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Restriction of Representations of $\mathrm{GL}(n+1,\mathbb{C})$ to $\mathrm{GL}(n,\mathbb{C})$ and Action of the Lie Overalgebra

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Abstract Consider a restriction of an irreducible finite dimensional holomorphic representation of $\mathrm{GL}(n+1,\mathbb{C})$ to the subgroup $\mathrm{GL}(n,\mathbb{C})$. We write explicitly formulas for generators of the Lie algebra $\mathfrak{gl}(n+1)$ in the direct sum of representations of $\mathrm{GL}(n,\mathbb{C})$. Nontrivial generators act as differential-difference operators, the differential part has order $n-1$, the difference part acts on the space of parameters (highest weights) of representations. We also formulate a conjecture about unitary principal series of $\mathrm{GL}(n,\mathbb{C})$.

Keywords Finite dimensional representations of GL · Restrictions of representations · Difference operators · Plucker identities · Zhelobenko operators

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Dedicated to Alexander Alexandrovich Kirillov in his $81 = 3^4$ birthday

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1 The Statement

1.1 The Gelfand–Tsetlin Formulas

It is well known that restrictions of finite dimensional holomorphic representations of the general linear group $\mathrm{GL}(n,\mathbb{C})$ to the subgroup $\mathrm{GL}(n-1,\mathbb{C})$ is multiplicity free. Considering a chain of restrictions

$$\mathrm{GL}(n,\mathbb{C}) \supset \mathrm{GL}(n-1,\mathbb{C}) \supset \mathrm{GL}(n-2,\mathbb{C}) \supset \cdots \supset \mathrm{GL}(1,\mathbb{C})$$

we get a canonical decomposition of our representation into a direct sum of one-dimensional subspaces. Taking a vector in each line we get a basis of the representation. In [7] Gelfand and Tsetlin announced formulas for action of generators of the Lie algebra $\mathfrak{gl}(n)$ in this basis. It turns out that the Lie algebra $\mathfrak{gl}(n)$ acts by difference operators in the space of functions on a certain convex polygon in the lattice $\mathbb{Z}^{n(n-1)/2}$. In particular, this gives an explicit realization of representations of the Lie algebra $\mathfrak{gl}(n)$.

For various proofs of the Gelfand–Tsetlin formulas and constructions of the bases, see [2, 3, 11, 17–19, 30, 31] (see more references in [17]), for other classical groups, see [8, 17, 28], for applications to infinite-dimensional representations, see [9, 13, 16, 27] for formulas on the group level, see [5, 9]. There are many other continuations of this story. However, our standpoint is slightly different.

1.2 Actions of Overalgebras in the Spectral Decompositions

We can write images of many operators under the classical Fourier transform. It is commonly accepted that Plancherel decompositions of representations are higher analogs of the Fourier transform.

Consider a group G and its subgroup H. Restrict an irreducible unitary representation of G to H. Generally, an explicit spectral decomposition of the restriction seems hopeless problem. However, there is a collection of explicitly solvable problems of this type [1]. In [20] it was conjectured that in such cases the action of the Lie algebra $\mathfrak{g}$ can be written explicitly as differential-difference operators. In fact, in [20] there was considered the tensor product $V_s \otimes V_s^*$ of a highest weight and a lowest weight unitary representations of $\mathrm{SL}(2,\mathbb{R})$ (i.e., $G \simeq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $H \simeq \mathrm{SL}(2,\mathbb{R})$ is the diagonal). This representation is a multiplicity free integral over the principal series of $\mathrm{SL}(2,\mathbb{R})$. It appears that the action of all generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ in the spectral decomposition can be written explicitly as differential-difference operators. The differential part of these operators has order two and the difference operators are difference operators in the imaginary direction.[2]

[1] Also, some explicitly solvable spectral problems in representation theory can be regarded as special cases of the restriction problem. In particular, a decomposition of L^2 on a classical pseudo-Riemannian symmetric space G/H can be regarded as a special case of the restriction of a Stein type principal series of a certain overgroup $\widetilde{G}$ to the symmetric subgroup G, [21]. So, for L^2 on symmetric spaces the problem of action of the overalgebra discussed below makes sense.

[2] On Sturm–Liouville in imaginary direction, see [23] and further references in that paper, see also [10]. The most of known appearances of such operators are related to representation theory and spectral decompositions of unitary representations.

Molchanov [12–15] solved several problems of this type related to rank-one symmetric spaces.[3] In all his cases the Lie overalgebra acts by differential-difference operators; difference operators act in the imaginary dimension. However, the order of the differential part in some cases is 4. In [25, 26] the author considered the Fourier transform on the group GL($2,\mathbb{R}$). There were evaluated the operators in the target space corresponding to multiplications by matrix elements and partial derivatives with respect to matrix elements. Formulas have similar structure, but in [26] they are simpler than in previous cases.

In the present paper, we consider restrictions of holomorphic finite-dimensional representations GL($n+1,\mathbb{C}$) to GL($n,\mathbb{C}$) and write explicit formulas for the action of the overalgebra in the spectral decomposition. Also, we formulate a conjecture concerning restrictions of unitary principal series.

1.3 Notation. The Group GL($n,\mathbb{C}$) and its Subgroups

Denote by GL($n,\mathbb{C}$) the group of invertible complex matrices of size n. By $\mathfrak{gl}(n)$ we denote its Lie algebra, i.e. the Lie algebra of all matrices of size n. Let $\mathrm{U}(n)\subset \mathrm{GL}(n,\mathbb{C})$ be the subgroup of unitary matrices.

By E_{ij} we denote the matrix whose ij's entry is 1 and other entries are 0. The unit matrix is denoted by 1 or 1_n, i.e. $1=\sum_j E_{jj}$. Matrices E_{ij} can be regarded as generators of the Lie algebra $\mathfrak{gl}_n$.

Denote by $N^+=N_n^+\subset \mathrm{GL}(n,\mathbb{C})$ the subgroup of all strictly upper triangular matrices of size n, i.e., matrices of the form

$$Z=\begin{pmatrix}1 & z_{12} & \dots & z_{1(n-1)} & z_{1n}\\ 0 & 1 & \dots & z_{2(n-1)} & z_{2n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & z_{(n-1)n}\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}.$$

By N_n^- we denote the group of strictly lower triangular matrices.

By $B_n^-\subset \mathrm{GL}(n,\mathbb{C})$ we denote the subgroup of lower triangular matrices, $g\in B_n^-$ if $g_{ij}=0$ if $i<j$. Let Δ be the subgroup of diagonal matrices, we denote its elements as $\delta=\mathrm{diag}(\delta_1,\dots,\delta_n)$.

Next, we need a notation for sub-matrices. For a matrix X denote by

$$[X]_{\alpha\beta}$$

the left upper corner of X of size $\alpha\times\beta$. Let $I=\{i_1,\dots,i_\alpha\}$, $J=\{j_1,\dots,j_\beta\}$ be collections of integers, we assume that their elements are ordered as $0<i_1<i_2<\dots i_\alpha$, $J: 1\leqslant j_1\leqslant\dots\leqslant j_\beta$. Denote by

$$\left[X\begin{pmatrix}I\\ J\end{pmatrix}\right]=\left[X\begin{pmatrix}i_1,\dots,i_\alpha\\ j_1,\dots,j_\beta\end{pmatrix}\right] \tag{1.1}$$

the matrix composed of entries x_{i_μ,j_ν}. By

$$[X(I)]=\left[X(i_i,\dots,i_\alpha)\right] \tag{1.2}$$

[3] In particular, he examined the restrictions of maximally degenerate principal series for the cases GL($n+1,\mathbb{R}$) $\supset$ O($n,1$), O(p,q) $\supset$ O($p-1,q$).

we denote matrix composed of $i_i, \dots, i_\alpha$-th rows of X (the order is not necessary increasing, also we allow coinciding rows).

1.4 Holomorphic Representations of $\mathrm{GL}(n, \mathbb{C})$

Recall that irreducible finite dimensional holomorphic representations ρ of the group $\mathrm{GL}(n, \mathbb{C})$ are enumerated by collections of integers (*signatures*)

$$\mathbf{p}: \; p_1 \geqslant \dots \geqslant p_n.$$

This means that there is a cyclic vector v (a *highest weight vector*) such that

$$\begin{aligned}\rho(Z)v &= v \qquad \text{for each } Z \in N_n^+,\\ \rho(\delta)v &= \prod_{j=1}^n \delta_j^{p_j} v, \qquad \text{where } \delta = \operatorname{diag}(\delta_1, \dots, \delta_n).\end{aligned}$$

Denote such representation by $\rho_{\mathbf{p}} = \rho_{\mathbf{p}}^n$. Denote the set of all signatures by Λ_n.

The dual (contragradient) representation to $\rho_{\mathbf{p}}$ has the signature

$$\mathbf{p}^* := (-p_n, \dots, -p_1). \tag{1.3}$$

1.5 Realizations of Representations by Differential Operators

Recall a model for irreducible finite dimensional holomorphic representations for details, see Section 2). We consider the space $\mathrm{Pol}(N_n^+)$ of polynomials in the variables z_{ij}. The generators of the Lie algebra $\mathfrak{gl}(n)$ act in this space via differential operators

$$E_{kk} = -\sum_{i<k} z_{ik}\partial_{ik} + p_k + \sum_{j>k} z_{kj}\partial_{kj}, \tag{1.4}$$

$$E_{k(k+1)} = \partial_{k(k+1)} + \sum_{i<k} z_{ik}\partial_{i(k+1)}, \tag{1.5}$$

$$\begin{aligned}E_{(k+1)k} &= \sum_{i<k} z_{i(k+1)}\partial_{ik} + (p_{k+1} - p_k) z_{k(k+1)} - z_{k(k+1)} \sum_{j>k} z_{kj}\partial_{kj}\\ &\quad + \sum_{m>k+1} \det \begin{pmatrix} z_{k(k+1)} & z_{km} \\ 1 & z_{(k+1)m} \end{pmatrix} \partial_{(k+1)m},\end{aligned} \tag{1.6}$$

where

$$\partial_{kl} := \frac{\partial}{\partial z_{kl}}.$$

All other generators can be expressed in the terms of E_{kk}, $E_{k(k+1)}$, $E_{(k+1)k}$ (also it is easy to write explicit formulas as it is explained in Section 2). In this way, we get a representation of the Lie algebra $\mathfrak{gl}(n)$.

There exists a unique finite dimensional subspace $V_{\mathbf{p}}$ invariant with respect to such operators. The representation $\rho_{\mathbf{p}}$ is realized in this space. The highest weight vector is $f(Z) = 1$. In the next subsection we describe this space more explicitly.

Remark This approach arises in [6]. Of course, this model is a coordinatization of the constructions of representations of $\mathrm{GL}(n, \mathbb{C})$ in sections of line bundles over the flag space $B_n^- \setminus \mathrm{GL}(n, \mathbb{C})$ (as in the Borel–Weil–Bott theorem). The space $N_n^+ \simeq \mathbb{C}^{n(n-1)/2}$ is an open dense chart on this space, the terms with ∂ in Eqs. 1.4–1.6 correspond to vector fields on the flag space, the zero order terms are corrections corresponding to the action in the bundle.

Elements of the space $V_{\mathbf{p}}^n$ are precisely polynomials, which are holomorphic as section of bundle on the whole flag space. However, we need explicit formulas and prefer a purely coordinate language.

1.6 Descriptions of the Space $V_{\mathbf{p}}$. Zhelobenko Operators

a) DESCRIPTION-1. Denote by $d\dot{Z}$ the standard Lebesgue measure on N_n^+

$$d\dot{Z} = \prod_{k<l} d\,\mathrm{Re}\, z_{kl}\, d\,\mathrm{Im}\, z_{kl}.$$

Denote by $d\mu(Z)$ the measure on N_n^+ given by the formula

$$d\mu_{\mathbf{p}}(Z) = d\mu_{\mathbf{p}}^n(Z) = \prod_{j=1}^{n-1} \det\bigl([Z]_{jn}([Z]_{jn})^*\bigr)^{-(p_j-p_{j-1})-2}\, d\dot{Z}. \tag{1.7}$$

Proposition 1.1 a) $V_{\mathbf{p}}^n = L^2(N_n^+,\mu_{\mathbf{p}}^n)\cap \mathrm{Pol}(N_n^+)$.
b) *The L^2-inner product in $V_{\mathbf{p}}^n$ is* $\mathrm{U}(n)$*-invariant.*

Proof is given in Section 2.

DESCRIPTION-2. We define the *Zhelobenko operators* by

$$R_{km} := \partial_{km} + \sum_{j>m} z_{mj}\,\partial_{kj}, \qquad m>k. \tag{1.8}$$

Theorem 1.2 *The space $V_{\mathbf{p}}^n$ consists of polynomials satisfying the conditions*

$$\Bigl\{\, R_{j(j+1)}^{p_j-p_{j+1}+1} f(Z) = 0.$$

See Zhelobenko, [29, 30], §6, Theorem 2, [32], Theorem 48.7.

DESCRIPTION-3. There is one more description of the space $V_{\mathbf{p}}^n$, which can be used with coordinate language.

Proposition 1.3 *The space $V_{\mathbf{p}}^n$ coincides with the space determined by the reproducing kernel*

$$\prod_{j=1}^{n-1} \det\Bigl([Z]_{jn}[Z]_{jn}^*\Bigr)^{p_j-p_{j-1}}.$$

On Hilbert spaces determined by reproducing kernels, see, e.g., [22], Section 7.1. The proposition is more-or-equivalent to the Borel–Weil theorem.

1.7 The Restriction of Representations GL($n+1,\mathbb{C}$) to GL($n,\mathbb{C}$)

Consider the representation $\rho_{\mathbf{r}}^{n+1}$ of $\mathrm{GL}(n+1,\mathbb{C})$ with a signature $\mathbf{r}=(r_1,\dots,r_{n+1})$. It is well-known (see Gelfand, Tsetlin [7]) that the restriction of $\rho_{\mathbf{r}}^{n+1}$ to $\mathrm{GL}(n,\mathbb{C})$ is multiplicity free and is a direct sum of all representations $\rho_{\mathbf{q}}$ of $\mathrm{GL}(n,\mathbb{C})$ with signatures satisfying the *interlacing conditions*

$$r_1 \geqslant q_1 \geqslant r_2 \geqslant q_2 \geqslant r_3 \geqslant \ldots \geqslant q_n \geqslant r_{n+1}. \tag{1.9}$$

Our purpose is to write explicitly the action of Lie algebra $\mathfrak{gl}(n+1)$ in the space $\oplus V_{\mathbf{q}}^n$.

1.8 Normalization

First, we intend to write a $\mathrm{GL}(n,\mathbb{C})$-invariant pairing

$$J_{\mathbf{p},\mathbf{q}} : V_{\mathbf{p}}^{n+1} \times V_{\mathbf{q}}^{n} \to \mathbb{C}.$$

as an integral

$$J_{\mathbf{p},\mathbf{q}}(\varphi,\psi) = \int_{N_{n+1}^{+}\times N_n^{+}} \overline{L_{\mathbf{p},\mathbf{q}}(U,Z)}\varphi(Z)\,\psi(U)\,d\mu_{\mathbf{p}}^{n+1}(Z)\,d\mu_{\mathbf{q}}^{n}(U), \tag{1.10}$$

where Z ranges in N_{n+1}^{+}, U ranges in N_n^{+}, and the kernel

$$L_{\mathbf{p},\mathbf{q}}(U,Z) \in V_{\mathbf{p}}^{n+1} \otimes V_{\mathbf{q}}^{n}$$

is a polynomial in the variables z_{ij}, u_{kl}.

Denote $Z^{cut} := [Z]_{(n+1)n}$, this matrix is obtained from Z by cutting of the last column. Denote by U^{ext} the $n\times(n+1)$-matrix obtained from U by adding the zero last column,

$$U^{ext} = \begin{pmatrix} 1 & u_{12} & \dots & u_{1n} & 0\\ 0 & 1 & \dots & u_{2n} & 0\\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{1.11}$$

Consider the $(n+1)\times(n+1)$-matrix composed from the first $(n+1-\alpha)$ rows of the matrix Z and the first α rows of the matrix U^{ext}. Denote by Φ_α the determinant of this matrix:

$$\Phi_\alpha(Z,U) = \det\begin{pmatrix} [Z]_{(n+1-\alpha)(n+1)}\\ [U^{ext}]_{\alpha(n+1)} \end{pmatrix}, \tag{1.12}$$

Consider the $n\times n$-matrix composed of the first $n-\alpha$ rows of the matrix Z^{cut} and the first α rows of the matrix U, denote by Ψ_α its determinant:

$$\Psi_\alpha(Z,U) = \det\begin{pmatrix} [Z^{cut}]_{(n-\alpha)n}\\ [U]_{\alpha n} \end{pmatrix}. \tag{1.13}$$

Proposition 1.4 *Consider signatures* $\mathbf{p} = (p_1,\dots,p_{n+1})$, $\mathbf{q} = (q_1,\dots,q_n)$ *such that* $\mathbf{p}$ *and* $\mathbf{q}^*$ *are interlacing. Then the expression* (1.10) *with the kernel*

$$L_{\mathbf{p},\mathbf{q}}(U,Z) = \Phi_1^{p_n+q_1}\Psi_1^{-p_n-q_2}\Phi_2^{p_{n-1}+q_2}\Psi_2^{-p_{n-1}-q_3}\dots\Phi_n^{p_1+q_n}. \tag{1.14}$$

determines a $\mathrm{GL}(n,\mathbb{C})$*-invariant nonzero pairing between* $V_{\mathbf{p}}^{n+1}$ *and* $V_{\mathbf{q}}^{n}$.

For instance, for $n=3$ we get

$$L_{\mathbf{p},\mathbf{q}}(U,Z) = \det\begin{pmatrix} 1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 1 & u_{12} & u_{13} & 0 \end{pmatrix}^{p_3+q_1} \det\begin{pmatrix} 1 & z_{12} & z_{13}\\ 0 & 1 & z_{23}\\ 1 & u_{12} & u_{13} \end{pmatrix}^{-p_3-q_2}$$
$$\times\det\begin{pmatrix} 1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 1 & u_{12} & u_{13} & 0\\ 0 & 1 & u_{23} & 0 \end{pmatrix}^{p_2+q_2} \det\begin{pmatrix} 1 & z_{12} & z_{13}\\ 1 & u_{12} & u_{13}\\ 0 & 1 & u_{23} \end{pmatrix}^{-p_2-q_3} \det\begin{pmatrix} 1 & z_{12} & z_{13} & z_{14}\\ 1 & u_{12} & u_{13} & 0\\ 0 & 1 & u_{23} & 0\\ 0 & 0 & 1 & 0 \end{pmatrix}^{p_1+q_3}.$$

Next we pass to the dual signature

$$\boxed{\mathbf{r} = \mathbf{p}^*}$$

(below $\mathbf{p}$ and $\mathbf{q}$ are rigidly linked by this restraint), and represent $L_{\mathbf{p},\mathbf{q}}$ in the form

$$L^{\mathbf{r}}_{\mathbf{q}}(U,Z) := L_{\mathbf{p},\mathbf{q}}(U,Z) = \Phi_1^{q_1-r_2}\Psi_1^{r_2-q_2}\Phi_2^{q_2-r_3}\Psi_2^{r_3-q_3}\dots\Phi_n^{q_n-r_{n+1}}. \tag{1.15}$$

1.9 Action of the overalgebra

Fix a signature $\mathbf{r}=\mathbf{p}^*\in\Lambda_{n+1}$ as in the previous subsection. Consider the space

$$\mathfrak{V}_{\mathbf{r}} := \bigoplus_{q_1,\dots,q_n:\, r_j\geqslant q_j\geqslant r_{j-1}} V^n_{\mathbf{q}}. \tag{1.16}$$

We can regard elements of this space as ‘expressions’

$$F(U,\mathbf{q})$$

of $n(n-1)/2$ continuous variables u_{kl}, where $1\leqslant k<l\leqslant n$, and of integer variables q_1, $\dots$, q_n. Integer variables range in the domain $r_j\geqslant q_j\geqslant r_{j+1}$. More precisely, for a fixed $\mathbf{q}$ the expression $F(U,\mathbf{q})$ is a polynomial in the variables u_{kl}, moreover, this polynomial is contained in $V_{\mathbf{q}}$.

Our family of forms $L_{\mathbf{p},\mathbf{q}}$ determines a duality between $V_{\mathbf{r}^*}$ and the space $\mathfrak{V}_{\mathbf{r}}$, hence we get a canonical identification of $V_{\mathbf{r}}$ and $\mathfrak{V}_{\mathbf{r}}$. Therefore, we have a canonically defined action of the Lie algebra $\mathfrak{gl}(n+1)$ in $\mathfrak{V}_{\mathbf{r}}$. We preserve the notation E_{kl} for operators in $V_{\mathbf{p}}$ and denote operators in $\mathfrak{V}_{\mathbf{r}}$ by F_{kl}. For $1\leqslant k,l\leqslant n$ the operators F^n_{kl} act in the space $\mathfrak{V}_{\mathbf{r}}$ by the first order differential operators in U with coefficients depending on $\mathbf{q}$ according the standard formulas (see Eqs. 1.4–1.6). For instance,

$$F_{kk} = -\sum_{i<k}u_{ik}\partial_{ik}+q_k+\sum_{j>k}u_{kj}\partial_{kj}, \tag{1.17}$$

$$F_{k(k+1)} = \partial_{k(k+1)}+\sum_{i<k}u_{ik}\partial_{i(k+1)}, \tag{1.18}$$

$$F_{(k+1)k} = \sum_{i<k}u_{i(k+1)}\partial_{ik}+(q_{k+1}-q_k)u_{k(k+1)}-u_{k(k+1)}\sum_{j>k}u_{kj}\partial_{kj} \tag{1.19}$$

$$+\sum_{m>k+1}\det\begin{pmatrix}u_{k(k+1)} & u_{km}\\ 1 & u_{(k+1)m}\end{pmatrix}\partial_{(k+1)m}, \tag{1.20}$$

where $\partial_{kl}:=\frac{\partial}{\partial u_{kl}}$.

The purpose of this work is to present formulas for the generators $F_{1(n+1)}$, $F_{(n+1)n}$. Together with Eqs. 1.17–1.20 they generate the Lie algebra $\mathfrak{gl}(n+1)$, formulas for the remaining generators $F_{j(n+1)}$, $F_{(n+1)j}$ consist of similar aggregates as below, but are longer.

Denote by $T_j^{\pm}$ the following difference operators

$$T_j^{\pm}F(\dots,q_j,\dots)=F(\dots,q_j\pm 1,\dots)$$

(the remaining variables do not change). We will write expressions, which are polynomial in u_{kl}, ∂_{kl}, linear in $T_j^{\pm}$ and rational in q_j. These expressions satisfy the commutation relations in $\mathfrak{gl}(n)$ and preserve the space $\mathfrak{V}_{\mathbf{r}}$.

For $1\leqslant k<l\leqslant n$ denote by $[k,l]$ the set

$$[k,l]:=\{k,k+1,\dots,l\}.$$

For a set

$$I:1\leqslant i_1<i_2<\dots<i_m$$

we write

$$I \lhd [k,l] \qquad \text{if } i_1 = k,\, i_m = l. \tag{1.21}$$

More generally, if $I \subset J \subset [1,n]$, we write $I \lhd J$ if the minimal (resp. maximal) element of I coincides with the minimal (resp. maximal) element of J.

For any $I \lhd [\alpha, \beta]$ we define the operator

$$R_I := R_{i_1 i_2} R_{i_2 i_3} \dots R_{i_{m-1} i_m} = R_{\alpha i_2} R_{i_2 i_3} \dots R_{i_{m-1}\beta},$$

where R_{ij} are the Zhelobenko operators. We also set

$$R_{kk} := 1.$$

Theorem 1.5 a) *The generator $F_{1(n+1)}$ acts by the formula*

$$F_{1(n+1)} = \sum_{m=1}^{n} A_m \left(\sum_{I \lhd [1,m]} \prod_{l \in [1,m] \setminus I} (q_m - q_l + l - m) \cdot R_I \right) T_m^-, \tag{1.22}$$

where coefficients A_m are given by

$$A_m = \frac{\prod_{j=m+1}^{n+1} (q_m - r_j + j - m - 1)}{\prod_{\alpha \ne m} (q_m - q_\alpha + \alpha - m)}.$$

b) *The generator $F_{(n+1)n}$ is given by the formula*

$$F_{(n+1)n} = -\sum_{m=1}^{n} B_m \left(\sum_{I \lhd [m,n]} \prod_{l \in [m,n] \setminus I} (q_m - q_l - m + l + 1) \cdot R_I \right) T_m^+, \tag{1.23}$$

where

$$B_m = \frac{\prod_{j=1}^{m} (q_m - r_j + j - m)}{\prod_{\alpha \ne m} (q_m - q_\alpha - m + \alpha)}.$$

1.10 Further Structure of the Paper

Section 2 contains preliminaries on holomorphic representations of $\mathrm{GL}(n, \mathbb{C})$. In Section 3, we verify the formula for the kernel $L(Z, U)$. Our main statement is equivalent to a verification of differential-difference equations for the kernel $L(Z, U)$, this is done in Section 4. In Section 5, we formulate a conjecture about analog of our statement for unitary representations.

2 Holomorphic Representations of GL(n, $\mathbb{C}$)

2.1 Realization in the Space of Functions on GL(n, $\mathbb{C}$)

For details and proof, see [30]. We say that a function $f(g)$ on $\mathrm{GL}(n, \mathbb{C})$ is a polynomial function if it can be expressed as a polynomial expression in matrix elements g_{ij} and $\det(g)^{-1}$. Denote the space of polynomial functions by $\mathbb{C}[\mathrm{GL}(n, \mathbb{C})]$. The group $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ acts in $\mathbb{C}[\mathrm{GL}(n, \mathbb{C})]$ by the left and right shifts':

$$\lambda_{left-right}(h_1, h_2) f(g) = f(h_1^{-1} g h_2).$$

This representation is a direct sum

$$\lambda \simeq \oplus_{\mathbf{p} \in \Lambda_n} \rho_{\mathbf{p}^*} \otimes \rho_{\mathbf{p}}.$$

A summand $V_{\mathbf{p}^*}\otimes V_{\mathbf{p}}$ is the $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$-cyclic span of the vector

$$\Delta[\mathbf{p}]=\prod_{j=1}^{n-1}\det[g]_{jj}^{p_j-p_{j+1}}\cdot(\det g)^{p_n}.$$

If $h_1\in N_n^-$ and $h_2\in N_n^+$, then $\lambda(h_1,h_2)\Delta[\mathbf{p}]=\Delta[\mathbf{p}]$.

Next, consider the space of polynomial functions invariant with respect to left shifts on elements of N_n^-

$$f(h^{-1}g)=f(g). \tag{2.1}$$

Equivalently, we consider the space of polynomial functions on $\mathbb{C}[N_n^-\setminus\mathrm{GL}(n,\mathbb{C})]$. Each $\rho_{\mathbf{p}^*}$ has a unique N_n^--invariant vector, therefore $\mathbb{C}[N_n^-\setminus\mathrm{GL}(n,\mathbb{C})]$ is a multiplicity free direct sum $\oplus_{\mathbf{p}}\rho_{\mathbf{p}}$.

Fix a signature $\mathbf{p}$ and consider the space of N_n^--invariant functions $H_{\mathbf{p}}$ such that for any diagonal matrix δ, we have

$$f(\delta g)=f(g)\prod_{j=1}^{n}\delta_j^{p_j}.$$

The group $\mathrm{GL}(n,\mathbb{C})$ acts in $H_{\mathbf{p}}$ by the right shifts,

$$\lambda_{right}(h)f(g)=f(gh). \tag{2.2}$$

This representation is irreducible and equivalent to $\rho_{\mathbf{p}}$. The vector $\Delta_{\mathbf{p}}$ is its highest weight vector.

2.2 Realization in the Space of Functions on N_n^+

An element $g\in\mathrm{GL}(n,\mathbb{C})$ satisfying the condition

$$\det[g]_{jj}\neq 0\qquad\text{for all } j$$

admits a unique Gauss decomposition

$$g=bZ,\qquad\text{where } b\in B_n^-,\ Z\in N_n^+.$$

Notice that

$$b_{jj}=\frac{\det[g]_{jj}}{\det[g]_{(j-1)(j-1)}}, \tag{2.3}$$

$$z_{ij}=\frac{\det\left[g\begin{pmatrix}1&2&\dots&i-1&j\\1&2&\dots&i&j\end{pmatrix}\right]}{\det\left[g\begin{pmatrix}1&2&\dots&i&j\\1&2&\dots&i&j\end{pmatrix}\right]}. \tag{2.4}$$

Restrict a function $f\in H_{\mathbf{p}}$ to the subgroup N_n^+. We get a polynomial in the variables z_{ij}, where $i<j$. By the Gauss decomposition, f is uniquely determined by this restriction. Therefore the space $V_{\mathbf{p}}$ can be regarded as a subspace of the space of polynomial in z_{ij}. The description of this space is given by the Zhelobenko Theorem 1.2.

Denote the factors in the Gauss decomposition of Zg in the following way

$$Zg=:b(Z,g)\cdot Z^{[g]},\qquad\text{where } b(Z,g)\in B_n^-,\ Z^{[g]}\in N_n^+, \tag{2.5}$$

Then

$$\rho_{\mathbf{p}}(g)f(Z)=f(Z^{[g]})\prod_{j=1}^{n}b_{jj}(Z,g)^{p_j}, \tag{2.6}$$

where $b_{jj}(Z,g)$ are the diagonal matrix elements of the lower triangular matrix $b(Z,g)$.

We present formulas for transformations $Z \mapsto Z^{[g(t)]}$, where $g(t)$ are standard one-parametric subgroups in $\mathrm{GL}(n,\mathbb{C})$.

1) Let $g(t)=\exp(1+tE_{kl})$, where $k<l$. Then we have transformation

$$\begin{aligned} z_{kl} &\mapsto z_{kl}+t; \\ z_{ml} &\mapsto z_{ml}+tz_{kl}, \qquad m<k; \\ z_{ij} &\mapsto z_{ij} \qquad \text{for other pairs } i,j. \end{aligned}$$

2) Let $g(t)=\exp(tE_{kk})$. We get

$$\begin{aligned} z_{kj} &\mapsto e^t z_{kj}, \qquad k<j; \\ z_{ik} &\mapsto e^{-t} z_{ik}, \qquad i<k; \\ z_{ij} &\mapsto z_{ij} \qquad \text{for other pairs } i,j. \end{aligned}$$

3) Let $g(t)=\exp(tE_{(k+1)k})$. Then

$$\begin{aligned} z_{ik} &\mapsto z_{ik}+tz_{i(k+1)}, \qquad i<k; \\ z_{kj} &\mapsto \frac{z_{kj}}{1+tz_{k(k+1)}}, \qquad j>k; \\ z_{(k+1)m} &\mapsto z_{(k+1)m}+t\det\begin{pmatrix} z_{k(k+1)} & z_{km} \\ 1 & z_{(k+1)m}\end{pmatrix}, \qquad m>k+1; \\ z_{ij} &\mapsto z_{ij} \qquad \text{for other pairs } i,j. \end{aligned}$$

4) Let $g(t)=\exp(tE_{1n})$, then

$$\begin{aligned} z_{1n} &\mapsto z_{1n}+t, \\ z_{ij} &\mapsto z_{ij} \qquad \text{for other pairs } i,j. \end{aligned}$$

Expressions (1.4)–(1.6) for the action of the Lie algebra $\mathfrak{gl}(n)$ easily follows from these formulas. Also, we get

$$E_{1n}=\partial_{1n}. \tag{2.7}$$

2.3 The Jacobian

Lemma 2.1 *The complex Jacobian $J(Z,g)$ of a transformation $Z\mapsto Z^{[g]}$ is*

$$J(Z,g)=\prod_{j=1}^{n-1}\det\big([Zg]_{jj}\big)^{-2}=\prod_{j=1}^{n-1}\det\big([Z]_{jn}[Z]_{nj}\big)^{-2}. \tag{2.8}$$

Proof We say that a function

$$c: N_n^+\times \mathrm{GL}(n,\mathbb{C})\to\mathbb{C}$$

satisfies the *chain identity* if

$$c(Z,gh)=c(Z,g)c(Z^g,h). \tag{2.9}$$

The Jacobian satisfies this identity. Let us verify that the right hand side of Eq. 2.8 also satisfies it. It suffices to consider one factor $\det[Zg]_{jj}^{-2}$.

Let us evaluate $\det[Z^{[g]}h]_{jj}$. Represent Z, g, h etc. as block $j+(n-j)$-matrices,

$$Z=\begin{pmatrix}Z_{11}&Z_{12}\\0&Z_{22}\end{pmatrix},\qquad g:=\begin{pmatrix}G_{11}&G_{12}\\G_{21}&G_{22}\end{pmatrix},\qquad h:=\begin{pmatrix}H_{11}&H_{12}\\H_{21}&G_{22}\end{pmatrix}. \tag{2.10}$$

Represent Zg as $C^{-1}U$, where $U\in N_n^+$, $C\in B_n^-$. Then

$$\begin{pmatrix}C_{11}&0\\C_{21}&C_{22}\end{pmatrix}\begin{pmatrix}Z_{11}G_{11}+Z_{12}G_{21}&Z_{11}G_{12}+Z_{12}G_{22}\\Z_{22}G_{21}&Z_{22}G_{22}\end{pmatrix}=\begin{pmatrix}U_{11}&U_{12}\\0&U_{22}\end{pmatrix}.$$

Since $\det U_{11}=1$, we have

$$\det C_{11}=\det(Z_{11}G_{11}+Z_{12}G_{21})^{-1}.$$

Next,

$$\begin{aligned}\det[Z^{[g]}h]_{jj}&=\det[Uh]_{jj}=\det[U_{11}H_{11}+U_{12}H_{21}]\\&=\det C_{11}\cdot\det\bigl((Z_{11}G_{11}+Z_{12}G_{21})H_{11}+(Z_{11}G_{12}+Z_{12}G_{22})H_{21}\bigr)\\&=\frac{\det\bigl(Z_{11}(G_{11}H_{11}+G_{12}H_{21})+Z_{12}(G_{21}H_{21}+G)\bigr)}{\det(Z_{11}G_{11}+Z_{12}G_{21})}=\frac{\det[Zgh]_{jj}}{\det[Zg]_{jj}},\end{aligned}$$

this proves the desired statement.

Since the both sides of Eq. 2.8 satisfy the chain identity (2.9), it suffices to verify the identities for a system of generators of GL$(n,\mathbb{C})$. Thus we can verify (2.8) for elements one-parametric subgroups $\exp(tE_{k(k+1)})$, $\exp(tE_{kk})$, $\exp(tE_{(k+1)k})$. Formulas for the corresponding transformations are present in the previous subsection. Only the case $\exp(tE_{(k+1)k})$ requires a calculation. In this case, the Jacobi matrix is triangular. Its diagonal values are

- $(1+tz_{k(k+1)})^{-2}$ for $z_{k(k+1)}$;
- $(1+tz_{k(k+1)})^{-1}$ for z_{km}, where $m>k+1$;
- $(1+tz_{k(k+1)})$ for $z_{(k+1)m}$, where $m>k+1$.

The Jacobian $(1+tz_{k(k+1)})^{-2}$.

On the other hand, in the product (2.8) the j-th factor is $(1+tz_{k(k+1)})^{-2}$, other factors are 1. □

2.4 Proof of Proposition 1.1

The U(n)*-invariance of the* $L^2(N_n^+,\mu_{\mathbf{p}})$*-inner product.* We must check the identity

$$\int_{N_n^+}|(\rho_{\mathbf{p}}(g)\varphi)(Z)|^2\,d\mu_{\mathbf{p}}(Z)=\int_{N_n^+}|\varphi(Z)|^2\,d\mu_{\mathbf{p}}(Z). \tag{2.11}$$

Fix $g\in$ U(n). Substituting $Z=U^{[g]}$ to the right-hand side, we get

$$\int_{N_n^+}\bigl|\varphi(U^{[g]})\bigr|^2\prod_{j=1}^{n-1}\det\Bigl([U^{[g]}]_{jn}\bigl(U^{[g]}]_{jn}\bigr)^*\Bigr)^{-(p_j-p_{j+1})-2}\,d\bigl(\dot{U^{[g]}}\bigr).$$

In notation (2.10) we have

$$Ug=\begin{pmatrix}U_{11}G_{11}+U_{12}G_{21}&U_{11}G_{12}+U_{12}G_{22}\\ *&*\end{pmatrix}.$$

Therefore $[U^{[g]}]_{jn}$ has the form

$$\big(C_{11}(U_{11}G_{11}+U_{12}G_{21})\ \ C_{11}(U_{11}G_{12}+U_{12}G_{22})\big)$$

where C_{11} is a lower-triangular matrix, and

$$\det\big(C_{11}(U_{11}G_{11}+U_{12}G_{21})\big)=1.$$

Hence

$$\begin{aligned}&\det\Big([U^{[g]}]_{jn}\big(U^{[g]}]_{jn}\big)^*\Big)\\ =\ &\det\big(\big([Ug]_{jn}\big([Ug]_{jn}\big)^*\big)\det(U_{11}G_{11}+U_{12}G_{21})^{-1}\det(U_{11}G_{11}+U_{12}G_{21})^{*-1}.\end{aligned}$$

Next,

$$\begin{aligned}&\big([U]_{jn}\big([U]_{jn}\big)^*\\ &=(U_{11}G_{11}+U_{12}G_{21})(G_{11}^*U_{11}^*+G_{21}^*U_{12}^*)+(U_{11}G_{12}+U_{12}G_{22})(G_{12}^*U_{11}^*+G_{22}^*U_{12}^*)\\ &=U_{11}(G_{11}G_{11}^*+G_{12}G_{12}^*)U_{11}^*+U_{11}(G_{11}G_{21}^*+G_{12}G_{22}^*)U_{12}^*\\ &\quad+U_{12}(G_{21}G_{11}^*+G_{22}G_{12}^*)U_{11}^*+U_{12}(G_{21}G_{21}+G_{22}G_{22}^*)U_{22}^*\end{aligned}$$

The matrix g is unitary, $gg^*=1$. Therefore the last expression equals to

$$U_{11}\cdot 1\cdot U_{11}^*+U_{11}\cdot 0\cdot U_{12}^*+U_{12}\cdot 0\cdot U_{11}^*+U_{12}\cdot 1\cdot U_{12}^*.$$

Thus

$$\det\Big([U^{[g]}]_{jn}\big(U^{[g]}]_{jn}\big)^*\Big)=\det\big(\big([U]_{jn}\big([U]_{jn}\big)^*\big)\big|\det(U_{11}G_{11}+U_{12}G_{21})\big|^{-2}.$$

Keeping in mind

$$d\big(\dot{U}^{[g]}\big)=\big|\det(U_{11}G_{11}+U_{12}G_{21})\big|^{-4}\,d\dot{Z},$$

we get

$$\begin{aligned}&\prod_{j=1}^{n-1}\det\Big([U^{[g]}]_{jn}\big(U^{[g]}]_{jn}\big)^*\Big)^{-(p_j-p_{j+1})-2}\ d\big(\dot{U}^{[g]}\big)\\ &=\sigma(U,g)\prod_{j=1}^{n-1}\det\big([U]_{jn}\big(U]_{jn}\big)^*\big)\ d\dot{U},\end{aligned}$$

where

$$\sigma(U,g)=\Bigg|\prod_{j=1}^{n-1}\det([Ug]_{jj})^{(p_j-p_{j+1})}\Bigg|^2.$$

Let us represent Ug in the form DW, where $D\in B_n^-$, $g\in N_n^+$. Then

$$\det([Ug]_{kk})\ =\ \prod_{i=1}^{k}b_{ii}(U,g),\quad k\leqslant n-1,\tag{2.12}$$

$$\det(g)\ =\ \prod_{i=1}^{n}b_{ii}(U,g),\tag{2.13}$$

here $b_{ii}(U,g)$ are the diagonal elements of Ug. Therefore,

$$\sigma(U,g)=\Bigg|\prod_{j=1}^{n-1}b_{jj}(U,g)^{p_j-p_{j+1}}\Bigg|^2$$

Since $g \in \mathrm{U}(n)$, we have $|\det(g)| = 1$. Keeping in mind (2.13), we come to

$$\sigma(U,g) = \left|\prod_{j=1}^{n} b_{jj}(U,g)^{p_j}\right|^2.$$

Therefore the right-hand side of Eq. 2.11 equals

$$\int_{N_n^+} \left| f(U^{[g]}) \prod_{j=1}^{n} b_{jj}(U,g)^{p_j}\right|^2 d\dot{U},$$

i.e., coincides with the left-hand side.

Thus the group $\mathrm{U}(n)$ acts in $L^2(N_n^+, d\mu_{\mathbf{p}})$ by the unitary operators (2.6).

Intersection of the space of L^2 *and the space of polynomials.* Denote this intersection by W.

First, W is $\mathrm{U}(n)$-invariant. Indeed, for $\psi \in W$ we have $\rho_{\mathbf{p}}(g)\psi \in L^2$. By definition, $\rho_{\mathbf{p}}(g)\psi$ is a rational holomorphic function. Represent it as an irreducible fraction $\alpha(Z)/\beta(Z)$. Let Z_0 be a non-singular point of the manifold $\beta(Z) = 0$, let $\mathcal{O}$ be neighborhood of Z_0. It is easy to see that

$$\int_{\mathcal{O}} |\rho_{\mathbf{p}}(g)\psi|^2 \, d\dot{Z} = \infty.$$

Therefore $\rho_{\mathbf{p}}(g)\psi$ is a polynomial.

Second, the space W is non-zero. For instance it contains a constant function. Indeed, $\int_{N_n^+} d\mu_{\mathbf{p}}$ was evaluated in [24] and it is finite.

The third, W is finite-dimensional. Indeed, our measure has the form $|r(Z)|^{-2}\, d\dot{Z}$, where $r(Z,\overline{Z})$ is a polynomial in $Z, \overline{Z}$. A polynomial $\varphi \in W$ satisfies the condition

$$\int_{N_n^+} \left| \frac{\varphi(Z)}{r(Z,\overline{Z})}\right|^2 d\dot{Z} < \infty.$$

Clearly, degree of φ is uniformly bounded.

Next, the operators (2.6) determine a unitary representation in $L^2(N_n^+)$ and this representation is an element of the principal non-unitary series (see, e.g., [6] or [30], Addendum). But a representation of the principal series can not have more than one finite-dimensional subrepresentation.

3 The Formula for the Kernel

In this section we prove Proposition 1.4.

3.1 Functions $\Phi[\dots]$, $\Psi[\dots]$

Consider the matrices Z, Z^{cut}, U, U^{ext} as in Section 1.8. We use notation (1.2) for their sub-matrices. Let $\alpha = 1,\dots,n-1$. Let I, J be sets of integers,

$$I:\ 0 < i_1 < \dots < i_{n+1-\alpha} \leqslant n+1, \qquad J:\ 0 < j_1 < \dots < j_\alpha \leqslant n.$$

Denote

$$\Phi_\alpha[I;J] = \Phi_\alpha[i_1,\dots,i_{n+1-\alpha}; j_1,\dots j_\alpha] := \det\begin{pmatrix} [Z(I)] \\ [U^{ext}(J)] \end{pmatrix}. \tag{3.1}$$

Let I, J be sets of integers,

$$I:\ 0 < i_1 < \cdots < i_{n-\alpha} \leqslant n+1, \qquad J:\ 0 < j_1 < \cdots < j_\alpha \leqslant n.$$

Denote

$$\Psi_\alpha[I;J] = \Psi_\alpha[i_1,\dots,i_{n-\alpha}; j_1,\dots j_\alpha] := \det\begin{pmatrix}[Z^{cut}(I)]\\ [U(J)]\end{pmatrix}. \tag{3.2}$$

We have two collections of variables z_{ij}, $u_{i'j'}$. Zhelobenko operators R_{kl} in z and u we denote by

$$R^z_{kl}, \qquad R^u_{kl}.$$

Lemma 3.1

$$R^z_{kl}\Phi_\alpha[I,J] = \begin{cases} 0, & \textit{if } k \notin I;\\ 0, & \textit{if } k\in I,\ l\in I;\\ (-1)^{\theta(I,k,l)}\Phi_\alpha[I^\circ;J], & \textit{if } k\in I,\ l\notin I. \end{cases} \tag{3.3}$$

where I° is obtained from I by replacing of k by l, and the corresponding change of order. If $k = i_s$ and $i_t < l < i_{t+1}$, then

$$I^\circ = (i_1,\dots,i_{s-1},i_{s+1},\dots,i_{t-1},l,i_{t+1},\dots,i_{n+1-\alpha})$$

and

$$\theta(I,k,l) = t-s.$$

The same properties (with obvious modifications) hold for

$$R^u_{kl}\Phi_\alpha[I;J], \qquad R^z_{kl}\Psi_\alpha[I;J], \qquad R^u_{kl}\Psi_\alpha[I;J].$$

Proof Recall that

$$R^z_{kl} = \partial_{kl} + \sum_{j>l} z_{lj}\partial_{kj}.$$

If $k\notin I$, then $\Phi_\alpha[I,J]$ does not depend on variables z_{km}. Therefore we get 0. If $k\in I$, these variables are present only in one row. Application of R^z_{kl} is equivalent to a change of the row

$$\begin{pmatrix}0 & \dots & 0 & 1 & z_{k(k+1)} & \dots & z_{kl} & z_{k(l+1)} & \dots & z_{k(n+1)}\end{pmatrix}$$

of the matrix $\begin{pmatrix}[Z(I)]\\ [U^{ext}(J)]\end{pmatrix}$ by the row

$$\begin{pmatrix}0 & \dots & 0 & 0 & 0 & \dots & 1 & z_{l(l+1)} & \dots & z_{k(n+1)}\end{pmatrix}.$$

If $l\in I$, then this row is present in the initial matrix, and again $\det = 0$. Otherwise, we come to $\Phi_\alpha[I^\circ;J]$ up to the order of rows. □

Corollary 3.2 *For any k, l, α,*

$$(R^z_{kl})^2\Phi_\alpha[I;J]=0, \quad (R^u_{kl})^2\Phi_\alpha[I;J]=0,$$
$$(R^z_{kl})^2\Psi_\alpha[I;J]=0, \quad (R^u_{kl})^2\Psi_\alpha[I;J]=0.$$

Corollary 3.3

$$R^z_{kl}R^z_{lm}\Phi_\alpha[I;J] = \begin{cases} -R^z_{km}\Phi_\alpha[I;J], & \textit{if } k\in I,\ l\notin I,\ m\notin I;\\ 0, & \textit{otherwise.}\end{cases} \tag{3.4}$$

The same property holds for operators R^u and for $\Psi_\alpha[I;J]$.

3.2 Verification of the Zhelobenko Conditions

Lemma 3.4 $L_{\mathbf{p},\mathbf{q}} \in V_{\mathbf{p}} \otimes V_{\mathbf{q}}$.

Proof By the interlacing conditions, $L_{\mathbf{p},\mathbf{q}}$ is a polynomial. We must verify the identities

$$(R^z_{k(k+1)})^{p_k-p_{k+1}+1}L=0, \qquad (R^u_{k(k+1)})^{q_k-q_{k+1}+1}L=0. \tag{3.5}$$

To be definite verify the first equality,

$$\begin{aligned}&(R^z_{k(k+1)})^{p_k-p_{k+1}+1}L_{\mathbf{p},\mathbf{q}}\\ &= (R^z_{k(k+1)})^{p_k-p_{k+1}+1}\left(\Psi_{n-k}^{-p_{k+1}-q_{n+1-k}}\Phi_{n+1-k}^{p_k+q_{n+1-k}}\cdot\Big\{\text{remaining factors}\Big\}\right)\\ &= (R^z_{k(k+1)})^{p_k-p_{k+1}+1}\left(\Psi_{n-k}^{-p_{k+1}-q_{n+1-k}}\Phi_{n+1-k}^{p_k+q_{n+1-k}}\right)\cdot\Big\{\text{remaining factors}\Big\}\end{aligned}$$

The sum of exponents (the both exponents are positive) is

$$\big(-p_{k+1}-q_{n+1-k}\big)+\big(p_k+q_{n+1-k}\big)=p_k-p_{k+1},$$

and we obtain the desired 0. If $k=n$ the factor Ψ_{n-k} is absent, we have

$$(R^z_{n(n+1)})^{p_n-p_{n+1}+1}\left(\Phi_1^{p_n+q_1}\right)\cdot\Big\{\text{remaining factors}\Big\}.$$

By the interlacing condition $-p_{n+1}\geqslant q_1$ and we again obtain 0. □

3.3 Invariance of the Kernel

Consider matrices Φ_α, Ψ_α defined by Eq. 1.13.

Lemma 3.5 *Let* $g\in \mathrm{GL}(n,\mathbb{C})$. *Denote* $\widetilde{g}=\begin{pmatrix} g&0\\0&1\end{pmatrix}\in \mathrm{GL}(n+1,\mathbb{C})$. *Then*

$$\Phi(Z^{[\widetilde{g}]},U^{[g]})=\Phi(Z,U)\det(g)\det\big([Z\widetilde{g}]_{(n+1-\alpha)(n+1-\alpha)}\big)^{-1}\det\big([Ug]_{\alpha\alpha}\big)^{-1}. \tag{3.6}$$

$$\Psi(Z^{[\widetilde{g}]},U^{[g]})=\Psi(Z,U)\det(g)\det\big([Z\widetilde{g}]_{(n-\alpha)(n-\alpha)}\big)^{-1}\det\big([Ug]_{\alpha\alpha}\big)^{-1}. \tag{3.7}$$

Proof We write the Gauss decompositions of $Z\widetilde{g}$, Ug,

$$Z\widetilde{g}=BP, \qquad Ug=CQ.$$

Then the Gauss decomposition of U^{ext} (see Eq. 1.11) is

$$U^{ext}\widetilde{g}=\begin{pmatrix} C&0\\0&1\end{pmatrix}\begin{pmatrix} Q&0\\0&1\end{pmatrix},$$

denote factors in the right-hand side by $\widetilde{C}$, $\widetilde{Q}$. Then

$$\begin{aligned}\begin{pmatrix}[Z^{[\widetilde{g}]}]_{(n+1-\alpha)(n+1)}\\ [(U^{ext})^{[\widetilde{g}]}]_{\alpha(n+1)}\end{pmatrix} &= \begin{pmatrix}[P]_{(n+1-\alpha)(n+1)}\\ [\widetilde{Q}]_{\alpha(n+1)}\end{pmatrix}\\ &= \begin{pmatrix}([\widetilde{B}]_{(n+1-\alpha)(n+1-\alpha)})^{-1} & 0\\ 0 & ([\widetilde{C}]_{\alpha\alpha})^{-1}\end{pmatrix}\begin{pmatrix}[Z]_{(n+1-\alpha)(n+1)}\\ [\widetilde{U}]_{\alpha(n+1)}\end{pmatrix}\cdot g\end{aligned}$$

We pass to determinants in the left-hand side and the right-hand side. Keeping in mind

$$\det\,[\widetilde{B}]_{(n+1-\alpha)(n+1-\alpha)} = \det[\widetilde{Z}\widetilde{g}]_{(n+1-\alpha)(n+1-\alpha)},$$
$$\det\,[\widetilde{C}]_{\alpha\alpha} = \det[U^{ext}\widetilde{g}]_{\alpha\alpha} = \det[Ug]_{\alpha\alpha},$$

we come to Eq. 3.6. Proof of Eq. 3.7 is similar. □

Proof of Proposition 1.4. Applying the lemma and Eq. 2.12, we get

$$L_{\mathbf{p},\mathbf{q}}(Z^{[\widetilde{g}]}, U^{[g]}) = L_{\mathbf{p},\mathbf{q}}(Z, U) \prod_{j=1}^{n} b_{jj}(Z, g)^{-q_1-p_j} \prod b_{jj}(U, g)^{-p_n-q_j} \det(g)^{p_1+q_n}.$$

Since

$$b_{(n+1)(n+1)}(Z, \widetilde{g}) = 1, \qquad \det(g) = \prod_{j=1}^{n} b_{jj}(Z, g) = \prod_{j=1}^{n} b_{jj}(U, g),$$

we come to

$$L_{\mathbf{p},\mathbf{q}}(Z^{[\widetilde{g}]}, U^{[g]}) = L_{\mathbf{p},\mathbf{q}}(Z, U) \prod_{j=1}^{n} b_{jj}(Z, g)^{-p_j} \prod b_{jj}(U, g)^{-q_j}.$$

Therefore,

$$(\rho_{\mathbf{p}}(\widetilde{g}) \otimes \rho_{\mathbf{q}}(g))L_{\mathbf{p},\mathbf{q}} = L_{\mathbf{p},\mathbf{q}}.$$

Thus $L_{\mathbf{p},\mathbf{q}}$ is a non-zero $\mathrm{GL}(n, \mathbb{C})$-invariant vector in $V_{\mathbf{p}} \otimes V_{\mathbf{q}}$. We also know that such a vector is unique up to a scalar factor. □

4 The Calculation

Below $\mathbf{q} \in \Lambda_n$, $\mathbf{r} = \mathbf{p}^* \in \Lambda_{n+1}$ are signatures. The signatures $\mathbf{r}$ and $\mathbf{q}$ are interlacing. The kernel L is the same as above (1.14),

$$L := L^{\mathbf{r}}_{\mathbf{q}}(Z, U) = \Phi_1^{q_1-r_2}\Psi_1^{r_2-q_2}\Phi_2^{q_2-r_3}\Psi_2^{r_3-q_3}\ldots\Phi_n^{q_n-r_{n+1}}.$$

Below

$$R_{kl} := R^u_{kl}.$$

We must verify the differential-difference equations

$$E_{1(n+1)}L = F_{1(n+1)}L, \qquad E_{(n+1)n}L = F_{(n+1)n}L,$$

$F_{1(n+1)}$, $F_{(n+1)n}$ are given by Eqs. 1.22–1.22. Evaluations of the left-hand sides is easy, it is contained in the next subsection. Evaluation of the right-hand sides is tricky and occupies the remaining part of the section.

4.1 Evaluation of $E_{1(n+1)}L$

Lemma 4.1

$$E_{1(n+1)}L = \left(-\frac{q_1 - r_2}{\Phi_1} + \sum_{\alpha=2}^{n} \frac{(q_\alpha - r_{\alpha+1})R_{1\alpha}\Psi_{\alpha-1}}{\Phi_\alpha}\right) \cdot L. \tag{4.1}$$

Remark For $\alpha = 1$ we immediately get $\frac{\partial\Phi_1}{\partial z_{1(n+1)}} = -1$. On the other hand, taking formally $\alpha = 1$ in the sum in Eq. 4.1 we get $R_{11}\Psi_0/\Phi_1$. At first glance, we must assume that $\Psi_0 = 1$

and that R_{11} is the identical operator. Under this assumption, $R_{11}\Psi_0/\Phi_1 = 1/\Phi_1$. However, this gives an incorrect sign.

In the following two proofs we need manipulations with determinants. To avoid huge matrices or compact notation, which are difficult for reading, we expose calculations for matrices having a minimal size that allows to visualize picture.

Proof We have (see Eq. 2.7)

$$E_{1(n+1)}L = \frac{\partial L}{\partial z_{1(n+1)}} = \left(\sum_{\alpha=1}^{n}(q_\alpha - r_{\alpha+1})\frac{\partial \Phi_\alpha}{\partial z_{1(n+1)}}\cdot \Phi_\alpha^{-1}\right)\cdot L.$$

Take $n=3$ and $\alpha=2$,

$$\Phi_2 = \det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 1 & u_{12} & u_{13} & 0\\ 0 & 1 & u_{23} & 0\end{pmatrix}. \tag{4.2}$$

Then

$$\frac{\partial\Phi_2}{\partial z_{14}} = -\det\begin{pmatrix}0 & 1 & z_{23}\\ 1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\end{pmatrix} = \det\begin{pmatrix}1 & z_{23}\\ 1 & u_{23}\end{pmatrix} = \det\begin{pmatrix}1 & z_{12} & z_{13}\\ 0 & 1 & z_{23}\\ 0 & 1 & u_{23}\end{pmatrix} = R_{12}\Psi_1.$$

□

Lemma 4.2

$$E_{(n+1)n}L = \left(z_{n(n+1)}(r_1-q_1) - \sum_{j=1}^{n-1}\frac{(r_{\alpha+1}-q_{\alpha+1})R_{(\alpha+1)n}\Phi_{\alpha+1}}{\Psi_\alpha}\right)\cdot L.$$

Proof Split $E_{(n+1)n}$ as a sum

$$E_{(n+1)n} = D + (r_1 - r_2)z_{n(n+1)}, \qquad D := \sum_{j=1}^{n-1} z_{j(n+1)}\partial_{jn} - z_{n(n+1)}^2\partial_{n(n+1)}.$$

Obviously,

$$E_{(n+1)n}L = \left((r_1-r_2)z_{n(n+1)} + \sum_{\alpha=1}^{n}\frac{(q_\alpha - r_{\alpha+1})D\Phi_\alpha}{\Phi_\alpha} + \sum_{\beta=1}^{n-1}\frac{(r_{\beta+1}-q_{\beta+1})D\Psi_\beta}{\Psi_\beta}\right)\cdot L$$

Let us evaluate all $D\Phi_\alpha$, $D\Psi_\alpha$.

1) $D\Phi_\alpha = 0$ for all $\alpha \neq 1$. Take $n=3$, $\alpha=2$,

$$D\Phi_2 = D\det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 1 & u_{12} & u_{13} & 0\\ 0 & 1 & u_{23} & 0\end{pmatrix} = \det\begin{pmatrix}1 & z_{12} & z_{14} & z_{14}\\ 0 & 1 & z_{24} & z_{24}\\ 1 & u_{12} & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = 0.$$

Variables z_{jn} (in our case z_{j4}) are present only in the next-to-last column. We apply the operator D to this column and come to a matrix with coinciding columns.

2) $D\Phi_1 = -z_{n(n+1)}\Phi_1$. Take $n = 3$,

$$D\Phi_1 = D\det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 1 & u_{12} & u_{13} & 0\end{pmatrix} = \det\begin{pmatrix}1 & z_{12} & z_{14} & z_{14}\\ 0 & 1 & z_{24} & z_{24}\\ 0 & 0 & 0 & z_{34}\\ 1 & u_{12} & 0 & 0\end{pmatrix} + z_{34}^2\det\begin{pmatrix}1 & z_{12} & z_{13}\\ 0 & 1 & z_{23}\\ 1 & u_{12} & u_{13}\end{pmatrix}$$

$$= -z_{34}\det\begin{pmatrix}1 & z_{12} & z_{14}\\ 0 & 1 & z_{24}\\ 1 & u_{12} & 0\end{pmatrix} + z_{34}^2\det\begin{pmatrix}1 & z_{12} & z_{13}\\ 0 & 1 & z_{23}\\ 1 & u_{12} & u_{13}\end{pmatrix} = -z_{34}\det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & 0\\ 1 & u_{12} & u_{13} & 0\end{pmatrix}$$

$$-z_{34}\det\begin{pmatrix}1 & z_{12} & z_{13} & 0\\ 0 & 1 & z_{23} & 0\\ 0 & 0 & 1 & z_{34}\\ 1 & u_{12} & u_{13} & 0\end{pmatrix} = -z_{34}\det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 1 & u_{12} & u_{13} & 0\end{pmatrix} = -z_{34}\Phi_1.$$

3) $D\Psi_\alpha = -R_{(\alpha+1)n}\Phi_{\alpha+1}$. Take $n = 4$, $\alpha = 1$.

$$D\Psi_1 = D\det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{24} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 1 & u_{12} & u_{13} & u_{14}\end{pmatrix} = \det\begin{pmatrix}1 & z_{12} & z_{13} & z_{15}\\ 0 & 1 & z_{24} & z_{25}\\ 0 & 0 & 1 & z_{35}\\ 1 & u_{12} & u_{13} & 0\end{pmatrix}$$

$$= -\det\begin{pmatrix}1 & z_{12} & z_{13} & z_{14} & z_{15}\\ 0 & 1 & z_{24} & z_{24} & z_{25}\\ 0 & 0 & 1 & z_{25} & z_{35}\\ 1 & u_{12} & u_{13} & u_{14} & 0\\ 0 & 0 & 0 & 1 & 0\end{pmatrix} = -R_{24}\Phi_2.$$

Thus we know all $D\Phi_\alpha$, $D\Psi_\alpha$. This gives us the desired statement. □

4.2 Quadratic relations

Now we wish to write a family of quadratic relations between different functions of type $R_{kl}\Phi_\alpha$, $R_{kl}\Psi_\beta$. We introduce notation

$$R_{kk}\Phi_{k-1} := +\Phi_{k-1}, \qquad \boxed{R_{kk}\Psi_{k-1} = -\Psi_{k-1}, \quad R_{11}\Psi_0 = -\Psi_0 = 1}. \tag{4.3}$$

We do not introduce operators R_{kk} and $R_{kk}\Phi_{k-1}$, $R_{kk}\Psi_{k-1}$ are only symbols used in formulas. In particular, formula Eq. 4.1 now can be written the form

$$E_{1(n+1)}L = L\cdot\sum_{\alpha=1}^{n}\frac{(q_\alpha - r_{\alpha+1})R_{1\alpha}\Psi_{\alpha-1}}{\Phi_\alpha}$$

without a term with abnormal sign. Below it allows to avoid numerous anomalies in the formulas, duplications of formulas and branchings of calculations.

Lemma 4.3 *a) Let $m < \alpha \leqslant \beta$,*

$$\sum_{j=\alpha+1}^{\beta+1} R_{mj}\Phi_\alpha\cdot R_{j(\beta+1)}\Psi_\beta = -\Phi_\alpha\cdot R_{m(\beta+1)}\Psi_\beta + \Phi_{\beta+1}\cdot R_{m\alpha}\Psi_{\alpha-1}. \tag{4.4}$$

b) Let $m < \alpha < \beta$. Then

$$\sum_{j=\alpha+1}^{\beta+1} R_{mj}\Psi_\alpha \cdot R_{j(\beta+1)}\Psi_\beta = -\Psi_\alpha \cdot R_{m(\beta+1)}\Psi_\beta. \tag{4.5}$$

c) Let $m < \alpha < \beta$. Then

$$\sum_{j=\alpha+1}^{\beta} R_{mj}\Phi_\alpha \cdot R_{j(\beta+1)}\Phi_\beta = -\Phi_\alpha \cdot R_{m(\beta+1)} - R_{m(\beta+1)}\Phi_\alpha \cdot \Phi_\beta. \tag{4.6}$$

Proof Denote by ξ the row $\begin{pmatrix}0 & 0 & \dots & 1\end{pmatrix}$ of the length $(n+1)$. We write a matrix $\Delta := \begin{pmatrix} Z \\ U^{ext} \\ \xi \end{pmatrix}$. Let us enumerate z-rows of these matrix by $1, 2, 3, \dots$ and u-rows by marks $\overline{1}, \overline{2}, \overline{3}, \dots$. Consider the following minors of Δ:

$$\begin{aligned} \Phi_\alpha &= \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{\alpha}]; \\ R_{m(\beta+1)}\Psi_\beta &= (-1)^{\beta-m} \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \beta+1; \xi]; \end{aligned} \tag{4.7}$$

Notice that we ξ in the last row allows to replace the last determinant by

$$\det\Big[\Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \beta+1; \xi]\Big]_{nn}.$$

and this proves (4.7).

We wish to apply one of the Plücker identities to the product of these minors (see, e.g., [4], Section 9.1). We take $\overline{m}$ from the collection $(1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{\alpha})$, exchange it with an element of the collection $(1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \beta+1; \xi)$ and consider the product of the corresponding minors. Next, we take the sum of all such products of Δ. In this way, we obtain

$$\det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{\alpha}] \cdot \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta+1}; \xi] \tag{4.8}$$
$$= \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, 1, \overline{m+1}, \dots, \overline{\alpha}] \tag{4.9}$$
$$\times \det \Delta[\overline{m}, 2, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta+1}; \xi] + \dots \tag{4.10}$$
$$+ \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{1}, \overline{m+1}, \dots, \overline{\alpha}] \tag{4.11}$$
$$\times \det \Delta[1, \dots, n-\beta; \overline{m}, \overline{2}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta+1}; \xi] + \dots \tag{4.12}$$
$$+ \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{\alpha+1}, \overline{m+1}, \dots, \overline{\alpha}] \tag{4.13}$$
$$\times \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\alpha}, \overline{m}, \overline{\alpha+2}, \dots, \beta+1; \xi] + \dots \tag{4.14}$$
$$+ \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{\beta}, \overline{m+1}, \dots, \overline{\alpha}] \tag{4.15}$$
$$\times \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta-1}, \overline{m}, \beta+1; \xi] \tag{4.16}$$
$$+ \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{\beta+1}, \overline{m+1}, \dots, \overline{\alpha}] \tag{4.17}$$
$$\times \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta}, \overline{m}; \xi] \tag{4.18}$$
$$+ \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \xi, \overline{m+1}, \dots, \overline{\alpha}] \tag{4.19}$$
$$\times \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta+1}; \overline{m}]. \tag{4.20}$$

Look to the summands of our expression up to signs.

1) After exchanging of $\overline{m}$ with $1, \dots, n-\beta$ we get a matrix $\Delta[\dots]$ with coinciding rows and therefore we obtain 0, see Eq. 4.9.
2) After exchanging of $\overline{m}$ with $\overline{1}, \dots, \alpha$ we again obtain 0, see Eq. 4.11.

3) The sum (4.13)–(4.16) corresponds to the sum in the left hand side of the the desired identity (4.4) with $j < \beta + 1$.
4) The summand (4.17)–(4.18) corresponds to the last term of the sum.
5) The summand (4.19)–(4.20) corresponds to the expression $\Phi_{\beta+1} \cdot R_{m\alpha}\Psi_{\alpha-1}$.
6) The term (4.8) is $\Phi_\alpha \cdot R_{m(\beta+1)}\Psi_\beta$.

Next, let us watch signs. Denote by $(i_1 i_2 \dots i_k)$ a cycle in a substitution. Denote by $\sigma(\cdot)$ the parity of a substitution.

Examine summands (4.13)–(4.16). For $j = \alpha+1, \dots, \beta+1$, we have

$$\det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{j}, \overline{m+1}, \dots, \overline{\alpha}] = \\ = \sigma\big(j\ (m+1)\ \dots\ \alpha\big) R_{mj}\Phi_\alpha = (-1)^{\alpha-m} R_{mj}\Phi_\alpha$$

For $j = \alpha+1, \dots, \beta$ we have

$$\det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{j-1}, \overline{m}, \overline{j+1}, \dots, \beta+1; \xi] \\ = \sigma\big(m \dots (j-1)\big) \cdot \sigma\big((j+1) \dots (\beta+1)\big) R_{j(\beta+1)}\Psi_\beta] = (-1)^{\beta-m+1} R_{j(\beta+1)}\Psi_\beta].$$

Next, in Eq. 4.17 we have

$$\det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta}, \overline{m}; \xi] = (-1)^{\beta-m}\Psi_\beta$$

and this gives the anomaly of a sign mentioned above.

Examine the last summand (4.19)–(4.20). We get

$$\det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \xi, \overline{m+1}, \dots, \overline{\alpha-1}, \overline{\alpha}] \\ = -\det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{\alpha}, \overline{m+1}, \dots, \overline{\alpha-1}, \xi] \\ - \det\Big[\Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{m-1}, \overline{\alpha}, \overline{m+1}, \dots, \overline{\alpha-1}]\Big]_{nn} = -R_{m\alpha}\Psi_{\alpha-1}$$

and

$$\det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta+1}; \overline{m}] \\ = \sigma\big(m\ \dots\ (\beta+1)\big)\Phi_{\beta+1} = (-1)^{\beta+1-m}\Phi_{\beta+1}.$$

Combining all signs we get the desired identity.

Proof of the statement B. We compose a new matrix $\Delta = \begin{pmatrix} Z^{cut} \\ U \end{pmatrix}$, consider the following minors of Δ,

$$\Psi_\alpha = \det \Delta[1, \dots, n-\alpha; \overline{1}, \dots, \overline{\alpha}], \\ R_{m(\beta+1)}\Psi_\beta = (-1)^{\beta-m} \det \Delta[1, \dots, n-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta}, \overline{\beta+1}],$$

and write the Plücker identity exchanging the $\overline{m}$-th row of Ψ_α with rows of $R_{m(\beta+1)}\Psi_\beta$.

Proof of the statement C. We compose a matrix $\Delta := \begin{pmatrix} Z \\ U^{ext} \end{pmatrix}$, take the minors

$$\Phi_\alpha = \det \Delta[1, \dots, n+1-\alpha; \overline{1}, \dots, \overline{\alpha}], \\ R_{m(\beta+1)}\Phi_\beta = (-1)^{\beta-m} \det \Delta[1, \dots, n+1-\beta; \overline{1}, \dots, \widehat{\overline{m}}, \dots, \overline{\beta}, \overline{\beta+1}].$$

and write the Plücker identities exchanging $\overline{m}$-th of Φ_α with rows of the second minor. □

4.3 A recurrence formula

Our next purpose is to derive a formula for $F_{1(n+1)}L$. Let us evaluate one of summands of this expression,

$$\Theta_{1n} := \Big(\sum_{I\lhd[1,n]}\prod_{l\in[1,n]\setminus I}(l-n+q_l-q_n)\cdot R_I\Big)T_n^- L. \tag{4.21}$$

Recall that the notation $J\lhd\{1,2,\dots n\}$ means that J is a subset in $\{1,2,\dots n\}$ containing 1 and n, see Section 1.9.

REMARK By definition, the right-hand side is a rational expression of the form

$$\frac{F}{\prod\Phi_\alpha^{s_\alpha}\prod\Psi_\beta^{t_\beta}}.$$

It is easy to observe (see below) that all the exponents s_α, t_β are equal 1. In a calculation below we decompose this function as a sum of 'prime fractions' with denominators Φ_α, Ψ_β and finally get an unexpectedly simple expression.

Let $1\leqslant k<l\leqslant n$, $\mu<n$. Denote

$$W_{kl}^\mu := \frac{(q_\mu-r_{\mu+1})R_{kl}\Phi_\mu}{\Phi_\mu}+\frac{(r_{\mu+1}-q_{\mu+1})R_{kl}\Psi_\mu}{\Psi_\mu}; \tag{4.22}$$

$$w_{kl}^\mu := (-1)^{l-k-1}\prod_{j=k+1}^{\mu}(q_n-q_j+j-n-1)\prod_{j=\mu+1}^{l-1}(q_n-q_j+j-n); \tag{4.23}$$

$$\zeta_{kl} = \sum_{\mu=k}^{l-1} w_{kl}^\mu W_{kl}^\mu + w_{k(n-1)}^{n-1}\frac{R_{kl}\Psi_{n-1}}{\Psi_{n-1}}. \tag{4.24}$$

Notice that the additional term in the last row is 0 if $l\neq n$.

Lemma 4.4

$$R_{kl}L = L\cdot\sum_{\mu:k\leqslant\mu<l}W_{kl}^\mu; \tag{4.25}$$

$$R_{kl}T_n^- L = L\cdot\left(\sum_{\mu:k\leqslant\mu<n}W_{kl}^\mu\right)\cdot\frac{\Psi_{n-1}}{\Phi_n}, \qquad \textit{for } l<n; \tag{4.26}$$

$$R_{kn}T_n^- L = L\cdot\left(\sum_{\mu:k\leqslant\mu<l}W_{kl}^\mu+\frac{R_{kn}\Psi_{n-1}}{\Psi_{n-1}}\right)\cdot\frac{\Psi_{n-1}}{\Phi_n}. \tag{4.27}$$

Proof Obviously,

$$R_{kl}L = L\left(\sum_{j=1}^{n}\frac{(q_\mu-r_{\mu+1})R_{kl\Phi_\mu}}{\Phi_\mu}+\sum_{j=1}^{n}\frac{(r_{\mu+1}-q_{\mu+1})R_{kl\Psi_\mu}}{\Psi_\mu}\right).$$

By Lemma 3.1, only terms with $k\leqslant\mu<l$ give a nonzero contribution. This gives the first row. Next,

$$T_n^- L = L\cdot\frac{\Psi_{n-1}}{\Phi_n}, \qquad R_{kl}T_n^- L = (R_{kl}L)\cdot\frac{\Psi_{n-1}}{\Phi_n}+L\cdot R_{kl}\left(\frac{\Psi_{n-1}}{\Phi_n}\right)$$

The second term in the right-hand side is 0 for $l < n$. For $l = n$ this term equals $R_{kn}\Psi_{n-1}/\Phi_n$. □

Lemma 4.5

$$\Theta_{1n} = \left(\sum_{J=\{j_1<j_2<\dots<j_s\}\lhd\{1,2,\dots n\}} \zeta_{j_1j_2}\zeta_{j_2j_3}\dots\zeta_{j_{s-1}j_s} \right) L \cdot \frac{\Psi_{n-1}}{\Phi_n}. \tag{4.28}$$

Proof First, we evaluate

$$R_I(T_n^- L)\cdot L^{-1}.$$

Each R_I is a product $R_{i_1i_2}\dots R_{i_{s-1}i_s}$. Each $R_{i_ti_{t+1}}$ is a first order differential operator without term of order zero, therefore we can expand $R_I L$ according the Leibniz rule.

Many summands of this expansion are zero by a priory reasons. Indeed, $R_{kl}\Phi_\mu$, $R_{kl}\Psi_\mu$ are nonzero only if $k \leqslant \mu < l$. Also, $R_{kl}R_{ab}\Phi_\mu = 0$ if $l < a$. In the case $l = a$ we have $R_{kl}R_{lb}\Phi_\mu = -R_{kb}\Phi_\mu$.

Therefor the expression $(R_I T_n^- L)/L$ is a sum of products of the following type

$$A[m_1,m_2]\,A[m_2,m_3]\dots A[m_{t-1},m_t]\cdot\frac{\Psi_{n-1}}{\Phi_n}, \tag{4.29}$$

where $M = \{m_1,\dots,m_t\} \lhd I$. and each factor $A[m_\tau,m_{\tau+1}] = A[i_a,i_c]$ has a form

$$\frac{(q_\mu - r_{\mu+1})\,R_{i_ai_{a+1}}R_{i_{a+1}i_{a+2}}\dots R_{i_{c-1}i_c}\Phi_\mu}{\Phi_\mu} \quad\text{or}\quad \frac{(r_{\mu+1} - q_{\mu+1})\,R_{i_ai_{a+1}}R_{i_{a+1}i_{a+2}}\dots R_{i_{c-1}i_c}\Psi_\mu}{\Psi_\mu},$$

where $i_{c-1} \leqslant \mu < i_c$. These expressions are equal correspondingly

$$\frac{(-1)^{c-a+1}(q_\mu - r_{\mu+1})\,R_{i_ai_c}\Phi_\mu}{\Phi_\mu} \quad\text{and}\quad \frac{(-1)^{c-a+1}(r_{\mu+1} - q_{\mu+1})\,R_{i_ai_c}\Psi_\mu}{\Psi_\mu}$$

Now fix M and consider the sum $\mathcal{S}[I,M]$ of all summands (4.29) with fixed M. It is easy to see that

$$\mathcal{S}[I,M] = (-1)^{c-a+1}\left[\prod_{\gamma=1}^{t-1}\left(\sum_{i_a\in I: m_\gamma\leqslant i_a<m_{\gamma+1}}\ \sum_{\mu: i_a\leqslant\mu<m_c} W^{\mu}_{i_am_{\gamma+1}}\right]\cdot L.\right.$$

Next, we represent Θ_{1n} as

$$\Theta_{1n} = \left(\sum_{I\lhd[1,n]}\prod_{l\in[1,n]\setminus I}(l-n+q_l-q_n)\cdot\sum_{M\lhd I}\mathcal{S}[I,M]\right)\cdot L\cdot\frac{\Psi_{n-1}}{\Phi_n} =$$

$$= \sum_{M\lhd[1,n]}\left(\sum_{I\supset M}\prod_{l\in[1,n]\setminus I}(l-n+q_l-q_n)\cdot\mathcal{S}[I,M]\right)\cdot L\cdot\frac{\Psi_{n-1}}{\Phi_n}.$$

Fix $M = \{m_1 < \dots < m_t\}$. Each summand in the big brackets splits into a product of the form

$$\prod_{\gamma=1}^{t-1} H[m_\gamma, m_{\gamma+1}],$$

where $H[m_\gamma, m_{\gamma+1}]$ is an expression, which depend on $m_\gamma, m_{\gamma+1}$ and does not depend on other m_i. Since the coefficients also are multiplicative in the same sense, the whole sum in the big brackets also splits in a product

$$\prod_{\gamma=1}^{t-1} \mathcal{Z}[m_\gamma, m_{\gamma_1}],$$

where each factor $\mathcal{Z}[m_\gamma, m_{\gamma+1}]$ depends only on m_γ, $m_{\gamma+1}$ and not on the remaining elements of a set M.

It remains to evaluate factors $\mathcal{Z}[\cdot]$,

$$\mathcal{Z}[k,l] = \sum_{\mu=k}^{l} \upsilon_{kl}^{\mu} W_{kl}^{\mu},$$

where

$$\begin{aligned}\upsilon_{kl}^{\mu} &= \sum_J \left(- \prod_{m\in[k,\mu]\setminus J} (-1)(m-n+q_n-q_m) \cdot \prod_{m=\mu+1}^{l-1} (m-n+q_n-q_m) \right) \\ &= \prod_{m=\mu+1}^{l-1} (m-n+q_n-q_m) \cdot \left(-\sum_J \prod_{m\in[k,\mu]\setminus J} (-1)(m-n+q_n-q_m) \right),\end{aligned}$$

the summation here is taken other all subsets $J \subset [k,\mu]$, containing k. In the brackets we get

$$\prod_{m=k+1}^{\mu} (-1)\big[1-(m-n+q_n-q_m)\big]$$

Thus $\mathcal{Z}_{kl}$ equals to ζ_{kl}. □

Let ζ_{kl} be as above (4.24). Denote

$$\Theta_{mn} := \left(\sum_{J=\{j_1<j_2<\cdots<j_s\}\triangleleft\{m,2,\ldots n\}} \zeta_{j_1 j_2}\zeta_{j_2 j_3}\ldots\zeta_{j_{s-1} j_s} \right) L \cdot \frac{\Psi_{n-1}}{\Phi_n}.$$

By the previous lemma, this notation is compatible with the earlier notation Θ_{1n}. Also, $\Theta_{nn} = \frac{\Psi_{n-1}}{\Phi_n}$.

Lemma 4.6 *The Θ_{mn} satisfies the following recurrence relation,*

$$\Theta_{mn} = \zeta_{m(m+1)}\Theta_{(m+1)n} + \zeta_{m(m+2)}\Theta_{(m+2)n} + \cdots + \zeta_{mn}\Theta_{nn}.$$

This statement is obvious.

4.4 Evaluation of Θ_{1n}

Denote

$$
\begin{aligned}
s^{\mu}_{m\tau} &= \prod_{j=m}^{\mu-1} (q_\tau - q_j + j - n)\\
\sigma^{\mu}_{\tau} &= (q_\mu - r_{\mu+1}) \prod_{i=\mu+2}^{\tau} (q_\tau - r_i - n - 1 + i).
\end{aligned}
$$

Lemma 4.7

$$
\Theta_{mn} L^{-1} = -\sum_{\mu=m}^{n} s^{\mu}_{mn} \sigma^{\mu}_{n} \frac{R_{m\mu}\Psi_{\mu-1}}{\Phi_\mu}.
$$

In particular, this gives an explicit expression for Θ_{1n}.

Proof We prove our statement by induction. Assume that for Θ_{nn}, $\Theta_{(n-1)n}$, $\dots$, $\Theta_{(m+1)n}$ the formula is correct. We must derive the equality

$$
\begin{aligned}
\Theta_{mn} \cdot L^{-1} &= \sum_{\gamma=m+1}^{n} \zeta_{m\gamma} \Theta_{(m+1)n}\\
&= \sum_{\gamma=m+1}^{n} \left(\sum_{\mu=m}^{\gamma-1} w^{\mu}_{m\gamma} W^{\mu}_{m\gamma} \right) \left(\sum_{\nu=\gamma}^{n} s^{\nu}_{\gamma n} \sigma^{\nu} \frac{R_{\gamma\nu}\Psi_{\nu-1}}{\Phi_\nu} \right).
\end{aligned}
$$

We have

$$
\begin{aligned}
\upsilon^{\mu}_{m\gamma} s^{n}_{\gamma n} &= \prod_{j=m+1}^{\mu} (q_n - q_j + j - n + 1) \cdot \prod_{j=\mu+1}^{\gamma-1} (q_n - q_j + j - n + 1) \prod_{j=\gamma}^{\nu-1} (q_n - q_j + j - n)\\
&= \prod_{j=m+1}^{\nu-1} (q_n - q_j + j - n + 1) \cdot \prod_{j=\mu+1}^{\gamma-1} (q_n - q_j + j - n + 1) =: \xi^{\mu}_{m\nu}.
\end{aligned}
$$

We stress that the expression $\xi^{\mu}_{m\nu}$ does not depend on γ. Our sum transforms to

$$
\sum_{\mu,\nu:\mu<\nu} \xi^{\mu}_{m\nu} \sigma^{\nu}_{n} \left[\sum_{\gamma=\mu+1}^{\nu} W^{\mu}_{m\gamma} \frac{R_{\gamma\nu}\Psi_{\nu-1}}{\Phi_\nu} \right] + \frac{\Psi_{n-1}}{\Phi_n}.
$$

Denote by $B^{\mu}_{m\nu}$ the expression in the square brackets and write it explicitly:

$$
\begin{aligned}
B^{\mu}_{m\nu} &= \sum_{\gamma=\mu+1}^{\nu} \left(\frac{(r_{\mu+1} - q_{\mu+1}) R_{m\gamma}\Psi_\mu}{\Psi_\mu} + \frac{(q_\mu - r_{\mu+1}) R_{m\gamma}\Phi_\mu}{\Phi_\mu} \right) \frac{R_{\gamma\nu}\Psi_{\nu-1}}{\Phi_\nu}\\
&= (r_{\mu+1} - q_{\mu+1}) \sum_{\gamma=\mu+1}^{\nu} \frac{R_{m\gamma}\Psi_\mu}{\Psi_\mu} \frac{R_{\gamma\nu}\Psi_{\nu-1}}{\Phi_\nu}\\
&\quad + (q_\mu - r_{\mu+1}) \sum_{\gamma=\mu+1}^{\nu} \frac{R_{m\gamma}\Phi_\mu}{\Psi_\mu} \frac{R_{\gamma\nu}\Psi_{\nu-1}}{\Phi_\nu}.
\end{aligned} \tag{4.30}
$$

Next, we apply the quadratic relations (4.4) and (4.5) and transform the last expression to

$$(r_{\mu+1}-q_{\mu+1})\frac{-R_{m\nu}\Psi_{\nu-1}}{\Phi_\nu}+(q_\mu-r_{\mu+1})\left(-\frac{R_{m\nu}\Psi_{\nu-1}}{\Phi_\nu}+\frac{R_{m\nu}\Psi_{\mu-1}}{\Phi_\mu}\right)$$
$$=(q_{\mu+1}-q_\mu)\frac{R_{m\nu}\Psi_{\nu-1}}{\Phi_\nu}+(q_\mu-r_{\mu+1})\frac{R_{m\nu}\Psi_{\mu-1}}{\Phi_\mu}.$$

Thus,

$$\Theta_{mn}:=\sum_{\mu,\nu:\mu<\nu}\xi^{\mu}_{m\nu}\sigma^{\nu}_{n}\left((q_{\mu+1}-q_\mu)\frac{R_{m\nu}\Psi_{\nu-1}}{\Phi_\nu}+(q_\mu-r_{\mu+1})\frac{R_{m\nu}\Psi_{\mu-1}}{\Phi_\mu}\right). \tag{4.31}$$

We collect similar terms and get

$$\Theta_{mn}:=\sum_\nu\Big(S_1+S_2\Big)\frac{R_{m\nu}\Psi_{\nu-1}}{\Phi_\nu},$$

where

$$S_1:=\sum_{\mu=1}^{\nu-1}\xi_{\mu\nu}\sigma_n^\nu(q_{\mu+1}-q_\mu), \tag{4.32}$$

$$S_2:=\sum_{\varkappa=\nu+1}^{n}\xi_{\nu\varkappa}\sigma_n^\varkappa(q_\nu-r_{\nu+1}). \tag{4.33}$$

First, we transform S_2,

$$S_2=(q_\nu-r_{\nu+1})\cdot\prod_{j=m+1}^{\nu}(q_n-q_j-n+j-1)$$
$$\times\sum_{\varkappa=\nu+1}^{n}\left(\prod_{j=\nu+1}^{\varkappa-1}(q_n-q_j+j-n)(q_\varkappa-r_{\varkappa+1})\cdot\prod_{j=2}^{n-\varkappa}(q_n-r_{n+2-j}-j)\right). \tag{4.34}$$

The sum in the second line is evaluated in the following lemma.

Lemma 4.8 *Let $n-k\geqslant\nu+1$. Then*

$$\sum_{\varkappa=n-k+1}^{n}\prod_{j=\nu+1}^{\varkappa-1}(q_n-q_j+j-n)\cdot(q_\varkappa-r_{\varkappa+1})\cdot\prod_{j=2}^{n-\varkappa}(q_n-r_{n+2-j}-j)$$
$$=\prod_{\nu+1}^{n-k}(q_n-q_j+j-n)\cdot\prod_{j=2}^{k}(q_n-r_{n+2-j}-k+1). \tag{4.35}$$

In particular,

$$\sum_{\varkappa=\nu+1}^{n}(\dots)=\prod_{j=2}^{n-\nu}(q_n-r_{n+2-j}-k+1),$$

and

$$(q_\nu-r_{\nu+1})\sum_{\varkappa=\nu+1}^{n}(\dots)=\sigma_n^\nu. \tag{4.36}$$

□

Proof of Lemma 4.8 We prove the statement by induction. Let for a given k the statement hold. Then

$$\begin{aligned}\sum_{\varkappa=n-k}^{n} &= \sum_{\varkappa=n-k+1}^{n} + \Big\{n-k\text{-th term}\Big\}\\
&= \prod_{\nu+1}^{n-k}(q_n - q_j + j - n)\cdot\prod_{j=2}^{k}(q_n - r_{n+2-j} - k + 1)\\
&\quad + \prod_{j=\nu+1}^{n-k-1}(q_n - q_j + j - n)\cdot(q_{n-k} - r_{n-k+1})\cdot\prod_{j=2}^{k}(q_n - r_{n+2-j} - j)\\
&= \prod_{j=\nu+1}^{n-k-1}(q_n - q_j + j - n)\prod_{j=2}^{k}(q_n - r_{n+2-j} - k + 1)\cdot((q_n - q_{n-k} - k) + (q_{n-k} - r_{n-k+1}))\end{aligned}$$

and the big bracket joins to the product $\prod_{j=2}^{k}$. □

Let us return to our calculation. We must evaluate $S_1 + S_2$,

$$\begin{aligned}S_1 + S_2 &= \sigma_n^{\nu}\sum_{\mu=1}^{\nu-1}\xi_{\mu\nu}(q_{\mu+1} - q_\mu) + S_2\\
&= \sigma_\nu\sum_{\mu=1}^{\nu-1}\left\{\prod_{j=m+1}^{\mu}(q_n - q_j + j - n - 1)\cdot\prod_{j=\mu+1}^{\nu-1}(q_n - q_j + j - n)\cdot(q_\mu - q_{\mu+1})\right\}\\
&\quad + \sigma_n^{\nu}\prod_{j=m+1}^{\nu}(q_n - q_j + j - n - 1). \qquad (4.37)\end{aligned}$$

An evaluation of this sum is similar to the proof of Lemma 4.8. We verify the following identity by induction,

$$\sigma_n^{\nu}\sum_{\nu-k}^{\nu-1}\xi_{\mu\nu}(q_{\mu+1} - q_\mu) + S_2 = \prod_{j=m+1}^{\nu-k+1}(q_n - q_j + j - 1 - n)\prod_{j=\nu-k+1}^{\nu-1}(q_n - q_j + j - n).$$

Substituting $k = \nu - 1$ we get the desired coefficient in Eq. 4.31.

It remains to notice that formulas for the coefficient in the front of $\frac{1}{\Phi_n}$ are slightly different. In this case, the sum S_2 is absent, but ζ_{mn} has an extra term $\frac{1}{\Phi_n}$. Starting this place, the calculation is the same, the extra term replaces the additional term in Eq. 4.37.

This completes a proof of Lemma 4.7.

4.5 The Extension of Calculations

Next, we must evaluate other summands in $F_{1(n+1)}L$. Denote

$$\Theta_{1\tau} := \sum_{I\triangleleft[1,\tau]}\prod_{l\in[1,\tau]\setminus I}(l - \tau + q_l - q_\tau)\cdot R_I\Big)T_m^- L.$$

Lemma 4.9

$$\Theta_{1\tau} = -\sum_{\mu=1}^{m} s_{1\tau}^{\mu}\sigma_{\tau}^{\mu}\frac{R_{1\mu}\Psi_{\mu-1}}{\Phi_{\mu}}. \tag{4.38}$$

Proof We decompose $T_{\tau}^{-}L$ as

$$T_{\tau}^{-}L = \left(\Phi_1^{-r_2+q_1}\Psi_1^{r_2-q_2}\dots\Psi_{\tau-1}^{r_\tau-q_\tau+1}\Phi_\tau^{-r_{\tau+1}+q_\tau-1}\right) \times\left(\Psi_{m-1}^{r_m-q_m}\Phi_\tau^{-r_{\tau+1}+q_\tau}\dots\Phi_n^{-r_{n+1}+q_n}\right).$$

Factors R_{kl} of operators R_I act as zero on the factors of the second bracket. Therefore this bracket can be regarded as a constant. After this, we get the same calculations as for Θ_{1n} but n is replaced by τ. □

4.6 The Summation

Lemma 4.10

$$F_{1(n+1)}L + E_{1(n+1)}L = 0.$$

Proof

$$F_{1(n+1)}L = \sum_{\tau=1}^{n} A_\tau \sum_{\alpha=1}^{\tau} s_{1\tau}^{\alpha}\sigma_{\tau}^{\alpha}\frac{R_{1\alpha}\Psi_{\alpha-1}}{\Phi_\alpha} = \sum_{\sigma=1}^{n}\left(\sum_{\tau=\alpha}^{n} A_\tau s_{1\tau}^{\alpha}\sigma_{\tau}^{\alpha}\right)\frac{R_{1\alpha}\Psi_{\alpha-1}}{\Phi_\alpha}.$$

We must verify the following identities

$$\sum_{\tau=\alpha}^{n} A_\tau s_{1\tau}^{\alpha}\sigma_{\tau}^{\alpha} = -(q_\alpha - r_{\alpha+1}). \tag{4.39}$$

A summand in the left-hand side equals

$$A_\tau s_{1\tau}^{\alpha}\sigma_{\tau}^{\alpha} = \frac{\prod_{j=1}^{n-\tau+1}(q_m - r_{n+2-j} + n - \tau + 1 - j)}{\prod_{j\neq\tau}(q_\tau - q_j + j - m)} \times(q_\alpha - r_{\alpha+1})\prod_{i=1}^{\alpha-1}(q_m - q_i + i - m)\cdot\prod_{k=n-m+2}^{n-\alpha}(q_m - r_{n+2-k} + n - \alpha - k + 1).$$

The factor $\prod_{i=1}^{\alpha-1}(q_m - q_i + i - m)$ cancels, the factors $\prod_{j=1}^{n-\tau+1}$ and $\prod_{k=n-m+2}^{n-\alpha}$ join together and we come to

$$A_\tau s_{1\tau}^{\alpha}\sigma_{\tau}^{\alpha} = (q_\alpha - r_{\alpha+1})\frac{\prod_{j=1}^{n-\sigma}(q_m - r_{n+2-j} + n - \tau + 1 - j}{\prod_{j\neq\tau, j>\sigma}(q_\tau - q_j + j - m)}.$$

Next, we write a rational function

$$V_\alpha(x) := \frac{\prod_{j=1}^{n-\sigma}(x + r_{n+2-j} + n - j + 1)}{\prod_{j=\sigma}^{n}(x - q_j + j)}$$

and evaluate the sum of its residues,

$$\sum_{j=\sigma}^{m} \mathrm{Res}_{x=q_j-j} V_\alpha(x) = -\mathrm{Res}_{x=\infty} V_\alpha(x) \tag{4.40}$$

Multiplying this identity by $(q_\alpha - r_{\alpha+1})$ we get (4.39). □

4.7 Invariance of the space $\mathfrak{V}_{\mathbf{r}}$

Lemma 4.11 *The space $\mathfrak{V}_{\mathbf{r}}$ (see* Eq. 1.16*) is invariant with respect to the operator $F_{1(n+1)}$.*

PROVE. Define elements $\ell_Z(U) \in \mathfrak{V}_{\mathbf{r}}$, where Z ranges in T^+_{n+1}, by

$$\ell_Z(U) = L(Z, U).$$

First, functions $\ell_Z(U)$ generate the space $\mathfrak{V}_{\mathbf{r}}$ (because the pairing $V_{\mathbf{p}} \times \mathfrak{V}_{\mathbf{r}} \to \mathbb{C}$ determined by L is nondegenerate). Next,

$$F_{1(n+1)}\ell_Z(U) = F_{1(n+1)}L(Z, U) = -E_{1(n+1)}L(Z, U) = -\frac{\partial}{\partial z_{1(n+1)}}\big(\ell_Z(U)\big).$$

Differentiating a family of elements of $\mathfrak{V}_{\mathbf{r}}$ with respect to a parameter we get elements of the same space $\mathfrak{V}_{\mathbf{r}}$.

4.8 Formula for $F_{(n+1)n}$

Here a calculation is more-or-less the same, we omit details (in a critical moment we use the quadratic identities (4.4), (4.6)).

5 Infinite Dimensional Case. A Conjecture

5.1 Principal Series

Now consider two collections of complex numbers $(p_1, \dots, p_n)$ and $(p_1^\circ, \dots, p_n^\circ)$ such that

$$p_j - p_j^\circ \in \mathbb{Z}, \qquad \mathrm{Re}(p_j + p_j^\circ) = -2(j-1).$$

Consider a representation $\rho^n_{\mathbf{p}|\mathbf{p}^\circ}$ of the group $\mathrm{GL}(n, \mathbb{C})$ in the space $L^2(N_n^+)$ determined by the formula

$$\rho_{\mathbf{p}|\mathbf{p}^\circ}(g)\varphi(g) = \varphi(Z^{[g]}) \cdot \prod_{j=1}^{n} b_{jj}(Z, g)^{p_j}\, \overline{b_{jj}(Z, g)}^{\,p_j^\circ}. \tag{5.1}$$

We get the unitary (nondegenerate) principal series of representations of $\mathrm{GL}(n, \mathbb{C})$ (see [6]). Denote by $\Lambda_n^{unitary}$ the space of all parameters p, p°.

REMARK. In formula (5.1), we have complex numbers in complex powers. We understand them in the following way:

$$b_{jj}^{p_j}\, \overline{b}_{jj}^{\,p_j^\circ} = |b_{jj}|^{2p_j}\, (\overline{b}_{jj}/b_{jj})^{p_j^\circ - p_j}.$$

In the right hand side, the first factor has a positive base of the power, the second factor has an integer exponent. Hence the product is well defined.

REMARK. Formula (5.1) makes sense if $p_j - p_j^\circ \in \mathbb{Z}$, and gives a non-unitary principal series of representations. The construction of holomorphic representation discussed above corresponds to $p \in \Lambda_n$, $p^\circ = 0$. If both

$$p,\ \ p^\circ \in \Lambda_n \tag{5.2}$$

then our representation contains a finite-dimensional (nonholomorphic) representation $\rho_{\mathbf{p}} \otimes \overline{\rho}_{\mathbf{p}^\circ}$, where $\overline{\rho}$ denotes the complex conjugate representation.

5.2 Restriction to the Smaller Subgroup

Consider a representation $\rho^{n+1}_{\mathbf{p}|\mathbf{p}^\circ}$ of the group GL($n+1,\mathbb{C}$). According [1], the restriction of $\rho_{\mathbf{p}|\mathbf{p}^\circ}$ to the subgroup GL($n,\mathbb{C}$) is a multiplicity free integral of all representations of $\rho^n_{\mathbf{p}|\mathbf{p}^\circ}$ of unitary nondegenerate principal series of GL($n,\mathbb{C}$). Moreover, the restriction has Lebesgue spectrum. Thus the restriction $U_{\mathbf{p}|\mathbf{p}^\circ}$ can be realized in the space $L^2(N_n^+ \times \Lambda_n^{unitary})$ by the formula

$$U_{\mathbf{p}|\mathbf{p}^\circ}(g)\psi(U,\mathbf{q}|\mathbf{q}^\circ) = \psi(Z^{[g]},\mathbf{q}|\mathbf{q}^\circ)\cdot \prod_{j=1}^{n} b_{jj}(Z,g)^{q_j}\,\overline{b_{jj}(Z,g)}^{\,q_j^\circ}.$$

5.3 Intertwining Operator

Next, we write an integral operator

$$L^2(N_{n+1}^+) \to L^2(N_n^+ \times \Lambda_n^{unitary})$$

by the formula

$$J\varphi(U,\mathbf{q}|\mathbf{q}^\circ) = \int_{N_{n+1}^+} \varphi(Z) L_{\mathbf{p},\mathbf{q}}(Z,U)\, L_{\mathbf{p}^\circ,\mathbf{q}^\circ}(Z,U)\, d\dot{Z}. \tag{5.3}$$

5.4 Additional operators

Denote the operators (1.22), (1.22) by $F^{\mathbf{p}}_{1(n+1)}$, $F^{\mathbf{p}}_{(n+1)n}$. Denote by $\overline{F}^{\mathbf{p}^\circ}_{1(n+1)}$, $\overline{F}^{\mathbf{p}^\circ}_{(n+1)n}$ the operators obtained from $F^{\mathbf{p}}_{1(n+1)}$, $F^{\mathbf{p}}_{(n+1)n}$ by replacing

$$\frac{\partial}{\partial z_{kl}} \mapsto \frac{\partial}{\partial \overline{z}_{kl}}, \qquad p_j \mapsto p_j^\circ, \qquad q_k \mapsto q_k^\circ.$$

Define the operators

$$F^{\mathbf{p}|\mathbf{p}^\circ}_{1(n+1)} = F^{\mathbf{p}}_{1(n+1)} + \overline{F}^{\mathbf{p}^\circ}_{1(n+1)},$$
$$F^{\mathbf{p}|\mathbf{p}^\circ}_{(n+1)n} = F^{\mathbf{p}}_{(n+1)n} + \overline{F}^{\mathbf{p}^\circ}_{(n+1)n}.$$

Conjecture 5.1 The operators $F^{\mathbf{p}|\mathbf{p}^\circ}_{1(n+1)}$, $F^{\mathbf{p}|\mathbf{p}^\circ}_{(n+1)n}$ are images of the operators $E_{1(n+1)}$, $E_{(n+1)n}$ under the integral transform (5.3).

The statement seems doubtless, since we have an analytic continuation of our finite-dimensional formulas from the set (5.2). However, this is not an automatic corollary of our result. In particular, it is necessary to find the Plancherel measure on $\Lambda_n^{unitary}$ (i.e. a measure making the operator J unitary).

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