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CORRECTION

Correction to: The heat flow for the full bosonic string

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Abstract We clarify several items in the original paper.

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As noted by Xuangzhi Cao and Qun Chen the term $|d\phi_t(e_1)\wedge d\phi_t(e_2)|^2$ was written as $|d\phi_t|^4$ at several places in [1], which we will correct below.

Regarding [1, Lemma 4.5], the corrected version looks like:

Lemma *Let $\phi_t\colon M\times[0,T_{\max})\to N$ be a smooth solution of* (4.1) *in* [1] *and $V\in C^2(N)$. Assume that N has negative sectional curvature. Then for all $t\in[0,T_{\max})$ the following inequalities hold:*

$$\frac{\partial}{\partial t}\frac{1}{2}|d\phi_t|^2\leq\Delta\frac{1}{2}|d\phi_t|^2+c_1|d\phi_t|^2+\left(\frac{1}{2}|Z|^2_{L^\infty}-2\kappa_N\right)|d\phi_t(e_1)\wedge d\phi_t(e_2)|^2 \tag{0.1}$$

and

$$\frac{\partial}{\partial t}\frac{1}{2}\left|\frac{\partial\phi_t}{\partial t}\right|^2\leq\Delta\frac{1}{2}\left|\frac{\partial\phi_t}{\partial t}\right|^2+\left(|\nabla Z|_{L^\infty}+\frac{1}{4}|Z|^2_{L^\infty}\right)|d\phi_t|^2\left|\frac{\partial\phi_t}{\partial t}\right|^2+c_2\left|\frac{\partial\phi_t}{\partial t}\right|^2 \tag{0.2}$$

with $c_1:=|\mathrm{Ric}^M|_{L^\infty}+|R|_{L^\infty}|\operatorname{Hess}V|_{L^\infty}$, $c_2:=|R|_{L^\infty}|\operatorname{Hess}V|_{L^\infty}$ and the positive number κ_N denotes an upper bound on the absolute value of the sectional curvature of N.

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Proof We have the following Bochner formula (formula (4.3) in [1])

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}|\mathrm{d}\phi_t|^2 = \Delta\frac{1}{2}|\mathrm{d}\phi_t|^2 - |\nabla \mathrm{d}\phi_t|^2 &+ \langle R^N(\mathrm{d}\phi_t(e_\alpha), \mathrm{d}\phi_t(e_\beta))\mathrm{d}\phi_t(e_\alpha), \mathrm{d}\phi_t(e_\beta)\rangle \\ &- \langle \mathrm{d}\phi_t(Ric^M(e_\alpha)), \mathrm{d}\phi_t(e_\alpha)\rangle + \langle Z(\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)), \tau(\phi_t)\rangle \\ &- R\,\mathrm{Hess}\, V(\mathrm{d}\phi_t, \mathrm{d}\phi_t).\end{aligned}$$

First, we estimate

$$|\langle Z(\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)), \tau(\phi_t)\rangle| \leq \sqrt{2}|Z|_{L^\infty}|\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)||\nabla \mathrm{d}\phi_t|,$$

which yields

$$\langle Z(\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)), \tau(\phi_t)\rangle - |\nabla \mathrm{d}\phi_t|^2 \leq \frac{1}{2}|Z|^2_{L^\infty}|\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)|^2.$$

Furthermore, we find (see also [2, Lemma 3.1])

$$\begin{aligned}\left\langle R^N(\mathrm{d}\phi_t(e_\alpha), \mathrm{d}\phi_t(e_\beta))\mathrm{d}\phi_t(e_\alpha), \mathrm{d}\phi_t(e_\beta)\right\rangle &= 2\left\langle R^N(\mathrm{d}\phi_t(e_1), \mathrm{d}\phi_t(e_2))\mathrm{d}\phi_t(e_1), \mathrm{d}\phi_t(e_2)\right\rangle \\ &= 2\left\langle Q^N(\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)), \mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)\right\rangle \\ &\leq -2\kappa^N|\mathrm{d}\phi_t(e_1)\wedge \mathrm{d}\phi_t(e_2)|^2.\end{aligned}$$

Here, Q^N denotes the curvature operator on N. By assumption N is compact and we can estimate the Hessian of the potential $V(\phi)$ by its maximum yielding the first claim.

Regarding the second statement, we want to mention that the original proof is correct. □

Via the maximum principle we thus obtain the following (which is Corollary 4.6 in [1] with adjusted constants).

Corollary *If* $\frac{1}{2}|Z|^2_{L^\infty} \leq \kappa^N$ *then for all* $t \in [0, T_{\max})$ *the following estimates hold:*

$$|\mathrm{d}\phi_t|^2 \leq |\mathrm{d}\phi_0|^2 e^{2c_1 t}, \tag{0.3}$$

$$\left|\frac{\partial\phi_t}{\partial t}\right|^2 \leq \left|\frac{\partial\phi_0}{\partial t}\right|^2 e^{\frac{\left(|\nabla Z|_{L^\infty}+\frac{1}{4}|Z|^2_{L^\infty}\right)|\mathrm{d}\phi_0|^2}{c_1}e^{2c_1 t}+c_2 t}. \tag{0.4}$$

We also correct a typo in the proof of [1, Lemma 4.8]. The equation on lines 5 and 6 on page 358 should be

$$\begin{aligned}\frac{\partial h}{\partial t} = \Delta h - |\mathrm{d}u - \mathrm{d}v|^2 &- \langle \mathrm{I\!I}_u(\mathrm{d}u, \mathrm{d}u) - \mathrm{I\!I}_v(\mathrm{d}v, \mathrm{d}v), u - v\rangle \\ &- \langle Z_u(\mathrm{d}u(e_1)\wedge \mathrm{d}u(e_2)) - Z_v(\mathrm{d}v(e_1)\wedge \mathrm{d}v(e_2)), u - v\rangle \\ &- R\langle \nabla V(u) - \nabla V(v), u - v\rangle.\end{aligned}$$

We also want to point out that one needs the following statement in the proofs of Lemma 4.15 and Lemma 4.16, which was not explicitly given in [1].

Lemma *Let* $\phi_t\colon M\times[0, T_{\max}) \to N$ *be a smooth solution of* (4.1) *in* [1] *with* $V \in C^2(N)$ *and* $\nabla Z = 0$. *Assume that* N *has negative sectional curvature. If* $\frac{1}{2}|Z|^2_{L^\infty} \leq \kappa^N$ *then for all* $t \in [0, T_{\max})$ *the following inequality holds*

$$\frac{\partial}{\partial t}\frac{1}{2}\left|\frac{\partial\phi_t}{\partial t}\right|^2 \leq \Delta\frac{1}{2}\left|\frac{\partial\phi_t}{\partial t}\right|^2 - \frac{1}{2}\left|\nabla\frac{\partial\phi_t}{\partial t}\right|^2 - R\,\mathrm{Hess}\, V\left(\frac{\partial\phi_t}{\partial t}, \frac{\partial\phi_t}{\partial t}\right). \tag{0.5}$$

Proof Here, we make use of the following Bochner formula ((4.4) in [1])

$$\frac{\partial}{\partial t}\frac{1}{2}\left|\frac{\partial\phi_t}{\partial t}\right|^2 = \Delta\frac{1}{2}\left|\frac{\partial\phi_t}{\partial t}\right|^2 - \left|\nabla\frac{\partial\phi_t}{\partial t}\right|^2 + \left\langle R^N\left(\mathrm{d}\phi_t(e_\alpha), \left(\frac{\partial\phi_t}{\partial t}\right)\mathrm{d}\phi_t(e_\alpha)\right), \frac{\partial\phi_t}{\partial t}\right\rangle$$
$$-\left\langle\frac{\nabla}{\partial t}Z(\mathrm{d}\phi_t(e_1)\wedge\mathrm{d}\phi_t(e_2)), \frac{\partial\phi_t}{\partial t}\right\rangle - R\,\mathrm{Hess}\,V\left(\frac{\partial\phi_t}{\partial t}, \frac{\partial\phi_t}{\partial t}\right).$$

We calculate

$$\frac{\nabla}{\partial t}Z(\mathrm{d}\phi_t(e_1)\wedge\mathrm{d}\phi_t(e_2)) = Z\left(\frac{\nabla}{\partial t}\mathrm{d}\phi_t(e_1)\wedge\mathrm{d}\phi_t(e_2)\right) + Z\left(\mathrm{d}\phi_t(e_1)\wedge\frac{\nabla}{\partial t}\mathrm{d}\phi_t(e_2)\right),$$

which allows us to estimate

$$\left\langle\frac{\nabla}{\partial t}Z(\mathrm{d}\phi_t(e_1)\wedge\mathrm{d}\phi_t(e_2)), \frac{\partial\phi_t}{\partial t}\right\rangle \leq |Z|_{L^\infty}\left|\frac{\nabla}{\partial t}\mathrm{d}\phi_t\right|\left|\mathrm{d}\phi_t\wedge\frac{\partial\phi_t}{\partial t}\right|,$$
$$\left\langle R^N(\mathrm{d}\phi_t(e_\alpha), \left(\frac{\partial\phi_t}{\partial t}\right)\mathrm{d}\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t}\right\rangle \leq -\kappa^N\left|\mathrm{d}\phi_t\wedge\frac{\partial\phi_t}{\partial t}\right|^2.$$

The result follows by applying Young's inequality. □

We also correct several typos in [1, Lemma 4.21]. The inequality given at the bottom should be

$$\begin{aligned}
E(\phi_1) - E(\phi_2) = &\int_0^1 \mathrm{d}\sigma\int_0^\sigma\left(\left|\nabla\frac{\partial\Phi}{\partial s}\right|^2 - \left\langle R^N\left(\mathrm{d}\Phi, \frac{\partial\Phi}{\partial s}\right)\mathrm{d}\Phi, \frac{\partial\Phi}{\partial s}\right\rangle\right.\\
&+\left\langle\frac{\partial\Phi}{\partial s}, Z\left(\frac{\nabla}{\partial s}\mathrm{d}\Phi(e_1)\wedge\mathrm{d}\Phi(e_2)\right)\right\rangle\\
&\left.+\left\langle\frac{\partial\Phi}{\partial s}, Z\left(\mathrm{d}\Phi(e_1)\wedge\frac{\nabla}{\partial s}\mathrm{d}\Phi(e_2)\right)\right\rangle + R\,\mathrm{Hess}\,V\left(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\right)\right)\mathrm{d}s\\
\geq &\int_0^1\mathrm{d}\sigma\int_0^\sigma\left(\left|\nabla\frac{\partial\Phi}{\partial s}\right|^2 + \kappa_N\left|\mathrm{d}\Phi\wedge\frac{\partial\Phi}{\partial s}\right|^2 - |Z|_{L^\infty}\left|\mathrm{d}\Phi\wedge\frac{\partial\Phi}{\partial s}\right|\left|\nabla\frac{\partial\Phi}{\partial s}\right|\right.\\
&\left.+R\,\mathrm{Hess}\,V\left(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\right)\right)\mathrm{d}s\\
\geq &\int_0^1\mathrm{d}\sigma\int_0^\sigma\left(\frac{1}{2}\left|\nabla\frac{\partial\Phi}{\partial s}\right|^2 + \left(\kappa^N - \frac{1}{2}|Z|^2_{L^\infty}\right)\left|\mathrm{d}\Phi\wedge\frac{\partial\Phi}{\partial s}\right|^2\right.\\
&\left.+R\,\mathrm{Hess}\,V\left(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\right)\right)\mathrm{d}s\\
> &\,0.
\end{aligned}$$

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References

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