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ORIGINAL ARTICLE

Time Reversal Symmetry and Collapse Models

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Abstract Dynamical collapse models embody the idea of a physical collapse of the wave function in a mathematically well-defined way. They involve modifications to the standard rules of quantum theory in order to describe collapse as a physical process. This appears to introduce a time reversal asymmetry into the dynamics since the state at any given time depends on collapses in the past but not in the future. Here we challenge this conclusion by demonstrating that, subject to specified model constraints, collapse models can be given a structurally time symmetric formulation in which the collapse events are the primitive objects of the theory. Three different examples of time asymmetries associated with collapse models are then examined and in each case it is shown that the same dynamical rule determining the collapse events works in both the forward and backward in time directions. Any physically observed time asymmetries that arise in such models are due to the asymmetric imposition of initial or final time boundary conditions, rather than from an inherent asymmetry in the dynamical law. This is the standard explanation of time asymmetric behaviour resulting from time symmetric laws.

Keywords Collapse of the wave function · Dynamical collapse models · Time reversal symmetry

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1 Introduction

A wave function undergoing collapse appears to be at odds with time reversal symmetry for the reason that a collapse corresponds to an updating of the wave function and the wave function at any given time only depends on collapse events in the past. For a physical collapse process this reflects an objective time asymmetry in the evolution of the wave function, undermining the common claim that the fundamental laws of physics are time symmetric.

Taking the idea of physical collapse seriously results in so-called dynamical collapse models. These are an attempt to adapt the dynamical rules of quantum theory in order to describe the collapse of the wave function as a physical process. The aim is to provide a mathematically well-defined dynamical description of the evolving wave function, capable of explaining both collapse and unitary behaviour in a way that is independent of *observers* or *measuring devices*. The equation of motion should be well approximated by the Schrödinger equation for systems involving few particles and it should naturally describe rapid collapse of macro pointer states in quantum experiment situations. The best known examples are the GRW model [1] and the CSL (continuous spontaneous localisation) model [2,3] (For general reviews of collapse models see [4,5]).

By treating the collapse of the wave function as a genuine physical process these models inherit a time reversal asymmetry in the wave function dynamics. Whilst this fits an intuitive view whereby the present state depends on the past and not on the future, it presents a problem in understanding the origin of this time asymmetry given that, aside from physical collapse, the fundamental laws of physics are time symmetric (taken to mean CPT symmetric in a standard model context). Furthermore, the famous result of Aharonov, Bergmann, and Lebowitz (ABL) [6] shows that the quantum mechanical estimates of measurement results can be presented in a time-symmetric way. The usual way in which quantum mechanics is applied involves the construction of ensembles of pre-selected states. This enables us to make predictions about future states. In the ABL formalism, states are both pre and post selected, and it is shown that expressions for estimates of measurement results in the intervening period are structurally time symmetric. Whilst this picture gives no account of how the time symmetry is exhibited in the evolution of the state, it indicates a time symmetry of quantum theory at the empirical level.

In light of these concerns we will re-examine the time asymmetry of collapse models. In fact we will show that collapse models can be understood without reference to a preferred direction of time. In order to do this we make use of a number of ideas:

◇ *Collapses happen randomly.* A typical feature of collapse models is that the wave function undergoes spontaneous collapses in some preferred basis. This process may be discrete or continuous. The probability for any given collapse to happen should be given by the Born rule. For example, in the GRW model [1] the preferred basis is the particle position basis and the collapse process is discrete. The wave function $\psi_t(x_1, \ldots, x_N)$ for N distinguishable particles usually satisfies the Schrödinger equation but from time to time, and with fixed probability per unit time for each particle, it makes a jump of the form

$$\psi_{t+} = j_i(z)\psi_t. \tag{1}$$

Each particle i has its own random sequence of jumps occurring at a different set of random times. The jump operator j is of the form

$$j_i(z) = \exp\left(-(z - x_i)^2/2a^2\right)/(\pi a^2)^{1/4}, \tag{2}$$

with a some fixed length scale. The action of j is therefore to quasi project the wave function for particle i about some position z. The collapse centre z is chosen randomly from a probability distribution

$$\frac{\int dx_1 \cdots dx_N |j(z - x_i)\psi_t|^2}{\int dx_1 \cdots dx_N |\psi_t|^2}. \tag{3}$$

This is precisely the Born rule probability for a quasi projection of the form j. The fixed rate of collapses can be chosen such that individual particles are rarely affected, but a bulk mass with large numbers of particles suffers frequent jumps. In this way macroscopic pointers rapidly commit to definite readings.

◇ *The set of collapse outcomes provide an empirically adequate description of the world.* The wave function is an object that does not exist in ordinary position space. However, the jumps in the GRW model are localised in space and time. The locations of the collapse centres z play the role of measurement outcomes in the effective position measurements that spontaneously occur in the GRW model. It therefore makes sense to treat them as the mathematical counterparts to real world events [7]. On a fine-grained scale the world appears as if composed from many discrete points. The local density of these points give a representation of the location of matter. In this picture the role of the wave function is in determining the probabilities for the various collapse locations. In general we will call z the collapse outcome or the collapse data. The collapse outcomes are the primitive objects in the theory. They alone are sufficient to form an empirically adequate description real world events.

In some other collapse models, notably the CSL model and the lattice model discussed below [8–10], the jumps act on a quantum field variable and occur throughout space and time. The resulting collapse outcomes take the form of a classical stochastic field. Again we expect to recover a picture of the world from this classical stochastic field by some coarse graining procedure.

◇ *The wave function encodes the past history of collapses giving maximal information about future collapse events.* As pointed out in Ref. [10], if a collapse model is considered as a stochastic law for generating the collapse centres, then the wave function can be relegated to the initial time from which it does not need to evolve. The collapsing evolution of the wave function corresponds to an updating, conditioned on the history of realised collapses, of the rule for determining the probability of future collapses. The evolving wave function is then just a convenient calculation tool for making the theory Markovian. Going further it can be speculated that the wave function might be abandoned altogether [11].

On the basis of these ideas we will present a manifestly time symmetric framework for a generic collapse model in which the primary objects are the collapse outcomes. This we do in Sect. 2. We will show that, given a valid set of collapse data, we can form empirically equivalent pictures of collapsing wave functions evolving both in the forward-in-time and backward-in-time directions, and in each case, the outcomes of subsequent collapses satisfy a probability rule of the same form. The only ways in which observable time asymmetry can appear in such a model is through the imposition of asymmetric boundary conditions at the initial and final times. We show that the usual forward-in-time wave function collapse dynamics exists within this framework given a constraint on the final time boundary condition of the Universe.

We then examine three different cases where collapse models might be thought to show distinctive time asymmetries: in Sect. 3 the lattice model of collapse proposed by Dowker, Henson, and Herbauts [8–10]; in Sect. 4 the QMUPL (quantum mechanics with universal position localisation) model of Diósi [12]; and in Sect. 5 the generic tendency of collapse models to increase energy over time.

The lattice model of Sect. 3 is a model for a complete universe described by a quantum field. We choose an initial condition, an initial wave function, and evolve forward in time in order to generate a set of collapses. We explicitly construct an equivalent, backward-in-time picture of an evolving wave function undergoing the same set of collapses in the reverse order. We then use statistical analysis to show that the collapses which were generated by the forward-in-time dynamics are distributed as though they had been generated by the backward-in-time collapse dynamics. In this particular model the probability distributions for collapse events are simple Bernoulli distributions. This allows us to find large numbers of random events and to confirm that the statistics work out in both time directions. The lattice model is an efficient model for this purpose, and is equivalent to the CSL model in $1 + 1$ dimensions.

In Sect. 4 we examine whether an individual isolated particle can reveal time asymmetries. For this we use the QMUPL model, which occurs as limit cases of both the GRW and CSL collapse models. The QMUPL model is particularly interesting for us, as the localised collapse dynamics are explicitly time asymmetric when expressed as a diffusion process for the central location of a stable localised wave packet. Once again we generate a set of collapses from forward-in-time evolving wave packets, and show how these collapse outcomes are still distributed as though they had been generated by a wave packet evolving in the opposite direction in time.

In the final case we will show how the apparent monotonic increase in energy is compatible with time symmetric dynamical laws and discuss the significance of asymmetric boundary conditions. Although the time asymmetries of the QMUPL model will be used for illustrative purposes, our arguments are expected to apply to generic collapse models. We end with a summary in Sect. 6.

2 General Proof of Time Reversal Symmetry

Here we demonstrate the time symmetric structure of collapse models. Further details can be found in Ref. [13]. We assume that a generic collapse model satisfies the rule that collapse events occur at certain discrete times. These times may be randomly

distributed or at regular intervals depending on the model. Continuous collapse models such as the CSL model can be obtained as a limit case of this construction (see Sect. VI of [3]). The effect on the quantum state of a collapse event at time t is given by

$$|\Psi_t\rangle \to |\Psi_{t+}\rangle = J(z_t)|\Psi_t\rangle, \tag{4}$$

where $J = J^\dagger$ is a jump operator. At all other times the state evolves unitarily with Hamiltonian $H = H^\dagger$ so that in the Schrödinger picture the state satisfies

$$|\Psi_t\rangle = U(t-s)|\Psi_s\rangle, \tag{5}$$

for $t > s$ if there are no collapse events between s and t, and where $U(t) = e^{-iHt}$.

The jump operators must satisfy the completeness property

$$\int dz J^2(z) = \mathbb{1}, \tag{6}$$

and the probability of a particular jump outcome z at time t, given the state $|\Psi_t\rangle$, is given by

$$\mathbb{P}(z) = \frac{\langle\Psi_t|J^2(z)|\Psi_t\rangle}{\langle\Psi_t|\Psi_t\rangle}. \tag{7}$$

The jump operators are generalised quantum measurement operators. The effect of the jumps on the quantum state is therefore equivalent to the effect of performing the corresponding measurement. The random variable z corresponds to the measurement outcome.

We see that this definition encompasses the GRW model where the jumps (2) correspond to quasi projections in particle position space.

Now suppose that there is a fixed initial state for the Universe ρ_I at time t_0 and a final constraint on the Universe, ρ_F at time t_n. These density matrix states represent the fact that the pure state of the Universe at the initial and final times is not precisely specified, and a probability distribution over realised pure states is given. The collapse events occur at discrete times t_i between t_0 and t_n with $0 < i < n$ such that $t_0 < t_1 < t_2 < \cdots < t_n$. Using the rules (4) and (5) we find that the (unnormalised) state at time t with $t_j < t \leq t_{j+1}$ is given by

$$\begin{aligned}\rho_t = {} & U(t-t_j)J(z_j)\cdots U_{2,1}J(z_1)U_{1,0}\rho_I \\ & U_{0,1}J(z_1)U_{1,2}\cdots J(z_j)U(t_j-t),\end{aligned} \tag{8}$$

where $U_{i,j} = U(t_i - t_j)$ and $z_i = z_{t_i}$. From (7) we can determine the probability for the complete set $\{z_i\} = \{z_1, z_2, \ldots, z_{n-1}\}$ of collapse data for the entire history of the Universe. Conditional on the initial and final states this is given by

$$\mathbb{P}(\{z_i\}|\rho_I, \rho_F) = \frac{\mathbb{P}(\{z_i\}, \rho_F|\rho_I)}{\mathbb{P}(\rho_F|\rho_I)} = \frac{\mathrm{Tr}\,[\rho_F \rho_n]}{\int dz_1 \cdots dz_{n-1} \mathrm{Tr}\,[\rho_F \rho_n]}, \tag{9}$$

with $\rho_n = \rho_{t_n}$ (and in general we denote $\rho_i = \rho_{t_i}$). Here the final condition of ρ_F is understood as being equivalent to a POVM measurement of ρ_F for final state ρ_n from the POVM basis $\{\rho_F, \mathbb{1} - \rho_F\}$. We note that a similar construction involving sets of projection operators in place of the sets of generalised measurement operators J used here, was used to discuss time symmetry in the context of quantum cosmology in the histories formulation [14].

We now show that (9) has time symmetric structure provided that there exists a complete set of basis states $\{|\phi_i\rangle\}$ such that

$$\begin{aligned} \langle\phi_i|U(t)|\phi_j\rangle^* &= \langle\phi_i|U(-t)|\phi_j\rangle; \\ \langle\phi_i|J(z)|\phi_j\rangle^* &= \langle\phi_i|J(z)|\phi_j\rangle. \end{aligned} \tag{10}$$

This is equivalent to the statement that there exists a basis in which both U and J are symmetric matrices. First we define ρ^* by

$$\langle\phi_i|\rho^*|\phi_j\rangle = \langle\phi_i|\rho|\phi_j\rangle^*, \tag{11}$$

from which it follows that

$$\mathbb{P}(\{z_i\}|\rho_I, \rho_F) = \frac{\mathrm{Tr}\left[\rho_n^* \rho_F^*\right]}{\int dz_1 \cdots dz_{n-1} \mathrm{Tr}\left[\rho_n^* \rho_F^*\right]}. \tag{12}$$

We then show using (10) that

$$\begin{aligned} \langle\phi_i|\rho_n^*|\phi_j\rangle &= \langle\phi_i|\rho_n|\phi_j\rangle^* \\ &= \langle\phi_i|U_{n,n-1}J(z_{n-1})\cdots U_{2,1}J(z_1)U_{1,0}\rho_I \\ &\quad U_{0,1}J(z_1)U_{1,2}\cdots J(z_{n-1})U_{n-1,n}|\phi_j\rangle^* \\ &= \langle\phi_i|U_{n-1,n}J(z_{n-1})\cdots U_{1,2}J(z_1)U_{0,1}\rho_I^* \\ &\quad U_{1,0}J(z_1)U_{2,1}\cdots J(z_{n-1})U_{n,n-1}|\phi_j\rangle. \end{aligned} \tag{13}$$

Next we adopt a convenient notation to describe the collapse events in the reverse ordering: we define a new time parameter by $\bar{t} = -t$; we relabel $\bar{t}_i = -t_{n-i}$ such that $\bar{t}_0 < \bar{t}_1 < \bar{t}_2 \cdots < \bar{t}_n$; and we write $\bar{z}_i = z_{i-n}$. We also denote $\bar{U}_{i,j} = U(\bar{t}_i - \bar{t}_j)$ so that

$$\begin{aligned} \rho_n^* = \bar{U}_{0,1}J(\bar{z}_1)\cdots\bar{U}_{n-2,n-1}J(\bar{z}_{n-1})\bar{U}_{n-1,n}\rho_I^* \\ \bar{U}_{n,n-1}J(\bar{z}_{n-1})\bar{U}_{n-1,n-2}\cdots J(\bar{z}_1)\bar{U}_{1,0}. \end{aligned} \tag{14}$$

Inserting this result into (12) we find

$$\mathbb{P}(\{z_i\}|\rho_I, \rho_F) = \frac{\mathrm{Tr}\left[\rho_I^* \bar{\rho}_n\right]}{\int d\bar{z}_1 \cdots d\bar{z}_{n-1} \mathrm{Tr}\left[\rho_I^* \bar{\rho}_n\right]}, \tag{15}$$

where we have used the cyclic property of the trace and

$$\begin{aligned}\bar{\rho}_n = {} & \bar{U}_{n,n-1} J(\bar{z}_{n-1}) \bar{U}_{n-1,n-2} \cdots J(\bar{z}_1) \bar{U}_{1,0} \rho_F^* \\ & \bar{U}_{0,1} J(\bar{z}_1) \cdots \bar{U}_{n-2,n-1} J(\bar{z}_{n-1}) \bar{U}_{n-1,n}.\end{aligned} \tag{16}$$

By comparison of (9) and (15) it becomes clear that the probability formula could equally well be interpreted as resulting from the reversed set of collapse outcomes $\{\bar{z}_1, \ldots, \bar{z}_{n-1}\}$, $\{\bar{t}_1, \ldots, \bar{t}_{n-1}\}$, with an initial state ρ_F^* at $\bar{t}_0$, and a final constraint ρ_I^* at $\bar{t}_n$, and with a dynamical rule that, if there is a collapse event at time $\bar{t}$ then,

$$|\bar{\Psi}_{\bar{t}}\rangle \to |\bar{\Psi}_{\bar{t}+}\rangle = J(\bar{z}_{\bar{t}})|\bar{\Psi}_{\bar{t}}\rangle, \tag{17}$$

and if there are no collapse events between $\bar{s}$ and $\bar{t} > \bar{s}$ then,

$$|\bar{\Psi}_{\bar{t}}\rangle = U(\bar{t} - \bar{s})|\bar{\Psi}_{\bar{s}}\rangle. \tag{18}$$

This is a backward-in-time time dynamics, identical in form to the forward-in-time dynamics. The collapse event at time $\bar{t}_j$ with outcome $\bar{z}_j$ corresponds to the forwards-in-time collapse event at time $t_{n-j}(= -\bar{t}_j)$ with outcome $z_{n-j}(= \bar{z}_j)$. The reverse time state $|\bar{\Psi}_{\bar{t}}\rangle$ suffers the same sequence of jumps $\{z_i\}$ as the forward-in-time state but in the reverse time order. Either the forward-in-time or backward-in-time dynamics can be used as the underlying state process and since (i) they each predict the same probability for a complete set of collapse outcomes $\{z_i\}$, and (ii) they are each empirically equivalent to the set of collapse outcomes which itself provides an empirically adequate description the world, then the formulation is structurally time symmetric.

We note that the conditions (10) are satisfied by the GRW model with $H = \sum_i p_i^2/2m + V(\{x_i\})$ if we choose the basis to be the particle position state basis.

From (9) we can recover the Born rule probability for a collapse event at time t_j conditional on the state at time t_j immediately prior to the collapse. The state at this time is known if we know the initial state and all collapse outcomes to the past of t_j. We therefore calculate

$$\begin{aligned}&\mathbb{P}(z_j|\rho_I, \rho_F, \{z_i|i < j\}) \\ &\quad = \frac{\int dz_{j+1} \cdots dz_{n-1} \mathbb{P}(\{z_i\}, \rho_F|\rho_I)}{\mathbb{P}(\{z_i|i < j\}, \rho_F|\rho_I)} \\ &\quad = \frac{\int dz_{j+1} \cdots dz_{n-1} \mathrm{Tr}\left[\rho_F \rho_n\right]}{\int dz_j \cdots dz_{n-1} \mathrm{Tr}\left[\rho_F \rho_n\right]} \\ &\quad = \frac{\int dz_{j+1} \cdots dz_{n-1} \mathrm{Tr}\left[\bar{\rho}_{n-j}^* J(z_j) \rho_j J(z_j)\right]}{\int dz_j \cdots dz_{n-1} \mathrm{Tr}\left[\bar{\rho}_{n-j}^* J(z_j) \rho_j J(z_j)\right]}.\end{aligned} \tag{19}$$

The expression in the last line can be recovered by conditioning on the state ρ_j, at time t_j, instead of conditioning on ρ_I and $\{z_i|i < j\}$, so that

$$\mathbb{P}(z_j|\rho_I, \rho_F, \{z_i|i < j\}) = \mathbb{P}(z_j|\rho_j, \rho_F). \tag{20}$$

Now if

$$\int d\bar{z}_1 \cdots d\bar{z}_{n-j-1}\bar{\rho}_{n-j} \propto \mathbb{1}, \tag{21}$$

then (19) becomes

$$\mathbb{P}(z_j|\rho_I, \rho_F, \{z_i|i < j\}) = \frac{\mathrm{Tr}\left[J(z_j)\rho_j J(z_j)\right]}{\mathrm{Tr}\left[\rho_j\right]}. \tag{22}$$

This is the Born rule probability for a collapse event $J(z_j)$ at time t_j given the state ρ_j at time t_j. If the initial state is pure we recover (7). One way to satisfy condition (21) is to choose the final state of the Universe to be the uniformly mixed state. Otherwise, for a general final condition ρ_F, (21) requires that for an hypothetical ensemble of Universes, the mixed state obtained by evolving using using the backward-in-time dynamics from time $\bar{t}_0$ to time $\bar{t}_{n-j} = -t_j$, is the uniformly mixed state. In this case the constraining effects of the final state of the Universe are shielded from the present by all future collapse events.

Since the formulation is time symmetric we can also write down the backward-in-time probability rule for the collapse event at time $\bar{t}_j$ conditional on all future collapse outcomes. This is

$$\begin{aligned}&\mathbb{P}(\bar{z}_j|\rho_I, \rho_F, \{\bar{z}_i|i < j\})\\ &= \frac{\int d\bar{z}_{j+1} \cdots d\bar{z}_{n-1}\mathrm{Tr}\left[\rho^*_{n-j}J(\bar{z}_j)\bar{\rho}_j J(\bar{z}_j)\right]}{\int d\bar{z}_j \cdots d\bar{z}_{n-1}\mathrm{Tr}\left[\rho^*_{n-j}J(\bar{z}_j)\bar{\rho}_j J(\bar{z}_j)\right]}.\end{aligned} \tag{23}$$

And if

$$\int dz_1 \cdots dz_{n-j-1}\rho_{n-j} \propto \mathbb{1}, \tag{24}$$

we recover the backward-in-time Born rule

$$\mathbb{P}(\bar{z}_j|\rho_I, \rho_F, \{\bar{z}_i|i < j\}) = \frac{\mathrm{Tr}\left[J(\bar{z}_j)\bar{\rho}_j J(\bar{z}_j)\right]}{\mathrm{Tr}\left[\bar{\rho}_j\right]}. \tag{25}$$

This generic and manifestly time symmetric formulation of collapse models includes the usual collapse model formulation as a special case of initial and final time boundary conditions. More generally it is seen that the Born rule applied in either time direction can be violated as a result of the boundary conditions. There is no inbuilt direction

of time. The only possible way in which time asymmetry can be introduced is via asymmetric boundary conditions of the Universe.

In the following sections we will investigate time reversal symmetry in some different examples of collapse models.

3 Lattice Model

Here we outline the lattice model of collapse proposed by Dowker & Henson [8] and further investigated by Dowker & Herbauts [9,10]. We demonstrate some pertinent features of the lattice model. In particular, given only the stochastic field of collapse data, it is not possible to observe that the state is in a superposition of different preferred basis states. This is important for our time symmetric picture since we should not be able to distinguish superposition states which appear in the wave function picture of one time direction but not the other. To discern the presence of some matter in a region, we must coarse grain to remove background noise. It turns out the scales on which we need to coarse grain are also the scales on which a superposition state collapses.

We will present the results of a statistical test, designed to demonstrate that the collapses are distributed as though generated by the backward-in-time collapse dynamics. This will provide an explicit demonstration of the results of the previous section.

The model is a modification of the massive Thirring model on a $(1+1)$D null lattice [15]. This is a unitary fermionic quantum field theory in 2D Minkowski space. The lattice is shown in Fig. 1.

In the time direction (vertical) the lattice extends arbitrarily far into the past or future. In the spatial dimension there are N vertices denoted by black circles and we impose periodic boundary conditions so that the space-time is in fact a cylinder. The centres of the links are indicated by black squares. These are the locations of the collapse events. A quantum state $|\Psi_\sigma\rangle$ is defined with reference to a space-like surface σ. This surface must cut through the links of the lattice in the manner shown. The surface must also satisfy the periodic boundary conditions.

We can divide the lattice into columns representing the spatial location which we label $i = 1, 2, \ldots, 2N$ as shown in Fig. 1. As we pass along a space-like surface

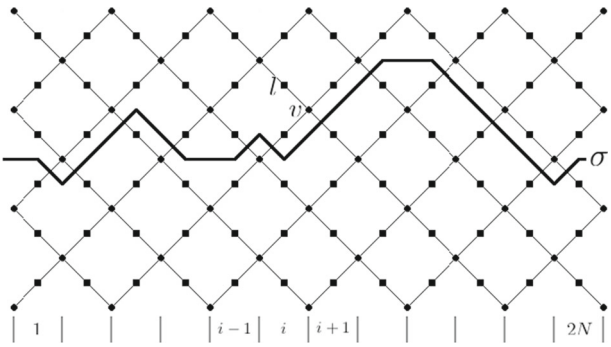

Fig. 1 The $(1+1)$D null lattice showing a space-like surface σ, a vertex v, and a link l

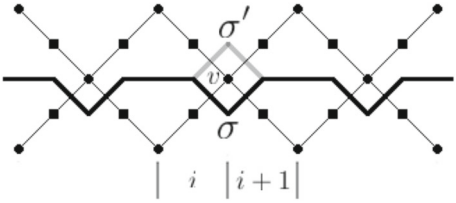

Fig. 2 Elementary evolution I: crossing a vertex

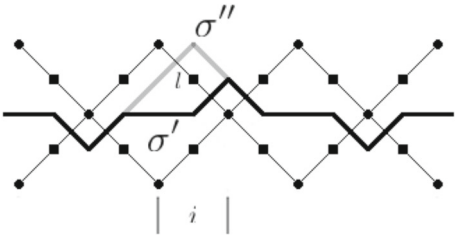

Fig. 3 Elementary evolution II: crossing the centre of a link

σ we cut through each of the $2N$ columns once. To each column we assign a qubit with basis states $|0\rangle$, corresponding to empty, and $|1\rangle$, corresponding to occupied; the full quantum state belongs to the tensor product space of all the qubit state spaces so that the basis states of the system are of the form $|u_1, u_2, \ldots, u_{2N}\rangle$ where the $u_i \in \{0, 1\}$ represent the individual qubit basis states. For example, a state of the form $|1, 0, 1, 1, 0, 0, 0, 1\rangle$ on a lattice of size $N = 4$ can be understood either as a particular quantum field configuration or a state of 4 'particles' at 4 specific locations. The vacuum, or unexcited quantum field state is $|0, 0, 0, 0, 0, 0, 0, 0\rangle$. We will refer to this basis as the preferred basis since it is the basis into which the system tends to collapse. The Hilbert space for the quantum state has dimension 2^{2N}.

The model also makes use of a classical stochastic field z taking values $z_l = 0$ or 1 at random on each link l of the lattice. These values are the binary collapse outcomes.

There are two types of elementary evolution of the state which occur when the surface σ advances in one of two different elementary ways. One corresponds the surface σ crossing a vertex. This is shown in Fig. 2 as the surface passes over the vertex v from σ to σ'. If the vertex involved in this elementary evolution is one which connects links at positions i and $i + 1$ (modulo $2N$), we define a unitary operator by

$$\mathbb{U}_v = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes U_{i,i+1} \otimes \mathbb{1}_{i+2} \otimes \cdots \otimes \mathbb{1}_{2N}, \tag{26}$$

where $U_{i,i+1}$ is some 4D unitary operator acting on the ith and $(i + 1)$th qubits. In crossing the vertex v the state changes according to

$$|\Psi_{\sigma'}\rangle = \mathbb{U}_v |\Psi_\sigma\rangle. \tag{27}$$

The second type of elementary evolution occurs when the surface passes over the centre of a link. This is shown in Fig. 3 where the surface passes the centre of the

link l from σ' to σ''. If the link is located at position i on the lattice we define a jump operator by

$$\mathbb{J}_l(z_l) = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes J_i(z_l) \otimes \mathbb{1}_{i+1} \otimes \cdots \otimes \mathbb{1}_{2N}. \tag{28}$$

In crossing the link l the state changes according to

$$|\Psi_{\sigma''}\rangle = \mathbb{J}_l(z_l)|\Psi_{\sigma'}\rangle. \tag{29}$$

The jump operator describes the collapse of the wave function. These jump operators occurring at each link l are equivalent to the generic jump operators occurring at points in time outlined in Sect. 2. If we assume some arbitrary total ordering of evolutions over links and vertices respecting the causal order of the lattice, then the set of jumps occur in an ordered sequence. This is the natural generalisation of the time ordered sequence of jumps presented earlier.

Without the jump operator this model is the light-cone lattice massive Thirring model of Ref. [15]. Depending on the value of z_l, the operator J_i acting on the ith qubit takes the form

$$\begin{aligned} J_i(0) &= \tfrac{1}{\sqrt{1+X^2}}\{|0_i\rangle\langle 0_i| + X|1_i\rangle\langle 1_i|\} \\ J_i(1) &= \tfrac{1}{\sqrt{1+X^2}}\{X|0_i\rangle\langle 0_i| + |1_i\rangle\langle 1_i|\} \end{aligned} \tag{30}$$

with X the fixed collapse parameter, $0 \leq X \leq 1$. The jump operators satisfy the completeness relation

$$J_i^2(0) + J_i^2(1) = \mathbb{1}_i. \tag{31}$$

If X is close to 0 then whenever they act they effectively perform a projective measurement on the qubit state in the 0, 1 basis. If X is close to 1 their action nudges the qubit state slightly towards either the 0 or 1 state with collapse requiring many such jump operations.

The value of the field variable z_l controls whether the collapse favours the $|0\rangle$ state or the $|1\rangle$ state. This field variable is chosen randomly and the probability that the field takes value z_l on the link l is given by the Born rule probability

$$\mathbb{P}(z_l|\Psi_{\sigma'}) = \frac{\langle\Psi_{\sigma'}|\mathbb{J}_l^2(z_l)|\Psi_{\sigma'}\rangle}{\langle\Psi_{\sigma'}|\Psi_{\sigma'}\rangle} = \frac{\langle\Psi_{\sigma''}|\Psi_{\sigma''}\rangle}{\langle\Psi_{\sigma'}|\Psi_{\sigma'}\rangle}. \tag{32}$$

From these elementary rules we can derive the rules for evolution from a general state on an initial surface σ_i to a final surface σ_f in the future. Suppose that in getting from σ_i to σ_f we must cross n links and m vertices. The links we label $\{l_1, \ldots, l_n\}$ and the vertices we label $\{v_1, \ldots, v_m\}$. It follows that the final state is of the form

$$|\Psi_{\sigma_f}\rangle = \mathcal{T}\left[\mathbb{J}_{l_1}(z_{l_1}) \cdots \mathbb{J}_{l_n}(z_{l_n})\mathbb{U}_{v_1} \cdots \mathbb{U}_{v_m}\right]|\Psi_{\sigma_i}\rangle, \tag{33}$$

where $\mathcal{T}$ is the time ordering operator. The probability for the field values $\{z_{l_1}, \ldots, z_{l_n}\}$ is given from (32) by

$$\mathbb{P}(z_{l_1}, \ldots, z_{l_n} | \Psi_{\sigma_i}) = \frac{\langle \Psi_{\sigma_f} | \Psi_{\sigma_f} \rangle}{\langle \Psi_{\sigma_i} | \Psi_{\sigma_i} \rangle}. \tag{34}$$

This probability is well defined given only the partial ordering of links and vertices imposed by the space-time causal order [8].

We shall work with a unitary operator of the form [15]

$$U_{i,i+1} = \begin{array}{c} \\ \cdot\ \cdot \\ \cdot\ \nearrow \\ \nwarrow\ \cdot \\ \nwarrow\ \nearrow \end{array} \begin{array}{c} \begin{array}{cccc} \cdot\ \cdot & \cdot\ \nwarrow & \nearrow\ \cdot & \nearrow\ \nwarrow \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sin\theta & \cos\theta & 0 \\ 0 & \cos\theta & i\sin\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \tag{35}$$

where $\nearrow\nwarrow$ denotes an incoming state where both the ith and the $(i+1)$th qubits are in the $|1\rangle$ state; $\nearrow\cdot$ denotes an incoming state where the ith qubit is in the $|1\rangle$ state and the $(i+1)$th qubit is in the $|0\rangle$ state; etc. Here, θ controls the speed of the 'particles' on the lattice with, e.g., $\theta = 0$ corresponding to light speed particles and $\theta = \pi/2$ corresponding to stationary particles. All excitations of the quantum field travel with the same speed although they may move either to the left or the right. This unitary operator conserves the number of particles (the number of occupied qubits).

We note that with these definitions, U and J are both symmetric matrices in the preferred basis and the conditions (10) are therefore satisfied. This means that the lattice theory has time reversal symmetry as described in Sect. 2.

According to Eq. (23) the Born rule should be observed statistically in the reverse time direction if the initial pure state is chosen from a uniformly mixed state. For a one particle state this means that we should draw the initial pure state with equal probability from the $2N$ dimensional basis of 1 particle vectors. In the preferred basis these vectors each take the form $|0, 0, \ldots, 1, \ldots, 0, 0\rangle$ with a single value 1 and all other entries zeros. We note that each of these basis vectors is related to any other by a symmetry transformation on the lattice (involving a translation and/or a parity transformation). Therefore choosing any individual single particle state as the initial state, we should statistically observe the Born rule for the reverse time dynamics.

We will therefore suppose that for a single particle system there exists some reverse time state $|\bar{\Psi}_\sigma\rangle$ which evolves as

$$|\bar{\Psi}_{\sigma_i}\rangle = \overline{\mathcal{T}}\left[\mathbb{J}_{l_1}(z_{l_1}) \cdots \mathbb{J}_{l_n}(z_{l_n}) \mathbb{U}_{v_1} \cdots \mathbb{U}_{v_m}\right] |\bar{\Psi}_{\sigma_f}\rangle. \tag{36}$$

where $\bar{\mathcal{T}}$ is the anti time ordering operator. This takes the same form as the forward-in-time rule (33). In general $|\bar{\Psi}_\sigma\rangle$ will not be the same as $|\Psi_\sigma\rangle$. Our aim is to demonstrate numerically that the field values $\{z_{l_1}, \ldots, z_{l_n}\}$ also satisfy the Born probability rule

$$\mathbb{P}(z_{l_1}, \ldots, z_{l_n} | \bar{\Psi}_{\sigma_f}) = \frac{\langle \bar{\Psi}_{\sigma_i} | \bar{\Psi}_{\sigma_i} \rangle}{\langle \bar{\Psi}_{\sigma_f} | \bar{\Psi}_{\sigma_f} \rangle}, \tag{37}$$

the counterpart to Eq. (34). If this is the case then the backward-in-time evolution uses precisely the same dynamical rule as the forward-in-time evolution. The wave functions for the two cases will differ in general on any given surface σ but the stochastic field z, the basis for an empirically adequate description of the world, will be consistent.

3.1 Coarse Graining and Collapse Time Scale

Given a set of collapse outcomes z, the forward-in-time wave function undergoing these collapses is likely to look (at least at the micro level) different from the backward-in-time wave function. If the time symmetric picture of physical collapse is to work, it must be the case that we are not able to observe the state of the wave function from the collapse data in a way which would allow us to distinguish these two cases.

In this section we adapt a calculation given in Ref. [9] and show that the time scale necessary to observe the presence of an excited quantum field on the lattice is the same time scale on which a superposition of such excited states will collapse. This means that it is not possible to directly observe a superposition in the preferred basis given only z.

Suppose that the system is in the vacuum state $|0, 0, \ldots, 0\rangle$. The evolution of the system defined by Eqs. (27) and (29) will not change the state. The stochastic field z will take value 0 with probability $1/(1 + X^2)$ and 1 with probability $X^2/(1 + X^2)$. We assume that X has a value close to 1. This is natural since we do not want single particle states to rapidly collapse. Consider a region of space-time R containing M links. Within this region the mean and variance of the field value are given by the binomial distribution

$$\mu = \frac{X^2}{1 + X^2}\ ,\ \sigma^2 = \frac{X^2}{M(1 + X^2)^2}. \tag{38}$$

Now suppose that there is some non-vacuum state with mean field value z_R in R. If we are to be able to observe z_R against the background noise we require

$$\sigma \ll |z_R - \mu|. \tag{39}$$

If we write $\epsilon = 1 - X$, taking ϵ to be small, we find that $|z_R - \mu|$ is at most $\mathcal{O}(\epsilon)$ (occurring when the state is maximally excited in the region R). Since $\sigma \sim M^{-1/2}$ we therefore must have

$$M \gg \epsilon^{-2}. \tag{40}$$

This determines the size of the region necessary to be able to identify an excited state.

Now consider a block of n qubits in the 1 state with all other qubits in the 0 state, superposed with a disjoint block of n qubits in the 1 state with all other qubits in the 0 state. For simplicity we assume that $\theta = \pi/2$ so that the unitary evolution has no effect. Write the state as $|A\rangle + |B\rangle$.

After we have evolved for a number m of time units the state will be of the unnormalised form $X^{M_B}|A\rangle + X^{M_A}|B\rangle$, where $M_A + M_B = 2nm = M$, and $M_{A/B}$ is the

number of links for which the field z_l corresponds to the A/B state qubit eigenvalue on that link wherever the A and B state qubit eigenvalues are different.

Since the value of ϵ is small, the field takes values 0 or 1 with probability $\sim 1/2$. The division of M into M_A and M_B has a distribution with standard deviation in $M_{A/B} \propto \sqrt{M}$. This means that one of the states will be suppressed with respect to the other by a factor of $X^{\sqrt{M}} \simeq \exp(-\epsilon\sqrt{M})$. Therefore there is exponential suppression when

$$M \sim \epsilon^{-2}. \tag{41}$$

We now put this together with result (40). If the stochastic field z is taken to represent our empirical access to the quantum world described by this model, then any feature observable against background noise requires coarse graining over a region containing $\gg \epsilon^{-2}$ links. Since a superposition state is only able to survive for $\sim\epsilon^{-2}$ links then we cannot directly observe the fluctuating densities during the collapse process.

Consider the example above and suppose that the state collapses to the $|B\rangle$ state after a certain period of time. If we then evolve the state backward in time it will simply stay in the $|B\rangle$ state. This can be consistent since the period during which the state was in a superposition of $|A\rangle$ and $|B\rangle$ when viewed forward in time is brief enough that it cannot be distinguished from $|B\rangle$ if we only have access to the stochastic field z.

3.2 Time Reversed Collapse on the Lattice

To test for consistency between forward-in-time and backward-in-time collapse dynamics on the lattice we apply the following test:

1. We generate some field data by starting with an initial one particle state $|\Psi_{\sigma_i}\rangle$ on an initial time slice $t = 0$ and evolving forward in time to reach a final state $|\Psi_{\sigma_f}\rangle$. This results in a random field z.
2. We then reverse the evolution in time using Eq. (36) and using $|\Psi^*_{\sigma_f}\rangle$ (where complex conjugation is taken with respect to the collapse basis) as a proxy for the unknown state $|\bar{\Psi}_{\sigma_f}\rangle$. We apply the collapses again in the reverse order using the same field data z. For each link l we make note of the probability that the field z_l takes the value 1 conditional on all z to the future of l. These probabilities can take values anywhere in the range [0, 1]. Once we have returned to time $t = 0$ we have, for each link on the lattice, a reverse time probability for the collapse outcome to take value 1 (alongside the realised value generated by the forward-in-time dynamics).
3. We divide the set of reverse time probabilities into a set of bins with boundaries $0 < p_1 < p_2 < \cdots < 1$. For each bin j we take the average probability as $\bar{p}_j = (p_{j-1} - p_j)/2$ and we count the number of collapse events in that bin, m_j. Using the binomial distribution we determine the mean number of times that we expect to see the stochastic field take value 1, μ_j, and the variance in this number, σ_j^2. We use a simple test to check that the data should be approximately normally distributed (essentially that m_j should be sufficiently large given the average probability $\bar{p}_j$ for the bin) and discard those bins where this is not the

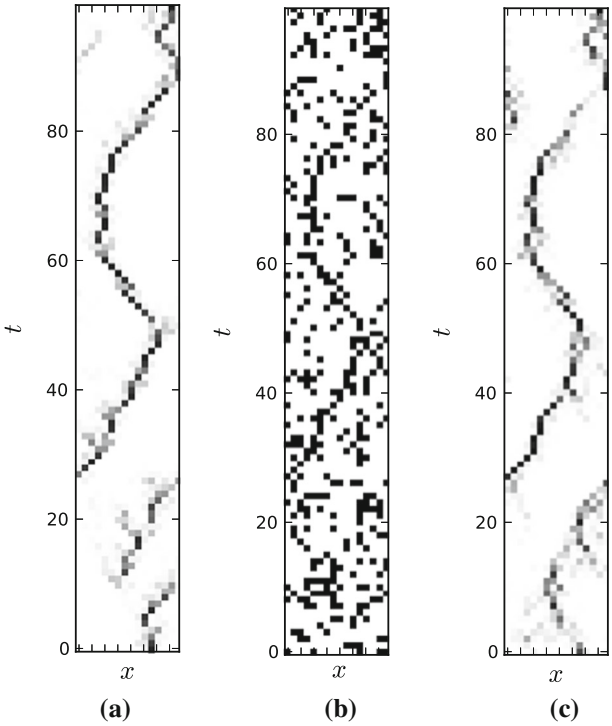

Fig. 4 **a** Quantum field expectation value for the forward-in-time state. **b** Stochastic field z. **c** Quantum field expectation value for the backward-in-time state

case. We then count the actual number of times n_j that the field z for each of those events realised the value 1. From these we calculate a chi-squared statistic

$$\chi^2 = \sum_j \frac{(n_j - \mu_j)^2}{\sigma_j^2}. \tag{42}$$

4. Using the theoretical distribution for the chi-squared statistic we calculate a p value giving the probability that the field data or something more extreme could have been generated by the reverse time probabilities. In statistics a p value of less than 0.5% is a typical standard for ruling out a hypothetical model.

Figure 4 shows a typical example of a simulation with $2N = 16$ in which we set $X = 0.5$ and $\theta = \pi/4$. The initial condition is of the form

$$|\Psi_{\sigma_i}\rangle = |0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0\rangle. \tag{43}$$

We evolve for 100 time steps. We choose an arbitrary total ordering of links and vertices such that elementary evolutions sweep from left to right (To be precise we start on a given time slice, we then evolve past the leftmost vertex, followed by the link above it to the left and then the link above it to the right. We then do the same

for the vertex to the right and so on until we have evolved the time slice by one time step. The procedure is then repeated). Each pixel in the figure represents a link on the lattice so that each graph is 16 pixels wide and 100 pixels high.

The left hand panel shows the expectation value $\langle\Psi_\sigma|\mathbb{A}_l|\Psi_\sigma\rangle$ where

$$\mathbb{A}_l = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes A_i \otimes \mathbb{1}_{i+1} \otimes \cdots \otimes \mathbb{1}_{2N}. \tag{44}$$

and

$$A_i = |1_i\rangle\langle 1_i|. \tag{45}$$

This is the expectation of the eigenvalue of the ith qubit. The surface σ is the surface immediately following evolution across the link l given the total ordering which we use. We use a grey scale to plot the values of $\langle\Psi_\sigma|\mathbb{A}_l|\Psi_\sigma\rangle$ with black corresponding to the value 1 and white corresponding to 0. A clear particle trajectory winding around the lattice is apparent.

The central panel shows the realised field values z on the lattice generated randomly using the probabilities calculated from the forward evolving state $|\Psi_\sigma\rangle$. Here black corresponds to $z_l = 1$ and white corresponds to $z_l = 0$. The particle trajectory is less apparent. As discussed in Sect. 3.1 a coarse graining procedure is necessary to eliminate background noise and establish the particle's whereabouts to within space time regions of greater than 1 pixel.

The right hand panel shows $\langle\bar{\Psi}_\sigma|\mathbb{A}_l|\bar{\Psi}_\sigma\rangle$, with the state evolved backward in time using the reversed total ordering and with collapses generated by the same field z. The particle trajectory is very similar to that in panel (a).

A careful observation of Fig. 4a reveals that dispersion occurs in the upward direction: localised particle states tend to become diffuse with increase in t before recollapsing with dead ends fading away. The same happens in the opposite time direction in Fig. 4c. Despite these differences the overall picture of a particle moving about on the lattice is broadly consistent.

As outlined above we calculate the chi-squared statistic (42) and a p value for the field data in panel (b) under the null hypothesis that data was generated by the backward-in-time collapse dynamics. In Fig. 5 we plot a histogram of the p values generated by running the simulation 500 times. The distribution of p values is approximately uniform over the range [0, 1]. This is as we would expect if the null hypothesis is correct.

We conclude that field data generated by forward-in-time collapse dynamics is distributed as if it had been generated by an equivalent backward-in-time collapse dynamics. The collapse dynamics satisfies time reversal symmetry.

4 Time Reversal Symmetry for a Localised Wave Packet

We now turn our attention to the QMUPL model [12] for a single particle. When the particle is localised the collapse dynamics reduce to a classical diffusion process for the average position and momentum of the packet. We demonstrate that this process

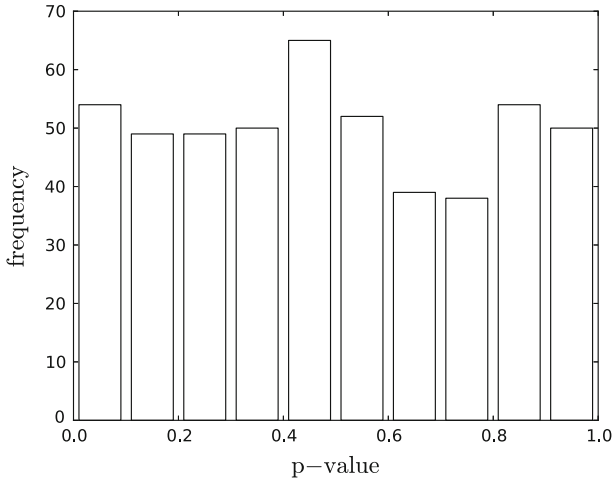

Fig. 5 Distribution of p values for 500 simulations of the lattice model

is asymmetric with respect to time reversal, but show it is still possible to construct an equivalent, backward-in-time process resulting from the same collapses occurring in the reverse order.

The QMUPL model can be thought of as a continuous version of GRW describing wave function collapse for distinguishable particles. It is also a limit of the CSL model when the particle density is low and the collapse length scale is large compared to the length scale of a wave packet [16]. In the QMUPL model the state vector for a single particle satisfies a quantum state diffusion of the form

$$d|\psi_t\rangle = \left\{-i\hat{H}dt - \frac{g^2}{8}(\hat{x} - \langle\hat{x}\rangle_t)^2 dt + \frac{g}{2}(\hat{x} - \langle\hat{x}\rangle_t)dB_t\right\}|\psi_t\rangle. \tag{46}$$

Here B_t is a standard Brownian motion and the collapse parameter g controls the rate at which collapse of the wave function occurs. In terms of the GRW collapse parameters $g = (2\lambda/a^2)^{1/2}$ where λ is the GRW collapse rate and a is the GRW length scale. The Hamiltonian $\hat{H} = \hat{p}/2m$.

Since the GRW model satisfies conditions (10) with the position state basis and the QMPUL model is a continuous limit of the GRW model, the QMPUL model has time reversal symmetry according to these criteria.

A typical feature of continuous collapse models is that after a sufficient period of time, the wave packet of an individual isolated particle achieves a stable localised shape. This happens when the dispersive effects of quantum theory balance with the localising effects of the collapses. The timescale on which the stable localised state of a wave packet is achieved is given by $t_{\text{loc}} \sim (\lambda\hbar/ma^2)^{-1/2}$ [12] where m is the particle mass (and we have included $\hbar$ for the purpose of this calculation). It is standard [17] to have λ scale as the mass of the particle squared, $\lambda = \lambda_0(m/m_0)^2$ where m_0 is the proton mass, and to use the GRW estimates [1] for the collapse parameters:

$\lambda_0 = 10^{-16}\text{s}^{-1}$; $a = 10^{-7}\text{m}$. This results in a localisation timescale $t_{\text{loc}} \sim 10^{-7}\text{s}$ for a composite particle of mass 1g.

For timescales longer than this, the dynamics are considerably simplified. The state can be characterised by only the central values of position $\langle \hat{x} \rangle_t = \langle \psi_t | \hat{x} | \psi_t \rangle = x_t$ and momentum $\langle \hat{p} \rangle_t = \langle \psi_t | \hat{p} | \psi_t \rangle = p_t$ for the wave packet. The collapse dynamics causes these phase space parameters to undergo a classical diffusion process. For the QMUPL model this process is given by [18–20]

$$dx_t = \frac{p_t}{m} dt + \frac{1}{\sqrt{m}} dB_t, \tag{47}$$

$$dp_t = \frac{g}{2} dB_t. \tag{48}$$

For the remainder of Sect. 4 we restrict our attention to the regime where this simplified model of a diffusing wave packet holds, to avoid the complication of an arbitrary wave function undergoing collapse.

We first show that Eqs. (47) and (48) do not have time reversal symmetry. The state of the system at any point in time is described by x, p, and B. Consider a sequence of two states S_1 and S_2 at times $t_1 = t$ and $t_2 = t + \Delta t$ respectively. We write

$$S_1 = \{x, p, B\} \tag{49}$$

$$S_2 = \{x + \Delta x, p + \Delta p, B + \Delta B\} \tag{50}$$

Consider the change as we go from state S_1 to S_2. The change in x is Δx, the change in p is Δp and the change in B is ΔB. From Eqs. (47) and (48) we expect that

$$\Delta x = \frac{p}{m} \Delta t + \frac{1}{\sqrt{m}} \Delta B, \tag{51}$$

$$\Delta p = \frac{g}{2} \Delta B. \tag{52}$$

We next define a time reversal transformation T. For a classical phase space trajectory this involves a change in the sign of p simply because playing the movie of these events backward in time, the particle appears to move in the opposite direction to when the movie is played forward in time. The time reversed states are therefore given by

$$S_1^T = \{x^T, p^T, B^T\} = \{x, -p, B\} \tag{53}$$

$$S_2^T = \{x + \Delta x, -p - \Delta p, B + \Delta B\} \tag{54}$$

Consider now the change in going from S_2^T to S_1^T. The change in x, $\Delta x^T = -\Delta x$, the change in p, $\Delta p^T = \Delta p$, and the change in B, $\Delta B^T = -\Delta B$. Inserting into Eqs. (51) and (52) results in

$$\Delta x^T = \frac{p^T}{m}\Delta t + \frac{1}{\sqrt{m}}\Delta B^T \tag{55}$$

$$\Delta p^T = -\frac{g}{2}\Delta B^T. \tag{56}$$

This is a different pair of equations from those which describe evolution forward in time. This would appear to mean that it would be possible to determine the forward direction of time from a given sequence of states. For example, we could simply observe Δx and Δp and see if there is a positive or negative correlation in the spontaneous jumps. If the correlation is positive then the evolution is forward in time; if the correlation is negative then the evolution is backward in time.

As with the lattice model we will show it is possible to understand what is going on in a time symmetric way. Equations (47) and (48) describe a diffusion process for the expectation values x_t and p_t: these are features of the wave function. The wave function is seen as a convenient way to encode the collapse history and it is this procedure which introduces the time asymmetry. For the remainder or Sect. 4 we return to the idea that it is not the diffusion of the wave function, but instead the locations of the collapses that are fundamental.

The reason for the stochastic motion of x_t and p_t is that collapses are occurring randomly on either side of the centre of the wave packet causing it to spontaneously jump about. By taking the continuous limit of the GRW model we can show that the locations of the collapse centres are given by

$$z_t = x_t + \frac{1}{g}\frac{\Delta B_t}{\Delta t}. \tag{57}$$

The collapse centres have a white noise distribution about the expected position x_t.

Now suppose that we only have access to the set $\{z_t\}$. We would like to confirm whether it is possible to give different but consistent pictures of the evolutions of x_t and p_t in either time direction, each satisfying the same dynamical law.

Referring to Eq. (23), in order to observe the Born rule backward in time, the initial condition should be a uniformly mixed state. In the context of the localised particle this is equivalent to a uniform distribution of initial values for x_0 and p_0. However, we note that for two different sets of initial values the particle diffusions are related by a Galilean transformation. The Born rule should therefore be statistically observed in the reverse time direction for a fixed initial localised state.

We perform the following test analogous to the test carried out in Sect. 3.2:

1. We generate some collapse centre data z_i, $i = 0, \ldots, n$ by starting with an initial localised state characterised by the central position x_0 and momentum p_0 of the wave packet and evolving forward in time for n discrete time steps using the rule

$$x_{i+1} = x_i + \frac{p_i}{m}\Delta t + \frac{1}{\sqrt{m}}\Delta B_i, \tag{58}$$

$$p_{i+1} = p_i + \frac{g}{2}\Delta B_i, \tag{59}$$

$$z_i = x_i + \frac{1}{g}\frac{\Delta B_i}{\Delta t}. \tag{60}$$

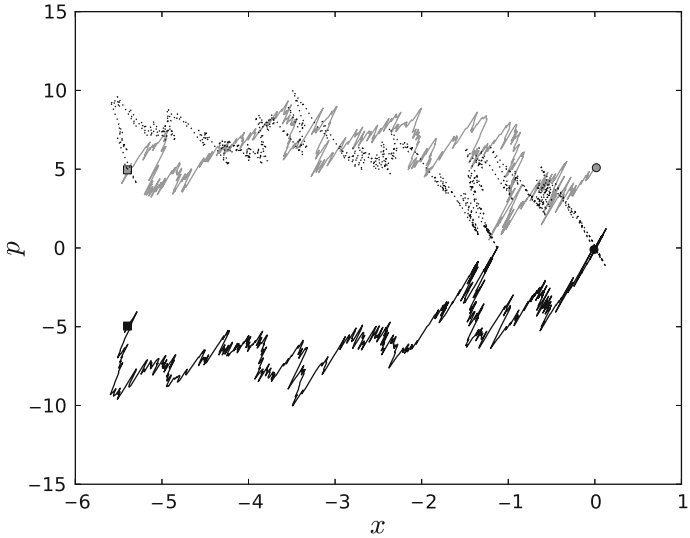

Fig. 6 Phase space trajectories forward in time (*black line*), forward in time with $p \to -p$ (*dotted line*), and backward in time (*grey line*). The distance unit is equivalent to $(400\,\hbar/\mathrm{mg}^2)^{1/4}$; the momentum unit is equivalent to $(\mathrm{mg}^2\hbar^3/400)^{1/4}$

Brownian increments ΔB_i are generated randomly. This results in a forward-in-time phase space trajectory x_i, p_i.

2. Next consider the collapses from stage 1, z_i, in reverse order. Denote the backward-in-time phase space trajectory by $\bar{x}_i$, $\bar{p}_i$. Starting at $\bar{x}_0 = x_n$ and $\bar{p}_0 = -p_n$ we use the dynamical law (where we use the notational conventions of Sect. 2 for the backward-in-time process)

$$\Delta \bar{B}_{i+1} = g\Delta t(z_{n-i-1} - \bar{x}_i), \tag{61}$$

$$\bar{x}_{i+1} = \bar{x}_i + \frac{\bar{p}_i}{m}\Delta t + \frac{1}{\sqrt{m}}\Delta \bar{B}_{i+1}, \tag{62}$$

$$\bar{p}_{i+1} = \bar{p}_i + \frac{g}{2}\Delta \bar{B}_{i+1}. \tag{63}$$

In the first equation we back-out the Brownian increments from the collapse centres using (57). The result is a backward-in-time phase space trajectory $\bar{x}_i$, $\bar{p}_i$.

3. We perform a Kolmogorov-Smirnov test on the set of implied increments $\Delta \bar{B}_i/\sqrt{\Delta t}$ to see if they fit a normal distribution. This results in p value for the reverse time $\Delta \bar{B}$ data. This tests whether the collapses occur with white noise distribution about $\bar{x}$. If they do then the backward-in-time trajectory satisfies the same dynamical law based on the collapsing wave function as the forward-in-time trajectory.

Figure 6 shows an example of a trajectory though phase space using $g = 20, m = 1$ and $\Delta t = 0.001$. The solid black line is the forward-in-time trajectory generated using Eqs. (51) and (52). The trajectory starts at time $t = 0$ at the black circle at $x = 0$,

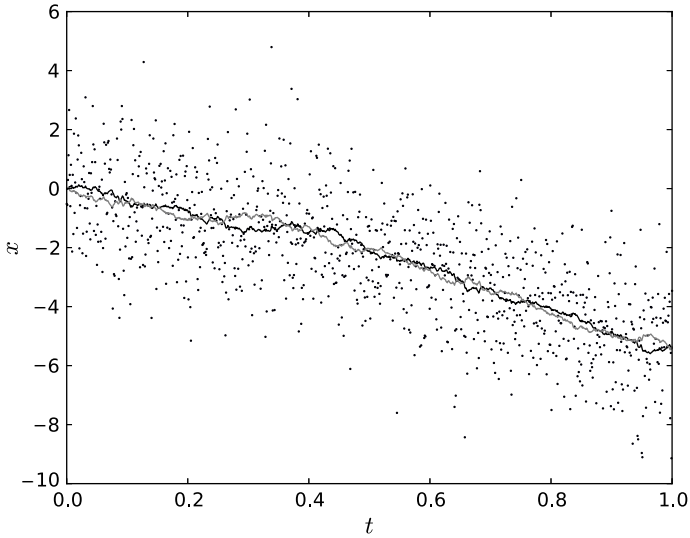

Fig. 7 Expected position with time. *Black line* is evolved forward in time; *grey line* is evolved backward in time. The distance unit is equivalent to $(400\,\hbar/\mathrm{mg}^2)^{1/4}$; the time unit is equivalent to $(400\,\mathrm{m}/\hbar\mathrm{g}^2)^{1/2}$

$p = 0$ and ends at time $t = 1$ at the black square. There is a clear positive correlation in the stochastic jumps in x and p. The dotted line is the straightforward time reverse of this trajectory obtained by transforming $p \to -p$. It is characteristically different from the forward-in-time trajectory. The correlation between stochastic jumps in x and p is negative. The grey line shows the reverse time trajectory determined by the procedure outlined above. This trajectory starts at the grey square at time $t = 1$ and ends at the grey circle at time $t = 0$. This trajectory approximates the dotted line but with positively correlated jumps in x and p. Visually, there is no way to distinguish the micro dynamics of the backward trajectory from the forward trajectory.

Figure 7 is from the same simulated trajectory. It shows how x changes with time. The dots are the collapse centres generated by the forward-in-time dynamics. The black line is the forward-in-time expectation x_t. It is seen that the collapse locations are distributed about this line and both the collapse locations and x_t follow the same trend. The grey line is the backward-in-time $\bar{x}_t$ which is slightly different from the forward-in-time x_t but which also sits well within the distribution of collapses.

Figure 8 shows a histogram of p values generated by running the simulation 5000 times. We find that the distribution of p values is reasonably uniform in the range [0, 1] indicating that indeed the increments $\Delta\bar{B}/\sqrt{\Delta t}$ belong to a normal distribution.

We conclude that the collapse data (the dots in Fig. 7) is distributed as though it had been generated by the backward-in-time collapse dynamics.

5 Equilibrium and Boundary Conditions

The examples in Sects. 3 and 4 consider two specific models of objective collapse, exhibiting properties that look like an inherent time asymmetry (collapse events affect-

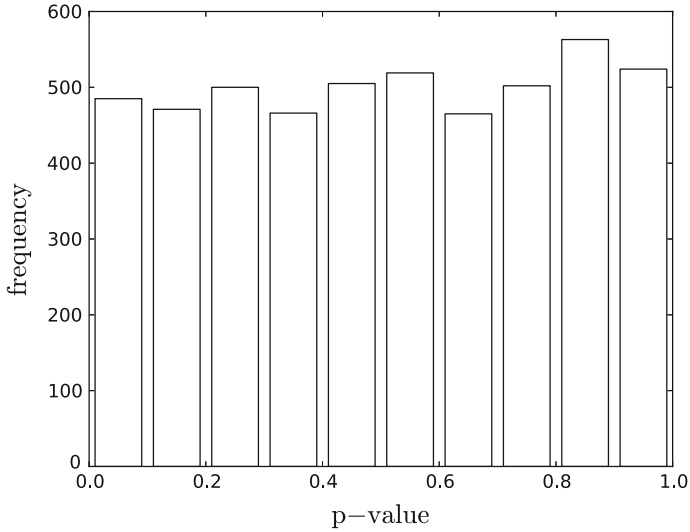

Fig. 8 Distribution of p values for 5000 simulations of the QMPUL localised particle model

ing the wave function after, but not before, the event; and positive correlation between the diffusion in mean position and momentum). Our third example looks at a more generic feature of collapse models: non-conservation of mean energy. This appears to give a directly observable time asymmetry. In doing so, we will shed light upon the role of initial and final boundary conditions in causing time asymmetric behaviour.

It is well known (see, e.g. [1]) that one of the physical effects predicted by collapse models is a gradual increase in energy. This effect can be easily seen in the QMUPL model. A monotonic mean energy increase follows simply from Eq. (48). The diffusion in momentum of an individual particle is such that momentum is just as likely to go up by some amount as it is to go down by the same amount. However, since energy is a convex function of momentum, $E \sim p^2$, the energy increases on average. It is important to understand how this can be the case with a time symmetric picture since this increase in mean energy only appears in one time direction.

Consider a set of free localised particles, initially each with $p = 0$ at time $t = 0$. The total energy is zero. After a period of diffusion, the particles will have a Gaussian distribution of momenta centred on zero, but with a non-zero total energy. Viewed as a process backward in time, the energetic particles would appear to simultaneously tend towards zero momentum and energy. If the backward evolving wave function were supposed to satisfy the Born rule, then the backward-in-time evolution appears unlikely. For example, we would find that the collapses undergone by particles with positive momentum would tend to cause them to reduce momentum, and the collapses undergone by particles with negative momentum would tend to cause them to increase momentum. By conditioning on the sign of the momentum, a reverse time observer would appear to see a conspiracy, that leads to the mean energy decreasing. By observing the change in mean energy over time, the direction of time could be deduced.

The conspiratorial behaviour for the reverse time picture comes about as a result, not of an inherent asymmetry in the dynamics, but of the time asymmetric use of boundary conditions. From the forward-in-time picture, we have imposed the initial condition at time $t = 0$ that $p = 0$, and then tracked the behaviour for times $t > 0$. The evolution of the system then proceeds according to the Born rule, with no future boundary condition imposed. Eventually this will result in a diffuse nearly uniform distribution.

Viewed in the reverse direction, we start with a diffuse distribution, evolving apparently according to the Born rule. We might expect this distribution to stay diffuse (or possibly spread out more if it is not in equilibrium). However, the behaviour we are tracking has a special feature that makes this impossible: these are pre-selected trajectories that must have come from localised $p = 0$ states at time $t = 0$. This condition on the trajectories introduces a pre-selective bias in the statistics of the reverse time collapse events, and leads to the apparent conspiratorial behaviour.

To understand this more clearly, we will now address a well known problem for any stochastic theory [21]: such a theory cannot exhibit time independent behaviour (such as that required by the Born rule) in both temporal directions, unless the probability distribution is in equilibrium.

The argument goes as follows: suppose a system follows a stochastic rule $R_{t_1|t_0}(S_j|S_i)$ for how it changes state from one time to another. At time t_0, the probability for state S_i is $P_{t_0}(S_i)$. At some later time $t_1 > t_0$, the stochastic evolution leads to the probability for state S_j of

$$P_{t_1}(S_j) = \sum_i R_{t_1|t_0}(S_j|S_i) P_{t_0}(S_i). \tag{64}$$

From Bayes' theorem, the system being in state S_j at time t_1 can be used to make retrodictions about the possible state at time t_0:

$$P_{t_0|t_1}(S_i|S_j) = R_{t_1|t_0}(S_j|S_i) \frac{P_{t_0}(S_i)}{P_{t_1}(S_j)}. \tag{65}$$

If the future directed stochastic evolution $R_{t_1|t_0}(S_j|S_i)$ is time independent (as we would expect from the Born rule), then adding any constant shift in time of τ will not affect the transition probabilities:

$$R_{t_1+\tau|t_0+\tau}(S_j|S_i) = R_{t_1|t_0}(S_j|S_i). \tag{66}$$

However, imposing the same condition on the retrodictive probabilities

$$P_{t_0+\tau|t_1+\tau}(S_i|S_j) = P_{t_0|t_1}(S_i|S_j) \tag{67}$$

gives

$$P_{t_0|t_1}(S_i|S_j) = R_{t_1|t_0}(S_j|S_i) \frac{P_{t_0+\tau}(S_i)}{P_{t_1+\tau}(S_j)}. \tag{68}$$

which can only hold if

$$\frac{P_{t_0}(S_i)}{P_{t_0+\tau}(S_i)} = \frac{P_{t_1}(S_j)}{P_{t_1+\tau}(S_j)} = f(\tau) \tag{69}$$

Normalisation of probabilities yields $f(\tau) = 1$. So the time independence condition cannot also hold in the backward direction, unless the system is in equilibrium with $P_{t_0}(S_i) = P_{t_0+\tau}(S_i) = P_E(S_i)$. This would give a backward-in-time retrodictive rule:

$$P_{t_0|t_1}(S_i|S_j) = R_{t_1|t_0}(S_j|S_i)\frac{P_E(S_i)}{P_E(S_j)}. \tag{70}$$

which is time independent.

We argue that this does not imply any sort of time asymmetry in the dynamics. Instead it is a result of the time asymmetric use of pre-selection statistics. Let us instead start with the equilibrium distribution $P_E(S_i)$, and then impose a pre-selection at time $t = 0$, that the system is in S_0. For times $t_f > 0$, we have, as before, the forward-in-time predictions $R_{t_f|0}(S_j|S_0)$. If we now try to retrodict from the occurrence of S_j at time t_f, to some intermediate time $0 < t_1 < t_f$, we get:

$$P_{t_1|t_f,0}(S_i|S_j, S_0) = \frac{R_{t_f|t_1}(S_j|S_i)R_{t_1|0}(S_i|S_0)}{\sum_{i'} R_{t_f|t_1}(S_j|S_{i'})R_{t_1|0}(S_{i'}|S_0)} \tag{71}$$

In general, this will not lead to the reverse time retrodictive rule Eq. (70), nor will it be time independent. Pre-selecting the state S_0 at time $t = 0$, biases the statistics of the retrodictive inferences in the time period $0 < t_1 < t_f$, as they *must* lead toward the state S_0 as $t_1 \to 0$.

However, exactly the same is true if we look at the forward-in-time statistics in the period $t_p < t_{-1} < 0$. First, if we impose the condition at time $t = 0$, that the system is in S_0, then at times $t_p < 0$ we have the retrodictive inference:

$$P_{t_p|0}(S_j|S_0) = R_{0|t_p}(S_0|S_j)\frac{P_E(S_j)}{P_E(S_0)}. \tag{72}$$

This leads to a time independent rule for $t_p < 0$. By contrast, if we try to make predictive inferences in the period $t_p < t_{-1} < 0$, based on the occurrence of state S_j at time t_p, we get

$$P_{t_{-1}|t_p,0}(S_i|S_j, S_0) = \frac{R_{0|t_{-1}}(0|S_i)R_{t_{-1}|t_p}(S_i|S_j)}{\sum_{i'} R_{0|t_{-1}}(0|S_{i'})R_{t_{-1}|t_p}(S_{i'}|S_j)} \tag{73}$$

Again, in general, this will not lead to the rule $R_{t_{-1}|t_p}(S_i|S_j)$ nor will it be time independent. Post-selecting the state at time $t = 0$, biases the forward-in-time statistics in the period $t_p < t_{-1} < 0$, as they *must* lead toward the state S_0 as $t_{-1} \to 0$.

There should be nothing terribly mysterious about the fact that an apparently time independent, future directed stochastic evolution law $R_{t_1|t_0}(S_j|S_i)$ still does not yield

time independent statistics when a post-selection is imposed at some time $t > t_1$. The post-selection biases the statistics. Equally so, therefore, there is nothing terribly mysterious that, when we start with a pre-selected ensemble, and look at the reverse time direction, the pre-selection introduces a bias in the retrodictive inferences. The apparent time asymmetry associated with the bias in the statistics is a result of the pre-selection, rather than anything fundamental to the dynamics.

We may now return to the problem of non-conservation of energy in collapse models. By pre-selecting a set of free localised particles, with $p = 0$ at time $t = 0$, as an initial condition, we are biasing the sample for the reverse time statistics when $t > 0$. An analogous situation can easily be constructed by starting with a wide and uniform distribution of initial momenta and post-selecting, at some time $t_f > t$, only those trajectories which end up with momentum equal to zero at $t = t_f$. The biases will now be present in the forward-in-time direction. Post-selected particles with positive momentum would tend to experience collapses which cause them to reduce momentum, and those with negative momentum would tend to increase.

Ideally, if the system has an equilibrium distribution, then to remove any biases we should start with samples whose distributions are approximately in equilibrium (see Ref. [22]). As in the case of the QMUPL model, if the momentum is not bounded, an equilibrium distribution of momentum can have an unbounded mean energy, even if the mean momentum is finite. Any selection at $t = 0$, with a finite mean energy, will be followed in the forward time direction by a diffusion towards equilibrium that increases the mean energy for $t > 0$. However, when looking at times $t < 0$, the selection at $t = 0$ is a post-selection. The mean energy will be seen to be *decreasing* in time, converging on the value fixed by the post-selection. Once again, we see that the apparent asymmetry results, not from any inherent asymmetry in the dynamics, but from the use of initial or final boundary conditions, with the ultimate pre and post selections provided by the initial and final time boundary constraints of the Universe. A time symmetric interpretation of the stochastic dynamical law is therefore not at odds with an observation of energy increase due to collapse. While we have used the QMUPL model as an illustrative example when needed, this argument will hold for any model in which the mean energy is unbounded for equilibrium distributions.

6 Summary

We have shown that collapse models can be viewed in a way in which they exhibit a time reversal symmetry. This is perhaps surprising given the apparent time-directedness of a collapse of the wave function. The key is to treat the collapses data as the fundamental stuff of the theory and the wave function as part of the dynamical law used to determine what the next collapse outcome will be. Indeed we have argued that for a given set of collapses there are two equivalent pictures of a collapsing wave function: one going forward in time and one going backward in time, each satisfying the same dynamical rules.

This idea brings a new perspective to the problems of quantum theory. For example, in the usual picture of measurement the preceding quantum state is replaced by an eigenvector of the observable which is measured. Consider a particle passing

through two misaligned polarisers. After passing through the first polariser the particle collapses to the first polarisation state. It stays in this state for the duration of its trajectory before it passes through the second polariser. In the backward-in-time picture the particle has the polarisation state of the second polariser at those points in time when it is situated between the two measurements. The resolution is to understand the wave function as an object which is part of the dynamical laws rather than part of the ontology. If we insist on understanding events in terms of the wave function then we must recognise that the wave function could be just as well determined by its future interactions as by its past [23].

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