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ERRATUM

Erratum to: On the geometry of spaces of oriented geodesics

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We correct some of the entries of Table 1 in our original paper.

1. Those entries of Table 1 in the original paper listing the numbers of geometric structures on the space of geodesics $L^{\epsilon}(S^{p,q})$ of the 3-dimensional space $S^{p,q}$ of non-zero constant curvature with $p+q=3$ are incorrect. In this note we give the correct values and arguments. In original paper we failed to note that the $(\mathfrak{so}(V)+\mathfrak{so}(V'))$-module $V\otimes V'$, where V, V' are 2-dimensional pseudo-Euclidian spaces, has not only the invariant (para)complex structures $J = J_V \otimes 1$, $J' = 1 \otimes J_{V'}$, but that also their product $J'' = JJ' = J_V \otimes J_{V'}$ is an invariant (para)complex structure. We next analyze the impact of this observation in detail and correct Table 1 accordingly. The authors would like to thank Henri Anciaux for drawing their attention to this oversight.

2. Let $E = E^{p+1,q}$ be a pseudo-Euclidean 4-dimensional vector space of signature $(p+1,q) = (4,0), (3,1), (2,2)$ or $(1,3)$ with an orthonormal basis e_1, e_2, e_3, e_4 with $e_4^2 := g(e_4, e_4) = 1$. We denote by $E' = e_4^{\perp}$ the orthogonal complement to e_4. The pseudo-sphere

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Table 1 Invariant geometric structures

	Symplectic structure	Complex		Para-complex		Kähler		ParaKähler	
		Int	Non	Int	Non	Int	Non	Int	Non
$L^{\pm}(E^{p+1,q})$									
$p+1+q \neq 3$	Symmetric	∅	∅	∅	∅	∅	∅	∅	∅
$p+1+q=3$		1	∅	∅	∅	1	∅	∅	∅
$L^{+}(S^{p,q})$									
$p+q>3$	1	1	∅	∅	∅	1	∅	∅	∅
$(p,q)=(3,0)$	$\mathbb{R}$	2	∅	1	∅	$\mathbb{R},\mathbb{R}$	∅	∅	∅
$(p,q)=(1,2)$	$\mathbb{R}$	2	∅	1	∅	$\mathbb{R},\mathbb{R}$	∅	∅	∅
$(p,q)=(2,1)$	$\mathbb{R}$	2	∅	1	∅	$\mathbb{R}$	∅	$\mathbb{R}$	∅
$L^{-}(S^{p,q})$									
$p+q>3$	1	∅	∅	1	∅	∅	∅	1	∅
$(p,q)=(0,3)$	$\mathbb{R}$	2	∅	1	∅	$\mathbb{R}$	∅	$\mathbb{R}$	∅
$(p,q)=(2,1)$	$\mathbb{R}$	2	∅	1	∅	$\mathbb{R}$	∅	$\mathbb{R}$	∅
$(p,q)=(1,2)$	$\mathbb{R}$	2	∅	1	∅	∅	∅	$\mathbb{R},\mathbb{R}$	∅
$L(\mathbb{C}P^n)$	$\mathbb{R}$	2	2	∅	∅	2	2	∅	∅
$L(\mathbb{H}P^n)$	1	1	∅	∅	∅	1	∅	∅	∅
$L(\mathbb{O}P^2)$	1	1	∅	∅	∅	1	∅	∅	∅
$L(\mathbb{C}H^n)$	$\mathbb{R}$	∅	∅	1	1	∅	∅	1	1
$L(\mathbb{H}H^n)$	1	∅	∅	1	1	∅	∅	1	1

of unit vectors $S^{p,q} = \{x \in E,\ x^2 = 1\}$ is the orbit $S^{p,q} = SO(E)e_4$. It has the reductive decomposition

$$\mathfrak{so}(E) = \mathfrak{so}(E') + e_4 \wedge E',$$

where $\mathfrak{so}(E') \simeq \mathfrak{so}_{3,0}, \mathfrak{so}_{2,1}, \mathfrak{so}_{1,2}, \mathfrak{so}_{0,3}$, respectively. The metric of E induces an $SO(E')$-invariant metric in the tangent space $T_0 S^{p,q} = e_4 \wedge E'$. The stability subalgebra of the geodesic $\ell(e_4 \wedge e_3)$ through the origin $o = e_4 \in S^{p,q}$ in the direction $e_4 \wedge e_3 \in T_o S^{p,q} = e_4 \wedge E'$ is $\mathfrak{so}(V) \times \mathfrak{so}(V^{\perp})$, where $V = \mathrm{span}(e_1, e_2)$ and $V^{\perp} = \mathrm{span}(e_3, e_4)$. The geodesic is spacelike if $\epsilon = e_3^2 = 1$ and timelike if $\epsilon = e_3^2 = -1$. The space of such geodesics is identified with $L^{\epsilon}(S^{p,q}) = SO(E)/SO(V) \times SO(V^{\perp})$ (up to a covering).

We now denote by g_V the unique (up to a scale) metric in 2-dimensional $SO(V)$-module V and by ω_V unique (up to scale) 2-form. Then for some positive λ, the complex structure $J_V = \lambda g_V^{-1} \circ \omega_V$ is the unique (up to sign) $\mathfrak{so}(V)$-invariant complex structure if the metric g_V has signature (2, 0) or (0, 2). It is the unique (up to sign) paracomplex structure if the signature of g_V is (1, 1). We use similar notations for $V^{\perp}$. The reductive decomposition of the symmetric space $L^{\epsilon}(S^{p,q})$ is given by

$$\mathfrak{so}(E) = (\mathfrak{so}(V) \oplus (\mathfrak{so}(V^{\perp})) + V \wedge V^{\perp}.$$

The tangent space $T_o L^{\epsilon}(S^{p,q})$ is identified with $\mathfrak{m} := V \wedge V^{\perp} \simeq V \otimes V^{\perp}$ with the natural action of the isotropy algebra $\mathfrak{h} = \mathfrak{so}(V) + \mathfrak{so}(V^{\perp})$. We have therefore proved the following.

Proposition *The $\mathfrak{h}$-invariant symmetric bilinear form (respectively, invariant 2-form) on $\mathfrak{m}$ are of the form*

$$h_{\lambda,\mu} = \lambda\omega_V \otimes \omega_{V^\perp} + \mu g_V \otimes g_{V^\perp}, \quad \omega_{\lambda,\mu} = \lambda g_V \otimes \omega_{V^\perp} + \mu\omega_V \otimes g_{V^\perp}$$

for $\lambda, \mu \in \mathbb{R}$.

The endomorphisms $J := J_V \otimes 1$, $J^\perp := 1 \otimes J_{V^\perp}$, $J' := J \circ J^\perp = J_V \otimes J_{V^\perp}$ are invariant (para)complex structure in $\mathfrak{m}$. Moreover, $J.J^\perp$ are skew-symmetric with respect to any metric $h_{\lambda,\mu}$ and the endomorphism J' is symmetric.

Since $S^{p,q}$ is an irreducible symmetric space, any invariant tensor field is parallel. We obtain

Corollary *Any non-degenerate form $h_{\lambda,\mu}$ defines an invariant pseudo-Riemannian metric, any non-degenerate 2-form $\omega_{\lambda,\mu}$ defines an invariant symplectic form and the endomorphisms $J, J^\perp, J'$ define integrable (para)complex structures on $L^\epsilon(S^{p,q})$. Furthermore, any pair of the form $(h_{\lambda,\mu}, J)$ or $(h_{\lambda.\mu}, J^\perp)$, where $h_{\lambda,\mu}$ is as above , defines a (para)Kähler structure on $\mathfrak{m}$.*

3. In the following table we indicate the signature of the metrics g_V and $g_{V^\perp}$ for different values of p and ϵ and the type of the endomorphisms $J = J_V \otimes 1$, $J^\perp = 1 \otimes J_{V^\perp}$, $J' = J_V \otimes J_V \perp$ (cx = complex structure, para = paracomplex structure). Note also that there is

p	ϵ	g_V	$g_{V^\perp}$	J	$J^\perp$	J'
3	+	(2,0)	(2,0)	cx	cx	para
2	+	(1,1)	(2,0)	para	cx	cx
2	−	(2,0)	(1,1)	cx	para	cx
1	+	(0,2)	(2,0)	cx	cx	para
1	−	(1,1)	(1,1)	para	para	cx
0	−	(0,2)	(1,1)	cx	para	cx

a natural isomorphism $L^\pm(S^{p,q} = L^\mp(S_-^{q,p})$, where $S_-^{q,p} = \{x \in E^{q,p+1}, x^2 = -1\}$. We finally reproduce Table 1 in its corrected form.