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ERRATUM

3-Quasi-Sasakian manifolds

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Abstract We correct the results in section 6 of [B. Cappelletti Montano, A. De Nicola, G. Dileo, 3-Quasi-Sasakian manifolds, Ann. Global Anal. Geom. 33 (2008), 397–409], concerning the corrected energy of the Reeb distribution of a compact 3-quasi-Sasakian manifold. The results are slightly different than what was originally claimed and they are obtained by using results in [B. Cappelletti Montano, A. De Nicola, G. Dileo, The geometry of a 3-quasi-Sasakian manifold, Int. J. Math., to appear, arXiv:0801.1818], where the geometry of these manifolds is more deeply investigated.

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Introduction

In [2] the authors study the geometry of 3-quasi-Sasakian manifolds, which include as special cases 3-Sasakian and 3-cosymplectic manifolds. A 3-quasi-Sasakian manifold is an almost 3-contact metric manifold $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ such that each almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is quasi-Sasakian. It is proven in [2] that the distribution generated by the Reeb vector fields ξ_1, ξ_2, ξ_3 is integrable, defining a canonical totally geodesic and Rie-

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mannian foliation of M^{4n+3}. The characteristic vector fields obey the commutation relations $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ for any even permutation (α, β, γ) of $\{1, 2, 3\}$ and some $c \in \mathbb{R}$. Furthermore, the ranks of the 1-forms η_1, η_2, η_3 coincide, so that 3-quasi-Sasakian manifolds are classified according to their well-defined rank, which is of the form $4l + 1$ in the Abelian case ($c = 0$), and $4l + 3$ in the non-Abelian one, $0 \leq l \leq n$. As a single application, we compute in [2] the corrected energy of the canonical foliation of a compact 3-quasi-Sasakian manifold, in the attempt to generalize a result of Blair and Turgut Vanli concerning 3-Sasakian manifolds ([1]).

The corrected energy $\mathcal{D}(\mathcal{V})$ of a p-dimensional distribution of a compact Riemannian manifold was defined by Chacón and Naveira in [4]. They also proved that the Reeb distribution of the natural 3-Sasakian structure on the sphere S^{4n+3} is a minimum of the corrected energy in the set of all integrable 3-dimensional distributions. In [1] Blair and Turgut Vanli tried to extend this result to the Reeb distribution of an arbitrary compact 3-Sasakian manifold. Unfortunately, as it is remarked by Perrone in [5], their demonstration does not prove the minimality of the corrected energy. As for the corrected energy of the Reeb distribution in a 3-quasi-Sasakian manifold, our demonstration of minimality in [2] contains the same gap as in [1].

In this erratum, we distinguish between 3-quasi-Sasakian manifolds of rank $4l + 1$ and those of rank $4l + 3$. We use the results contained in [3], where the geometry of these manifolds is more deeply investigated. Indeed, a 3-quasi-Sasakian manifold of rank $4l + 1$ turns out to be a 3-cosymplectic manifold and in this case, supposing the manifold to be compact, the corrected energy of the Reeb distribution vanishes. As regards compact 3-quasi-Sasakian manifolds of rank $4l + 3$, we prove that the Reeb distribution represents a minimum for the corrected energy among a suitable subset of all integrable 3-dimensional distributions.

Corrected energy of 3-quasi Sasakian manifolds

The corrected energy $\mathcal{D}(\mathcal{V})$ of a p-dimensional distribution $\mathcal{V}$ on a compact oriented Riemannian manifold (M^m, g) is defined as (cf. [4])

$$\mathcal{D}(\mathcal{V}) = \int_M \left(\sum_{a=1}^{m} \|\nabla_{e_a}\xi\|^2 + q(q-2)\|\overrightarrow{H}_{\mathcal{H}}\|^2 + p^2\|\overrightarrow{H}_{\mathcal{V}}\|^2 \right) d\text{vol},$$

where $\{e_1, \dots, e_m\}$ is a local orthonormal adapted frame with $e_1, \dots, e_p \in \mathcal{V}_x$ and $e_{p+1}, \dots, e_{m=p+q} \in \mathcal{H}_x = \mathcal{V}_x^\perp$, and $\xi = e_1 \wedge \dots \wedge e_p$ is a p-vector which determines the distribution $\mathcal{V}$ regarded as a section of the Grassmann bundle $G(p, M^m)$ of oriented p-planes in the tangent spaces of M^m. Finally $\overrightarrow{H}_{\mathcal{H}}$ and $\overrightarrow{H}_{\mathcal{V}}$ are the mean curvatures of the distributions $\mathcal{H}$ and $\mathcal{V}$ (see [4] and [2] for the details). It is proven in [4] that if $\mathcal{V}$ is integrable then

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{i,\alpha} c_{i\alpha} d\text{vol}, \tag{1}$$

where $c_{i\alpha} = K(e_i, e_\alpha)$ is the sectional curvature of the plane spanned by $e_i \in \mathcal{H}$ and $e_\alpha \in \mathcal{V}$. Moreover, the equality in (1) holds if and only if $\mathcal{V}$ is totally geodesic and $e_1, \dots, e_p$ are $\mathcal{H}$-conformal, that is $(\mathcal{L}_{e_i} g)(X, Y) = f_i g(X, Y)$, for any $X, Y \in \mathcal{H}$ and $i \in \{1, \dots, p\}$, where $\mathcal{L}_{e_i}$ denotes the Lie derivative and f_i is a function on M.

Now, let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a compact 3-quasi Sasakian manifold and let ξ denote the Reeb distribution determined by the 3-vector $\xi_1 \wedge \xi_2 \wedge \xi_3$. It is proven in [2] that the corrected energy of ξ is given by

$$\mathcal{D}(\xi) = \int_M \left(\sum_{\alpha=1}^{3} \|\nabla \xi_\alpha\|^2 - \frac{3}{2}c^2 \right) d\text{vol}. \tag{2}$$

If M^{4n+3} is a 3-quasi-Sasakian manifold of rank $4l + 1$, then it is necessarily a 3-cosymplectic manifold (see [3]). Therefore, $\nabla \xi_\alpha = 0$ and $c = 0$. Using (2), it follows that the corrected energy $\mathcal{D}(\xi)$ vanishes.

Now, let us consider a 3-quasi-Sasakian manifold $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ of rank $4l + 3$, with $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$, $c \neq 0$. It is proven in [3] that M^{4n+3} is locally the Riemannian product of a 3-α-Sasakian manifold M^{4l+3}, where $\alpha = \frac{c}{2}$, and a hyper-Kähler manifold M^{4m}, with $m = n - l$. In particular, M^{4m} is a leaf of the distribution $\mathcal{E}^{4m} := \{X \in TM \mid \text{ for any } \alpha \in \{1, 2, 3\}\ i_X \eta_\alpha = 0 \text{ and } i_X d\eta_\alpha = 0\}$, while M^{4l+3} is a leaf of the orthogonal distribution $\mathcal{E}^{4l+3}$. Moreover, the Ricci tensor of M^{4n+3} is given by

$$\text{Ric}(X, Y) = \begin{cases} \frac{c^2}{2}(2l + 1)g(X, Y), & \text{if } X, Y \in \Gamma(\mathcal{E}^{4l+3}); \\ 0, & \text{elsewhere.} \end{cases} \tag{3}$$

We can prove the following.

Theorem 1 *Let M^{4n+3} be a compact 3-quasi-Sasakian manifold of rank $4l+3$. Then, among the integrable 3-dimensional distributions $\mathcal{V}$ of M^{4n+3} such that $\mathcal{V} \subset \mathcal{E}^{4l+3}$ and $K(\mathcal{V}) \leq \frac{3}{4}c^2$, the Reeb distribution ξ minimizes the corrected energy $\mathcal{D}(\mathcal{V})$, where $K(\mathcal{V}) := K(e_1, e_2) + K(e_1, e_3) + K(e_2, e_3)$ is the curvature of the distribution $\mathcal{V}$. Moreover $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\xi)$ if and only if $K(\mathcal{V}) = \frac{3}{4}c^2$, $\mathcal{V}$ is totally geodesic and e_1, e_2, e_3 are $\mathcal{H}$-conformal.*

Proof We compute the corrected energy $\mathcal{D}(\xi)$ of the canonical distribution given by (2). Since for a quasi-Sasakian structure $\|\nabla \xi_\alpha\|^2 = \text{Ric}(\xi_\alpha, \xi_\alpha)$, applying (3), we have $\mathcal{D}(\xi) = 3c^2 l \text{vol}(M^{4n+3})$. Now, let $\mathcal{V}$ be a 3-dimensional integrable distribution such that $\mathcal{V} \subset \mathcal{E}^{4l+3}$ and $K(\mathcal{V}) \leq \frac{3}{4}c^2$. We prove that $\mathcal{D}(\mathcal{V}) \geq \mathcal{D}(\xi)$. Let $\{e_1, \ldots, e_{4n+3}\}$ be a local orthonormal adapted frame with $e_1, e_2, e_3 \in \mathcal{V}$ and $e_4, \ldots, e_{4n+3} \in \mathcal{H} = \mathcal{V}^\perp$. Using (3) again, we get

$$\begin{aligned}
\sum_{\alpha=1}^{3} \sum_{i=1}^{4n} K(e_i, e_\alpha) &= \sum_{\alpha=1}^{3} \sum_{i=1}^{4n+3} K(e_i, e_\alpha) - \sum_{\alpha,\beta=1}^{3} K(e_\alpha, e_\beta) \\
&= \sum_{\alpha=1}^{3} \text{Ric}(e_\alpha, e_\alpha) - 2K(\mathcal{V}) \\
&= \frac{3}{2}c^2(2l + 1) - 2K(\mathcal{V}).
\end{aligned} \tag{4}$$

Arguing as in [5], $K(\mathcal{V})$ depends only on the distribution, in the sense that it is invariant under adapted orthonormal frame changes. Moreover, supposing $K(\mathcal{V}) \leq \frac{3}{4}c^2$ and applying (1), we have

$$\mathcal{D}(\mathcal{V}) \geq 3c^2 l \text{vol}(M^{4n+3}) = \mathcal{D}(\xi),$$

and the equality holds if and only if $K(\mathcal{V}) = \frac{3}{4}c^2$, $\mathcal{V}$ is totally geodesic and e_1, e_2, e_3 are $\mathcal{H}$-conformal. □

In the above theorem, if $l < n$, since the distribution $\mathcal{E}^{4l+3}$ defines a Riemannian foliation, then $(\mathcal{L}_{e_i} g)(X, Y) = 0$ for any $i \in \{1, 2, 3\}$ and $X, Y \in \mathcal{E}^{4m}$. Therefore, e_1, e_2, e_3

are $\mathcal{H}$-conformal if and only if the distribution $\mathcal{V}$ defines a Riemannian foliation. As for 3-quasi-Sasakian manifolds of maximal rank $4n+3$, they are necessarily 3-α-Sasakian manifolds, with $\alpha = \frac{c}{2}$ (see [3, Corollary 4.4]). Hence, we obtain the following.

Corollary 2 *Let M^{4n+3} be a compact* 3-α*-Sasakian manifold. Then, among the integrable* 3*-dimensional distributions $\mathcal{V}$ of M^{4n+3} with curvature $K(\mathcal{V}) \leq 3\alpha^2$, the Reeb distribution ξ minimizes the corrected energy $\mathcal{D}(\mathcal{V})$. Moreover $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\xi)$ if and only if $\mathcal{V}$ is totally geodesic, e_1, e_2, e_3 are $\mathcal{H}$-conformal and $K(\mathcal{V}) = 3\alpha^2$.*

The sphere $S^{4n+3}(r)$ of radius r can be canonically endowed with a 3-α-Sasakian structure $(\phi_\delta, \xi_\delta, \eta_\delta, g)$ with $\alpha = \frac{1}{r}$ ([3]) Since for any 3-dimensional distribution $\mathcal{V}$, $K(\mathcal{V}) = 3\alpha^2$, then the Reeb distribution ξ minimizes the corrected energy among the integrable 3-dimensional distributions of $S^{4n+3}(r)$.

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