

$$x := \left[\varphi \left(\frac{1}{4}|X_2|^2 + Z_2 + \zeta + \frac{1}{2}\beta_2(v, X_2) \right) \right]^{-1}.$$

Using (3.1) we find

$$\begin{aligned} \text{ht}(gz) &= \text{Re} \left(\frac{1}{4}|X_1|^2 + Z_1 + xt^{-1} - \frac{1}{2}t^{-1}\beta_2(x\psi(X_2 + v), X_1) \right) \\ &\quad - \frac{1}{4}|X_1 - xt^{-1/2}\psi(X_2 + v)|^2 \\ &= \frac{1}{4}|X_1|^2 + t^{-1} \text{Re}(x) - \frac{1}{2}t^{-1/2} \langle x\psi(X_2 + v), X_1 \rangle \\ &\quad - \frac{1}{4}|X_1 - xt^{-1/2}\psi(X_2 + v)|^2 \\ &= t^{-1} \text{Re}(x) - \frac{1}{4}t^{-1}|x|^2 \cdot |X_2 + v|^2 \\ &= t^{-1}|x|^2 \left[\text{Re}(|x|^{-2}x) - \frac{1}{4}|X_2 + v|^2 \right]. \end{aligned}$$

Since $\text{Re}(\eta) = \text{Re} \bar{\eta}$ for each $\eta \in C$, it follows that

$$\begin{aligned} \text{Re}(|x|^{-2}x) &= \text{Re}(\bar{x}^{-1}) = \text{Re}(x^{-1}) \\ &= \text{Re} \left(\frac{1}{4}|X_2|^2 + Z_2 + \zeta + \frac{1}{2}\beta_2(v, X_2) \right) \\ &= \frac{1}{4}|X_2|^2 + \text{Re} \zeta + \frac{1}{2}\langle v, X_2 \rangle \\ &= \frac{1}{4}|X_2 + v|^2 + \text{Re} \zeta - \frac{1}{4}|v|^2. \end{aligned}$$

Thus,

$$\text{ht}(gz) = t^{-1}|x|^2 \left[\text{Re} \zeta - \frac{1}{4}|v|^2 \right] = t^{-1}|x|^2 \text{ht}(z).$$

Lemma 3.10 shows that

$$\rho((\zeta, v), g^{-1}\infty) = \left| \frac{1}{4}|X_2|^2 + Z_2 + \zeta + \frac{1}{2}\beta_2(v, X_2) \right|^{1/2} = |x|^{-1/2}.$$

Since $R(g) = t^{-1/4}$, we have

$$\text{ht}(z) = t|x|^{-2} \text{ht}(gz) = \left(\frac{\rho(z, g^{-1}\infty)}{R(g)} \right)^4 \text{ht}(gz).$$

□

3.3 Fundamental regions

A subgroup Γ of G is said to be *of type (O)* if

$$\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g) = D \setminus \overline{\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g)}.$$

For a subset S of G let $\langle S \rangle$ denote the subgroup of G generated by S . Then S is said to be of *of type (F)*, if for each $z \in D$ the maximum of the set

$$\{\text{ht}(gz) \mid g \in \langle S \rangle\}$$

exists.

Theorem 3.18 *Let Γ be a subgroup of G of type (O) such that $\Gamma \setminus \Gamma_\infty$ is of type (F). Suppose that \mathcal{F}_∞ is a fundamental region for Γ_∞ in D satisfying*

$$\overline{\mathcal{F}_\infty} \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{ext } I(g)} = \overline{\mathcal{F}_\infty \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)}.$$

Then

$$\mathcal{F} := \mathcal{F}_\infty \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)$$

is a fundamental region for Γ in D .

Proof Since Γ is of type (O), the set $\bigcap \{\text{ext } I(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$ is open. Then \mathcal{F}_∞ being open as a fundamental region for Γ_∞ in D yields that \mathcal{F} is open.

Now let $z \in \mathcal{F}$ and $g \in \Gamma \setminus \{\text{id}\}$. If $g \in \Gamma_\infty$, then $gz \notin \mathcal{F}_\infty$ since the Γ_∞ -translates of \mathcal{F}_∞ are pairwise disjoint. If $g \in \Gamma \setminus \Gamma_\infty$, then $z \in \text{ext } I(g)$. Lemma 3.13 states that $gz \in \text{int } I(g^{-1})$, and hence $gz \notin \text{ext } I(g^{-1})$. Thus, in each case, $gz \notin \mathcal{F}$. This shows that the Γ -translates of \mathcal{F} are pairwise disjoint.

It remains to prove that $D \subseteq \Gamma \cdot \mathcal{F}$. To that end let $z \in D$ and set

$$A := \bigcap \left\{ \overline{\text{ext } I(g)} \mid g \in \Gamma \setminus \Gamma_\infty \right\}.$$

Since $\Gamma \setminus \Gamma_\infty$ is of type (F), the set $\langle \Gamma \setminus \Gamma_\infty \rangle z$ contains an element of maximal height, say w . We claim that $w \in A$. Seeking a contradiction assume that $w \notin A$. Proposition 3.12 yields the existence of $h \in \Gamma \setminus \Gamma_\infty$ with $w \in \text{int } I(h)$. Then, by definition,

$$\rho(w, h^{-1}\infty) < R(h).$$

This and Lemma 3.17 show that $\text{ht}(hw) > \text{ht}(w)$, which is a contradiction to the choice of w . Thus, $w \in A$.

Since $\Gamma_\infty \overline{\mathcal{F}_\infty} = D$, there is $k \in \Gamma_\infty$ such that $kw \in \overline{\mathcal{F}_\infty}$. Corollary 3.16 implies that $kw \in A$. Finally,

$$kw \in \overline{\mathcal{F}_\infty} \cap A = \overline{\mathcal{F}_\infty \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)} = \overline{\mathcal{F}}.$$

This completes the proof. □

Remark 3.19 Let Γ be a subgroup of G and suppose that the set

$$\{\text{int } I(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$$

of interiors of all isometric spheres is locally finite. Then

$$\overline{\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g)} = \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)}.$$

by [20, Hilfssatz 7.14]. Lemma 3.11(iii) implies that

$$\mathbb{C} \left(\overline{\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \text{int } I(g)} \right) = \mathbb{C} \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{int } I(g)} \right) = \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \mathbb{C} \overline{\text{int } I(g)} = \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g).$$

Hence, if the set of interiors of isometric spheres is locally finite, then Γ is of type (O).

A special case of Theorem 3.18 is the following corollary.

Corollary 3.20 *Let Γ be a subgroup of G of type (O) and $\Gamma_\infty = \{\text{id}\}$. Further suppose that $\Gamma \setminus \Gamma_\infty$ is of type (F). Then*

$$\mathcal{F} := \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)$$

is a fundamental region for Γ in D .

The following proposition serves to show that in many situations the fundamental region in Theorem 3.18 is actually a fundamental domain (see Corollary 3.23 below).

Proposition 3.21 *Suppose that $\mathfrak{v} = \{0\}$. Then the set $\text{ext } I(g)$ is geodesically convex for each $g \in G \setminus G_\infty$.*

Proof Let $z_1, z_2 \in \text{ext } I(g), z_1 \neq z_2$. Then there exist $s_2 > s_1 > 0$ and $h \in G$ such that $z_1 = ha_{s_1} \cdot o_D$ and $z_2 = ha_{s_2} \cdot o_D$. W.l.o.g. we may assume that h is either of the form $h = n \in N$ (if $h \in G_\infty$) or $h = n'\sigma n \in N\sigma N$ (if $h \in G \setminus G_\infty$). The geodesic segment connecting z_1 and z_2 is given by

$$c := \{ha_r \cdot o_D \mid s_1 \leq r \leq s_2\}.$$

In the following we will show that $c \subseteq \text{ext } I(g)$ by examining separately the two cases for h .

Suppose that $g = n_1\sigma ma_t n_2$ with $n_1, n_2 = (1, Z_s) \in N, a_t \in A$ and $m \in M$ and $r \in [s_1, s_2]$. At first let $h = n = (1, Z) \in N$. Then

$$\begin{aligned} \rho(na_r \cdot o_D, g^{-1}\infty) &= \rho((r, Z), g^{-1}\infty) = |Z_s + Z + r|^{1/2} = (|Z_s + Z|^2 + r^2)^{1/4} \\ &\geq (|Z_s + Z|^2 + s_1^2)^{1/4} = \rho(na_{s_1} \cdot o_D, g^{-1}\infty) > R(g). \end{aligned}$$

Hence $ha_r \cdot o_D \in \text{ext } I(g)$.

Now let $h = n'\sigma n \in N\sigma N$ with $n' = (1, Z')$ and $n = (1, Z)$. A straightforward calculation shows that

$$\rho(ha_r \cdot o_D, g^{-1}\infty) > R(g) = t^{-1/4}$$

is equivalent to

$$0 < ar^4 + br^2 + c \tag{3.2}$$

where $Z'_s := Z_s + Z'$ and

$$\begin{aligned} a &:= |Z'_s|^2 - t^{-1} \\ b &:= 1 + 2|Z|^2|Z'_s|^2 - 2\langle Z, Z'_s \rangle - 2|Z|^2t^{-1} \\ c &:= |Z|^4|Z'_s|^2 - 2|Z|^2\langle Z, Z'_s \rangle + |Z|^2 - |Z|^4t^{-1}. \end{aligned}$$

Solving (3.2) separately for $a = 0$, $a > 0$ and $a < 0$ yields that it is satisfied for r . Therefore, $ha_r \cdot o_D \in \text{ext } I(g)$. This completes the proof. \square

Remark 3.22 Using the classification in [15] of J^2C -module structures and the statement [5, Proposition 2.5.1] on the (non-)existence of totally geodesic submanifolds of codimension one in rank one Riemannian symmetric spaces we see that Proposition 3.21 in general becomes false if $\mathfrak{v} \neq \{0\}$.

Theorem 3.18 and Proposition 3.21 immediately imply the following statement.

Corollary 3.23 *Let $\mathfrak{v} = \{0\}$ and let Γ be a subgroup of G of type (O) such that $\Gamma \setminus \Gamma_\infty$ is of type (F). Suppose that \mathcal{F}_∞ is a convex fundamental domain for Γ_∞ in D satisfying*

$$\overline{\mathcal{F}_\infty} \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \overline{\text{ext } I(g)} = \overline{\mathcal{F}_\infty} \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g).$$

Then

$$\mathcal{F} := \mathcal{F}_\infty \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)$$

is a convex fundamental domain for Γ in D .

4 Projective models

The purpose of this section is to show that the existing definitions of isometric spheres and results concerning the existence of isometric fundamental regions in literature are essentially covered by the definitions and results in Sect. 3. The reason for the reservation towards a confirmation to cover all existing definitions and results is twofold: On the one hand the author cannot guarantee to be aware of all existing results. On the other hand, at least for the real hyperbolic plane, the literature contains non-equivalent definitions of isometric spheres. Moreover, the existence results of isometric fundamental regions by Ford are proved for a weaker notion of fundamental region than the one used here. Section 4.7 contains a detailed discussion of the latter issues.

Let (C, V, J) be a J^2C -module structure. In Sect. 4.1 we introduce the structure of division algebras on C following [15] and [16]. For C being an associative division algebra, we redo, in Sects. 4.3 and 4.4, the classical projective construction of hyperbolic spaces in terms of the J^2C -module structure. A long part of Sect. 4.4 is devoted to a detailed study of the relation between the isometry group G of the symmetric space and the natural “matrix” group on the projective space. This investigation will show that the matrix group is isomorphic to a certain subgroup G^{res} of G . In Sect. 4.5, we use these results to provide a characterization of the isometric sphere of $g \in G^{\text{res}}$ via a cocycle. In Sect. 4.6, we prove that a special class of subgroups Γ of G^{res} are of type (O) with $\Gamma \setminus \Gamma_\infty$ being of type (F) and use Theorem 3.18 to show the existence of an isometric fundamental domain for Γ . Finally, in Sect. 4.7, we bring together these investigations for a comparison with the existing literature.

4.1 Division algebras induced by J^2C -module structures

Let (C, V, J) be a J^2C -module structure. Suppose that $V \neq \{0\}$ and fix an element $v \in V \setminus \{0\}$. From (M3) and (M2) it follows that for each pair $(\zeta, \eta) \in C \times C$ there exists a unique element $\tau \in C$ such that

$$J(\zeta, J(\eta, v)) = J(\tau, v).$$

Hence, each choice $v \in V \setminus \{0\}$ equips C with a multiplication $\cdot_v : C \times C \rightarrow C$ via

$$\zeta \cdot_v \eta := \tau \iff J(\zeta, J(\eta, v)) = J(\tau, v).$$

Equation (2.9) shows that the inverse of $\zeta \in C \setminus \{0\}$ is $\zeta^{-1} = |\zeta|^{-2}\bar{\zeta}$, independent of the choice of $v \in V \setminus \{0\}$. The following properties of (C, \cdot_v) are shown in [15].

Proposition 4.1 (Proposition 1.1 and Corollary 1.5 in [15])

- (i) For each $v \in V \setminus \{0\}$, the Euclidean vector space C with the multiplication \cdot_v is a normed, not necessarily associative, division algebra.
- (ii) The multiplication \cdot_v on C is independent of the choice of $v \in V \setminus \{0\}$ if and only if (C, \cdot_v) is associative for one (and hence for all) v .

We call the J^2C -module structure (C, V, J) associative if (C, \cdot_v) is associative for some (and hence each) $v \in V \setminus \{0\}$ and otherwise non-associative. If (C, \cdot_v) is associative, we omit the subscript v of the multiplication \cdot_v .

Remark 4.2 Only associative J^2C -module structures are modules in the sense of [4].

Remark 4.3 Suppose that (C, V, J) is an associative J^2C -module structure. The classification in [15] (or [16, Sect. 4]) shows that C is real or complex or quaternionic numbers.

4.2 C -sesquilinear hermitian forms

From now on let (C, V, J) be an associative J^2C -module structure. Suppose that M is a (left) C -module. A map $\Phi : M \times M \rightarrow C$ is said to be a C -sesquilinear hermitian form if Φ is C -hermitian and C -linear in the first variable, that is, if the following properties are satisfied:

- (SH1) $\Phi(\zeta_1 x_1 + \zeta_2 x_2, y) = \zeta_1 \Phi(x_1, y) + \zeta_2 \Phi(x_2, y)$ for all $\zeta_1, \zeta_2 \in C$ and $x, y \in M$,
- (SH2) $\Phi(x, y) = \overline{\Phi(y, x)}$ for all $x, y \in M$.

A C -sesquilinear hermitian form Φ is called non-degenerate if $\Phi(m_1, \cdot) = 0$ implies that $m_1 = 0$. It is called indefinite if there exist $m_1, m_2 \in M$ such that $\Phi(m_1, m_1) < 0$ and $\Phi(m_2, m_2) > 0$.

Proposition 4.4 The map $\beta_1 : C \times C \rightarrow C, \beta_1(x, y) := x\bar{y}$, is C -sesquilinear hermitian. Further $\text{Re } \beta_1(x, y) = \langle x, y \rangle$ for all $x, y \in C$.

Proof Obviously, β_1 is \mathbb{R} -bilinear. For each $y \in C$, the C -linearity of $\beta_1(\cdot, y)$ is exactly the left-sided distribution law of the division algebra C . To show (SH2) let $x, y \in C$. From (2.7) it follows that

$$J_{y\bar{x}} = J_{y\bar{x}}^* = J_x^* J_y^* = J_x J_y = J_{x\bar{y}}.$$

Therefore,

$$\beta_1(x, y) = x\bar{y} = \overline{y\bar{x}} = \overline{\beta_1(y, x)}.$$

Hence β_1 is C -sesquilinear hermitian. Finally, polarization of $\beta_1(x, x) = |x|^2$ (see (2.8)) over \mathbb{R} implies the remaining statement. \square

Remark 4.5 An immediate consequence of Proposition 4.4 is that conjugation and multiplication in C anticommute, i. e., $\bar{x}\bar{y} = \overline{y\bar{x}}$ for all $x, y \in C$. In particular, for a C -sesquilinear hermitian form Φ on the C -module M we have

$$\Phi(m_1, \zeta m_2) = \Phi(m_1, m_2)\bar{\zeta}$$

for all $m_1, m_2 \in M$ and all $\zeta \in C$. Moreover, we have $\Phi(m, m) = \overline{\Phi(m, m)}$ for each $m \in M$ and therefore, $\Phi(m, m) \in \mathbb{R}$.

Lemma 4.6 For all $\zeta, \eta, \xi \in C$ we have $\langle \zeta, \xi\eta \rangle = \langle \bar{\xi}\zeta, \eta \rangle$.

Proof Let $v \in V \setminus \{0\}$. Using (2.6) and (2.7) we find

$$2\langle \zeta, \xi\eta \rangle |v|^2 = 2\langle \zeta v, \xi\eta v \rangle = 2\langle \bar{\xi}\zeta v, \eta v \rangle = 2\langle \bar{\xi}\zeta, \eta \rangle |v|^2,$$

hence $\langle \zeta, \xi\eta \rangle = \langle \bar{\xi}\zeta, \eta \rangle$. \square

Proposition 4.7 The map $\beta_2: V \times V \rightarrow C$ is C -sesquilinear hermitian.

Proof Because of Proposition 2.11 it remains to show that β_2 is C -linear in the first variable. Let $v, u \in V$ and $\zeta \in C$. Lemma 4.6 and (2.7) yield that for all $\eta \in C$ we have

$$\langle \zeta\beta_2(v, u), \eta \rangle = \langle \beta_2(v, u), \bar{\zeta}\eta \rangle = \langle \bar{\zeta}\eta u, v \rangle = \langle \eta u, \zeta v \rangle = \langle \beta_2(\zeta v, u), \eta \rangle.$$

Hence $\zeta\beta_2(v, u) = \beta_2(\zeta v, u)$. \square

A finite sequence (v_1, \dots, v_n) in V is called an *orthonormal C -basis* of V if

(CON1) $|v_j| = 1$ for each $j \in \{1, \dots, n\}$,

(CON2) for each pair $(i, j) \in \{1, \dots, n\}^2, i \neq j$, the sets Cv_i and Cv_j are orthogonal,

(CON3) if we use for each $j \in \{1, \dots, n\}$ the bijection $C \rightarrow Cv_j, \zeta \mapsto \zeta v_j$, to equip Cv_j with the structure of a Euclidean vector space and a C -module, then V is isomorphic as C -module and Euclidean vector space to the direct sum $\bigoplus_{j=1}^n Cv_j$ of the Euclidean spaces and C -modules $Cv_j, j = 1, \dots, n$.

The following lemma yields that V is a free C -module.

Lemma 4.8 There is an orthonormal C -basis of V .

Proof Clearly, one finds a sequence (v_1, \dots, v_n) in V which satisfies (CON1) and (CON2) such that V is isomorphic as Euclidean vector space to $\bigoplus_{j=1}^n Cv_j$. From Propositions 4.7 and 2.11 it follows that

$$\psi: \begin{cases} V \rightarrow \bigoplus_{j=1}^n Cv_j \\ v \mapsto \sum_{j=1}^n \beta_2(v, v_j)v_j \end{cases}$$

is an isomorphism (of Euclidean vector spaces) from V to $\bigoplus_{j=1}^n Cv_j$. Let $v \in V, \eta \in C$ and suppose that v is isomorphic to $(\zeta_1 v_1, \dots, \zeta_n v_n) \in \bigoplus_{j=1}^n Cv_j$. Then

$$\psi(\eta v) = \sum_{j=1}^n \beta_2(\eta v, v_j)v_j = \sum_{j=1}^n (\eta\beta_2(v, v_j)) v_j = \eta \left(\sum_{j=1}^n \beta_2(v, v_j)v_j \right) = \eta\psi(v).$$

This shows that ψ is indeed an isomorphism of C -modules. \square

4.3 The C -projective space $P_C(E)$

Let (C, V, J) be an associative J^2C -module structure. Further let $W := C \oplus V$ be the Euclidean direct sum of C and V , and $E := C \oplus W = C \oplus C \oplus V$ be that of C and W . We define a C -multiplication on E by

$$\because \begin{cases} C \times E & \rightarrow E \\ (\tau, (\zeta, \eta, v)) & \mapsto (\tau\zeta, \tau\eta, \tau v). \end{cases}$$

Because (C, V, J) is associative and hence $\sigma(\tau v) = (\sigma\tau)v$ for each $v \in V$ and all $\sigma, \tau \in C$ (or equivalently, since the definition of the product in C does not depend on the choice of $v \in V \setminus \{0\}$, see Proposition 4.1), E becomes a C -module. Since V is a free C -module by Lemma 4.8, also E is a free C -module.

Two elements z_1, z_2 of $E \setminus \{0\}$ are called *equivalent* ($z_1 \sim z_2$) if there is $\tau \in C$ such that $\tau z_1 = z_2$. Note that τ is actually in $C \setminus \{0\}$. Then E being a C -module implies that \sim is indeed an equivalence relation on $E \setminus \{0\}$. The C -projective space $P_C(E)$ of E is defined as the set of equivalence classes of \sim ,

$$P_C(E) := (E \setminus \{0\}) / \sim,$$

endowed with the induced topology and the differential structure generated by the following (standard) charts: Let $\{v_1, \dots, v_{n-1}\}$ be an orthonormal C -basis of V . Then E is isomorphic to C^{n+1} both as Euclidean vector space and C -module. For each $j = 1, \dots, n + 1$ the set

$$U_j := \{[(\zeta_1, \dots, \zeta_{n+1})] \in P_C(E) \mid \zeta_j \neq 0\}$$

is open, and the maps $\varphi_j : U_j \rightarrow C^n$

$$\varphi_j([(\zeta_1, \dots, \zeta_{n+1})]) := \zeta_j^{-1}(\zeta_1, \dots, \widehat{\zeta_j}, \dots, \zeta_{n+1})$$

are pairwise compatible in the sense that they are real differentiable (they are not C -differentiable unless C is commutative). Here, $\widehat{\zeta_j}$ means that ζ_j is omitted, hence $(\zeta_1, \dots, \widehat{\zeta_j}, \dots, \zeta_{n+1}) = (\zeta_1, \dots, \zeta_{j-1}, \zeta_{j+1}, \dots, \zeta_{n+1}) \in C^n$. Obviously, the differential structure is independent of the choice of the orthonormal C -basis of V , and $P_C(E)$ is a real smooth manifold of dimension $n \cdot \dim_{\mathbb{R}} C$.

Let Φ be a C -sesquilinear hermitian form on E . Recall from Remark 4.5 that $\Phi(z, z) \in \mathbb{R}$ for each $z \in E$. Suppose that q is its *associated quadratic form*, that is

$$q : \begin{cases} E \rightarrow \mathbb{R} \\ z \mapsto \Phi(z, z). \end{cases}$$

Then we define the following sets:

$E_-(\Phi) := q^{-1}((-\infty, 0))$	the set of Φ -negative vectors,
$E_0(\Phi) := q^{-1}(0) \setminus \{0\}$	the set of Φ -zero vectors, and
$E_+(\Phi) := q^{-1}((0, \infty))$	the set of Φ -positive vectors.

For each $\tau \in C$ and $z \in E \setminus \{0\}$ we have

$$q(\tau z) = \Phi(\tau z, \tau z) = \tau \Phi(z, z) \bar{\tau} = \Phi(z, z) \tau \bar{\tau} = q(z) |\tau|^2. \tag{4.1}$$

Lemma 4.9 *Let Φ be a C -sesquilinear hermitian form on E . The set $P_C(E)$ equals the disjoint union $P_C(E_-(\Phi)) \cup P_C(E_0(\Phi)) \cup P_C(E_+(\Phi))$.*

Proof Clearly, $P_C(E) = P_C(E_-(\Phi)) \cup P_C(E_0(\Phi)) \cup P_C(E_+(\Phi))$. Hence it remains to prove that this union is disjoint. Suppose that q denotes the quadratic form associated to Φ . If we assume for contradiction that

$$P_C(E_-(\Phi)) \cap P_C(E_0(\Phi)) \neq \emptyset,$$

then there are equivalent $z_1, z_2 \in E \setminus \{0\}$ such that $q(z_1) < 0$ and $q(z_2) = 0$. But then there is $\tau \in C \setminus \{0\}$ such that

$$0 = q(z_2) = q(\tau z_1) = q(z_1)|\tau|^2 < 0,$$

which is a contradiction. Hence $P_C(E_-(\Phi)) \cap P_C(E_0(\Phi)) = \emptyset$. Analogously we see that $P_C(E_-(\Phi)) \cap P_C(E_+(\Phi)) = \emptyset$ and $P_C(E_+(\Phi)) \cap P_C(E_0(\Phi)) = \emptyset$. \square

Let $\pi : E \setminus \{0\} \rightarrow P_C(E)$ denote the projection on the equivalence classes. Since $C \setminus \{0\}$ acts homeomorphically on $E \setminus \{0\}$, the projection π is open. Further π is continuous by the definition of the topology on $P_C(E)$.

Proposition 4.10 *Let $\Phi : E \times E \rightarrow C$ be a C -sesquilinear hermitian form, which is indefinite and non-degenerate. Then*

$$\partial P_C(E_-(\Phi)) = P_C(E_0(\Phi)).$$

Proof Let $U \subseteq P_C(E_-(\Phi))$. All complements, closures, interiors, and boundaries of subsets of $E \setminus \{0\}$ are taken in $E \setminus \{0\}$. At first we show that

$$\mathbb{C}\pi(\mathbb{C}\partial\pi^{-1}(U)) = \partial U. \tag{4.2}$$

To that end let $M := \pi^{-1}(U)$. Then $\mathbb{C}M = \pi^{-1}(\mathbb{C}U)$. From π being open and continuous it follows that $(\mathbb{C}U)^\circ = \pi((\mathbb{C}M)^\circ)$. This yields

$$\pi(\mathbb{C}\overline{M}) = \pi((\mathbb{C}M)^\circ) = (\mathbb{C}U)^\circ = \mathbb{C}\overline{U},$$

hence $\mathbb{C}\pi(\mathbb{C}\overline{M}) = \overline{U}$. Again from π being open and continuous we get $U^\circ = \pi(M^\circ)$. Therefore,

$$\begin{aligned} \mathbb{C}\pi(\mathbb{C}\partial\pi^{-1}(U)) &= \mathbb{C}\pi(\mathbb{C}\partial M) \\ &= \mathbb{C}\pi(\mathbb{C}(\overline{M} \cap \mathbb{C}M^\circ)) \\ &= \mathbb{C}(\pi(\mathbb{C}\overline{M}) \cup \pi(M^\circ)) \\ &= \mathbb{C}\pi(\mathbb{C}\overline{M}) \cap \mathbb{C}\pi(M^\circ) \\ &= \overline{U} \cap \mathbb{C}U^\circ = \partial U. \end{aligned}$$

Let q denote the quadratic form associated to Φ . Then q is smooth. Since Φ is non-degenerate, 0 is a regular value for $q|_{E \setminus \{0\}}$. Hence $E_0(\Phi) = q^{-1}(0) \setminus \{0\}$ is the boundary of the bounded submanifold $q^{-1}((-\infty, 0]) \setminus \{0\}$ and therefore also of $E_-(\Phi) = q^{-1}((-\infty, 0))$. Then the statement follows from (4.2) and Lemma 4.9 with $U := P_C(E_-(\Phi))$. \square

4.4 The C -hyperbolic spaces $P_C(E_-(\Psi_1))$ and $P_C(E_-(\Psi_2))$

Let (C, V, J) be an associative J^2C -module structure. We define two specific non-degenerate, indefinite C -sesquilinear hermitian forms Ψ_1 and Ψ_2 and consider the manifolds $P_C(E_-(\Psi_j))$ defined in Sect. 4.3. For each space we choose a set of representatives and a Riemannian metric on it such that $P_C(E_-(\Psi_1))$ is essentially the ball model from Sect. 2.5

and $P_C(E_-(\Psi_2))$ becomes the Siegel domain model for hyperbolic space, which is essentially the model D from Sect. 2.4.

We denote by $GL_C(E)$ the group of all C -linear invertible maps $E \rightarrow E$. Set

$$U(\Psi_j, C) := \{g \in GL_C(E) \mid \forall z_1, z_2 \in E: \Psi_j(gz_1, gz_2) = \Psi_j(z_1, z_2)\},$$

and let $Z(\Psi_j, C)$ denote the center of $U(\Psi_j, C)$. Recall that G denotes the full isometry group of B , respectively D . In this section, we establish a natural and explicit isomorphism between the quotient group $PU(\Psi_j, C) := U(\Psi_j, C)/Z(\Psi_j, C)$ and a subgroup G^{res} of G . Moreover, we explicitly characterize G^{res} . In Sect. 4.5, this isomorphism is used to show that the definition of isometric spheres in literature is subsumed by our definition.

Let $\beta_3: W \times W \rightarrow C$ be the sum of β_1 and β_2 , that is,

$$\beta_3((\eta_1, v_1), (\eta_2, v_2)) := \beta_1(\eta_1, \eta_2) + \beta_2(v_1, v_2).$$

We define the maps $\Psi_j: E \times E \rightarrow C$ ($j = 1, 2$) by

$$\begin{aligned} \Psi_1((\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2)) &:= -\beta_1(\zeta_1, \zeta_2) + \beta_1(\eta_1, \eta_2) + \beta_2(v_1, v_2) \\ &= -\beta_1(\zeta_1, \zeta_2) + \beta_3((\eta_1, v_1), (\eta_2, v_2)) \end{aligned}$$

and

$$\Psi_2((\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2)) := -\beta_1(\zeta_1, \eta_2) - \beta_1(\eta_1, \zeta_2) + \beta_2(v_1, v_2).$$

Let q_j denote the associated quadratic forms. For all $(\zeta, w) = (\zeta, \eta, v) \in E$ we have

$$q_1((\zeta, w)) = q_1((\zeta, \eta, v)) = -|\zeta|^2 + |\eta|^2 + |v|^2 = -|\zeta|^2 + |w|^2.$$

Employing Proposition 4.4, we see that

$$q_2((\zeta, \eta, v)) = -2 \operatorname{Re} \beta_1(\zeta, \eta) + |v|^2 = -2\langle \zeta, \eta \rangle + |v|^2.$$

Lemma 4.11 *For $j = 1, 2$, the map Ψ_j is a C -sesquilinear hermitian form on E which is non-degenerate and indefinite.*

Proof Propositions 4.4 and 4.7 imply that Ψ_j is C -sesquilinear hermitian. Suppose that there is $(\zeta, \eta, v) \in E$ such that for all $(\sigma, \tau, u) \in E$ we have

$$\Psi_1((\zeta, \eta, v), (\sigma, \tau, u)) = 0.$$

Propositions 4.4 and 2.11 show that

$$0 = \Psi_1((\zeta, \eta, v), (0, \eta, v)) = |\eta|^2 + |v|^2.$$

Hence $(\eta, v) = (0, 0)$. Further

$$0 = \Psi_1((\zeta, \eta, v), (\zeta, 0, 0)) = -|\zeta|^2,$$

and therefore $\zeta = 0$. This shows that Ψ_1 is non-degenerate. Suppose now that there is $(\zeta, \eta, v) \in E$ such that for all $(\sigma, \tau, u) \in E$ we have

$$\Psi_2((\zeta, \eta, v), (\sigma, \tau, u)) = 0.$$

Then

$$0 = \Psi_2((\zeta, \eta, v), (0, 0, v)) = |v|^2$$

and therefore $v = 0$. Moreover,

$$0 = \Psi_2 ((\zeta, \eta, v), (0, \zeta, 0)) = -|\zeta|^2.$$

Thus $\zeta = 0$ and analogously $\eta = 0$. Hence Ψ_2 is non-degenerate. Finally, for (any) $v \in V \setminus \{0\}$,

$$\begin{aligned} q_1 ((1, 0, 0)) &= -1 & \text{and} & & q_1 ((0, 0, v)) &= |v|^2 > 0, \\ q_2 ((1, 1, 0)) &= -2 & \text{and} & & q_2 ((0, 0, v)) &= |v|^2 > 0. \end{aligned}$$

Therefore, Ψ_1 and Ψ_2 are indefinite. □

4.4.1 Set of representatives for $\overline{P_C(E_-(\Psi_1))}$

Proposition 4.10 shows that we have

$$\overline{P_C(E_-(\Psi_j))} = P_C(E_-(\Psi_j)) \cup P_C(E_0(\Psi_j))$$

for $j \in \{1, 2\}$. If $z = (\zeta, w) \in E_-(\Psi_1) \cup E_0(\Psi_1)$, then $\zeta \neq 0$. Therefore, the element $[z] \in P_C(E_-(\Psi_1)) \cup P_C(E_0(\Psi_1))$ is represented by $(1, \zeta^{-1}w)$, and this is the unique representative of the form $(1, *)$. If $z \in E_-(\Psi_1)$, then

$$0 > q_1(1, \zeta^{-1}w) = -1 + |\zeta^{-1}w|^2.$$

This shows that $1 > |\zeta^{-1}w|^2$. Conversely, if $w \in W$ with $|w|^2 < 1$, then $[(1, w)]$ is an element of $P_C(E_-(\Psi_1))$. Recall the unit ball $B = \{w \in W \mid |w| < 1\}$ in W . Then $P_C(E_-(\Psi_1))$ and B are in bijection via the map

$$[(\zeta, w)] \mapsto \zeta^{-1}w.$$

The same argument shows that $\partial P_C(E_-(\Psi_1)) = P_C(E_0(\Psi_1))$ is bijective to ∂B via this map. We define $\tau_B: \overline{P_C(E_-(\Psi_1))} \rightarrow \overline{B}$ as

$$\tau_B([(\zeta, w)]) := \zeta^{-1}w.$$

Its inverse is

$$\tau_B^{-1}(w) = [(1, w)].$$

In comparison with Sect. 4.3, we see that $\overline{P_C(E_-(\Psi_1))}$ is a subset of U_1 and τ_B a restriction of φ_1 . Therefore, τ_B is a diffeomorphism between the manifolds $P_C(E_-(\Psi_1))$ and B , and also between the manifolds with boundary $\overline{P_C(E_-(\Psi_1))}$ and \overline{B} .

4.4.2 Riemannian metric on B

Since B is an open subset of the vector space W , we may and shall identify the tangent space at a point of B with W . We define a Riemannian metric $\tilde{\rho}$ on B by

$$\begin{aligned} \tilde{\rho}(p)(X, Y) &:= \frac{\langle X, Y \rangle}{1 - |p|^2} + \frac{\langle \beta_3(X, p), \beta_3(Y, p) \rangle}{(1 - |p|^2)^2} \\ &= \frac{1}{(1 - |p|^2)^2} \operatorname{Re}((1 - |p|^2)\beta_3(X, Y) + \beta_1(\beta_3(X, p), \beta_3(Y, p))) \end{aligned}$$

for all $p \in B$ and all $X, Y \in T_p B = W$. Proposition 4.13 below shows that $\tilde{\rho}$ essentially coincides with the Riemannian metric defined in (2.11).

Lemma 4.12 *Let $x, y \in W$. Then $\beta_3(x, y) = 0$ if and only if $y \in (Cx)^\perp$.*

Proof Let $\zeta, \eta, \xi \in C$. Proposition 4.4 yields

$$\langle \zeta \bar{\eta}, \xi \rangle = \operatorname{Re} \beta_1(\zeta \bar{\eta}, \xi) = \operatorname{Re}(\zeta \bar{\eta} \bar{\xi}) = \operatorname{Re}(\zeta \bar{\xi} \bar{\eta}) = \operatorname{Re} \beta_1(\zeta, \xi \bar{\eta}) = \langle \zeta, \xi \bar{\eta} \rangle.$$

By Riesz’ Representation Theorem we know that $\beta_3(x, y) = 0$ if and only if

$$\langle \beta_3(x, y), \xi \rangle = 0 \quad \text{for all } \xi \in C.$$

Supposing that $x = (\zeta, v), y = (\eta, u) \in C \oplus V$, it follows from the definitions of $\beta_3, \beta_1, \beta_2$ and the inner product on W that

$$\begin{aligned} \langle \beta_3(x, y), \xi \rangle &= \langle \beta_1(\zeta, \eta), \xi \rangle + \langle \beta_2(v, u), \xi \rangle = \langle \zeta \bar{\eta}, \xi \rangle + \langle \xi u, v \rangle \\ &= \langle \zeta, \xi \bar{\eta} \rangle + \langle v, \xi u \rangle = \langle \bar{\xi} \zeta, \eta \rangle + \langle \bar{\xi} v, u \rangle \\ &= \langle \bar{\xi} x, y \rangle. \end{aligned}$$

Hence $\beta_3(x, y) = 0$ if and only if $y \in (Cx)^\perp$. □

Proposition 4.13 *The map $\tilde{\rho}$ coincides, up to a multiplicative factor of 4, with the Riemannian metric given by (2.11) on B .*

Proof For $p = 0$ and all $X, Y \in T_0B$ we have

$$\tilde{\rho}(0)(X, Y) = \langle X, Y \rangle = \frac{1}{4} \langle X, Y \rangle_{0--}.$$

Suppose that $p = (\zeta, v) \in B \setminus \{0\}$. We claim that the equivalence class Cp (see Sect. 2.5) coincides with the C -orbit $C \cdot p$ of p . For $\zeta = 0$ this is obviously true. Suppose that $\zeta \neq 0$. For each $\tau \in C \setminus \{0\}$, we have $\tau p = (\tau \zeta, \tau v)$ and

$$(\tau \zeta)^{-1}(\tau v) = \zeta^{-1} \tau^{-1} \tau v = \zeta^{-1} v.$$

Hence $(\zeta, v) \sim (\tau \zeta, \tau v)$. If $\tau = 0$, then $\tau p = 0 \in Cp$ by definition. Conversely, suppose that $(\eta, u) \in Cp \setminus \{0\}$. Then

$$\eta \zeta^{-1} p = \eta \zeta^{-1}(\zeta, v) = (\eta, \eta \zeta^{-1} v) = (\eta, \eta \eta^{-1} u) = (\eta, u).$$

Thus, $(\eta, u) \in C \cdot p$. Clearly, $0 \in Cp \cap C \cdot p$. This shows that $Cp = C \cdot p$.

Now let $p \in B \setminus \{0\}$ and $X, Y \in T_pB$. Suppose first that $X, Y \in (Cp)^\perp$. Then Lemma 4.12 shows that

$$\tilde{\rho}(p)(X, Y) = \frac{\langle X, Y \rangle}{1 - |p|^2} = \frac{1}{4} \langle X, Y \rangle_{p--}.$$

Suppose now that $X \in Cp$ and $Y \in (Cp)^\perp$ (or vice versa). Then

$$\tilde{\rho}(p)(X, Y) = \frac{\langle X, Y \rangle}{1 - |p|^2} = 0 = \frac{1}{4} \langle X, Y \rangle_{p--}.$$

Finally suppose that $X, Y \in Cp$. Then $X = \tau_1 p$ and $Y = \tau_2 p$ for some $\tau_1, \tau_2 \in C$. From Lemma 2.12, Propositions 4.4 and 4.7 it follows that

$$\begin{aligned} \langle X, Y \rangle &= \operatorname{Re} \beta_3(X, Y) = \operatorname{Re} \beta_3(\tau_1 p, \tau_2 p) = \operatorname{Re}(\tau_1 \beta_3(p, p) \bar{\tau}_2) = |p|^2 \operatorname{Re}(\tau_1 \bar{\tau}_2) \\ &= |p|^2 \langle \tau_1, \tau_2 \rangle. \end{aligned}$$

Then

$$\begin{aligned}
 \tilde{\rho}(p)(X, Y) &= \frac{\langle X, Y \rangle}{1 - |p|^2} + \frac{\langle \beta_3(\tau_1 p, p), \beta_3(\tau_2 p, p) \rangle}{(1 - |p|^2)^2} \\
 &= \frac{|p|^2 \langle \tau_1, \tau_2 \rangle}{1 - |p|^2} + \frac{\langle \tau_1 \beta_3(p, p), \tau_2 \beta_3(p, p) \rangle}{(1 - |p|^2)^2} \\
 &= \frac{|p|^2 \langle \tau_1, \tau_2 \rangle}{1 - |p|^2} + \frac{|p|^4 \langle \tau_1, \tau_2 \rangle}{(1 - |p|^2)^2} \\
 &= \frac{|p|^2 \langle \tau_1, \tau_2 \rangle}{(1 - |p|^2)^2} \\
 &= \frac{\langle X, Y \rangle}{(1 - |p|^2)^2} = \frac{1}{4} \langle X, Y \rangle_{p--}.
 \end{aligned}$$

This completes the proof. □

4.4.3 Induced Riemannian isometries on B

Let $\pi_B := \tau_B \circ \pi : E_-(\Psi_1) \rightarrow B$ and suppose that $g \in U(\Psi_1, C)$. Since g is C -linear, it induces a (unique) map \tilde{g} on B by requiring the diagram

$$\begin{array}{ccc}
 E_-(\Psi_1) & \xrightarrow{g} & E_-(\Psi_1) \\
 \pi_B \downarrow & & \downarrow \pi_B \\
 B & \xrightarrow{\tilde{g}} & B.
 \end{array}$$

to commute. Our next goal is to show that each such induced map is a Riemannian isometry on B .

To simplify proofs we fix an orthonormal C -basis $\mathcal{B}(V) := \{v_1, \dots, v_{n-1}\}$ for V . Then $\mathcal{B}(E) := \{e_1, e_2, \dots, e_{n+1}\}$, given by

$$e_1 := (1, 0, 0), \quad e_2 := (0, 1, 0), \quad e_k := (0, 0, v_{k-2}) \quad \text{for } k = 3, \dots, n + 1,$$

is an orthonormal C -basis for E . In the following we will identify each element in E, W and V , and each map $g \in U(\Psi_1, C)$ with its representative with respect to $\mathcal{B}(E)$. Since E is a left C -module, the representing vector of $z \in E$ is a row, and the application of a map $g \in U(\Psi_1, C)$ to z corresponds to the multiplication of the row vector of z to the matrix of g (and not vice versa, as usually in linear algebra). Further we use the notation z^* for \bar{z}^T (the conjugate transpose of z).

Lemma 4.14 *Let $v = (\zeta_1, \dots, \zeta_{n-1}), u = (\eta_1, \dots, \eta_{n-1}) \in V = C^{n-1}$. Then*

$$\beta_2(v, u) = \sum_{j=1}^{n-1} \zeta_j \bar{\eta}_j = vu^*.$$

Proof Let $v_j, v_k \in \mathcal{B}(V)$. If $j = k$, then $\beta_2(v_j, v_k) = |v_j|^2 = 1$. If $j \neq k$, then v_j is orthogonal to Cv_k , hence

$$\langle \beta_2(v_j, v_k), \xi \rangle = \langle \xi v_k, v_j \rangle = 0$$

for each $\xi \in C$, hence $\beta_2(v_j, v_k) = 0$. The claim now follows by the C -sesquilinearity of β_2 . □

Remark 4.15 Let $g \in U(\Psi_1, C)$ and suppose that

$$g = \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix}$$

w.r.t. to $\mathcal{B}(E)$, where $a \in C, b, c \in C^n$ and $A \in C^{n \times n}$. Then the induced map $\tilde{g}: B \rightarrow B$ is given by

$$\tilde{g}(p) = (a + pc^\top)^{-1}(b + pA). \tag{4.3}$$

Further, the representing matrix of Ψ_1 is $\begin{pmatrix} -1 & \\ & I \end{pmatrix}$, where I denotes the $n \times n$ identity matrix. Since g preserves Ψ_1 , we find the conditions

$$\begin{aligned} \begin{pmatrix} -1 & \\ & I \end{pmatrix} &= \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix} \begin{pmatrix} -1 & \\ & I \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ b^* & A^* \end{pmatrix} \\ &= \begin{pmatrix} -|a|^2 + |b|^2 & -a\bar{c} + bA^* \\ -c^\top \bar{a} + Ab^* & -c^\top \bar{c} + AA^* \end{pmatrix} \end{aligned} \tag{4.4}$$

on the entries of g .

The following proposition is proved by a long but straightforward calculation. We omit this proof here.

Proposition 4.16 *Let $g \in U(\Psi_1, C)$. The induced map on B is a Riemannian isometry.*

In the following we will determine which elements in $U(\Psi_1, C)$ induce the same isometry on B . We denote the center of C by $Z(C)$ and set

$$Z^1(C) := \{a \in Z(C) \mid |a| = 1\},$$

which are the central elements in C of unit length. Further we let $Z(\Psi_1, C)$ denote the center of $U(\Psi_1, C)$.

Lemma 4.17 *We have $Z(\Psi_1, C) = \{a \text{ id}_E \mid a \in Z^1(C)\}$.*

Proof Clearly, $\{a \text{ id}_E \mid a \in Z^1(C)\} \subseteq Z(\Psi_1, C)$. For the converse inclusion relation let

$$g = \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix} \in Z(\Psi_1, C).$$

For each $d \in C, |d| = 1$, and each matrix $D \in C^{n \times n}, DD^* = I$, the matrix

$$h = \begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix}$$

is in $U(\Psi_1, C)$. So necessarily,

$$\begin{pmatrix} ad & bD \\ c^\top d & AD \end{pmatrix} = gh = hg = \begin{pmatrix} da & db \\ Dc^\top & DA \end{pmatrix}.$$

The left upper entries show that $a \in Z(C)$. Choosing different values for d , but the same (invertible) D , we find $b = c = 0$. If D runs through all permutation matrices and all rotation matrices, then we see that A is a diagonal matrix $\text{diag}(x, \dots, x)$ with $x \in Z(C)$. By (4.4), $|a| = 1 = |x|$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & xI \end{pmatrix}.$$

Let $h = \begin{pmatrix} d & u \\ w^\top & B \end{pmatrix} \in U(\Psi_1, C)$. Then

$$\begin{pmatrix} ad & au \\ xw^\top & xB \end{pmatrix} = gh = hg = \begin{pmatrix} da & ux \\ w^\top a & Bx \end{pmatrix}.$$

Therefore, $au = ux = xu$. For $u \neq 0$ it follows that $a = x$. Hence $g = a \text{id}_E$ for some $a \in Z^1(C)$. □

Proposition 4.18 *Let $g_1, g_2 \in U(\Psi_1, C)$. Then g_1 and g_2 induce the same isometry on B if and only if $g_1h = g_2$ for some $h \in Z(\Psi_1, C)$.*

Proof It suffices to show that exactly the elements in $Z(\Psi_1, C)$ induce id_B . Let g be an element of $U(\Psi_1, C)$ and suppose that

$$g = \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix}$$

w.r.t. $\mathcal{B}(E)$, where $a \in C, b, c \in C^n$ and $A \in C^{n \times n}$. Then the induced isometry on B is

$$\tilde{g}: \begin{cases} B \rightarrow B \\ p \mapsto (a + pc^\top)^{-1}(b + pA). \end{cases}$$

Suppose that $\tilde{g} = \text{id}_B$. Then

$$0 = \tilde{g}(0) = a^{-1}b,$$

which yields that $b = 0$ and, by (4.4), $|a| = 1$. Hence

$$g = \begin{pmatrix} a & 0 \\ c^\top & A \end{pmatrix}.$$

Now (4.4) shows that $0 = -a\bar{c}$, which implies that $c = 0$. Thus,

$$\tilde{g}(p) = a^{-1}pA$$

for all $p \in B$. Suppose that $A = (a_{ij})_{i,j=1,\dots,n}$. Let $j \in \{1, \dots, n\}$ and consider the vector $p = (p_i)_{i=1,\dots,n}$ with $p_j = \frac{1}{2}$ and $p_i = 0$ for $i \neq j$. Then $p \in B$ and

$$p = \tilde{g}(p) = \frac{1}{2}a^{-1}(a_{j1}, \dots, a_{jn}).$$

Therefore, $a_{ij} = 0$ for $i \neq j$ and $a_{jj} = a$. This shows that $A = aI$.

Now let $\zeta \in C$ with $|\zeta| < 1$. Then $(\zeta, 0) \in B$ and therefore,

$$(\zeta, 0) = \tilde{g}(\zeta, 0) = a^{-1}(\zeta, 0)a.$$

Hence $a\zeta = \zeta a$. By scaling, this equality holds for all $\zeta \in C$. Thus $a \in Z^1(C)$, which yields that $g \in Z(\Psi_1, C)$. Conversely, each element of $Z(\Psi_1, C)$ clearly induces id_B on B . □

Let

$$\text{PU}(\Psi_1, C) := U(\Psi_1, C)/Z(\Psi_1, C)$$

and denote the coset of $g \in U(\Psi_1, C)$ by $[g]$. As before we use the notation \tilde{g} for the isometry on B induced by $g \in U(\Psi_1, C)$. Recall that G is the full isometry group of B .

Corollary 4.19 *The map*

$$j_{\Psi_1} : \begin{cases} \text{PU}(\Psi_1, C) \rightarrow G \\ [g] \mapsto \tilde{g} \end{cases}$$

is a monomorphism of groups.

The next goal is to characterize the subgroup of G to which $\text{PU}(\Psi_1, C)$ is isomorphic. For this it is most convenient to work with the space $P_C(E_-(\Psi_2))$.

4.4.4 *Set of representatives for $\overline{P_C(E_-(\Psi_2))}$*

If $z = (\zeta, \eta, v) \in E_-(\Psi_2)$, then

$$0 > q_2((\zeta, \eta, v)) = -2\langle \zeta, \eta \rangle + |v|^2,$$

which shows that $\zeta \neq 0$. Thus, $[z] \in P_C(E_-(\Psi_2))$ is represented by $(1, \zeta^{-1}\eta, \zeta^{-1}v)$ and this is the unique representative of the form $(1, *, *)$. We note that

$$0 > q_2((1, \zeta^{-1}\eta, \zeta^{-1}v)) = -2\langle 1, \zeta^{-1}\eta \rangle + |\zeta^{-1}v|^2 = -2\text{Re}(\zeta^{-1}\eta) + |\zeta^{-1}v|^2,$$

and therefore

$$\text{Re}(\zeta^{-1}\eta) > \frac{1}{2}|\zeta^{-1}v|^2.$$

Hence, if we define

$$H := \left\{ (\tau, u) \in C \oplus V \mid \text{Re}(\tau) > \frac{1}{2}|u|^2 \right\},$$

then

$$(\zeta^{-1}\eta, \zeta^{-1}v) \in H.$$

Conversely, if $(\tau, u) \in H$, then $[(1, \tau, u)] \in P_C(E_-(\Psi_2))$.

If $z = (\zeta, \eta, v) \in E_0(\Psi_2)$, then either $\zeta \neq 0$ or $z = (0, \eta, 0)$ with $\eta \neq 0$. Applying the previous argumentation we see that the set of elements in $P_C(E_0(\Psi_2))$ which have a representative $(\zeta, \eta, v) \in E_0(\Psi_2)$ with $\zeta \neq 0$ is bijective to

$$\left\{ (\tau, u) \in C \oplus V \mid \text{Re}(\tau) = \frac{1}{2}|u|^2 \right\}$$

via

$$[(\zeta, \eta, v)] \mapsto (\zeta^{-1}\eta, \zeta^{-1}v).$$

If $z = (0, \eta, 0)$ with $\eta \neq 0$, then $[z]$ is represented by $(0, 1, 0)$. Let

$$\overline{H}^g := \left\{ (\tau, u) \in C \oplus V \mid \text{Re}(\tau) \geq \frac{1}{2}|u|^2 \right\} \cup \{\infty\}$$

denote the closure of H in the one-point compactification $(C \oplus V) \cup \{\infty\}$ of $C \oplus V$. Then the map $\tau_H : \overline{P_C(E_-(\Psi_2))} \rightarrow \overline{H}^g$,

$$\tau_H([(\zeta, \eta, v)]) := \begin{cases} (\zeta^{-1}\eta, \zeta^{-1}v) & \text{if } \zeta \neq 0, \\ \infty & \text{if } \zeta = 0, \end{cases}$$

is a bijection with inverse map

$$\tau_H^{-1}(\infty) = [(0, 1, 0)] \quad \tau_H^{-1}(\tau, u) = [(1, \tau, u)].$$

Since $\tau_H|_{P_C(E_-(\Psi_2))}$ is a restriction of the chart map φ_1 from Sect. 4.3, τ_H is a diffeomorphism between $P_C(E_-(\Psi_2))$ and H .

4.4.5 Riemannian metric and induced isometries on H

Recall the model D from Sect. 2.4. The map

$$\beta : \begin{cases} \overline{H}^g & \rightarrow \overline{D}^g \\ \infty & \mapsto \infty \\ (\zeta, v) & \mapsto (\zeta, \sqrt{2}v) \end{cases}$$

is clearly a diffeomorphism between the manifolds \overline{H}^g and \overline{D}^g with boundary, and hence a diffeomorphism between H and D . We endow H with a Riemannian metric by requiring that β be an isometry.

Let $\pi_H := \tau_H \circ \pi : E_-(\Psi_2) \rightarrow H$. As before, each $g \in U(\Psi_2, C)$ defines a (unique) map \tilde{g} on H which makes the diagram

$$\begin{array}{ccc} E_-(\Psi_2) & \xrightarrow{g} & E_-(\Psi_2) \\ \pi_H \downarrow & & \downarrow \pi_H \\ H & \xrightarrow{\tilde{g}} & H \end{array}$$

commutative.

Let $Z(\Psi_2, C)$ denote the center of $U(\Psi_2, C)$ and set

$$PU(\Psi_2, C) := U(\Psi_2, C)/Z(\Psi_2, C).$$

In the following we will use the results from the previous subsections to show that $PU(\Psi_2, C)$ is isomorphic to a subgroup of the full isometry group G of H . We consider the map

$$T : \begin{cases} E & \rightarrow E \\ (\zeta, \eta, v) & \mapsto \left(\frac{1}{\sqrt{2}}(\zeta - \eta), \frac{1}{\sqrt{2}}(\zeta + \eta), v \right). \end{cases}$$

Then T is C -linear and invertible with inverse map $T^{-1} : E \rightarrow E$,

$$T^{-1}(\zeta, \eta, v) = \left(\frac{1}{\sqrt{2}}(\zeta + \eta), \frac{1}{\sqrt{2}}(-\zeta + \eta), v \right).$$

The following lemma is shown by a straightforward calculation.

Lemma 4.20 *The map T transforms Ψ_2 into Ψ_1 , i. e., $\Psi_2 \circ (T \times T) = \Psi_1$. Further, we have $T(E_-(\Psi_1)) = E_-(\Psi_2)$ and $T(E_0(\Psi_1)) = E_0(\Psi_2)$.*

Proposition 4.21 *The map*

$$\lambda : \begin{cases} U(\Psi_1, C) & \rightarrow U(\Psi_2, C) \\ g & \mapsto T \circ g \circ T^{-1} \end{cases}$$

is an isomorphism with $\lambda(Z(\Psi_1, C)) = Z(\Psi_2, C)$.

Proof Let $g \in U(\Psi_1, C)$. Since $T \in GL_C(E)$, we have $\lambda(g) \in GL_C(E)$. Lemma 4.20 shows that

$$\begin{aligned} \Psi_2 \circ (TgT^{-1} \times TgT^{-1}) &= \Psi_2 \circ (T \times T) \circ (gT^{-1} \times gT^{-1}) \\ &= \Psi_1 \circ (g \times g) \circ (T^{-1} \times T^{-1}) = \Psi_1 \circ (T^{-1} \times T^{-1}) \\ &= \Psi_2. \end{aligned}$$

Hence $\lambda(g) \in U(\Psi_2, C)$. Clearly, λ is an isomorphism of groups, which shows that $\lambda(Z(\Psi_1, C)) = Z(\Psi_2, C)$. □

For the proof of the following proposition recall the Cayley transform $\mathcal{C}: B \rightarrow D$ from Sect. 2.8. For each $\zeta \in C \setminus \{1\}$ we have

$$\begin{aligned} (1 - \zeta)^{-1}(1 + \zeta) &= |1 - \zeta|^{-2}(1 - \bar{\zeta})(1 + \zeta) = |1 - \zeta|^{-2}(1 - \bar{\zeta} + \zeta - |\zeta|^2) \\ &= |1 - \zeta|^{-2}(1 + 2 \operatorname{Im} \zeta - |\zeta|^2). \end{aligned}$$

Hence

$$\mathcal{C}(\zeta, v) = (1 - \zeta)^{-1}(1 + \zeta, 2v).$$

Proposition 4.22 *Let $g \in U(\Psi_2, C)$. Then the induced map \tilde{g} on H is an isometry. Further, the map*

$$j_{\Psi_2}: \begin{cases} \operatorname{PU}(\Psi_2, C) & \rightarrow G \\ [g] & \mapsto \tilde{g} \end{cases}$$

is well-defined and a monomorphism of groups.

Proof The map T induces the map $\hat{T}: P_C(E_-(\Psi_1)) \rightarrow P_C(E_-(\Psi_2))$ given by

$$\hat{T}([\zeta, \eta, v]) = \left[\left(\frac{1}{\sqrt{2}}(\zeta - \eta), \frac{1}{\sqrt{2}}(\zeta + \eta), v \right) \right]$$

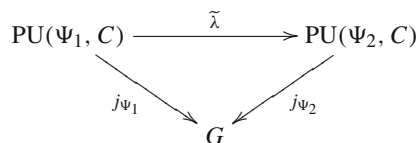
and the map $\tilde{T}: B \rightarrow H$ given by

$$\begin{aligned} \tilde{T}((\eta, v)) &= \tau_H(\hat{T}[(1, \eta, v)]) = \tau_H\left(\left[\frac{1}{\sqrt{2}}(1 - \eta), \frac{1}{\sqrt{2}}(1 + \eta), v\right]\right) \\ &= \tau_H\left(\left[\left(1, (1 - \eta)^{-1}(1 + \eta), (1 - \eta)^{-1}\sqrt{2}v\right)\right]\right) \\ &= (1 - \eta)^{-1}(1 + \eta, \sqrt{2}v). \end{aligned}$$

Therefore, $\tilde{T} = \beta^{-1} \circ \mathcal{C}$. From Propositions 2.9 and 4.13 it follows that the isometry group of B and that of H are identical. Then Propositions 4.16 and 4.21 imply that for each $g \in U(\Psi_2, C)$, the induced map \tilde{g} is an isometry on H . Moreover, Proposition 4.21 shows that λ factors to a map

$$\tilde{\lambda}: \operatorname{PU}(\Psi_1, C) \rightarrow \operatorname{PU}(\Psi_2, C).$$

Recall the map j_{Ψ_1} from Corollary 4.19. Then the diagram



commutes, which completes the proof. □

4.4.6 Lifted isometries

In this section, we determine which isometries on H are induced from an element in $U(\Psi_2, C)$. To that end we need explicit formulas for the action of an element $g \in G$ on H , which are provided by the following remark.

Remark 4.23 Let $g \in G$. Section 2.8 provides explicit formulas for the action of g on D . Using the isometry $\beta: H \rightarrow D$, the action of g on H is given by

$$g^H := \beta^{-1} \circ g \circ \beta: H \rightarrow H.$$

Evaluating this formula, we find the following action laws. For the geodesic inversion σ we have

$$\sigma^H(\zeta, v) = \zeta^{-1}(1, -v).$$

For $a_s \in A$ we get

$$a_s^H(\zeta, v) = (s\zeta, s^{1/2}v).$$

For $n = (\xi, w) \in N$ it follows

$$n^H(\zeta, v) = \left(\zeta + \xi + \frac{1}{4}|w|^2 + \beta_2 \left(v, \frac{1}{\sqrt{2}}w \right), \frac{1}{\sqrt{2}}w + v \right).$$

For $m = (\varphi, \psi) \in M$ we have

$$m^H(\zeta, v) = (\varphi(\zeta), \psi(v)).$$

Proposition 4.24 *A representative of σ^H in $U(\Psi_2, C)$ is*

$$g(\zeta, \eta, v) = (\eta, \zeta, -v).$$

Proposition 4.25 *Let $a_s \in A$. Then*

$$g(\zeta, \eta, v) = (s^{-1/2}\zeta, s^{1/2}\eta, v)$$

is a representative of a_s in $U(\Psi_2, C)$.

Proof Obviously, g is C -linear and induces a_s on B . Further

$$\begin{aligned} \Psi_2(g(\zeta_1, \eta_1, v_1), g(\zeta_2, \eta_2, v_2)) &= \Psi_2((s^{-1/2}\zeta_1, s^{1/2}\eta_1, v_1), (s^{-1/2}\zeta_2, s^{1/2}\eta_2, v_2)) \\ &= -s^{-1/2}\zeta_1\overline{\eta_2}s^{1/2} - s^{-1/2}\eta_1\overline{\zeta_2}s^{1/2} + \beta_2(v_1, v_2) \\ &= -\zeta_1\overline{\eta_2} - \eta_1\overline{\zeta_2} + \beta_2(v_1, v_2) \\ &= \Psi_2((\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2)). \end{aligned}$$

Hence $g \in U(\Psi_2, C)$. □

Proposition 4.26 *Let $n = (\xi, w) \in N$. A representative of n in $U(\Psi_2, C)$ is*

$$g(\zeta, \eta, v) = \left(\zeta, \zeta \left(\xi + \frac{1}{4}|w|^2 \right) + \eta + \frac{1}{\sqrt{2}}\beta_2(v, w), \frac{1}{\sqrt{2}}\zeta w + v \right).$$

Proof The map g is clearly C -linear and induces n on B . To see that Ψ_2 is g -invariant, we calculate

$$\begin{aligned} &\Psi_2(g(\zeta_1, \eta_1, v_1), g(\zeta_2, \eta_2, v_2)) \\ &= \Psi_2\left[\left(\zeta_1, \zeta_1\left(\xi + \frac{1}{4}|w|^2\right) + \eta_1 + \frac{1}{\sqrt{2}}\beta_2(v_1, w), \frac{1}{\sqrt{2}}\zeta_1 w + v_1\right), \right. \\ &\quad \left. \left(\zeta_2, \zeta_2\left(\xi + \frac{1}{4}|w|^2\right) + \eta_2 + \frac{1}{\sqrt{2}}\beta_2(v_2, w), \frac{1}{\sqrt{2}}\zeta_2 w + v_2\right)\right] \\ &= -\zeta_1\left[\left(-\xi + \frac{1}{4}|w|^2\right)\bar{\zeta}_2 + \bar{\eta}_2 + \frac{1}{\sqrt{2}}\beta_2(w, v_2)\right] \\ &\quad - \left[\zeta_1\left(\xi + \frac{1}{4}|w|^2\right) + \eta_1 + \frac{1}{\sqrt{2}}\beta_2(v_1, w)\right]\bar{\zeta}_2 + \beta_2\left(\frac{1}{\sqrt{2}}\zeta_1 w + v_1, \frac{1}{\sqrt{2}}\zeta_2 w + v_2\right) \\ &= \zeta_1\xi\bar{\zeta}_2 - \frac{1}{4}|w|^2\zeta_1\bar{\zeta}_2 - \zeta_1\bar{\eta}_2 - \frac{1}{\sqrt{2}}\zeta_1\beta_2(w, v_2) - \zeta_1\xi\bar{\zeta}_2 - \frac{1}{4}|w|^2\zeta_1\bar{\zeta}_2 \\ &\quad - \eta_1\bar{\zeta}_2 - \frac{1}{\sqrt{2}}\beta_2(v_1, w)\bar{\zeta}_2 + \frac{1}{2}|w|^2\zeta_1\bar{\zeta}_2 + \frac{1}{\sqrt{2}}\zeta_1\beta_2(w, v_2) \\ &\quad + \frac{1}{\sqrt{2}}\beta_2(v_1, w)\bar{\zeta}_2 + \beta_2(v_1, v_2) \\ &= \Psi_2((\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2)). \end{aligned}$$

This completes the proof. □

The remaining part of this section is devoted to the discussion which elements of M have a representative in $U(\Psi_2, C)$. The situation for M is much more involved than the proofs of Propositions 4.24–4.26. In particular, it will turn out that in general not every element of M can be lifted to $U(\Psi_2, C)$.

Lemma 4.27 *Let $m = (\varphi, \psi) \in M$ and suppose that φ is an inner automorphism of C . Then*

$$\{\beta_2(\psi(v_1), \psi(v_2)) \mid v_1, v_2 \in V\} = C.$$

Proof Let $a \in C \setminus \{0\}$ and suppose that $\varphi(\zeta) = a^{-1}\zeta a$ for all $\zeta \in C$. Choose $v \in V$ with $|v| = 1$. For each $\zeta \in C$ we find

$$\begin{aligned} \beta_2(\psi(a\zeta a^{-1}v), \psi(v)) &= \beta_2(\varphi(a\zeta a^{-1})\psi(v), \psi(v)) = \beta_2(\zeta\psi(v), \psi(v)) \\ &= \zeta\beta_2(\psi(v), \psi(v)) = \zeta|\psi(v)|^2 = \zeta|v|^2 \\ &= \zeta. \end{aligned}$$

Therefore,

$$C \subseteq \{\beta_2(\psi(v_1), \psi(v_2)) \mid v_1, v_2 \in V\}.$$

The converse inclusion relation clearly holds by the range of β_2 . □

Let $i_H: \bar{H}^g \rightarrow E_-(\Psi_2) \cup E_0(\Psi_2)$ be any section of

$$\pi_H = \tau_H \circ \pi: E_-(\Psi_2) \cup E_0(\Psi_2) \rightarrow \bar{H}^g.$$

Let $\tilde{g} \in G$ and recall from Proposition 2.14 that g extends continuously to \overline{H}^g . If the element $g \in U(\Psi_2, C)$ is a representative of \tilde{g} , then the diagram

$$\begin{array}{ccc}
 E_-(\Psi_2) \cup E_0(\Psi_2) & \xrightarrow{g} & E_-(\Psi_2) \cup E_0(\Psi_2) \\
 \uparrow i_H & & \downarrow \pi_H \\
 \overline{H}^g & \xrightarrow{\tilde{g}} & \overline{H}^g
 \end{array}$$

commutes. We will make use of this fact in the proof of Proposition 4.28 below. For convenience we define i_H by

$$i_H(\infty) := (0, 1, 0) \quad \text{and} \quad i_H(\eta, v) := (1, \eta, v). \tag{4.5}$$

Proposition 4.28 *Let $m = (\varphi, \psi) \in M$. Then there exists a representative of m in $U(\Psi_2, C)$ if and only if $\varphi = \text{id}$. In this case,*

$$g(\zeta, \eta, v) = (\zeta, \eta, \psi(v))$$

is such a representative.

Proof Suppose first that $m = (\varphi, \psi)$ with $\varphi = \text{id}$. We will show that

$$g: \begin{cases} E & \rightarrow E \\ (\zeta, \eta, v) & \mapsto (\zeta, \eta, \psi(v)) \end{cases}$$

is an element of $U(\Psi_2, C)$. To that end let $(\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2) \in E$ and $\zeta \in C$. Then

$$\begin{aligned}
 g(\zeta(\zeta_1, \eta_1, v_1) + (\zeta_2, \eta_2, v_2)) &= (\zeta\zeta_1 + \zeta_2, \zeta\eta_1 + \eta_2, \psi(\zeta v_1 + v_2)) \\
 &= (\zeta\zeta_1, \zeta\eta_1, \psi(\zeta v_1)) + (\zeta_2, \eta_2, \psi(v_2)) \\
 &= (\zeta\zeta_1, \zeta\eta_1, \varphi(\zeta)\psi(v_1)) + g(\zeta_2, \eta_2, v_2) \\
 &= (\zeta\zeta_1, \zeta\eta_1, \zeta\psi(v_1)) + g(\zeta_2, \eta_2, v_2) \\
 &= \zeta g(\zeta_1, \eta_1, v_1) + g(\zeta_2, \eta_2, v_2).
 \end{aligned}$$

This shows that g is C -linear. The map g is obviously invertible, hence $g \in \text{GL}_C(E)$. Lemma 3.14(ii) implies that Ψ_2 is invariant under g . Hence $g \in U(\Psi_2, C)$. Clearly, g induces m .

Suppose now that $m = (\varphi, \psi) \in M$ and that there is a representative g of m in $U(\Psi_2, C)$. We have to show that $\varphi = \text{id}$. Since $m(0) = 0$, it follows that

$$g(i_H(0)) = g(1, 0, 0) \in \pi_H^{-1}(0) = (C \setminus \{0\}) \times \{0\} \times \{0\}.$$

Thus, there is $a \in C \setminus \{0\}$ such that $g(1, 0, 0) = (a, 0, 0)$. Further $m(\infty) = \infty$. The same argument shows that there is $b \in C \setminus \{0\}$ such that

$$g(i_H(\infty)) = g(0, 1, 0) = (0, b, 0).$$

Then for each $\zeta \in C$ with $\text{Re } \zeta \geq 0$ we have

$$\begin{aligned}
 (\varphi(\zeta), 0) &= m(\zeta, 0) = \pi_H(g(i_H(\zeta, 0))) = \pi_H(g(1, \zeta, 0)) = \pi_H((a, \zeta b, 0)) \\
 &= (a^{-1}\zeta b, 0).
 \end{aligned}$$

Thus $\varphi(\zeta) = a^{-1}\zeta b$ for all $\zeta \in C$ with $\text{Re } \zeta \geq 0$. Now

$$1 = \varphi(1) = a^{-1}b$$

and therefore $b = a$. Hence $\varphi(\zeta) = a^{-1}\zeta a$ for all $\zeta \in C$ with $\operatorname{Re} \zeta \geq 0$. Suppose now that $(\eta, v) \in H$ is arbitrary. Then

$$\begin{aligned} g(i_H(\eta, v)) &= g(1, \eta, v) = g(1, \eta, 0) + g(0, 0, v) \\ &= (a, \eta a, 0) + (\sigma, \tau, u) = (a + \sigma + \eta a + \tau, u) \end{aligned}$$

where (σ, τ, u) depends only on v . From

$$\begin{aligned} (a^{-1}\eta a, \psi(v)) &= (\varphi(\eta), \psi(v)) = \pi_H(a + \sigma, \eta a + \tau, u) \\ &= ((a + \sigma)^{-1}(\eta a + \tau), (a + \sigma)^{-1}u) \end{aligned}$$

it follows that

$$a^{-1}\eta a = (a + \sigma)^{-1}\eta a + (a + \sigma)^{-1}\tau$$

for all η with $\operatorname{Re}(\eta) > \frac{1}{2}|v|^2$. Pick $k \in \mathbb{R}$ with $k > \frac{1}{2}|v|^2$. Then

$$k = a^{-1}ka = k(a + \sigma)^{-1}a + (a + \sigma)^{-1}\tau$$

and

$$2k = 2k(a + \sigma)^{-1}a + (a + \sigma)^{-1}\tau.$$

Solving this system for σ and τ , we find $\sigma = \tau = 0$. Hence

$$g(1, \eta, v) = (a, \eta a, a\psi(v)).$$

Since g has to be C -linear, it follows that

$$g(\zeta, \eta, v) = (\zeta a, \eta a, a\psi(v))$$

for all $(\zeta, \eta, v) \in E$. We derive further properties of a . Let $v \in V \setminus \{0\}$. Then g being in $U(\Psi_2, C)$ yields that

$$|v|^2 = q_2((0, 0, v)) = q_2(g(0, 0, v)) = |a\psi(v)|^2 = |a|^2|\psi(v)|^2 = |a|^2|\psi(v)|^2,$$

hence $|a|^2 = 1$. Again using that $g \in U(\Psi_2, C)$ we find that for all $v_1, v_2 \in V$

$$\begin{aligned} \beta_2(\psi(v_1), \psi(v_2)) &= \Psi_2((0, 0, v_1), (0, 0, v_2)) \\ &= \Psi_2(g(0, 0, v_1), g(0, 0, v_2)) = \Psi_2((0, 0, a\psi(v_1)), (0, 0, a\psi(v_2))) \\ &= \beta_2(a\psi(v_1), a\psi(v_2)) = a\beta_2(\psi(v_1), \psi(v_2))\bar{a} \\ &= a\beta_2(\psi(v_1), \psi(v_2))a^{-1}. \end{aligned}$$

Before we can apply Lemma 4.27 we have to show that $\varphi(\zeta) = a^{-1}\zeta a$ for all $\zeta \in C$. Let $\zeta \in C$ with $\operatorname{Re} \zeta < 0$ and consider the decomposition $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \mathbb{R}$ and $\zeta_2 \in C'$. Then the \mathbb{R} -linearity of φ yields

$$\begin{aligned} \varphi(\zeta) &= \varphi(\zeta_1 + \zeta_2) = \varphi(\zeta_1) + \varphi(\zeta_2) = -\varphi(-\zeta_1) + a^{-1}\zeta_2 a \\ &= -a^{-1}(-\zeta_1)a + a^{-1}\zeta_2 a = a^{-1}\zeta_1 a + a^{-1}\zeta_2 a = a^{-1}(\zeta_1 + \zeta_2)a = a^{-1}\zeta a. \end{aligned}$$

Then Lemma 4.27 implies that $a \in Z(C)$. Therefore, $\varphi = \operatorname{id}$. □

We set

$$M^{\text{res}} := \{(\varphi, \psi) \in M \mid \varphi = \text{id}\} \text{ and } G^{\text{res}} := N\sigma M^{\text{res}} AN \cup M^{\text{res}} AN.$$

Further we define a map $\varphi_H: G^{\text{res}} \rightarrow \text{PU}(\Psi_2, C)$ as follows: For $\tilde{g} = \sigma, \tilde{g} \in A, \tilde{g} \in N$ or $\tilde{g} \in M^{\text{res}}$ we set $\varphi_H(\tilde{g}) := [g]$, where g is the lift of \tilde{g} as in Propositions 4.24–4.26 or 4.28, respectively. For $\tilde{g} = n_2\sigma ma_s n_1 \in N\sigma M^{\text{res}} AN$ we define

$$\varphi_H(\tilde{g}) := \varphi_H(n_2)\varphi_H(\sigma)\varphi_H(m)\varphi_H(a_s)\varphi_H(n_1), \tag{4.6}$$

and likewise for $\tilde{g} = ma_s n \in M^{\text{res}} AN$ we set

$$\varphi_H(\tilde{g}) := \varphi_H(m)\varphi_H(a_s)\varphi_H(n).$$

In other words, we extend φ_H to a group homomorphism. Since the Bruhat decomposition of an element $\tilde{g} \in G^{\text{res}}$ is unique and the Bruhat decomposition of G^{res} can be directly transferred to $\text{PU}(\Psi_2, C)$, the map φ_H is indeed well-defined. By our previous considerations, φ_H is even a group isomorphism.

The following remark shows that M^{res} is not necessarily all of M .

Remark 4.29 The classification [15, Theorem 3.1] of division algebras arising from associative J^2C -module structures (C, V, J) shows that C is either real or complex or quaternionic numbers. In the following we show that $M = M^{\text{res}}$ for $C = \mathbb{R}$, but $M \neq M^{\text{res}}$ for $C = \mathbb{C}$ or $C = \mathbb{H}$ (quaternions).

- (i) Let $C = \mathbb{R}$ and suppose that $m = (\varphi, \psi) \in M$. We claim that $\varphi = \text{id}$. Since $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a norm-preserving endomorphism of \mathbb{R} , the map φ is either id or $-\text{id}$. Assume for contradiction that $\varphi = -\text{id}$. Let $(\zeta, v) \in C \times V$ such that $\zeta \neq 0$ and $v \neq 0$. Then

$$\varphi(\zeta)\psi(v) = -\zeta\psi(v) = -\psi(\zeta v) \neq \psi(\zeta v).$$

Hence $m \notin M$. This is a contradiction and therefore $\varphi = \text{id}$.

- (ii) Let $C = \mathbb{C} = V$ and suppose that $\varphi = \psi$ are complex conjugation. For all elements $(\zeta, v) \in C \oplus V = \mathbb{C}^2$ we have $\varphi(\zeta)\psi(v) = \bar{\zeta}\bar{v} = \zeta v = \psi(\zeta v)$. Clearly, φ, ψ are \mathbb{R} -linear endomorphisms of the Euclidean vector space \mathbb{C} . Therefore, $m = (\varphi, \psi) \in M$, but $m \notin M^{\text{res}}$.
- (iii) Let $C = \mathbb{H} = V$. Define $\varphi: C \rightarrow C$ and $\psi: V \rightarrow V$ by

$$\begin{aligned} \varphi(a + ib + jc + kd) &:= a - ib - jc + kd \\ \psi(a + ib + jc + kd) &:= -a + ib + jc - kd \end{aligned}$$

for $a + ib + jc + kd \in \mathbb{H}$. Clearly, φ, ψ are \mathbb{R} -linear endomorphisms of the Euclidean vector space \mathbb{H} . We claim that $J \circ (\varphi \times \psi) = \psi \circ J$. To that end let $\zeta = a_1 + ib_1 + jc_1 + kd_1 \in C$ and $v = a_2 + ib_2 + jc_2 + kd_2 \in V$. Then

$$\begin{aligned} \zeta v &= a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) \\ &\quad + j(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) + k(a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(\zeta v) &= -a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) \\ &\quad + j(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) + k(-a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2). \end{aligned}$$

On the other side we get

$$\begin{aligned} \varphi(\zeta)\psi(v) &= (a_1 - ib_1 - jc_1 + kd_1)(-a_2 + ib_2 + jc_2 - kd_2) \\ &= -a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) \\ &\quad + j(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) + k(-a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2) \\ &= \psi(\zeta v). \end{aligned}$$

Therefore, $m = (\varphi, \psi) \in M$, but $m \notin M^{\text{res}}$.

4.5 Isometric spheres via cocycles

For an element $g \in G^{\text{res}} \setminus G_\infty$, we give a characterization of the isometric sphere and its radius via a cocycle. Using the isometry $\beta: H \rightarrow D$ we find for the height function on H the formula

$$\text{ht}^H(\zeta, v) = \text{Re } \zeta - \frac{1}{2}|v|^2,$$

and for the Cygan metric ($z_j = (\zeta_j, v_j)$)

$$\rho^H(z_1, z_2) = \left| \frac{1}{2}|v_1|^2 + \frac{1}{2}|v_2|^2 + \left| \text{ht}^H(z_1) - \text{ht}^H(z_2) \right| + \text{Im } \zeta_1 - \text{Im } \zeta_2 - \beta_2(v_1, v_2) \right|^{1/2} \tag{4.7}$$

The basis point in H is $o^H = (1, 0)$. The horospherical coordinates of $z \in \overline{H}^g \setminus \{\infty\}$, $z = (\zeta, v)$, are

$$\left(\text{ht}^H(z), \text{Im } \zeta, \frac{1}{\sqrt{2}}v \right)_h.$$

In the following we derive a formula for isometric spheres which uses a cocycle.

Let $Z^1(C)$ act diagonally on E and suppose that $\pi^{(1)}: E \rightarrow E/Z^1(C)$ is the canonical projection. Further set

$$\pi_H^{(1)} := \tau_H \circ \pi \circ \left(\pi^{(1)} \right)^{-1} : (E_-(\Psi_2) \cup E_0(\Psi_2)) / Z^1(C) \rightarrow \overline{H}^g.$$

Recall the section i_H of π_H from (4.5) and set

$$i_H^{(1)} := \pi^{(1)} \circ i_H : \overline{H}^g \rightarrow E/Z^1(C).$$

Further recall the isomorphism $\varphi_H: G^{\text{res}} \rightarrow \text{PU}(\Psi_2, C)$ from (4.6). For each $g \in G^{\text{res}}$ the map $\varphi_H(g): E \rightarrow E$ induces canonically a map $E/Z^1(C) \rightarrow E/Z^1(C)$, which we denote by $\varphi_H(g)$ as well. For all $g \in G^{\text{res}}$ the diagram

$$\begin{array}{ccc} E/Z^1(C) & \xrightarrow{\varphi_H(g)} & E/Z^1(C) \\ i_H^{(1)} \uparrow & & \downarrow \pi_H^{(1)} \\ \overline{H}^g & \xrightarrow{g} & \overline{H}^g \end{array}$$

commutes, but the diagram

$$\begin{array}{ccc} E/Z^1(C) & \xrightarrow{\varphi_H(g)} & E/Z^1(C) \\ i_H^{(1)} \uparrow & & \uparrow i_H^{(1)} \\ \overline{H}^g & \xrightarrow{g} & \overline{H}^g \end{array}$$

in general not. The second diagram gives rise to the cocycle

$$j: G^{\text{res}} \times \overline{H}^g \rightarrow C^\times / Z^1(C)$$

defined by

$$\varphi_H(g) \left(i_H^{(1)}(z) \right) = j(g, z) i_H^{(1)}(gz) \quad \forall g \in G^{\text{res}} \quad \forall z \in \overline{H}^g.$$

Lemma 4.30 *Let $g = n_2 \sigma a_t n_1 \in G^{\text{res}} \setminus G_\infty$. Then*

$$j(g^{-1}, \infty) = t^{1/2} \pmod{Z^1(C)}.$$

Further, $R(g) = |j(g^{-1}, \infty)|^{-1/2}$.

Proof Suppose that $n_1 = (\xi_1, w_1)$. From $n_1^{-1} = (-\xi_1, -w_1)$ and $g^{-1} = n_1^{-1} \sigma a_t m n_2^{-1}$ it follows that

$$g^{-1} \infty = n_1^{-1} \sigma \infty = n_1^{-1} 0 = \left(-\xi_1 + \frac{1}{4} |w_1|^2, -\frac{1}{\sqrt{2}} w_1 \right).$$

Further

$$\varphi_H(g^{-1})(i_H(\infty)) = \left(t^{1/2}, t^{1/2} \left(-\xi_1 + \frac{1}{4} |w_1|^2 \right), -\frac{1}{\sqrt{2}} t^{1/2} w_1 \right) \pmod{Z^1(C)}.$$

Hence $j(g^{-1}, \infty) = t^{1/2} \pmod{Z^1(C)}$. Then $R(g) = t^{-1/4} = |j(g^{-1}, \infty)|^{-1/2}$. □

Proposition 4.31 *Let $z_1, z_2 \in \overline{H}^g \setminus \{\infty\}$, $z_j = (\zeta_j, v_j)$. Then*

$$\rho^H(z_1, z_2) = \left| \Psi_2(i_H(z_1), i_H(z_2)) + 2 \min(\text{ht}^H(z_1), \text{ht}^H(z_2)) \right|^{1/2}.$$

Proof For all $k_1, k_2 \in \mathbb{R}$ we have

$$k_1 + k_2 - |k_1 - k_2| = 2 \min(k_1, k_2).$$

This and the definition of Ψ_2 show that

$$\begin{aligned} & \left| \Psi_2(i_H(z_1), i_H(z_2)) + 2 \min(\text{ht}^H(z_1), \text{ht}^H(z_2)) \right| \\ &= \left| \Psi_2((1, \zeta_1, v_1), (1, \zeta_2, v_2)) + \text{ht}^H(z_1) + \text{ht}^H(z_2) - \left| \text{ht}^H(z_1) - \text{ht}^H(z_2) \right| \right| \\ &= \left| -\beta_1(1, \zeta_2) - \beta_1(\zeta_1, 1) + \beta_2(v_1, v_2) + \text{ht}^H(z_1) + \text{ht}^H(z_2) - \left| \text{ht}^H(z_1) - \text{ht}^H(z_2) \right| \right| \\ &= \left| \zeta_1 - \text{ht}^H(z_1) + \bar{\zeta}_2 - \text{ht}^H(z_2) - \beta_2(v_1, v_2) + \left| \text{ht}^H(z_1) - \text{ht}^H(z_2) \right| \right|. \end{aligned}$$

Since

$$\zeta_1 - \text{ht}^H(z_1) = \zeta_1 - \text{Re } \zeta_1 + \frac{1}{2} |v_1|^2 = \text{Im } \zeta_1 + \frac{1}{2} |v_1|^2$$

and

$$\bar{\zeta}_2 - \text{ht}^H(z_2) = \bar{\zeta}_2 - \text{Re } \zeta_2 + \frac{1}{2}|v_2|^2 = -\text{Im } \zeta_2 + \frac{1}{2}|v_2|^2,$$

it follows that

$$\begin{aligned} & \left| \Psi_2(i_H(z_1), i_H(z_2)) + 2 \min(\text{ht}^H(z_1), \text{ht}^H(z_2)) \right| \\ &= \left| \frac{1}{2}|v_1|^2 + \text{Im } \zeta_1 + \frac{1}{2}|v_2|^2 - \text{Im } \zeta_2 - \beta_2(v_1, v_2) + \left| \text{ht}^H(z_1) - \text{ht}^H(z_2) \right| \right| \\ &= \rho^H(z_1, z_2)^2. \end{aligned}$$

Since $\rho^H(z_1, z_2)$ is nonnegative, the claim follows. □

For the proof of the following proposition we note that the map

$$\Psi_2^{(1)}: \begin{cases} E/Z^1(C) \times E/Z^1(C) & \rightarrow C/Z^1(C) \\ ([z_1], [z_2]) & \mapsto [\Psi_2(z_1, z_2)] \end{cases}$$

is well-defined. In particular, we have

$$\left| \Psi_2^{(1)}([z_1], [z_2]) \right| = |\Psi_2(z_1, z_2)|$$

for all $z_1, z_2 \in E$. Further, let

$$\partial_g H := \left\{ (\zeta, v) \in C \oplus V \mid \text{Re}(\zeta) = \frac{1}{2}|v|^2 \right\} \cup \{\infty\}$$

denote the boundary of H in \overline{H}^g .

Proposition 4.32 *Let $g \in G^{\text{res}} \setminus G_\infty$ and $z \in \overline{H}^g \setminus \{\infty, g^{-1}\infty\}$. Then*

$$|j(g, z)|^{1/2} = |j(g^{-1}, \infty)|^{1/2} \rho^H(z, g^{-1}\infty) = \frac{\rho^H(z, g^{-1}\infty)}{R(g)}.$$

Proof The second equality is proved by Lemma 4.30. For the proof of the first equality recall from Proposition 4.31 that

$$\rho^H(z, g^{-1}\infty) = \left| \Psi_2(i_H(z), i_H(g^{-1}\infty)) + 2 \min(\text{ht}^H(z), \text{ht}^H(g^{-1}\infty)) \right|^{1/2}.$$

From $g^{-1}\infty \in \partial_g H \setminus \{\infty\}$ it follows that $\text{ht}(g^{-1}\infty) = 0$. Hence

$$\rho(z, g^{-1}\infty)^2 = \left| \Psi_2(i_H(z), i_H(g^{-1}\infty)) \right|.$$

Suppose that $gz = (\zeta, v)$. Then

$$\begin{aligned} \Psi_2(i_H(gz), i_H(\infty)) &= \Psi_2((1, gz), (0, 1, 0)) \\ &= -\beta_1(1, 1) - \beta_1(\zeta, 0) + \beta_2(v, 0) = -1. \end{aligned}$$

It follows that

$$\begin{aligned}
 \rho^H(z, g^{-1}\infty)^2 &= |\Psi_2(i_H(z), i_H(g^{-1}\infty))| = \left| \Psi_2^{(1)}\left(i_H^{(1)}(z), i_H^{(1)}(g^{-1}\infty)\right) \right| \\
 &= \left| \Psi_2^{(1)}\left(i_H^{(1)}(z), j(g^{-1}, \infty)\varphi_H(g^{-1})i_H^{(1)}(\infty)\right) \right| \\
 &= |j(g^{-1}, \infty)|^{-1} \left| \Psi_2^{(1)}\left(i_H^{(1)}(z), \varphi_H(g^{-1})i_H^{(1)}(\infty)\right) \right| \\
 &= |j(g^{-1}, \infty)|^{-1} \left| \Psi_2^{(1)}\left(\varphi_H(g)i_H^{(1)}(z), i_H^{(1)}(\infty)\right) \right| \\
 &= |j(g^{-1}, \infty)|^{-1} |j(g, z)| |\Psi_2(i_H(gz), i_H(\infty))| \\
 &= |j(g^{-1}, \infty)|^{-1} |j(g, z)|.
 \end{aligned}$$

This proves the claim. □

Proposition 4.33 *Let $g \in G^{\text{res}} \setminus G_\infty$. Then*

$$\begin{aligned}
 I(g) &= \{z \in H \mid |j(g, z)| = 1\}, \\
 \text{ext } I(g) &= \{z \in H \mid |j(g, z)| > 1\}, \\
 \text{int } I(g) &= \{z \in H \mid |j(g, z)| < 1\}.
 \end{aligned}$$

Proof This claim follows immediately from comparing Definition 3.9 and Proposition 4.32. □

4.6 A special instance of Theorem 3.18

The most frequent appearance of isometric fundamental regions in the literature is for properly discontinuous subgroups Γ of G^{res} for which ∞ is an ordinary point and the stabilizer Γ_∞ is trivial. We show that under these conditions Γ is of type (O) and $\Gamma \setminus \Gamma_\infty$ of type (F). Theorem 3.18 implies the existence of an isometric fundamental region for Γ , which will be seen to actually be a fundamental domain.

Definition 4.34 Let U be a subset of $\overline{H^g}$ and Γ a subgroup of G . Then Γ is said to *act properly discontinuously* on U if for each compact subset K of U , the set $K \cap gK$ is nonempty for only finitely many g in Γ . The group Γ is said to be *properly discontinuous* if Γ acts properly discontinuously on H .

Definition and Remark 4.35 Let Γ be a properly discontinuous subgroup of G and $z \in H$. The *limit set* $L(\Gamma)$ of Γ is the set of accumulation points of the orbit Γz . Since Γ is properly discontinuous, $L(\Gamma)$ is a subset of $\partial_g H$. The combination of Propositions 2.9 and 1.4 in [9] shows that $L(\Gamma)$ is independent of the choice of z . The *ordinary set* or *discontinuity set* $\Omega(\Gamma)$ of Γ is the complement of $L(\Gamma)$ in $\partial_g H$, hence $\Omega(\Gamma) := \partial_g H \setminus L(\Gamma)$.

Remark 4.36 Let $g, h \in G^{\text{res}}$ and $z \in \overline{H^g}$. Since φ_H is a group isomorphism and $\varphi_H(g)$ is C -linear, it follows that

$$\begin{aligned}
 \varphi_H(gh)(i_H(z)) &= \varphi_H(g)(\varphi_H(h)(i_H(z))) \\
 &= \varphi_H(g)(j(h, z)i_H(hz)) \\
 &= j(h, z)\varphi_H(g)(i_H(hz)) \\
 &= j(h, z)j(g, hz)i_H(hgz).
 \end{aligned}$$

Further,

$$\varphi_H(gh)(i_H(z)) = j(gh, z)i_H(ghz).$$

Thus,

$$j(gh, z) = j(h, z)j(g, hz).$$

The proof of the following lemma proceeds along the lines of [10, Section 17].

Lemma 4.37 *Let Γ be a properly discontinuous subgroup of G^{res} such that $\Gamma_\infty = \{id\}$ and $\infty \in \Omega(\Gamma)$. Then*

- (i) *the set of radii of the isometric spheres of Γ is bounded from above.*
- (ii) *the number of isometric spheres with radius exceeding a given positive quantity is finite.*

Proof We start by proving some relations between radii and distances of centers of isometric spheres. Let $g, h \in \Gamma \setminus \Gamma_\infty$ such that $g \neq h^{-1}$. The cocycle relation shows that

$$j((gh)^{-1}, \infty) = j(g^{-1}, \infty)j(h^{-1}, g^{-1}\infty).$$

Proposition 4.32 yields

$$\begin{aligned} |j((gh)^{-1}, \infty)|^{-1/2} &= |j(g^{-1}, \infty)|^{-1/2}|j(h^{-1}, g^{-1}\infty)|^{-1/2} \\ &= |j(g^{-1}, \infty)|^{-1/2}|j(h, \infty)|^{-1/2}\rho^H(g^{-1}\infty, h\infty)^{-1}. \end{aligned}$$

Note that $gh \neq id$ and therefore $gh \notin \Gamma_\infty$. Then Lemma 4.30 shows that

$$R(gh) = \frac{R(g)R(h^{-1})}{\rho^H(g^{-1}\infty, h\infty)}. \tag{4.8}$$

Using the same arguments, we find

$$j(g^{-1}, \infty) = j(h(gh)^{-1}, \infty) = j((gh)^{-1}, \infty)j(h, (gh)^{-1}\infty)$$

and therefore

$$\begin{aligned} R(g) &= |j(g^{-1}, \infty)|^{-1/2} = |j((gh)^{-1}, \infty)|^{-1/2} |j(h, (gh)^{-1}\infty)|^{-1/2} \\ &= R(gh) |j(h, (gh)^{-1}\infty)|^{-1/2}. \end{aligned} \tag{4.9}$$

Proposition 4.32 shows that identity

$$|j(h, (gh)^{-1}\infty)|^{1/2} = R(h)^{-1}\rho^H((gh)^{-1}\infty, h^{-1}\infty). \tag{4.10}$$

Because $R(h) = R(h^{-1})$, it follows from (4.8) to (4.10) that

$$\begin{aligned} \rho^H((gh)^{-1}\infty, h^{-1}\infty) &= |j(h, (gh)^{-1}\infty)|^{1/2} R(h) = \frac{R(gh)R(h)}{R(g)} \\ &= \frac{R(h)^2}{\rho^H(g^{-1}\infty, h\infty)}. \end{aligned} \tag{4.11}$$

Since Γ is properly discontinuous and $\infty \in \Omega(\Gamma)$, [9, Proposition 8.5] shows that there exists an open neighborhood U of ∞ in $\overline{H^g}$ such that $(\Gamma \setminus \Gamma_\infty) \subseteq \overline{H^g} \setminus U$. Since $\overline{H^g} \setminus U$ is compact, we find $m > 0$ such that

$$\rho^H(a\infty, b\infty) < m$$

for all $a, b \in \Gamma \setminus \Gamma_\infty$. Then (4.11) shows that

$$R(h)^2 = \rho^H((gh)^{-1}\infty, h^{-1}\infty) \rho^H(g^{-1}\infty, h\infty) < m^2. \tag{4.12}$$

Thus, for each $a \in \Gamma \setminus \Gamma_\infty$ we have $R(a) < m$, which proves (i).

Now let $k > 0$ and suppose that there are $a, b \in \Gamma \setminus \Gamma_\infty$ such that $I(a) \neq I(b^{-1})$ and $R(a), R(b) > k$. Since $a \neq b^{-1}$, (4.8) shows in combination with (4.12) that

$$\rho^H(a^{-1}\infty, b\infty) = \frac{R(a)R(b)}{R(ab)} > \frac{k^2}{m}.$$

This means that the distance between the centers of isometric spheres whose radii exceed k is bounded from below by k^2/m . The centers of all these isometric spheres are contained in the compact set $\partial_g H \setminus U$, which is bounded in $C \times V$ and hence also w. r. t. ρ^H . It follows that there are only finitely many spheres with radius exceeding k . This proves (ii). \square

Proposition 4.38 *Suppose that Γ is a properly discontinuous subgroup of G^{res} such that $\Gamma_\infty = \{\text{id}\}$ and $\infty \in \Omega(\Gamma)$. Then Γ is of type (O) and $\Gamma \setminus \Gamma_\infty$ of type (F). Moreover,*

$$\mathcal{F} := \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)$$

is a fundamental domain for Γ in H .

Proof For each $z \in H$, the map $\rho^H(\cdot, z): H \rightarrow \mathbb{R}$ is continuous. Therefore, each ρ^H -ball is open in H . Lemma 4.37(ii) implies that the set $\{\text{int } I(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$ is locally finite. Then Remark 3.19 shows that Γ is of type (O). Note that here the subgroup $\langle \Gamma \setminus \Gamma_\infty \rangle$ of Γ which is generated by $\Gamma \setminus \Gamma_\infty$ is exactly Γ . Let $z \in H$. Lemma 5 before Theorem 5.3.4 in [21] states that Γz is a closed subset of H . Since $\infty \in \Omega(\Gamma)$, Proposition 8.5 in [9] shows that we find an open neighborhood U of ∞ in $\overline{H^g}$ such that $\Gamma z \subseteq \overline{H^g} \setminus U$. Now $\overline{H^g} \setminus U$ is compact and therefore Γz is so. The height function is continuous, which shows that the maximum of $\{\text{ht}(gz) \mid g \in \Gamma\}$ exists. Thus, $\Gamma \setminus \Gamma_\infty$ is of type (F). By Theorem 3.18, \mathcal{F} is a fundamental region for Γ in H . By Lemma 4.37(i) the radii of the isometric spheres of Γ are uniformly bounded from above. Hence there is $R > 0$ such that the arc-connected set $\{z \in H \mid \text{ht}^H(z) = R\}$ is contained in \mathcal{F} . Let $w_1, w_2 \in \mathcal{F}$. Then $w_1 + [0, R - \text{ht}^H(w_1)]$ and $w_2 + [0, R - \text{ht}^H(w_2)]$ are contained in \mathcal{F} by Lemma 3.11(v). Hence, there is an arc in \mathcal{F} from w_1 to w_2 . This shows that \mathcal{F} is arc-connected, and thus connected. \square

4.7 Isometric spheres and isometric fundamental regions in the literature

For real hyperbolic spaces, definitions of isometric spheres are given at several places, e. g., the original definition of Ford in [10] for the plane, in [13] for the upper half plane model and the disk (ball) model of two-dimensional real hyperbolic space, in [17] for three-dimensional space and in [2] (or, earlier, in [1]) for arbitrary dimensions. The definition for the upper half plane model is not equivalent to that in the disk model (see [17]). Ford’s definition is that for the upper half plane model. His definition has been directly generalized to higher dimensions. Ford and Apanasov show the existence of isometric fundamental regions for a huge class of groups.

For complex hyperbolic spaces, the (to the knowledge of the author) only existing definitions of isometric spheres are given by Parker in [18], Goldman in [11], and Kamiya in [12]. Kamiya also discusses the existence of isometric fundamental regions for certain groups.

For quaternionic hyperbolic spaces, an investigation of isometric fundamental regions does not seem to exist. A definition of isometric spheres is provided by [14].

In this final section, we discuss the relation of the existing definitions of isometric spheres (for real hyperbolic space only exemplarily) to our definition of isometric spheres, and we compare the existing statements on the existence of isometric fundamental regions to Theorem 3.18.

4.8 Real hyperbolic spaces

Ford [10] shows the existence of isometric fundamental regions for groups of isometries acting on real hyperbolic plane (see [10, Theorems 15 and 22 in Ch. III]). Ford’s definition of fundamental region is not equivalent to our definition. In fact, each fundamental region in sense of our definition is a fundamental region in the sense of Ford, but not the other way round. In particular, the translates of a fundamental region in sense of Ford are not required to cover the whole space. For this reason the hypothesis of [10, Theorems 15, 22] are weaker than that of Theorem 3.18. However, the following discussion shows that the definition of isometric spheres in [10] is subsumed by our definition.

In [2], the model

$$D' = \{(t, Z) \in \mathbb{R} \times \mathfrak{J} \mid t > 0\}$$

of real hyperbolic space is used, and the isometric sphere for an element $g \in G^{\text{res}}$, acting on D' , is defined as

$$I(g) := \{z \in D' \mid |g'(z)| = 1\}.$$

More precisely, Apanasov (as all other references) uses the coordinates in the order $\mathfrak{J} \times \mathbb{R}$. In particular, his model space for two-dimensional real hyperbolic space is the upper half plane, whereas D' is the right half plane. Clearly, this difference has no affect on his definition of isometric sphere. Lemma 4.39 below will show that this definition of isometric spheres is subsumed by our definition.

The space D' is the symmetric space in Sect. 2.4 constructed from the abelian H -type algebra $\mathfrak{n} = (\mathfrak{n}, \{0\}) = (\mathfrak{J}, \{0\})$. In Sects. 4–4.6 we had to work with the ordered decomposition $(\{0\}, \mathfrak{v}) = (\{0\}, \mathfrak{n})$ of \mathfrak{n} . Hence we used the symmetric space

$$D = \left\{ (u, X) \in \mathbb{R} \times \mathfrak{v} \mid u > \frac{1}{4}|X|^2 \right\},$$

which is isometric to D' . According to [7], the isometry from D' to D is

$$\nu: \begin{cases} D' & \rightarrow D \\ (t, Z) & \mapsto (t^2 + |Z|^2, 2Z). \end{cases}$$

Then the action of an isometry $g \in G^{\text{res}}$ on D' is given by $\nu_G(g) := \nu^{-1} \circ g \circ \nu$. Recall the isometry $\beta: H \rightarrow D$ from Sect. 4.4.1.

Lemma 4.39 *Let $z \in D'$ and $g \in G^{\text{res}} \setminus G_\infty^{\text{res}}$. Then*

$$|j(g, \beta^{-1} \circ \nu(z))|^{-1} = |\nu_G(g)'(z)|.$$

Proof Let $z = (t, Z) \in D'$ and $g = n_2 \sigma m a_s n_1 \in G^{\text{res}} \setminus G_\infty^{\text{res}}$ with $n_j = (1, w_j)$, $j = 1, 2$, and $m = (\text{id}, \psi)$. At first we calculate the value of $|j(g, \beta^{-1} \circ \nu(z))|$. We have

$$\begin{aligned} \varphi_H(g) i_H(\beta^{-1} \circ \nu(z)) &= \varphi_H(g) \left(1, t^2 + |Z|^2, \sqrt{2}Z\right) \\ &= \varphi_H(n_2) \varphi_H(\sigma) \varphi_H(m) \varphi_H(a_s) \varphi_H(n_1) \left(1, t^2 + |Z|^2, \sqrt{2}Z\right) \\ &= \left(s^{1/2} \left(\frac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1)\right), *, *\right). \end{aligned}$$

Thus

$$|j(g, \beta^{-1} \circ \nu(z))| = s^{1/2} \left| \frac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1) \right|.$$

Note that Z and $\beta_2(Z, w_1)$ have to be seen as element of D (not of D'). Hence $\beta_2(Z, w_1) \in \mathbb{R}$. Therefore, $\beta_2(Z, w_1) = \langle Z, w_1 \rangle$ and further

$$|j(g, \beta^{-1} \circ \nu(z))| = s^{1/2} \left| t^2 + \left| \frac{1}{2}w_1 + Z \right|^2 \right| = s^{1/2} \left| \left(t, \frac{1}{2}w_1 + Z \right) \right|^2.$$

Let $(u, W) \in \mathbb{R} \times \mathfrak{g}$. For the derivative $\nu_G(g)'(z)$ we find

$$\nu_G(g)'(z)(u, W) = -s^{-1/2} \left(t, \psi \left(\frac{1}{2}w_1 + Z \right) \right)^{-1} (u, \psi(W)) \left(t, \psi \left(\frac{1}{2}w_1 + Z \right) \right)^{-1}.$$

Then

$$\begin{aligned} |\nu_G(g)'(z)(u, W)| &= s^{-1/2} \left| \left(t, \psi \left(\frac{1}{2}w_1 + Z \right) \right) \right|^{-1} \cdot |(u, \psi(W))| \\ &\quad \cdot \left| \left(t, \psi \left(\frac{1}{2}w_1 + Z \right) \right) \right|^{-1} \\ &= s^{-1/2} \left| \left(t, \frac{1}{2}w_1 + Z \right) \right|^{-2} |(u, W)|. \end{aligned}$$

Thus,

$$|\nu_G(g)'(z)| = s^{-1/2} \left| \left(t, \frac{1}{2}w_1 + Z \right) \right|^{-2} = |j(g, \beta^{-1} \circ \nu(z))|^{-1}.$$

This completes the proof. □

Lemma 4.39 and Proposition 4.33 immediately imply the following characterization of exteriors of isometric spheres.

Proposition 4.40 *Let $g \in G^{\text{res}} \setminus G_\infty^{\text{res}}$. Then*

$$\text{ext } I(g) = \{z \in D' \mid |\nu_G(g)'(z)| < 1\}.$$

Hence [2, Theorem 2.30] is a special case of Proposition 4.38. Lemma 2.31 in [2] states an extension of Theorem 2.30 for subgroups Γ of G^{res} with $\Gamma_\infty \neq \{\text{id}\}$. Unfortunately, the hypotheses of [2, Lemma 2.31] are not completely stated, for which reason we cannot compare this lemma with Theorem 3.18.

4.9 Complex hyperbolic spaces

The isometric spheres for isometries of complex hyperbolic spaces in [12] are identical to those in [11]. Kamiya uses the model H and defines the Cygan metric by formula (4.7). Recall the map φ_H from Sect. 4.4.6. Let $f \in \text{PU}(1, n; \mathbb{C}) = \text{PU}(\Psi_2, \mathbb{C})$ such that $\varphi_H^{-1}(f)$ does not fix ∞ . Further suppose that $(a_{ij})_{i,j=1,\dots,n+1}$ is a matrix representative of f . Then Kamiya defines the isometric sphere of f to be the set

$$I(f) := \left\{ z \in H \mid \rho(z, \varphi_H^{-1}(f^{-1})\infty) = R_f \right\}$$

where $R_f := |a_{12}|^{-1/2}$. One easily proves that this definition does not depend on the choice of the matrix representative. The following lemma shows that our definition of isometric spheres covers this one.

Lemma 4.41 *Suppose that $f = (a_{ij})_{i,j=1,\dots,n+1} \in \text{PU}(1, n; \mathbb{C})$ with $\varphi_H^{-1}(f)(\infty) \neq \infty$. Then $R_f = R(\varphi_H^{-1}(f))$.*

Proof Set $g := \varphi_H^{-1}(f)$. We have

$$f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ \vdots \\ a_{n+1,2} \end{pmatrix}.$$

Therefore,

$$|j(g, \infty)| = |a_{12}|.$$

Since $|j(g^{-1}, \infty)| = |j(g, \infty)|$, Lemma 4.30 shows that $R(g) = |a_{12}|^{-1/2}$. □

[12, Theorem 3.1] states the existence of isometric fundamental domains for discrete subgroups Γ of G^{res} for which, after possible conjugation of Γ , we have $\infty \in \Omega(\Gamma)$ and $\Gamma_\infty = \{\text{id}\}$. By [21, Theorem 5.3.5], Γ is discrete if and only if Γ is properly discontinuous. Therefore, Kamiya’s Theorem is a special case of Proposition 4.38.

In [18], Parker uses a section of the projection map from $E \setminus \{0\}$ to horospherical coordinates which is reminiscent of the ball model. Therefore, we expect that, as in the real case, this definition is not equivalent to the definition from [12].

4.10 Quaternionic hyperbolic spaces

In [14], Kim and Parker propose a definition of isometric spheres for isometries in $G^{\text{res}} \setminus G_\infty^{\text{res}}$ of quaternionic hyperbolic space. They use the model H and horospherical coordinates for the definition.

With the bijection

$$\begin{cases} \mathfrak{H} \times \mathfrak{v} & \rightarrow \mathfrak{H} \times \mathfrak{v} \\ (Z, X) & \mapsto (Z, \frac{1}{2}X), \end{cases}$$

our Heisenberg group N and our Cygan metric is transferred into their one. After shuffling coordinates, they characterize (see [14, Proposition 4.3]) the isometric sphere $I(g)$ of an element $g = (a_{ij})_{i,j=1,\dots,n+1}$ in $\text{PSp}(n, 1; \mathbb{H})$ with $\varphi_H^{-1}(g)(\infty) \neq \infty$ as

$$I(g) = \left\{ z \in H \mid \rho(z, \varphi_H^{-1}(g^{-1})\infty) = \sqrt{2} \cdot |a_{12}|^{1/2} \right\}.$$

They use a slightly different indefinite form on E for the definition of the hyperbolic space. Despite this difference we can apply the calculation in Sect. 4.9 we see that our definition of isometric sphere provides $|a_{12}|^{1/2}$ as radius. The factor $\sqrt{2}$ in [14] is due to the factor $\frac{1}{2}$ in their choice of the section of the projection from $E_-(\Psi_2)$ to H .

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