Gravitational Energy in Newtonian Gravity: A Response to Dewar and Weatherall

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Abstract

The paper investigates the status of gravitational energy in Newtonian Gravity (NG), developing upon recent work by Dewar and Weatherall. The latter suggest that gravitational energy is a gauge quantity. This is potentially misleading: its gauge status crucially depends on the spacetime setting one adopts. In line with Møller-Nielsen's plea for a motivational approach to symmetries, we supplement Dewar and Weatherall's work by discussing gravitational energy–stress in Newtonian spacetime, Galilean spacetime, Maxwell-Huygens spacetime, and Newton–Cartan Theory (NCT). Although we ultimately concur with Dewar and Weatherall that the notion of gravitational energy is problematic in NCT, our analysis goes beyond their work. The absence of an explicit definition of gravitational energy–stress in NCT somewhat detracts from the force of Dewar and Weatherall's argument. We fill this gap by examining the supposed gauge status of prima facie plausible candidates—NCT analogues of gravitational energy–stress pseudotensors, the Komar mass, and the Bel-Robinson tensor. Our paper further strengthens Dewar and Weatherall's results. In addition, it sheds more light upon the subtle link between sufficiently rich inertial structure and the definability of gravitational energy in NG.

Keywords Gravitational energy · Newtonian gravitation theory · Newton–Cartan theory · General relativity · Pseudotensors

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1 Introduction

Energy is a pivotal concept in all of physics. Ubiquitous—not least via the First Law of Thermodynamics—it has even been argued (by e.g. [7]) to be the salient metaphysical property of matter. It's therefore enticing to inquire into the status of the energy associated with the gravitational degrees of freedom in Newtonian Gravity (NG): Does NG admit of a meaningful definition of gravitational energy?

The question is of interest for at least three reasons. First, one would like to learn what makes gravity special, vis-à-vis other physical entities—perhaps already at the pre-general-relativistic level. Second, given that NG is an action-at-a-distance theory, one may wonder: Does this fact impinge upon the definition of a local notion of gravitational energy? With general relativity (GR), as a local field theory, in mind, one may ask: To what extent is NG free from the conceptual and interpretative difficulties of quasi-local notions of gravitational energy in GR (cf. e.g. [67])? Third, it's well known (e.g. [44], Ch. 12) that NG can be cast in a purely geometrical form, analogous to GR: In it, gravitational phenomena are re-conceptualised as manifestations of a non-flat spacetime geometry. This geometrisation seems to be linked to GR's notorious conceptual difficulties with respect to finding a meaningful notion of gravitational energy (cf. [48]). Echoing Bunge's suggestion, one might think, these difficulties in defining gravitational energy in GR intimate that gravity isn't a *matter* field, i.e. of the same type as the electromagnetic one (cf., for instance, Sotiriou, [66], Sect. 5.2). Studying NG in its geometrised form in closer detail thus promises to help us to better understand such conceptual difficulties—especially given that gravitational energy in NG's un-geometrised form is well-understood and unproblematic. Or so it appears.

In a recent paper [15], Dewar and Weatherall have challenged this. They assert that gravitational energy in Newtonian gravitational theories fails to be well-defined: that it's gauge-variant. The current paper responds to this claim. Notwithstanding our agreement with Dewar and Weatherall's overall conclusion, we feel that their reasoning leaves a few more things to be said—both formally and in substance. In particular, they don't attend to the question whether the spacetime setting makes a difference to the status of gravitational energy in NG. Furthermore, as Dewar and Weatherall (op.cit, p. 13) expressly acknowledge, their reasoning is predicated on a particular view of what counts as a gauge transformation. According to this view, Newton–Cartan Theory *just is* a gauge-invariant reformulation of NG. We explore the consequences of a different understanding gauge—one which, we contend, is closer to orthodoxy in the philosophy of physics literature (for better or worse).

We proceed as follows. Section 2 clarifies some preliminaries about gauge-invariance and models (Sect. 2.1), and then reviews three non-geometrised classical spacetime settings for Newtonian Gravity: Newtonian spacetime (Sect. 2.2), Galilean spacetime (Sect. 2.3), and Maxwell-Huygens spacetime (Sect. 2.4); for each, the status of gravitational energy is assessed in detail. Section 3 focuses on Newton–Cartan Theory (NCT). Section 3.1 outlines NCT's basics. In Sect. 3.2, we explain why Dewar and Weatherall's objections against gravitational energy in NCT are specious. In Sect. 3.3, we try to fill the gap in their reasoning. Section 4 discusses the results achieved, and their relation to Dewar and Weatherall's own conclusions.

2 Gravitational Energy in Classical Spacetimes

Dewar and Weatherall raise a deep question about the status of gravitational energy in Newtonian Gravity (NG): Is it a well-defined physical quantity? An answer isn't straightforward: Depending on which space–time setting one adopts, *different* variants of NG ensue [21, 43, 74]. As a result, the status of gravitational energy shouldn't be expected to be the same ab initio. After preliminaries about gauge-invariance (Sect. 2.1), this section reviews NG within the three non-geometrised classical space–time settings: Newtonian space–time (Sect. 2.2), Galilean space–time (Sect. 2.3) and Maxwell-Huygens space–time (Sect. 2.4), respectively.

2.1 Models and Gauge-Invariance

In this section, we clarify two concepts germane to the subsequent analysis: the classes of models relevant for us, and the notion of gauge equivalence.

Consider a given classical (non-quantum) theory T. Its associated models consist of n-tuples of geometrical objects, defined on a space–time manifold $\mathcal{M}$ (for details, see [2, 70]). Some of these represent matter variables, e.g. particle positions or field configurations. The remaining objects represent space–time structure, encoding e.g. chronogeometric or inertial structure.[1]

The various space–time settings correspond to different choices for the space–time structure. Each choice ought to conform to Earman's adequacy condition: The space-time symmetries should match the dynamical symmetries [19]. That is: The maximal group of diffeomorphisms under which the dynamical matter variables are invariant should coincide with those under which the spacetime structures are invariant.[2] These matter variables are introduced as follows.

A theory's most general class of models is called its "kinematically possible models" (KPMs). According to Curiel [9], they specify two things.

1. A specification of the theory's ontology (in the sense of [55]): The KPMs individuate possible *kinds* of objects to which the theory T is applicable—e.g. a viscous fluid or an electromagnetic field;

2. a specification of the theory's ideology (in the sense of Quine, ibid.): The KPMs enumerate (without determining) the degrees of freedom that form the complete state of possible objects of that kind at a point in time.

The laws which according to T relate the entities in the KPMs are given by dynamical equations. In particular, these laws fix their law-like interrelations (e.g. diachronic evolution). They pick out of the KPMs the *dynamically* possible models (DPMs). (One may conceive of the KPMs as representing T's *metaphysically* possible worlds, say, of viscous fluids or electromagnetic fields. The DPMs describe *nomologically* possible worlds; in them, the laws of nature prescribed by T hold. Models representing

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[1] We needn't embroil ourselves in the question of whether such a matter/space–time dichotomy can be upheld categorically (e.g. [6, 29, 35, 40, 42, 58, 60, 62]).

[2] The transformation isn't to be applied to fixed fields in the theory. Here, a field is called 'fixed', if it is identically the same in every kinematically possible model (to be defined further below), cf. [4], fn 137.

particular worlds—e.g. the actual one we inhabit—are obtained, if one further restricts the DPMs by boundary (or initial) conditions.)

Occasionally, one may wish to interpret a multiplicity of DPMs as representing the *same* world. This constitutes a gauge redundancy. On a mainstream view (which we won't call into question in this paper),[3] for an object to represent a meaningful physical quantity, it must be gauge-independent. Else, it lacks intelligible identity conditions: The properties of an object violating gauge-independence are unclear.

The present paper isn't concerned with discussing the criteria of *when* to physically identify two models (cf. e.g. [12, 13, 27, 40, 45])—nor with the pondering on the question when to identify two *theories* (cf., for instance, [57]). While we won't critically discuss different positions on these matters, we'll nonetheless adopt a cautious stance. Regarding the identification of two models, we'll follow Møller-Nielsen's "motivational approach" (see below). Regarding the identity of gravitational theories, Dewar and Weatherall adopt the latter's own criterion [75]: Two empirically equivalent theories are merely reformulations of the same theory if they are categorically equivalent to one another. This stands in opposition to a more traditional view of theory equivalence, such as Glymour's [24, 25]. When discussing the variants of NG in the various space–time settings, we'll side with the received view: Contrary to Dewar and Weatherall, we'll treat them as *different* theories. To our minds, the absence of any consensus on such conundrums about theory equivalence (cf. [10, 32]) commends cautious conservatism. A given formalism can be interpreted in multiple ways. Whether it's to be regarded as equivalent to a theory couched in a different formalism depends on this interpretation.

Dewar and Weatherall aver that gravitational energy density in NG lacks gauge-independence. This claim deserves scrutiny in each of the canonical non-geometric space–time settings. The remainder of the section will tackle this.

2.2 Newtonian Spacetime

Let's first consider NG set in Newtonian space–time (NST), NG_{NST}. Its KPMs consist of the 7-tuple

$$\langle \mathcal{M}, t_{ab}, h^{ab}, \sigma^a, \nabla, \varphi, \rho \rangle.$$

Here, $\mathcal{M}$ denotes the smooth, 4-dimensional differentiable manifold of events in space–time.

t_{ab} and h^{ab} are smooth, symmetric tensor fields on $\mathcal{M}$, of signature $(1, 0, 0, 0)$ and $(0, 1, 1, 1)$, respectively. That is (see [39]): $\forall p \in \mathcal{M} : \exists \left(\xi^a_{(b)} \right)_{b=0,\ldots,3} \in T\mathcal{M}$ such that $t_{ab} \xi^a_{(c)} \xi^b_{(d)} = \delta_{c,d} \delta_{c,0}$.

Analoguously, for the spatial metric, $\forall p \in \mathcal{M} : \exists \left(\sigma^{(a)}_b \right)_{a=0,\ldots,3} \in T^*\mathcal{M}$ such that $h^{ab} \sigma^{(c)}_a \sigma^{(d)}_b = \delta^{c,d} \left(\delta^{c,1} + \delta^{c,2} + \delta^{c,3} \right)$. The two fields represent a temporal and a spatial "metric", respectively. Due to their degeneracy, they aren't metrics proper. In

particular, whilst being able to raise indices with h^{ab}, we can't lower them with an inverse metric. For these two metrics, the following three conditions hold ("orthogonality" and "temporal" and "spatial metric compatibility", respectively):

$$h^{ab}t_{bc} = 0$$

$$\nabla_c h^{ab} = 0 \quad \& \quad \nabla_c t_{ab} = 0.$$

Given a vector field ξ^a, its temporal length is defined via $\left(t_{bc}\xi^b\xi^c\right)^{1/2}$. The vector field is called time-like or space-like, if its temporal length is positive or zero, respectively. (For the analogous spatial "metric" we refer to [39]. The details subsequently play no important role.)

The vector field σ^a is time-like (in the sense that $t_{ab}\sigma^a \neq 0$). Its integral curves represent the persisting points of absolute space. It grounds a standard of absolute rest/motion.

∇ is a flat derivative operator on $\mathcal{M}^4$:

$$R^a_{bcd} = 0.$$

It supplies the geodesic/inertial structure—loosely speaking: a standard of straightness-in terms of which inertial motion is defined.[5]

The gravitational potential and its source, the mass density, are represented by the smooth scalar fields φ and ρ. (For simplicity, we'll ignore in the following trivial gauge transformations of the potential, $\varphi \mapsto \varphi + \varphi_0$, for constant φ_0.[6])

Throughout, we'll assume that classical space–times are temporally orientable. That is: There exists a continuous, globally defined covector field t_a such that $t_{ab} = t_a t_b$. A time-like vector ξ^a is future-directed, if $\xi^a t_a > 0$. Otherwise, it's past-directed. In conjunction with the orthogonality and metric compatibility conditions, orientability allows us to slice up a spacetime into simultaneity hypersurfaces (see [39]).

In DPMs of NG$_{NST}$, the gravitational potential φ obeys the Newton–Poisson Equation,

$$h^{ab}\nabla_a\nabla_b\varphi = 4\pi\rho.$$

Consider now Galilean (static and kinematic) shifts[7]:

$$\Gamma : \begin{cases} t \\ x^i \end{cases} \mapsto \begin{cases} t + t_0 \\ x_0 + R^i_j x^j + v^i t \end{cases}.$$

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[4] Recall that this means that the concatenation of parallel transporting a vector ξ^a commutes:

$$\nabla_{[a}\nabla_{b]}\xi^c \equiv R^c_{dab}\xi^d = 0.$$

[5] It deserves to be underlined that the role of inertial structure isn't *exhausted* by explaining (or grounding) inertial motion (see e.g. [54], Sect. 5.2).

[6] Strictly speaking, in order for this shift to be regard as trivial, one must embrace a form of anti-quidditism about properties. See Martens and Read [41], for details.

[7] This terminology follows Huggett [31].

They comprise uniform time (t_0) and space translations (x_0), time-independent spatial rotations (R^i_j), and constant velocity boosts ($t v^i$) in absolute space. In geometric terms this translates into linear transformations of the type $\sigma^a \mapsto S^a_b \sigma^b + \sigma^a_0$ for a constant vector field σ^a_0 and a constant orthogonal matrix S^a_b with $\det(S^a_b) = 1$ ([19], Ch. 2).

In NG$_{\text{NST}}$, kinematic shifts reflect meaningful differences. (Throughout, we'll adopt the position known as sophisticated substantivalism, see e.g. [54]. It denies that static shifts—uniform time and space translations—correspond to physically distinct possibilities.) They describe distinct, velocity-boosted worlds. A material reference body in kinematically shifted models moves at different velocities v^i relative to the persisting points of absolute space.

Already Newton himself, in his Corollary V,[8] acknowledged the symmetry of models of NG$_{\text{NST}}$ under Galilean shifts: Its laws remain invariant under them; with respect to the laws, Galilei-shifted models are indistinguishable.

Dynamical shifts generalise kinematic ones. They allow for an arbitrary time-dependent translation $\vec{d}(t)$, concomitant with a re-scaling of the potential:

$$\Delta : \begin{cases} \vec{x} \\ \varphi \end{cases} \mapsto \begin{cases} \vec{x} + \vec{d}(t) \\ \varphi - \ddot{\vec{d}} \cdot \vec{x} + f(t). \end{cases}$$

In a dynamical shift, one subjects the system to a uniform acceleration, $\ddot{\vec{d}}(t)$, and adds a force that remains constant on simultaneity surfaces. (The question of how to translate this into the coordinate-free language of differential geometry needn't distract us here; we'll return to it in Sect. 3.1). A fortiori, dynamical shifts mediate meaningful differences: They represent universes in which some material reference body moves at different (uniform) accelerations with respect to the persisting points of absolute space.

Two models of NG$_{\text{NST}}$, related via dynamical shifts, thus also represent distinct worlds.[9] (We'll see presently that they are nonetheless *observationally* indiscernible.)

According to Dewar and Weatherall, dynamical shifts threaten the gauge-invariance of gravitational energy. To see how, let's first introduce the energy density of the gravi-

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[8] "The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion" (Newton 1729).

[9] We'll set aside here the question of whether dynamically shifted models still constitute solutions of NG$_{\text{NST}}$. Potential doubts might arise from the fact that the "sourceless sources", driving such shifts, are inimical to the Newtonian framework. At least as it stands, this argument doesn't sway us. First, the historical Newtonian framework has no direct bearing on the systematic question at hand. Secondly, and more importantly, to assess the question from a systematic angle, one must spell out what one means by, and what is included in the "Newtonian framework". An explicit argument must then be given why "sourceless sources" are indeed prohibited within it. (For instance, it's not obvious that the Newton's Third Law is applicable: It refers only to forces—and one may deny that dynamical shifts constitute forces proper. With forces being arguably causes, we have hereby touched on a subtle question in the metaphysics of causation within Newtonian physics.) Given the lack of a robust consensus on the details of the metaphysical framework appropriate to Newtonian physics—and the ineluctable disputes concomitant with metaphysical frameworks quite general—we deem it prudent to remain neutral on whether dynamical shifts don't preserve solutions of NG$_{\text{NST}}$. We thank an anonymous referee for pressing us on this important subtlety.

tational potential φ as the Noether-current associated with time-translation invariance of the NG Lagrangian,

$$E^{(NG_{NST})} = -\frac{1}{8\pi} h^{ab} \nabla_a \varphi \nabla_b \varphi.$$

It's easily verified to be invariant exactly under Galilei-shifts: Static and kinematic shifts don't alter the gravitational energy density. By contrast, due to the scaling in the gravitational potentials, dynamical shifts do: Two NG_{NST} models related via dynamical shifts *differ on* their gravitational energy density.

Should this disconcert us? Clearly—no: The two NST models, related via kinematic (Γ) or dynamic shifts (Δ) describe *distinct* worlds. Consequently, gravitational energy density, set within NST, isn't a gauge-quantity—contra Dewar and Weatherall. That gravitational energy doesn't vary between such worlds is irrelevant.

In conclusion: NST *has* sufficient structure to ward off the spectre of gauge-dependence for gravitational energy density.

To be sure, NST is an objectionable space–time setting for NG. Via its unobservable absolute standard of rest, its dynamical and space–time symmetries don't align. This flouts Earman's adequacy conditions. Yet, one mustn't conflate the flaws of a *space–time setting* with the (alleged) shortcomings of a quantity-gravitational energy—*defined within this space–time setting*.

How does the situation look in space–time settings that amend this defect of NST? We next discuss Galilean space-time.

2.3 Galilean Space–Time (GST)

GST ameliorates (some of) NST's shortcomings: It drops the assumption of absolute rest, i.e. the vector field σ^a. Thereby, one can pare down redundant structure. The points of space in GST no longer persist: Their diachronic identity is jettisoned. In NG_{GST}, one identifies all DPMs of NG_{NST} that differ only through Galilean shifts Γ.[10] Thus, GST retains an absolute standard of straightness of paths between two events: Whether a path is straight—a geodesic with respect to the flat derivative operator ∇_b—is an absolute matter of fact. (In modal language: Those DPMs of NG_{NST} in which the spatio-temporal paths of all possible test matter are parallel are identified as describing the same world.)[11] Contrariwise, as in NST, dynamically shifted DPMs remain distinct: In NG_{GST}, dynamical shifts *aren't* gauge-transformations.

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[10] In the case of static shifts, as discussed above, one may invoke sophisticated substantivalism, i.e. the denial that worlds are distinct that differ only with regard to which spacetime points exhibit which metrical properties (cf., for instance, Pooley ([54], Sect. 7).

In the case of kinematic shifts, the symmetry arguably only motivates the search for a more perspicuous ontology that can metaphysically elucidate the identity of kinematically shifted worlds [46]. This is provided by GST's transition from a 3-dimensional to the 4-dimensional picture of reality (cf., for instance, [43], pp. 54).

[11] This follows from the fact that the totality of geodesics on a manifold uniquely determine a derivative operator.

What does this imply for gravitational energy density? The expression for gravitational energy density for NG$_{GST}$ carries over from Sect. 2.3:

$$E^{(NG_{GST})} = -\frac{1}{8\pi} h^{ab} \nabla_a \varphi \nabla_b \varphi.$$

It didn't depend on σ^a, anyway. Under dynamical shifts it changes.

But as before, this is harmless: Dynamically shifted DPMs describe distinct worlds. Hence, that they differ on their gravitational energy, doesn't render the latter gauge-variant. (To be sure, dynamically shifted DPMs are empirically indistinguishable. All relational quantities remain unaltered. So, an observer in one of several dynamically shifted worlds couldn't ascertain which is hers. This predicament may be metaphysically lamentable—but it's not a shortcoming of gravitational energy.)

As in the NG$_{NST}$ case, Dewar and Weatherall's diagnosis of the gauge-dependence of gravitational energy is therefore unfounded in NG$_{GST}$.

The rebuttals of Dewar and Weatherall's claim so far may appear trivial. After all, GST—and a fortiori NST—arguably aren't the most perspicuous space–time settings for NG.[12] The analysis becomes more interesting for the two possible improvements on NG$_{GST}$, Maxwell-Huygens Gravity (NG$_{MHST}$) and Newton-Cartan Theory (NCT), respectively. We'll conclude this section with the former, before turning to the latter in Sect. 3.

2.4 Maxwell-Huygens Spacetime (MHST)

In light of his Corollary VI,[13] Newton may be credited with recognising the empirical indistinguishability of models of NG related via uniform accelerations, $\vec{x} \mapsto \vec{x}' = \vec{x} + \vec{d}$ (t) [64]. Here, $\vec{d}$ is a twice-differentiable function, representing an accelerational boost. Nonetheless, Newton persevered in his belief in absolute space. In rational (albeit historically incorrect, see Hoefer and Huggett [30], Sect. 6; [63]) reconstructions, he is frequently (e.g. [43], Ch. 2) imputed an invocation of an inference to the best explanation for inertial effects. Consider, for instance, the surface of a water-filled pail. It's (observably!) concave, if and only if the bucket is rotating. Is this rotation best conceptualised as rotation in absolute space, with the latter understood at least at the level of NG$_{GST}$? At first blush, it might appear so. But in fact, NG$_{MHST}$ further whittles down NG$_{GST}$'s structure by exploiting the symmetry of the Poisson Equation under uniform accelerations. NG$_{GST}$ only preserves an absolute sense of non-linear acceleration (equivalently[14]: rotation), evinced in inertial effects, such as in the above bucket experiment.

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[12] It's all the more surprising that GST is the only space–time setting (apart from NCT) for NG which Dewar and Weatherall explicitly consider.

[13] "If bodies, anyhow moved among themselves, are urged in the direction of parallel lines by equal accelerative forces; they will all continue to move among themselves, after the same manner as if they had been urged by no such forces" (Newton 1729).

[14] To be sure: A time-dependent linear acceleration—a linear acceleration that is still an arbitrary function of time, i.e. of the form $\xi^b \nabla_b \xi^a = \alpha^a(t) + \beta(t)\xi^a$—isn't equivalent to a rotation. But any non-linear acceleration is.

In NG$_{\text{MHST}}$, as we understand it (see below), one identifies NG$_{\text{GST}}$ models related via uniform accelerations[15]: One stipulates that they describe the same world. In contrast to uniform accelerations in NST and GST, in MHST they *are* gauge-transformations.

Recall that within the geometric framework of classical space–times, derivative operators encode inertial structure. Hence, they define standards of accelerations. For MHST, one must thus identify those derivative operators that correspond to the same standards of non-rotational acceleration, up to uniform-accelerational transformations. This translates into the following condition for any two such standards of acceleration ∇ and ∇', and all unit, time-like vector-fields ξ^a (see [39] for details):

$$\nabla^{[a}\xi^{b]} = 0 \Leftrightarrow \nabla'^{[a}\xi^{b]} = 0.$$

One can envision this condition as the requirement that the verdict whether trajectories of free particles in those space–times are not twisted be independent of the choice of the standard of acceleration. Such space–times are rotationally equivalent, with rotation understood as the twisting of possible free-fall trajectories.

Formally, one can now quotient out those of NG$_{\text{GST}}$'s KPMs which differ only by uniform accelerations (for details, see [73]). A KPM in NG$_{\text{MHST}}$ thus takes the form of the following 6-tuple:

$$\langle \mathcal{M}, t_a, h^{ab}, [\nabla], \varphi, \rho \rangle.$$

The only novel object, unfamiliar from GST is $[\nabla]$, the "standard of rotation". It's the equivalence class of flat, metrically compatible, rotationally equivalent derivative operators in GST[16]:

$$[\nabla] = \left\{ \nabla' : \mathrm{R'}^a_{bcd} = 0 \,\&\, \nabla'_a t_b \,\&\, \nabla'_c h^{ab} = 0 \,\&\, \left[|\xi^a t_a| = 1 \Rightarrow \left(\nabla'^{[a}\xi^{b]} = 0 \Leftrightarrow \nabla^{[a}\xi^{b]} = 0 \right) \right] \right\}$$

Here, R'^a_{bcd} is the Riemann tensor associated with ∇'.[17]

It's straightforward to show that the Newton-Poisson Equation in GST, $h^{ab}\nabla_a\nabla_b\varphi = 4\pi\rho$, remains invariant under changes of rotationally equivalent derivative operators [14]. DPMs in NG$_{\text{MHST}}$ can thus be obtained by identifying rotationally equivalent, but otherwise identical, DPMs of NG$_{\text{GST}}$.

To invest NG$_{\text{MHST}}$ with empirical content, we still need equations of motion for matter under the influence of gravity. They group together the equations of motion for NG$_{\text{GST}}$ within a standard of rotation, such that their (time-like, unit) solutions

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[15] Again, considering the Poisson equation only. Particle equations of motion will be considered below.

[16] Speaking of an equivalence class of derivative operators, rather than (e.g.) defining a primitive standard of rotation (as does e.g. Weatherall [73]), invokes Dewar's 'sophistication' about symmetries [14]. A recent skeptical attitude towards said 'sophistication' can be found in Martens and Read [41]. Since we share the latters' skepticism, ultimately we would find it preferable to work with Weatherall's standard of rotation, rather than an equivalence class of operators. Nevertheless, for continuity with the literature, we set such concerns aside in the remainder of this article.

[17] Two rotationally equivalent derivative operators ∇ and ∇' in this class are related via $\nabla' = (\nabla, \eta^a t_b t_c)$ for some spacelike vector field η^a, satisfying $\nabla^b \eta^a = 0$ [14].

$\xi, \xi' \in T\mathcal{M}$ define accelerations (with respect to the *same* derivative operators) that differ only by linear accelerations[18]:

$$\left\{ \xi \in T\mathcal{M} : \exists \nabla' \in [\nabla] \text{such that } \xi^b \nabla'_b \xi^a + \nabla'^a \varphi = 0 \right\} = \bigcup_{\nabla' \in [\nabla]} \left\{ \xi \in T\mathcal{M} : \xi^b \nabla'_b \xi^a + \nabla'^a \varphi = 0 \right\}.$$

The (class of) equations of motion picking out this solution set, $\left\{ \xi^b \nabla'_b \xi^a + \nabla^a \varphi = 0 : \nabla' \in [\nabla] \right\}$, is trivially invariant under uniform accelerations.

What might be candidates for gravitational energy in NG_{MHST}, $E^{(NG_{MHST})}$? The most natural one is defined as the equivalence class of all gravitational energy densities of rotationally equivalent NG_{GST} models. As the action of two derivative operators upon a scalar is the same, $\nabla \varphi = \nabla' \varphi$, this equivalence class is well-defined: The gravitational energy densities of two Galilean spacetimes with rotationally equivalent derivative operators ∇' and ∇ coincide,

$$E' = -\frac{1}{8\pi} h^{ab} \nabla'_a \varphi \nabla'_b \varphi = -\frac{1}{8\pi} h^{ab} \nabla_a \varphi \nabla_b \varphi = E.$$

Consequently, within a model of NG_{MHST}, gravitational energy density is a well-defined quantity. It's *not* gauge-dependent.[19] (Note that while uniform accelerations are gauge transformations in NG_{MHST}, dynamical shifts, which also include a transformation of the potential, aren't.)

Again, Dewar and Weatherall's proclamation of the gauge-dependence of gravitational energy doesn't apply to NG_{MHST}. Like in NG_{GST}, that dynamically shifted models of NG_{MHST} differ in their gravitational energy densities is benign: They describe distinct worlds.

One may, however, repudiate this formulation of NG within MHST for two reasons: its implausible conceptual prerequisites and its radicalness, respectively.

Firstly, it's unsatisfactory that in order to define MHST via an equivalence class of derivative operators of GST, one draws on structure that ultimately one *doesn't* attribute to the space-time (cf. Dewar [13], Martens and Read [41], Weatherall [73]).[20]

Compare the transition from NST to GST: There, the standard of absolute rest, represented by the time-like vector field σ^a, could simply be excised: It played only an otiose role in the formulation of NG.) Indeed, elsewhere Weatherall [73] proffers an alternative characterisation of MHST without reference to derivative operators. Absent

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[18] We wish to underscore that, given our cautious approach to theory identity, our version of NG_{MHST} *differs* from the ones, primarily considered in the literature (e.g. [14]; Weatherall [73]). Our version's ontology includes a gravitational field $\nabla^a \varphi$. By contrast, in Weatherall's [73] treatment, "we (do not) need to interpret the gravitational potential or corresponding gravitational field, $\nabla \varphi$, as representing facts about force or a field-like entity" (p. 88). Similarly, in Dewar's treatment there is also no (privileged) choice of gravitational potential or gravitational field.

We thank an anonymous referee for pressing us on this.

[19] Note that on Dewar's version of NG_{MHST} the gravitational energy density *would* count as gauge-variant, as the potential would change.

[20] One could also rephrase this objection in terms of physical degrees of freedom. Concepts natural to a theory reflect these, as it were, carving nature at its joints. For NCT, the true physical quantities are U(1)-invariants—rather than boosts, parameterized by $U(1)$ [68].

a derivative operator, though, how to define gravitational energy density? Evidently, the standard definition is no longer available in that case.

Gravitational energy density could well turn out not to be definable at all (as Dewar and Weatherall themselves admit)! That would certainly be grist to Dewar and Weatherall's mills—but for reasons other than those they cite. (It would be desirable to investigate whether gravitational energy density could be defined without derivative operators. We'll not pursue this, here, though.)

Elsewhere, Weatherall ([73], Sect. 5) draws attention also to a second blemish of MHST: It's more revisionary than at first blush it appears. Forces as they figure on the l.h.s. of Newton's 2nd Law are absolute: They are formulated in terms of *one* derivative operator. It's unclear whether all of (non-gravitational) physics can be reformulated on MHST. Think, for instance, of the Abraham–Lorentz–Dirac force, describing the recoil force of accelerated charged particles due to radiation (see [59]): It's manifestly *not* invariant under uniform accelerations. Hence, adopting MHST as the space-time setting for Newtonian physics necessitates a revision of the mathematical and conceptual foundations of much of classical physics. This may seem gratuitously radical.[21]

Another response to NG$_{GST}$'s redundancy is therefore appealing. In conjunction with the equivalence of inertial and gravitational mass, its symmetry under dynamical shifts motivates a geometrisation of NG: Like in General Relativity, gravitational effects are absorbed into the space-time's non-flat inertial structure. The result is known as Newton-Cartan Theory (NCT). To this we turn next.

3 Newton–Cartan Theory

We'll now investigate Dewar and Weatherall's claim that gravitational energy isn't well-defined in NCT. This section first (Sect. 3.1) reviews the basics of NCT. Next (Sect. 3.2), we expound why Dewar and Weatherall's arguments are specious. In Sect. 3.3, we try to fill the gap in their reasoning.

3.1 Geometrised NG

In this section, we review the basics of NCT, as contained in Trautman's Geometrisation Lemma and its converse Recovery Theorem (for all details, see [39], Ch. 4.2).

In NCT, the gravitational potential of NG is absorbed into NCT's (non-flat) derivative operator. This is encapsulated in Trautman's Geometrisation Lemma.

Let $\langle \mathcal{M}, t_a, h^{ab}, \nabla_a \rangle$ be a Galilean (henceforth: "classical") spacetime. (The derivative operator ∇_a is assumed to be flat; its associated Riemann tensor vanishes, $R^a_{bcd} = 0$.) Let furthermore φ and ρ be smooth, real-value scalar fields on $\mathcal{M}$ which

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[21] The revisionary nature of MHST also crops up with respect to its interpretation. Recall that (considering the gravitational field equations only) two GST models correspond to the same Maxwell-Huygens spacetime, if and only if they differ merely up to uniform accelerations. Consequently, two DPMs of GST that, albeit rotationally equivalent, differ merely in their potentials, count as distinct. That raises the question of how to interpret the scalar in MHST: What is its ontological status? Is it a real physical field, on a par with, say, the electromagnetic one? On which space does it live?

obey the Poisson Equation, $h^{ab}\nabla_a\nabla_b\varphi = 4\pi\rho$. Finally, let $\tilde{\nabla}_a = \left(\nabla, -t_a t_b h^{cd}\nabla_d\varphi\right)$.[22] Then, the following three propositions hold:

1. $\langle\mathcal{M}, t_a, h^{ab}, \tilde{\nabla}_a\rangle$ is a classical spacetime.
2. $\tilde{\nabla}_a$ is the (unique) derivative operator such that for all time-like curves on $\mathcal{M}$ with 4-velocity ξ^a: $\xi^a\tilde{\nabla}_a\xi^b = 0 \Leftrightarrow \xi^a\nabla_a\xi^b = -h^{bc}\nabla_c\varphi$.
3. The Riemann curvature $\tilde{R}^a_{bcd}$ associated with $\tilde{\nabla}_a$ satisfies the

   a. the "geometrised" Poisson Equation $\tilde{R}_{ab} := \tilde{R}^c_{acb} = 4\pi\rho t_a t_a$,
   b. and the curvature conditions $\tilde{R}^{a\,b}_{\ c\,d} = 0$ & $\tilde{R}^{a\,c}_{\ b\,d} = \tilde{R}^{c\,a}_{\ d\,b}$.

The second proposition states an equivalence between geodesic/un-accelerated/inertial motion with respect to one derivative operator, and particular accelerated/non-inertial motion with respect to another: Exactly those curves are geodesics with respect to $\tilde{\nabla}_a$ that describe accelerated motion that is the result of the Newtonian gravitational force, with respect to ∇_a. In this sense gravity is geometrised—or rather "inertialised" (cf. [47], Ch. 9; [38], esp. §4): The deviation from inertial trajectories, defined via ∇_a, due to the gravitational force is reconceptualised as a manifestation of (non-flat) inertial structure, defined via $\tilde{\nabla}_a$. (The interpretation of the curvature conditions shan't concern us here. Instead, we refer to [39], Ch. 4.3.)

Via the Recovery Theorem, we can re-translate geometrised NCT gravity back into non-geometrised NG$_{\text{GST}}$.

Let the classical spacetime $\langle\mathcal{M}, t_a, h^{ab}, \tilde{\nabla}_a\rangle$ satisfy the geometrised Poisson Equation $\tilde{R}_{ab} = 4\pi\rho t_a t_a$ for some smooth scalar field ρ on $\mathcal{M}$, and the Trautmann curvature conditions $\tilde{R}^{a\,b}_{\ c\,d} = 0$ & $\tilde{R}^{a\,c}_{\ b\,d} = \tilde{R}^{c\,a}_{\ d\,b}$. Then, in the neighbourhood of any point a real-valued scalar φ and a derivative operator ∇ exist, such that the following propositions hold:

1. ∇ is compatible with t_a and h^{ab}.
2. ∇ is flat. (Its associated Riemann tensor vanishes, $R^a_{bcd} = 0$.)
3. For all time-like curves on $\mathcal{M}$ with 4-velocity ξ^a: $\xi^a\tilde{\nabla}_a\xi^b = 0 \Leftrightarrow \xi^a\nabla_a\xi^b = -h^{bc}\nabla_c\varphi$.
4. φ satisfies the Poisson Equation: $h^{ab}\nabla_a\nabla_b\varphi = 4\pi\rho$.

Via the Recovery Theorem, we can "de-geometrise" NCT spacetimes: Geodesic/inertial motion with respect to $\tilde{\nabla}$, which was force-free, is now re-conceptualised as accelerated/non-inertial motion with respect to ∇, subject to the gravitational force.

The de-geometrisation isn't unique. A second pair φ' and ∇' for which

$$h^{ab}\nabla_a\nabla_b\left(\varphi - \varphi'\right) = 0 \quad\&\quad \nabla' = \left(\nabla, t_a t_b h^{cd}\nabla_d\left(\varphi - \varphi'\right)\right)$$

also satisfies the conditions 1–4 of the Recovery Theorem.

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[22] That is (see [39], Ch. 1.7): Let ∇ and ∇' be two derivative operators on a manifold $\mathcal{M}$. Then, they are uniquely related via a symmetric tensor field C^a_{bc}: For any tensor $\alpha^{a_1...a_r}_{b_1...b_s}$ of rank (r, s) on $\mathcal{M}$,

$$\left(\nabla'_m - \nabla_m\right)\alpha^{a_1...a_r}_{b_1...b_s} = \alpha^{a_1...a_r}_{nb_2...b_s}C^n_{mb_1} + \alpha^{a_1...a_r}_{b_1 nb_3...b_s}C^n_{mb_3} + \cdots - \alpha^{da_2...a_r}_{b_1...b_s}C^{a_1}_{md} - \alpha^{a_1 da_3...a_r}_{b_1...b_s}C^{a_2}_{md} - \cdots.$$

The transformations between any pair (φ, ∇) and (φ', ∇') that each satisfies the two non-uniqueness conditions are the dynamical shifts, mentioned in Sect. 2.2. Consequently, two models of NG_{GST} related via dynamical shifts are "de-geometrisations" of the *same* NCT spacetime. It has therefore been argued—e.g. by Pooley ([54], Sect. 6.1.1) or Knox [34]—that the gravitational scalar and the derivative operator of ungeometrised NG—i.e. NG_{GST}—are merely gauge-dependent quantities; geometrised NG—i.e. NCT—provides a gauge-free formulation of NG. With its dynamical symmetries matching its spacetime symmetries, and hence conforming to Earman's adequacy conditions, NCT is a satisfactory theory of gravity.

In summary: NCT allows us to re-conceptualise gravitational effects as manifestations of non-flat spacetime geometry (inertial structure). Models of NG_{GST} related via dynamical shifts can be identified as the same NCT spacetime.

3.2 Dewar and Weatherall on Gravitational Energy in NCT

Let's now assess Dewar and Weatherall's principal argument against gravitational energy in NCT. Its logical form can be reconstructed as follows:

(1) The natural expression for gravitational energy in NG_{GST} isn't invariant under dynamical shifts.
(2) In NCT, one identifies those DPMs of NG_{GST} that are related via dynamical shifts as physically equivalent; they are gauge.
(3) *Therefore*, gravitational energy in NCT isn't gauge-invariant.

Our authors correctly observe (1) and (2). However, their conclusion—(3)—is objectionable for a simple reason: Nowhere do Dewar and Weatherall *explicitly* define the object that is supposed to most naturally represent gravitational energy in NCT.

This is a crucial shortcoming. It renders their argument both formally and substantively incomplete. After all, Trautman's Geometrisation Lemma and Recovery Theorem (Sect. 3.1) only equip us with a translation between the 6-tuple $\langle \mathcal{M}, t_a, h^{ab}, \nabla_a, \rho, \varphi \rangle$ of non-geometrised NG_{GST} quantities, and the 5-tuple $\langle \mathcal{M}, t_a, h^{ab}, \tilde{\nabla}_a, \rho \rangle$ of geometrised NCT quantities; both are silent on any other quantities.

For Dewar and Weatherall's above syllogism to *formally* go through, premise (1) needs to be superseded by

(1') The (most natural) *NCT counterpart* of the Galilean gravitational energy isn't invariant under dynamical shifts.

With this, the conjunction of all three premises entails the conclusion:

$$(1') \& (2) \to (3).$$

But why believe that (1') is true? It's far from clear—as Dewar and Weatherall concede themselves—whether the NCT counterpart of Galilean gravitational energy *even exists*—and if it does, whether it indeed fails to be invariant under dynamical shifts.

(To be sure, if either could be negated, this would be grist to Dewar and Weatherall's mills. Their *conclusion* would remain intact. But it would follow from *different reasons*: namely those against the existence of the most natural NCT counterpart of Galilean gravitational energy, rather than the gauge-dependence of an actually existing NCT gravitational energy.) In short: It's one thing to doubt the definability of gravitational energy; it's another to doubt its physical meaningfulness (or *well*-definedness). Dewar and Weatherall focus on the latter.

Even if one charitably grants that the meaning of "most natural candidate" is clear, one may impugn the very existence of an NCT counterpart of Galilean gravitational energy. As the Geometrisation Lemma discloses, Galilean gravitational energy contains terms absent in NCT. In the latter's DPMs, a gravitational potential doesn't appear; it has been absorbed by NCT's non-flat connection. Furthermore, Galilean gravitational energy is defined via the (flat) derivative operator of GST. Which derivative operator should then enter the NCT counterpart of Galilean gravitational energy? An intuitive choice would, of course, be NCT's (non-flat) derivative operator. But this is scarcely compelling.

If thus gravitational energy in NG_{GST} essentially hinges on terms absent in NCT, then why assume that it can be defined at all in NCT?

In conclusion: Unless the possible candidate for NCT's gravitational energy is explicitly defined, Dewar and Weatherall's criticism of the latter's (alleged) gauge-dependence forfeits much of its force.

To fill this lacuna, we'll now discuss various concrete options.

3.3 Candidates for Gravitational Energy in NCT

In the preceding section, we argued that Dewar and Weatherall's criticism of gravitational energy in NCT is vitiated by their lack of an explicit definition of gravitational energy in NCT. Here, we'll examine a number of natural candidates: 1. Pseudotensors, 2. Komar energy, 3. Lorentz and Levi–Civita's proposal, 4. The Bel-Robinson tensor. 5. Pittsification. Rather than suffering from gauge-dependence, these proposals will be argued to be either not well-defined, or to yield trivial gravitational energy.

Dewar and Weatherall ([15], fn. 30) enjoin such an examination of explicit proposals. It has two kinds of merits. After all, in empirically equivalent theories, radically different objects can play the same role. (Think of Starobinski's original model of cosmic inflation (see, e.g., [11], Sects. 2, 3 for details.) In one formulation, the latter is driven by a scalar, hence arguably a matter field *on* spacetime. In an equivalent formulation, inflation is merely a manifestation *of* spacetime curvature deviating from what it should be according to GR). Furthermore, comprehending the various possibilities in which a conceptually rich theory such as NCT can fail to exhibit a certain feature considerably enhances our understanding of it. In particular, this broadening of our repertoire of instruments is likely to pay off in comparing NCT to other theories in its theoretical vicinity, such as GR. (In the apt terms of Pitts [52]: Spacetime philosophy should aspire to "modal cosmopolitanism"—rather than "modal provincialism".)

3.3.1 Pseudotensors

In this subsection, we evaluate the natural NCT counterparts of the general-relativistic pseudotensors as possible candidates for gravitational energy. They are found to trivialise the latter.

The standard approach to gravitational energy in GR proceeds via the Noether theorems.[23] The absence of a (tractable, natural) Lagrangian or Hamiltonian formulation of NCT encumbers this road, though.[24] It's straightforward to find a Lagrangian *with* suitable multipliers. But the latter are, of course, under-determined. Canonical gravitational energy–momentum for (i.e. the Noether current attributed to) the (non-flat) NCT metric would depend on the Lagrange multipliers, and hence would be ill-defined.

One might, however, take the definitions of pseudotensors, as familiar from GR, and just *stipulate* their formal NCT analogues. What encourages such a procedure is that pseudotensors—at least in GR—arguably satisfy natural desiderata for local gravitational energy, e.g. a conservation law, the dependence only on first derivatives of the field variables, or the reduction to the familiar Newtonian potential energy in the weak-field limit ([17], Sect. 3.2).

Following Goldberg [26], an infinitely large class of pseudotensor densities (of arbitrary weights $n+1$, $n \in \mathbb{N}^{\geq 0}$) can be constructed as follows. (We restrict ourselves to mixed indices—one up, one down.)

$$\vartheta_\mu^{(n)\nu} = |g|^{\frac{n}{2}} \left\{ \vartheta_\mu^\nu + \frac{n}{2} U_\mu^{[\nu\sigma]} \partial_\sigma \ln|g| \right\}$$

Here, $|g|$ denotes the modulo of the determinant of GR's metric. $U_\mu^{[\nu\sigma]}$ denotes a so-called super-potential. (The details needn't detain us here.)

For $n = 0$, we obtain the weight-one density of the Einstein-pseudotensor t_μ^ν[25]:

$$\vartheta_\mu^{(0)\nu} = \sqrt{|g|} t_\mu^\nu := 2\sqrt{|g|} G_\mu^\nu + \partial_\sigma \left(|g|^{-\frac{1}{2}} g_{\mu\lambda} \partial_\rho \left(|g| g^{\lambda[\nu} g^{\sigma]\rho} \right) \right).$$

Together with the matter energy–momentum tensor $|g|^{\frac{n+1}{2}} T_\nu^\mu$, (of weight $n+1$), the pseudotensors—representing gravitational energy–momentum—form the system's *total* energy–momentum $\mathcal{T}_\mu^{(n)\nu} := |g|^{\frac{n+1}{2}} T_\nu^\mu + \vartheta_\mu^{(n)\nu}$. The latter satisfies the continuity equation:

$$\partial_\nu \mathcal{T}_\mu^{(n)\nu} = 0.$$

Albeit not a tensor equation, this continuity equation holds in all coordinate systems. Hence, total energy–momentum can be said to be (locally/differentially) conserved.

For the NCT counterparts to the general-relativistic pseudotensors, it's tempting to replace the general-relativistic metric in the above expressions by NCT's spatial or

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[23] Historically too, this was Einstein's route—avant la lettre (Brading [5]).

[24] In private correspondence, Nic Teh has conjectured that the non-existence of a Lagrangian or Hamiltonian formulation without Lagrange multipliers of NCT is even provable (cf. [28]).

[25] Alternate formulations can be found in e.g. Dirac ([16], Ch. 31, 32) or Ohanian and Ruffini ([49], A5).

temporal pseudo-metric, h^{ab} and $t_{ab} = t_a t_b$ respectively. In fact, it can be shown [3] that NCT doesn't admit of a non-degenerate metric with which the NCT connection is compatible. Hence, the subsequent discussion is without loss of generality.

However, due to their degeneracy, i.e. vanishing determinant, this is a non-starter: One can easily verify that the resulting NCT pseudotensors either are trivial or nor defined at all. The latter is the case for $\vartheta_\mu^{(0)v}$, i.e. $n = 0$[26]; the former is the case for $\vartheta_\mu^{(n)v}$ s for $n > 0$.

In conclusion: The natural NCT counterparts to GR's standard pseudotensor weights either are either ill-defined, or they yield a trivial notion of gravitational energy. While consonant with Dewar and Weatherall's conclusions, this result has nothing to do with a lack of gauge-invariance.[27]

3.3.2 Komar Mass

This section is devoted to a plausible definition of total energy of NCT spacetimes via the Komar integral. Like pseudotensors, it trivialises gravitational energy.

The most natural path to a global notion of gravitational energy in GR proceeds via the Noetherian route or, equivalently, the Hamiltonian formalism. As mentioned in the preceding section, for NCT this path is blocked. For static spacetimes in GR, an alternative exists: the Komar integral. (In GR, it coincides with the Hamiltonian definition, see e.g. [53], Ch. 4.3.)

Consider a static spacetime, i.e. one with a(n asymptotically normalised) time-like Killing field ξ, satisfying $\nabla_{(a}\xi_{b)} = 0$. For such a spacetime, there exists a natural definition of "holding an object in place" via ξ's orbit (see [71] for details). This gives rise to a likewise natural notion of acceleration with respect to this orbit. Via this acceleration, a force can be defined that an observer at infinity must exert in order to keep a unit mass in place. Analogously to the characterisation of the total energy of the electrostatic field in terms of its asymptotic properties, we thus arrive—after various manipulations, for which we refer to the literature (ibid.)—at the following expression for the energy enclosed in the topological 2-sphere $\mathcal{S}_t$ in the hypersurface orthogonal to ξ:

$$E = -8\pi \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} d\sigma^{ab} \nabla_a \xi_b.$$

Here, $d\sigma^{ab}$ denotes the surface element on $\mathcal{S}_t$. This integral can serve as a definition of total energy in general-relativistic static spacetimes. It turns out to be conserved.

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[26] Dirac's affine form of the weight-1 Einstein pseudotensor density also yields a vanishing result for a singular metric.

[27] GR's pseudotensors are usually regarded as tainted by the problem of coordinate dependence (cf., for instance, Weyl [77], p. 273). By contrast, NCT's pseudotensors are free from that evil: Whenever they are defined, the NCT pseudtensor densities vanish coordinate-independently. In the same vein, they are—albeit *trivially*—invariant under dynamical symmetries.

Given that NCT spacetimes are static in a natural sense,[28] it's now tempting to stipulate the NCT counterpart of the Komar integral as a candidate for the total energy of NCT spacetimes as well. To that end, one plausibly replaces the Killing field in the Komar expression's integrand by NCT's time covector, $\xi_a \rightarrow t_a$. This already suffices to trivialise the proposal: Due to the compatibility condition of NCT's time pseudo-metric, $\nabla_a t_b = 0$, the NCT counterpart of the Komar integral vanishes. Consequently, the total energy of a NCT spacetime would be zero. Gravitational energy—understood as the energy left after subtracting the energy contributions of ordinary matter—would then always exactly counterbalance matter energy. This is implausible for reasons that we'll explain in the next subsection, in which we'll discuss Lorentz and Levi–Civita's proposal.

3.3.3 Lorentz and Levi–Civita's Proposal

Lorentz and Levi–Civita proposed the Einstein tensor, $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ (or, for reasons of dimensionality, $-\frac{1}{2\kappa} G_{ab}$, with $\kappa := \frac{4\pi G}{c^4}$) as a representation of gravitational energy in GR (for details, see [8], Sects. 5–11). Is this convincing for the NCT case? For reasons again both general and specific to NCT, we argue that this isn't the case.

Three facts commend Lorentz and Levi–Civita's proposal. (1) In contrast to pseudotensorial approaches, the Einstein tensor is a bona fide tensor. (2) It obeys a bona fide covariant conservation law: the contracted Bianchi identity, $\nabla_b G^{ab} \equiv 0$. The attendant total energy–momentum $_{(LLC)}\mathfrak{T}^{ab} := -\frac{1}{2\kappa} G^{ab} + T^{ab}$, satisfies both an ordinary and covariant continuity equation, $\partial_b\big(_{(LLC)}\mathfrak{T}^{ab}\big) = \nabla_b\big(_{(LLC)}\mathfrak{T}^{ab}\big) = 0$. (3) The Einstein tensor is the exact *gravitational* counterpart of the *matter* energy–momentum tensor: Whereas the latter is defined variationally as $T_{ab} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{ab}}\big(\sqrt{|g|}\mathcal{L}_{(m)}\big)$, one obtains the Einstein tensor (up to a proportionality factor) by replacing the matter Lagrangian by the purely gravitational Einstein-Hilbert Lagrangian,

$$G_{ab} \propto \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^{ab}}\big(\sqrt{|g|}R\big).$$

The first two features carry over to NCT. The third one, however, doesn't: The absence of a natural Lagrangian formulation of NCT's *full* gravitational sector (cf. [28])—including the two Trautmann conditions imposed on curvature—weakens the analogy between the Einstein tensor and the matter energy–momentum tensor.

But there are stronger reasons to question Lorentz and Levi–Civita's proposal: physical implausibility and vacuity, respectively (cf. [50], fns 180–181). Firstly, consider the Einstein Equations in vacuum. This, on Lorentz and Levi–Civita's proposal, yields *vanishing* gravitational energy, $G_{ab} = 0$. But that's counterintuitive: Since the Einstein tensor is constructed from traces of the Riemann tensor, a solution of the vacuum Einstein Equations has in general non-vanishing Weyl structure.[29] The latter encapsulates gravitational radiation. Prima facie, one would expect it to possess

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[28] That is: Its defining partial differential equations are elliptic. Hence, information about variations in a region propagates instantaneously.

[29] Dewar and Weatherall ([15], Sect. 4) show that for NCT spacetimes, one can indeed define a (non-trivial) Weyl tensor (cf. [18, 20, 72]).

gravitational energy—contrary to Lorentz and Levi–Civita's proposal (cf. [17] for a critique). Equally implausibly, it purports that there are no differences between gravitational energy in the exterior of a static and, say, rotating black hole, respectively: In either case, gravitational energy would be zero. For NCT, the objection needs to be slightly adapted. NCT's Poisson Equation is elliptic. Hence its solutions can't propagate. In that sense, there is of course no gravitational radiation. Still, one would expect different NCT spacetimes with non-vanishing Weyl structure—i.e. different homogenous solutions of the Poisson Equation—to differ in their gravitational energy. (Recall that the Weyl tensor measures tidal deformations in the shape of extended spacetime regions.)

Besides such doubts regarding its physical plausibility, it seems mysterious and contrived that, on Lorentz and Levi–Civita's proposal, any matter energy–momentum is exactly counterbalanced by gravitational energy (in the GR case): In *all* possible spacetimes, the *total* energy always vanishes, $-\frac{1}{2\kappa}G_{ab} + 2\kappa T_{ab} = 0$. It's elusive what positing such an entity would help *explain*. As Levi–Civita conceded in a letter to Einstein, the proposal is sterile in that "[…] the energy principle would lose all its heuristic value, because no physical process (or almost none) could be excluded a priori. In fact, [in order to get any physical process] one only has to associate with it a suitable change of the [gravitational field]". For NCT, this sterility is exacerbated by the fact that the Einstein tensor reduces to the Ricci tensor, and that the latter vanishes for mixed indices,

$$^{(NCT)}G_a^b \equiv {}^{(NCT)}R_a^b \equiv 0.$$

In other words: Lorentz and Levi–Civita's proposal yields only a trivial gravitational energy–momentum flux along some direction $\xi^a : {}^{(NCT)}G_a^b\xi^a \equiv 0$.

In conclusion: The Einstein tensor isn't suited for representing gravitational energy in both GR and NCT; it lacks physical informativeness and plausibility. The issue of gauge-dependence under dynamical shifts doesn't arise in any form.

Let's turn next to another tensorial proposal, Bel and Robinson's superenergy tensor.

3.3.4 The Bel–Robinson Tensor

In this subsection, we examine the NCT counterpart of the Bel–Robinson tensor as a candidate for NCT's gravitational energy.

Recall the energy–momentum tensor of electrodynamics:

$$4\pi T_{(em)}^{\mu\nu} = F_\lambda^\mu F^{\lambda\nu} - \frac{1}{4}g^{\mu\nu} * \left(F^{\kappa\lambda}\right) * \left(F_{\kappa\lambda}\right),$$

with the Faraday tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and its dual $*(F^{\mu\nu}) = \epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$. (We use the latter—rather than the non-dual—in the second term of the energy–momentum tensor to render its structural similarity with the Bel-Robinson Tensor more transparent.) In an analogous manner, one can construct a tensor from the Riemann

tensor,[30] mimicking the symmetric electromagnetic energy–momentum tensor (see Garecki [22] for details). The result is the so-called "superenergy tensor":

$$T^{abcd} := R^{aefc}R^b{}_{ef}{}^d + *\left(R^{aefc}\right)*\left(R^b{}_{ef}{}^d\right) = R^{aefc}R^b{}_{ef}{}^d + R^{aefd}R^b{}_{ef}{}^d - \frac{1}{2}g^{ab}R^{efgc}R_{efg}{}^d.$$

(Here, $*$ denotes the usual dual operation: $*(R_{abcd}) = \frac{1}{2}\epsilon_{abef}R^e{}_{cd}{}^f$.) Bel and Robinson proposed it as a candidate for gravitational energy in GR.

As a consequence of the Bianchi identities (and hence, independently of the Einstein Equations), its covariant divergence vanishes:

$$\nabla_a T^{abcd} \equiv 0.$$

Note that due to the Einstein Equations, in vacuum the Riemann tensor can be replaced by the Weyl tensor. The latter encodes gravitational degrees of freedom that can propagate through vacuum. In light of this, the Bel-Robinson tensor seems apt for describing energy associated with gravitational radiation.

What makes it of particular interest is that the Bel-Robinson tensor appears in the expansion of the Einstein pseudotensor at a point, when evaluated in normal coordinates for some other point (see So, Nester and Chen [65] for details).

Due to the flatness of NCT spacetimes (in the sense of $R^{ab}{}_{cd} = 0$, Sect. 3.1), a non-trivial Bel-Robinson tensor in NCT must be defined as a tensor of rank (1,3):

$$^{(NCT)}T^i{}_{klm} := R^i{}_{abl}R^b{}_{mk}{}^a + *\left(R^i{}_{abl}\right)*\left(R^b{}_{mk}{}^a\right),$$

with the Riemann tensors (and their duals), associated with the NCT connection. As the Bianchi identities also hold in NCT, also $\nabla_i{}^{(NCT)}T^i{}_{klm} \equiv 0$ obtains.

However, $^{(NCT)}T^i{}_{klm}$ isn't a convincing proposal for gravitational energy in NCT for reasons both general and specific to NCT.

Generally (and like in GR), it has the dimensions $length^{-4}$. So, neither the Bel-Robinson tensor nor any of its powers have the right dimension, unless one introduces a novel constant of nature. But this seems ad-hoc.

Moreover, the Bel-Robinson tensor is linked to *differences* in pseudotensorial gravitational energy (and hence, on a standard interpretation of pseudotensors: to differences in gravitational energy simpliciter), rather than to the latter directly (ibid.). So, its physical interpretation would presuppose a non-trivial notion of pseudotensorial gravitational energy. But as we saw in Sect. 3.3.2, the most immediate NCT counterparts to pseudotensors are vacuous.

In conclusion: As a proposal for gravitational energy in NCT, the Bel-Robinson tensor is both formally, as well as in absence of its connection to non-vacuous pseudotensorial gravitational energy, unsuitable. Contra Dewar and Weatherall, gauge-variance isn't the issue here, though.

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[30] This may be motivated by Synge's suggestion that GR's gravitational field is represented by the Riemann curvature tensor (cf. [36] for a critical discussion). That is: According to Synge, one should view the Riemann tensor as the GR counterpart to the Faraday/field strength tensor—a view backed up by the perspective from the fibre bundle formalism [76].

We conclude our perusal of candidates for gravitational energy in NCT with a non-tensorial proposal, due to Pitts.

3.3.5 Pittsification

Pitts [51] has recently propounded an astute solution to the problem of coordinate/gauge-dependence of pseudotensors in GR: Take your favorite pseudotensor, say the Einstein pseudotensor ϑ_a^b, and declare the *totality* of its values in all possible coordinate systems (at neighbourhood of a point) *one* object. Symbolically:

$$\left\{ (\forall \text{ coordinate systems CS}) \left(\vartheta_\mu^\nu \right)_{CS} \right\}.$$

It has (uncountably) infinite components. Each corresponds to the pseudotensor's value in one possible coordinate system.

There are two ways to transfer this idea to NCT. The first one takes the NCT counterparts of pseudotensors, and "Pittsifies" them as in Pitts' original proposal for GR. But this is of little interest, as the NCT counterparts of pseudotensors are either trivial or not defined (Sect. 3.3.1).

More auspicious is another option. It starts from NG_{GST}'s gravitational energy. As described in Sect. 3.2, a DPM in NCT $\mathfrak{M}$ can be de-geometrised into an equivalence class of GST models $GST_\alpha(\mathfrak{M})$ for some index set $\alpha \in \mathcal{A}$. For any two $\alpha, \alpha' \in \mathcal{A}$, the models $GST_\alpha(\mathfrak{M})$ and $GST_{\alpha'}(\mathfrak{M})$ differ only up to dynamical shifts. Now Pittsify the gravitational energies of all these $GST_\alpha(\mathfrak{M})$s. This yields the (Pittsified) NCT gravitational energy, symbolically:

$$E(\mathfrak{M}) := \{ (\forall \alpha \in \mathcal{A}) E[GST_\alpha(\mathfrak{M})] \}.$$

Each component of this object corresponds to one possible GST de-geometrisation. By construction, it's gauge-invariant under dynamical shifts. (Recall: De-geometrisations of an NCT spacetime are all related via dynamical shifts.)

Pittsification welds together into one well-defined, formal object the gravitational energies of those GST spacetimes that correspond to the same NCT spacetime. It's not obvious, though, that it provides a satisfactory representation for gravitational energy in NCT: Firstly, its conceptual prerequisites seem alien to NCT; secondly, one may have qualms about its physical meaningfulness.

The Pittsified NCT gravitational energy is constructed from the gravitational energies of those NG_{GST} spacetimes the geometrisation of which yields the same NCT spacetime. On the one hand, this yields a formally well-defined object—even a geometric one.[31] On the other hand, one may wonder: Is it legitimate to introduce into a theory quantities built from terms that belong to, and are meaningful only within, a *different* theory? That is: Are we allowed to use quantities prima facie intelligible only in NG_{GST} in order to define a quantity supposedly meaningful in NCT?

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[31] If one is willing to extend the standard meaning of geometrical objects to objects with infinitely many components ([51], Sects. 1–2).

Perhaps such a worry might be allayed by the thought that the *individual* de-geometrised NG$_{GST}$ spacetimes lack meaning in NCT; only their *totality* accrues it. Consider the gauge-quantities of electromagnetism, the 4-potentials. By themselves, they don't possess physical significance, either (perhaps setting aside potential subtleties for the Aharonov–Bohm effect); only a suitable combination of them—i.e. the Faraday tensor—does. By analogy, one might argue that only the Pittsified NCT gravitational energy as a whole is meaningful; its individual components—the NG$_{GST}$ quantities—aren't. One could counter by questioning the whole procedure: Isn't Pittsification too cheap a trick to procure gauge-invariant quantities? Finding gauge-invariant quantities is a formidable task in ongoing research in (non-Abelian) gauge theories. One would like more than a *merely* formal object: How to ensure that the Pittsified gravitational energy *actually* possesses physical significance?[32] (Consider, by analogy, the Pittsification of the electromagnetic 4-potentials, i.e. the infinite-component object made up of all 4-potentials in all possible gauges. In a formal sense, it's evidently gauge-independent. One would baulk, however, at attributing it physical content, as expressed in the electromagnetic fields.)

In conclusion: Via Pittsification, we can define a formal candidate for gravitational energy of a NCT spacetime from the gravitational energies of its corresponding NG$_{GST}$ de-geometrisations. Reasons to object to this proposal don't include gauge-dependence; rather, they consist in doubts about its physical significance and conceptual adequacy.[33]

4 Discussion

Dewar and Weatherall [15] conclude their paper with "an important lesson for how to understand energy in geometrized theories. [...] (T)here is a deep relationship between the classical notions of energy, work, force, and inertia. Energy is a measure of the ability to do work [...] But in theories in which gravitation is 'geometrized' in the sense that gravitation is understood as an inertial effect in curved spacetime, we should not think of gravitation as a force at all—and so, in particular, it is not the sort of thing that does work. To the contrary, work makes sense only as a measure of the deviation from inertial motion over some distance."

Our discussion illustrates this insight in slightly more detail.[34] It shows explicitly that *non-geometrised* variants of NG in the above sense—NG$_{NST}$ and NG$_{GST}$—*do* allow for a well-defined notion of gravitational energy. Contrariwise, for NG in spacetime settings where inertial structure has absorbed gravity—MHST and NCT—-

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[32] This is squarely related to the question of inferences from symmetries to reality: Declaring the physical equivalence between symmetry-related models of a theory remains merely formal and verbal, unless a metaphysically perspicuous explication of the corresponding ontological picture is forthcoming [46].

[33] This criticism mirrors the one mounted against Pittsification of pseudotensors in GR ([17], Sect. 3.3).

[34] We plan to complement our and Dewar and Weatherall's results by an investigation from the viewpoint of teleparallelisation (for a conceptual introduction, see [33]). Recently, Teh and Read [69] have shown that the Trautman Recovery Theorem is an instance of teleparallelisation. It will be interesting to study whether further illuminating insights into gravitational energy in NCT and NG can be gained by applying the machinery of teleparallelisation.

gravitational energy faces several obstacles.[35] The status of a prima facie central concept such as the energy associated with Newtonian gravitational degrees of freedom crucially depends on how Newtonian Gravity is *interpreted*.

Dewar and Weatherall make a farther-reaching suggestion: "[…] (W)e should understand the energy density of Yang-Mills fields [including electromagnetism, the authors] as *relative* to some background structure—namely, the inertial structure determined by the spacetime metric in general relativity" (p. 27, their emphasis). We concur with this. But it's worthwhile stressing that it implies a minor rectification of Dewar and Weatherall's conclusion, cited above: It's less the geometrisation of gravity itself—the absorption of gravitational effects by inertial structure—that is responsible for the difficulties in defining gravitational energy. Rather, it's the existence of (sufficiently rich) inertial structure *simpliciter* that seems to be a prerequisite for a meaningful (or at least, robust, cf. [56]) definition of field energies.[36] In fact, one may construe the main problem diagnosed in Sect. 2.4 for gravitational energy for NG$_{MHST}$ as a violation of this requirement: MHST's inertial structure is too impoverished to even allow us to define gravitational energy; for that, the derivative operator had to be imported from GST.

Our discussion also emphasised an additional difficulty for gravitational energy for NCT: the absence of a natural Lagrangian (or Hamiltonian) formulation. Despite the similarities with respect to geometrizing gravity, this makes its status *more* precarious than in GR. Energy is arguably a cluster concept. But it wouldn't be too much of a stretch, either, to regard the definition of energy within the Lagrangian/Hamiltonian framework as the primary meaning of energy in field theories. Hence, we propose, not only will the comparison with Yang-Mills theories be rewarding, as Dewar and Weatherall suggest; it will likewise be illuminating to investigate the status of field energies in non-Lagrangian theories.

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[35] It's important that the notion of geometrisation relevant here is specific—the absorption of gravitational effects into inertial structure. *Other* notions of geometrisation (see [37], Ch. 9) don't seem relevant. Reichenbach's geometrised toy unification of gravity and electromagnetism [23], for instance, admits of the standard GR electromagnetic energy–stress tensor.

[36] This point seems pertinent for all theories in which the notion of inertial trajectories becomes questionable. In fact, this is the case for Bohmian Mechanics (see [1]): Energy is no longer a fundamental concept.

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